RECURRANCE FORMULA FOR THE VENEZIANO MODEL N-POINT FUNCTIONS

Koichi Mano

Air Force Cambridge Research Laboratories
L. G. Hanscom Field, Massachusetts

1 May 1972
A recurrence formula is derived for a function which reduces to the Veneziano model \((n + 3)\)-point function. It is shown that the formula is equivalent to, but is more self-contained than, the Ilapinson and Plahte formula in that it does not require the prescription for the parameters involved.

KEYWORDS: Veneziano-type amplitudes
Recurrence Formula for the Veneziano Model $N$-Point Functions

Koichi Mano
Air Force Cambridge Research Laboratories, Bedford, Massachusetts 01730
(Received 1 May 1972)

A recurrence formula is derived for a function which reduces to the Veneziano model $(n+3)$-point function. It is shown that the formula is equivalent to, but is more self-contained than, the Hopkinson and Plate formula in that it does not require the prescription for the parameters involved.

The extension to the $n$-point functions of the Veneziano's four-point function was accomplished either by generalizing the integral representation for the beta function which comprises the essential ingredient of the four-point function or by generating a recurrence formula for the $n$-point function. In the latter approach, the authors attempted to justify the formula for arbitrary value of $n$ after showing, through introduction of the integral representation for the beta function, that the formula produces the already known integral expressions for the cases of $n = 5, 6,$ and 7. The recurrence formula which has apparently been discovered on a heuristic basis is not necessarily very transparent, as the authors themselves admit if, especially in connection with the definition of their variables $x_{ij}$.

Recently it was pointed out that the generalized Veneziano amplitudes may be regarded as the boundary values of a class of generalized hypergeometric functions that are Radon transforms of products of linear forms. In a work by the present author which shows that the amplitudes possess a structure similar to that of the Lauricella's hypergeometric functions, he has made an iterative use of a recurrence formula for arbitrary value of $n$ without requiring any prescription for the parameters involved therein.

To begin our discussion, let us consider a function $V_n$ of variables $r_{ij}$ defined as below:

$$V_n(\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{n-1}; \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}; \beta_0, \beta_1, \beta_2, \ldots, \beta_{n-1})$$

where the parameters $\alpha_{ij}$ are regarded as functions of the momenta $p_i$, $i = 1, 2, \ldots, n+3$, of the external particles, it will be seen that $V_n$ for $r_{ij} = 1$ can readily be related to the known integral representation for the $(n+3)$-point function.

In carrying out the multiple integrations in Eq. (1), use has been made in Ref. 6 of the following recurrence formula:

$$V_n(\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{n-1}; \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}; \beta_0, \beta_1, \beta_2, \ldots, \beta_{n-1})$$

$$= \sum_{\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{n-1}} (\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{n-1}) \cdot (\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1})$$

$$\times (\beta_0, \beta_1, \beta_2, \ldots, \beta_{n-1})$$

In Eq. (2) the summation is over the integers between 0 and $\sigma$ of the indexes $r_{ij}$, $\beta_{ij}$ stands for the beta function, and $\beta_{ij}$ are given by

$$\beta_{ij} = \sum_{(p_1, p_2, \ldots, p_n)} r_{ij}^{p_{ij}}$$

where

$$x_{ij} = -\sigma(s_{ij})$$

and $s_{ij}$ are defined according to certain rules (given in a tabular form) which will not be reproduced here. More noteworthy of the present formula is the fact that Eq. (2) is self-contained such that in contrast to Ref. 3, there is required no prescription for defining the parameters of the function $V_{n+1}$ which corresponds to $B_{n+1}$ of Eq. (4).

In order to achieve the above we have to establish the relation between our parameters $\alpha_{ij}$ and those of Ref. 3. For this purpose let us note first that the external lines are labeled $1, 2, \ldots, n+3$ both for the $(n+3)$-point function in Ref. 1 and for $V_n$ in the present paper, while they are labeled $0, 1, \ldots, n+2$ for the $(n+3)$-point function $B_{n+1}$ of Ref. 7. Further, we note that the integration variables $u_{1}, u_{2}, \ldots, u_{n+1}$ in Ref. 7 and the present paper may be made to correspond to $u_{1}, u_{2}, \ldots, u_{n+1}$ in Ref. 1 with this in mind one can compare Eq. (1) with the corresponding expression that follows from the representation for $B_{n+1}$ of Ref. 1 through rearrangement of the integrand.

Namely, by introducing $x_{ij} = -\sigma(s_{ij})$ from Ref. 1, we write $\sigma_{ij}$ for $\sigma(s_{ij})$ of Ref. 1, it becomes possible to express our $\alpha_{ij}$ in terms of $\sigma_{ij}$. If we further write $\xi_{ij} = -\sigma_{ij}$ with $\xi_{ij} = 0$ and

$$\xi_{ij} = \xi_{ij} - \xi_{i+1,j} - \xi_{i,j-1} + \xi_{i+1,j+1},$$

where $\xi_{ij}$ and $\xi_{ij}$ stand for $x_{ij}$ and $z_{ij}$, respectively, of Ref. 3, it follows that
\[ \alpha_{0i} = \xi_{1,i+1} \quad \text{for} \quad i = 1, 2, \ldots, n \quad (6) \]
\[ \alpha_{1i} = \xi_{1,i+2} \quad \text{for} \quad i = 1, 2, \ldots, n \quad (7) \]

That the integral in Eq. (1) reduces for \( n_{ij} = 1 \) to \( b_{n-1} \) of Ref. 7 can be seen from the observation that our \( p_{ji}, i = 1, 2, \ldots, n + 3 \) correspond to \( p_{ji}, i = 0, 1, \ldots, n + 2 \) of Ref. 7 and through specialization of the relation \( \alpha''(r_{n,j}) = \alpha''_{ij} + \alpha''_{i} \) of Ref. 1 to the form \( \alpha''_{ij} + n \) as is done in Ref. 7.

Although we have connected our parameters to those of Ref. 3 in Eqs. (6) and (7), the precise correspondence between Eqs. (2) and (4) cannot be considered complete until the arrangement of \( \xi_{ij} \) in \( B_{n}(x) \) of Eq. (4) is unambiguously established. In Ref. 3 this arrangement has been left out unstated, which fact is responsible in part for requiring the somewhat troublesome rules for determining \( x' \) in \( B_{n-1}(x') \) which should have really been unnecessary, as will be shown below.

Let us suppose that the correspondence between \( V_{n} \) in this paper and \( B_{n} \) for \( N = n + 3 \), of Ref. 3 is given by the following:

\[ V_{n}(\sigma_{01}, \ldots, \sigma_{0n}; \sigma_{11}, \ldots, \sigma_{1n}; \sigma_{21}, \ldots, \sigma_{2,n-1}; \ldots; \sigma_{n1}, \sigma_{n2}, \ldots, \sigma_{n,n-1}) \]
\[ \equiv B_{n}(\xi_{12}, \xi_{23}, \ldots, \xi_{n-1,n}; \xi_{13}, \xi_{14}, \xi_{25}, \ldots, \xi_{n-1,n+2}; \ldots; \xi_{1,n+1}, \xi_{2,n+2}). \quad (8) \]

Then the transition from \( V_{n} \) to \( B_{n} \), and vice versa can be effected on a firm basis by referring to Eqs. (6) and (7).

With the help of Eq. (8) we now can translate Eq. (2) into a formula which is given in terms of the function \( B_{n} \):

\[ B_{n}(\xi_{12}, \xi_{23}, \ldots, \xi_{n-1,n}; \xi_{13}, \xi_{24}, \ldots, \xi_{n-1,n+1}; \xi_{1,n+1}, \xi_{2,n+2}) \]

Note that we wrote \( B_{n} \) for \( B \) and used was made of the following relation in obtaining Eq. (9):

\[ B(\sigma_{0n}, \sigma_{01}, \sigma_{1n}, \alpha_{1n}) = B(\sigma_{0n}, \sigma_{01}, \alpha_{1n}, \beta_{n-1}). \]

That Eq. (9) is identical to Eq. (4) with \( N = n + 3 \), \( x \rightarrow \xi \), and \( z \rightarrow \xi \), can be checked easily. This establishes, therefore, that the order in which \( x_{ij} \) appears in \( B_{n}(x) \) of Eq. (4), which was not stated explicitly in Ref. 3, should be exactly as is displayed in \( B_{n+1}(x) \) of Eq. (8).

We emphasize that the recurrence formula for the \( (n + 3) \)-point function, Eq. (9), as derived from Eq. (2) is complete as it stands and requires no rules for defining the parameters of the function \( B_{n+2} \). In connection with the table for \( x_{ij} \) of \( B_{n+1}(x') \) in Ref. 3, we note that not all the entries in the table are actually needed for the recurrence formula. In fact, what is needed is only that portion of the table for \( i = 1, j < N - 2 \) and \( i \geq j + 1 \), because as may be seen from the arguments of \( B_{n+2} \) in Eq. (9), we require only \( \xi_{1j} \) for \( j < N - 2 = n + 1 \). Moreover, there arises no need for including in the table the relation \( x_{1j} = x_{1j}^{*} + \sum_{j=1}^{n-3} k_{i,j-1} \), for \( j = N - 2 = n + 1 \), unless we unnecessarily rewrite the argument \( x_{1j}^{*} + \sum_{j=1}^{n-3} k_{i,j-1} \) in \( B_{n} \). Finally, it is noted further that \( x_{1j} \) for \( j > 1 \) and \( j = N - 1 \) should have not been included in the table since no such variables are actually involved in the recurrence formula for \( B_{n}(x) \).