A NOTE ON EXCHANGES IN MATROID BASES
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A Note on Exchanges in Matroid Bases

In a matroid with bases B and B', a B-exchange is a pair of elements e, e', where B - e + e' is a base. A serial exchange of B into B' is a sequence of pairs e₁, e'₁, for i = 1, ..., n, such that eᵢ, e'ᵢ, is a Bᵢ-₁-exchange, where B₀ = B, Bᵢ = Bᵢ-₁ - eᵢ + e'ᵢ, and Bₙ = B'. This paper shows there is a one-to-one correspondence between elements of B and B' such that corresponding elements e, e' give B-exchanges; furthermore, the pairs eᵢ, e'ᵢ can be sequenced to give a serial exchange of B into B'. A symmetric exchange is a pair of elements e, e' such that e, e' is a B-exchange and e', e is a B'-exchange. Any element of B can be symmetrically exchanged with at least one element of B'. But in contrast to B-exchanges, it is not always possible to make a correspondence between B and B' so corresponding elements give symmetric exchanges.
Unclassified

<table>
<thead>
<tr>
<th>Mutroids</th>
<th>Spanning Trees</th>
<th>Telecommunications</th>
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A NOTE ON EXCHANGES IN
MATROID BASES

by

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ABSTRACT

In a matroid with bases B and B', a B-exchange is a pair of elements e, e', where B - e + e' is a base. A serial exchange of B into B' is a sequence of pairs e_i, e_i', for i = 1, ..., n, such that e_i, e_i' is a B_i-1-exchange, where B_0 = B, B_1 = B_1-1 - e_i + e_i', and B_n = B'. This paper shows there is a one-to-one correspondence between elements of B and B' such that corresponding elements e, e' give B-exchanges; furthermore, the pairs e, e' can be sequenced to give a serial exchange of B into B'. A symmetric exchange is a pair of elements e, e' such that e, e' is a B-exchange and e', e is a B'-exchange. Any element of B can be symmetrically exchanged with at least one element of B'. But in contrast to B-exchanges, it is not always possible to make a correspondence between B and B' so corresponding elements give symmetric exchanges.
Many network and linear programming problems are solved by repeatedly exchanging elements of a base. The pivot step in linear programming is a general example. The existence of such exchanges can be taken as a defining property of a matroid [2]. This note presents results concerning several types of matroid base exchanges.

First we define three types of exchanges. Let \( M \) be a matroid, with bases \( B \) and \( B' \). For example, Figure shows the graphic matroid on four nodes. One base consists of the solid arcs 1, 2, 3; another base consists of the dotted arcs 4, 5, 6.

An ordered pair of elements \( e, e' \) is a \( B \)-exchange if \( B - \{e\} + \{e'\} \) is a base. Table 1 shows the possible \( B \)-exchanges for each element in \( B = \{1, 2, 3\} \).

A serial exchange of \( B \) into \( B' \) is a sequence of ordered pairs, \( e_1, e'_1; e_2, e'_2; \ldots; e_n, e'_n \), such that for all \( i \) in \( 1 \leq i \leq n \), a base is formed by the set

\[
B_i = B - \{e_1, \ldots, e_i\} + \{e'_1, \ldots, e'_i\}
\]

Furthermore, \( B_n = B' \). The definition implies each pair \( e_i, e'_i \) is a \( B_{i-1} \) exchange. Hence the sequence of exchanges can be executed serially. Figure 2 shows a serial exchange of the base \( \{1, 2, 3\} \) into \( \{4, 5, 6\} \).

A symmetric exchange is an ordered pair of elements \( e, e' \) such that the sets \( B - \{e\} + \{e'\} \) and \( B' - \{e'\} + \{e\} \) are bases. Equivalently, the pair \( e, e' \) is a \( B \)-exchange and \( e', e \) is a \( B' \)-exchange. Table 2 shows the possible symmetric exchanges for each element in \( B = \{1, 2, 3\} \).

To characterize these exchanges, we introduce notation for some well-known matroid concepts [2]. For a base \( B \) and an element \( f \notin B \), \( B(f) \) denotes the unique circuit in the set \( B + f \). In Figure 1, for base \( B = \{1, 2, 3\} \), \( B(5) = \{2, 3, 5\} \).
For a set of elements $D$, $sp(D)$ denotes the span of $D$. This set is defined as the smallest superset of $D$ such that for any element $f$, if $sp(D) + f$ contains a circuit containing $f$, then $f \in sp(D)$. In Figure 1, $sp \{4, 6\} = \{2, 4, 6\}$.

**Lemma 1:** For elements $e \in B$, $e' \notin B$, these conditions are equivalent:

1. $e$, $e'$ is a $B$-exchange
2. $e \in B(e')$.
3. $e' \notin sp(B-e)$.

**Proof:** An immediate consequence of the definitions.

**Corollary 1:** For elements $e \in B - B'$, $e' \in B' - B$, these conditions are equivalent:

1. $e$, $e'$ is a symmetric exchange.
2. $e' \in B'(e) - sp(B - e)$

It is apparent from the lemma that any element $e \in B$ gives a $B$-exchange with at least one element of $B'$. We show the same is true for symmetric exchanges.

**Theorem 1:** For any element $e \in B$, there is an element $e' \in B$ such that $e$, $e'$ is a symmetric exchange.

**Proof:** Consider any element $e \in B$. If $e \in B'$, then clearly $e$, $e'$ is a symmetric exchange. So assume $e \notin B'$.

Since $B$ is a base, element $e \notin sp(B-e)$. Thus the circuit $B'(e)$ is not contained in $sp(B - e) + e$, that is,

$$B'(e) = e \notin sp(B - e).$$

Now corollary 1 shows there is a symmetric exchange for $e$, completing the proof.
In Figure 1, we can pair the elements of $B = \{1, 2, 3\}$ and $\{4, 5, 6\}$ so each pair gives a $B$-exchange: $1, 6; 2, 5; 3, 4$. Figure 2 shows these pairs, in the given sequence, are a serial exchange of $\{1, 2, 3\}$ into $\{4, 5, 6\}$. Now we show such a pairing can be made in general.

**Theorem 2:** There is a one-to-one correspondence between elements of $B$ and $B'$, such that corresponding elements $e, e'$ give a $B$-exchange. Furthermore, the pairs $e, e'$ can be sequenced to give a serial exchange of $B$ into $B'$.

**Proof:** Denote the bases by

$$B = \{e_1, e_2, \ldots, e_n\}, \quad B' = \{e'_1, e'_2, \ldots, e'_n\}.$$  

We assert indices can be chosen in $B$ so for all $i$ in $1 \leq i \leq n$, the pair $e_i, e'_i$ is a $B$-exchange; furthermore, a base is formed by the set

$$B'_i = \{e_1, e_2, \ldots, e_i, e'_i, e'_{i+1}, e'_{i+2}, \ldots, e'_n\}.$$  

Note the assertion implies the theorem. For the pairs $e_i, e'_i$ give a correspondence of $B$-exchanges, and the sequence $e_n, e'_n; e_{n-1}; \ldots; e_1, e'_1$ is a serial exchange of $B$ into $B'$. The assertion is proved by induction on $i$. The initial step, $i = 0$, is obvious, since $B'_0 = B$ is a base. For the inductive step, suppose $B'_i$ is a base. We prove the assertion for $i + 1$, as follows. Element $e'_{i+1}$ of base $B'_i$ gives a symmetric exchange with some element of base $B$. This element cannot be $e_j$, for $j$ in $1 \leq j \leq i$, since $e_j \in B'_i$. With proper choice of indices, we can assume $e_{i+1}, e'_{i+1}$ is a symmetric exchange. Thus $e_{i+1}, e'_{i+1}$ is a $B$-exchange, and $B'_{i+1}$ is a base. This completes the induction.

The proof of Theorem 2 gives a constructive procedure for finding the one-to-one correspondence of $B$-exchanges. The theorem itself, specialized to graphic matroids, is useful in finding minimum weight spanning trees with specified degree at one node [1].
It is natural to try to generalize Theorem 2 to symmetric exchanges. However Table 2 shows it is not always possible to pair the elements of two bases so each pair is a symmetric exchange.
Figure 1. Bases \{1, 2, 3\} and \{4, 5, 6\} in a graphic matroid.

Table 1. B-\textit{ax}changes for e, B = \{1, 2, 3\}.

<table>
<thead>
<tr>
<th>e</th>
<th>e'</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4, 6</td>
</tr>
<tr>
<td>2</td>
<td>5, 6</td>
</tr>
<tr>
<td>3</td>
<td>4, 5, 6</td>
</tr>
</tbody>
</table>

Table 2. Symmetric exchanges for e.

<table>
<thead>
<tr>
<th>e</th>
<th>e'</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>4, 5, 6</td>
</tr>
</tbody>
</table>

Figure 2. Serial exchange of \{1, 2, 3\} into \{4, 5, 6\}.
REFERENCES
