The Design of a Tunable Notch Filter
With Variable Q

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THE DESIGN OF A TUNABLE NOTCH FILTER WITH VARIABLE Q

In electro-mechanical servo systems one often encounters undesired resonances which contribute to the system instability. One method used to combat such a resonance is to place a notch filter in the feedback loop to reduce gain at the resonance frequency, while allowing frequencies outside the notch to pass without appreciable phase shift or attenuation. The notch filter described in this paper is particularly useful in this application since it can be tuned to the resonant frequency while in the circuit, the Q can be adjusted to cover the resonance, and the notch attenuation can be adjusted for essentially zero transmission at the center frequency.

The filter is of simple design and easily constructed. The most critical restriction is that the operational amplifiers used in the circuit be stable at unity gain. However, this is not particularly difficult to realize.
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ABSTRACT

In electro-mechanical servo systems one often encounters undesired resonances which contribute to the system instability. One method used to combat such a resonance is to place a notch filter in the feedback loop to reduce gain at the resonance frequency, while allowing frequencies outside the notch to pass without appreciable phase shift or attenuation. The notch filter described in this paper is particularly useful in this application since it can be tuned to the resonant frequency while in the circuit, the Q can be adjusted to cover the resonance, and the notch attenuation can be adjusted for essentially zero transmission at the center frequency.

The filter is of simple design and easily constructed. The most critical restriction is that the operational amplifiers used in the circuit be stable at unity gain. However, this is not particularly difficult to realize.

PROBLEM STATUS

This is a final report on one phase of a problem, already closed.

AUTHORIZATION

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INTRODUCTION

In electro-mechanical servo systems one often encounters undesired resonances which contribute to the system instability. One method used to combat such a resonance is to place a notch filter in the feedback loop to reduce gain at the resonance frequency, while allowing frequencies outside the notch to pass without appreciable phase shift or attenuation. The notch filter described in this paper is particularly useful in this application since it can be tuned to the resonant frequency while in the circuit, the Q can be adjusted to cover the resonance, and the notch attenuation can be adjusted for essentially zero transmission at the center frequency.

The filter is of simple design and easily constructed. The most critical restriction is that the operational amplifiers used in the circuit be stable at unity gain. However, this is not particularly difficult to realize.

BASIC THEORY

In Figure 1, the (+) input is the non-inverting input to the operational amplifier; the (-) input, the inverting input. In the following discussion it will be assumed that the operational amplifiers used are "ideal"\(^{(1)}\); i.e.

1. \(Z_{in} = \infty\)
2. \(Z_{out} = 0\)
3. Gain = \(\infty\)
4. Zero voltage difference between the two input terminals gives a zero output, (infinite common mode rejection and zero offset voltage).

Typical operational amplifiers have \(Z_{in}\) of \(10^6\) ohms, \(Z_{out}\) of less than \(5 \times 10^3\) ohms, gains greater than 80 db, common mode rejection of greater than 80 db, and offset voltages less than 1 millivolt. Such operational amplifiers cause deviation of only a few percent from the results calculated using ideal models.

Referring to Figure 1, and measuring the voltages with respect to ground, the output of the filter is given by

1
For the bridged-T network, it can be shown (see appendix) that for low source impedance,

\[ e_1 = e_2 \left( \frac{1}{6} \left( 5 + e^{i\alpha} \right) \right) \quad (2a) \]

where

\[ \cos \alpha = \frac{x^2 - 9}{x^2 + 9} \]
\[ \sin \alpha = \frac{6x}{x^2 + 9} \]

and

\[ x = \frac{w}{w_0} - \frac{w_0}{w} \quad (2b) \]

\[ w_0 = 1/RC, \ w = \text{input frequency}. \]

From (2a) it may be seen that

\[ e_2 - e_1 = e_2 \left( \frac{1}{6} \left( 1 - e^{i\alpha} \right) \right) \quad (2c) \]

Inserting (2a, c) into (1),

\[ e_{\text{out}} = -\frac{R_5}{R} (e_2 - e_1) + \frac{R_4 + R_5}{R_4} \left[ e_2 \left( \frac{1}{6} \left( 1 + e^{i\alpha} \right) \right) \right] \quad (3a) \]

When \( e_{\text{out}} = 0 \) at \( e^{i\alpha} = -1 \), using (3a),

\[ \frac{R_5}{R_3} = 2 \left( \frac{R_4 + R_5}{R_4} \right) \quad (3b) \]

With (3b) and (2a), (1) reduces to

\[ e_{\text{out}} = \frac{R_4 + R_5}{R_4} \frac{e_2}{2} \left[ 1 + e^{i\alpha} \right] \quad (4) \]
For the first amplifier the output is

$$e_2 = - \left[ \frac{R_2}{R_1} e_{in} + \frac{R_2}{R_6} e_{out} \right]. \quad (5)$$

Finally, combining (4) and (5),

$$\frac{e_{out}}{e_{in}} = - \left[ \frac{R_2}{R_1} \right] \left[ \frac{R_4 + R_5}{R_4} \right] \left[ \frac{1 + e^{i\alpha}}{2 + \frac{R_2}{R_6} \left( \frac{R_4 + R_5}{R_4} \right) \left( 1 + e^{i\alpha} \right)} \right]. \quad (6)$$

To remove the imaginary terms from the denominator and convert (6) to a more usable form, let

$$B = \frac{R_2}{R_6} \left( \frac{R_4 + R_5}{R_4} \right) \quad (7a)$$

and

$$\phi = \tan^{-1} \frac{\sin \alpha}{\left( 1 + B \right) (1 + \omega \alpha)}$$

$$\quad = \tan^{-1} \left( \frac{-3}{X(1 + B)} \right) \quad (7b)$$

so that

$$\frac{1 + e^{i\alpha}}{2 + B + Be^{i\alpha}} = \left[ \frac{1 + \omega \alpha}{B^2(1 + \omega \alpha) + 2B(1 + \omega \alpha) + 2} \right]^{\frac{1}{2}} e^{i\phi}. \quad (7c)$$

Then, finally the output vs the input of the filter is

$$\frac{e_{out}}{e_{in}} = - \frac{R_6}{R_1} B \left[ \frac{1 + \omega \alpha}{B^2(1 + \omega \alpha) + 2B(1 + \omega \alpha) + 2} \right]^{\frac{1}{2}} e^{i\phi}. \quad (8)$$
To define the characteristics of the filter better, the theoretical gain and $Q$ must be established. Equation (2b) shows that as $\omega$ gets much greater or much smaller than $\omega_0$, $x$ approaches $+\infty$ or $-\infty$ respectively. For either value, $\omega \varphi$ approaches 1 while $\varphi$ approaches zero. Setting the gain of the filter equal to the gain in these limits, and applying the limits to (8),

$$A = \lim_{\omega \to 0} \frac{e_{out}}{e_{in}} = \left| -\frac{R_6}{R_1} \frac{B}{B + 1} \right|. \quad (9)$$

Since at the notch frequency the filter output is zero, I now chose to define a $\tilde{Q}$ in terms of the bandwidth at the points where

$$\left| \frac{e_{out}}{e_{in}} \right| = A/2:$$

$$\tilde{Q} = \frac{\omega_+}{\omega_+ - \omega_-}. \quad (10)$$

$\omega_+$ is the frequency greater than $\omega_0$ such that $\left| \frac{e_{out}}{e_{in}} \right| = A/2$; $\omega_-$ is the frequency less than $\omega_0$ such that $\left| \frac{e_{out}}{e_{in}} \right| = A/2$.

Solving (8) for these 6 db points,

$$\frac{\omega_+ - \omega_0}{\omega_0} = \pm \left[ \frac{3}{4(B+1)^2} \right]^{1/2} \left[ \sum_{n=1}^{\infty} \frac{(2n-3)!!}{(2n)!!} \left( \frac{-3}{4(B+1)^2} \right)^n \right]. \quad (11a)$$
and
\[ Q = \frac{B + 1}{\sqrt{3}} \] \hspace{1cm} (11b)

where

\[ (2n-3)!! \equiv 1.3.5.7 \ldots \ldots (2n-3) \]
\[ (2n)!! \equiv 2.4.6 \ldots \ldots (2n) \]

and for
\[ \sum_{n=1}^{\infty} \frac{(2n-3)!!}{(2n)!!} \left( \frac{-3}{4(B+1)^2} \right)^n = 1 - \left( 1 + \frac{3}{4(B+1)^2} \right)^{\frac{1}{2}} = E. \]

\[ E \] is a measure of the lack of symmetry of the 6 db points about the notch frequency; it represents a 33% shift in \((w_n - w_0)\) for \(B = 0\), but only about 4% for \(B = 2\). Therefore, for \(B\) greater than 2, the notch is essentially symmetrical in nature.

The phase shift at \(w_n\) is \(\mp 90^\circ\); at \(w = 0, 180^\circ\); at \(w >> w_n, 180^\circ\). Setting
\[ x_\pm = \left( \frac{w_+}{w_0} - \frac{w_-}{w_0} \right) = \pm \frac{\sqrt{3}}{B+1} \]
and using (7b),
\[ \phi_\pm = \tan^{-1} \sqrt{3} = \pm 60^\circ. \]

Thus for all \(B, \phi_\pm = 60^\circ\) and the overall phase shift is \(180^\circ \pm 60^\circ\). Thus the overall gain should be less than 2 to insure stability of the circuit.

If \(180^\circ\) of phase shift is unacceptable in the circuit and an odd number of filters is to be used, the circuit in Figure 1 may be replaced by circuit in Figure 2. For this circuit, equation 5 becomes
\[ e_2 = \left[ \frac{R_6 + R_2}{R_6} \frac{R_8}{R_7 + R_8} e_{in} + \frac{R_2}{R_6} e_{out} \right] \] \hspace{1cm} (12)
Equation (12) may be rewritten as

\[ e_2 = -\left[ \frac{R_2}{R_1} e_{\text{in}} + \frac{R_2}{R_6} e_{\text{out}} \right] \]

where

\[ R'_1 = -\frac{R_6 (R_7 + R_8)}{R_8 (R_2 + R_6) R_2}. \]

With this modification equation (8) becomes

\[ \frac{e_{\text{out}}}{e_{\text{in}}} = \frac{R_8 (R_2 + R_6)}{R_6 (R_7 + R_8)} B \left[ \frac{1 + \omega \Omega}{B^2 (1 + \Omega) + 2B (1 + \Omega) + 2} \right]^{1/2} e^{i \phi}, \quad (13) \]

introducing the desired 180° phase shift. The gain at \( w \rightarrow 0, \Omega \) now becomes

\[ A_{\text{non-invert}} = \lim_{w \rightarrow \infty} \frac{e_{\text{out}}}{e_{\text{in}}} = \frac{R_8 (R_2 + R_6)}{R_6 (R_7 + R_8)} \frac{B}{B + 1}. \quad (14) \]

All the rest of the argument remains identical.

**SUMMARY**

1. Notch frequency = \( w_n = 1/RC \)
2a. Gain = \( A_1 = -\frac{R_6}{R_1} \frac{B}{B + 1} \) (inverting)
2b. Gain = \( A_{NI} = \frac{R_7 (R_2 + R_6)}{R_2 (R_7 + R_8)} \) (non-inverting)
3. The 50% gain points occur at $w_{\pm}$, where

$$w_{\pm} = w_0 \pm \frac{w_0}{2} \left[ 13 \sum_{n=1}^{\infty} \frac{[2n-3]!!}{(2n)!!} \frac{(-3)^n}{4(B+1)^n} \right].$$

4. The condition for null at the notch frequency is

$$\frac{1}{R_3} = 2 \left( \frac{1}{R_4} + \frac{1}{R_5} \right).$$

5a. The notch frequency is set by $R$
b. The $Q$ is set by $R_6$
c. Inverting gain is set by $R_1$
d. Non-inverting gain is set by $R_7$ and $R_8$.

6. $Q = \frac{w_0}{w_+ - w_-} = \frac{B+1}{\sqrt{3}}$

DESIGN PROCEDURE

1. Choose the desired frequency range, and select suitable operational amplifiers.
2. Choose, $R_2$, $R_4$, and $R_5$ compatible with the op amps being used and the circuit being driven. If more drive is needed, a buffer stage may be added.
3. From equation 10 and 11b, and using the center frequency $w_0$ and the average 6 db width desired, $B$ may be calculated:

$$B = \frac{\sqrt{3}}{w_+ - w_-} - 1, \quad Q = \frac{B + 1}{\sqrt{3}}.$$

4. Using the values in 2 and 3 and knowing the desired gain (which should be less than 2), the equation for $R_3$, $R_6$, and $R_1$ are:
\[ R_3 = \frac{\frac{R_4 R_5}{R_4 + R_5}}{\frac{1}{2}} \quad \text{(from (3b))} \]

\[ R_6 = \frac{R_2}{B} \frac{R_4 + R_5}{R_4} \quad \text{(from (7a))} \]

\[ R_1 = \frac{R_6}{A} \frac{B}{B + 1} \quad \text{(from (9))} \]

5. Compute RC for the desired frequency. Since ganged pots track about \( \pm 5\% \), it is best to use a fixed R which gives the maximum frequency desired and a dual pot which when added to the fixed R gives the minimum frequency desired. The capacitors should be of the low drift mica type.

6. In order to compensate for resistor tolerances and to set the notch center frequency output to zero, about \( 1/4 \) of \( R_3 \) should be obtained using a variable pot of value \( R_3/2 \).

7. Making \( R_6 \) variable allows one to adjust the Q of the notch filter. To compensate for the effect of \( R_6 \) on the overall gain, \( R_1 \) should be made variable also.

**DESIGN EXAMPLES**

1. The amplifiers to be used are Philbrick type K2-W with \( \pm 100 \) volts output at \( \pm 1 \) mA, \( f_0 = 100 \) cps, gain of 2, input signal \( < \pm 10 \) volts.

2. Set \( R_2 = 30 \) K

\[ R_4 = 50 \text{ K} \]

\[ R_5 = 200 \text{ K} \]

3. For 6 db points at \( f_0 \pm 10 \) cps,

\[ B = \frac{\sqrt{3} w_+}{w_+ - w_-} - 1 = \sqrt{3} \frac{100}{20} - 1 = 7.66; \quad Q = 5.00 \]

4. \( R_3 = \frac{1}{2} \frac{R_4 R_5}{R_4 + R_5} = \frac{1}{2} \frac{10,000}{250} \text{ K} = 20 \text{ K} \) (use a 25 K pot)
\[
\frac{R_6 - \frac{R_2}{2}}{\frac{R_4 + R_5}{R_4}} = \frac{1}{7.66} \times \frac{30}{50} = 250 \text{ K} = 19.58 \text{ K}
\]

\[
R_1 = \frac{19.58 \text{ K}}{2} \times \frac{7.66}{8.66} = 9.66 \text{ K}
\]

5. \[C = 0.09 \mu\text{fd (measured)}\]

\[R = \frac{107}{180} = 17.7 \text{ K (use a 25 K ganged pot.)}\]

**B.1. Experimental Results**

1. At \(+R = \text{Max} \approx 25 \text{ K}, f_0 = 71 \text{ Hz}\)
2. At \(+R \approx 4.2 \text{ K}, f_0 \approx 400 \text{ Hz}\).
3. For \(R < 4 \text{ K}, \) the ganged pots did not track well, and \(Q\) and notch attenuation began to fall off.

**2. For** \(f_0 = 100 \text{ Hz}, R \approx 17.5 \text{ K}\)

- For \(\text{Gain} = 2.0\) @ 10 Hz and 1 KHz, \(R_1 \approx 8.4 \text{ K}\).
- For notch attenuation > 50 db, \(R_3 \approx 20 \text{ K}\)
- For \(\text{Gain} = 1, f \approx 90 \text{ Hz} \) and 112 Hz; \(Q = 4.6\).

**II.A.1. Two notch filters are to be built using integrated circuits. The amplifiers to be used are Fairchild \(\mu\text{A739}'s. The filters are to be unity gain with frequencies of 96 Hz respectively.**

2. Set \(R_2 = 30 \text{ K}\)

\[R_4 = 51 \text{ K}\]

\[R_5 = 100 \text{ K}\]

3. For the first filter, the 6 db points are at \(\approx \pm 6 \text{ Hz};\) for the second, \(\approx \pm 11 \text{ Hz}\):

\[B_1 = \sqrt{3} \frac{96}{12} - 1 = 12.85\]

\[B_2 = \sqrt{3} \frac{187}{22} - 1 = 13.72\]

Use \(B = 13.3, \bar{Q} = 8.3\).
4. \( R_3 = \frac{1}{2} \cdot \frac{5100}{151} = 16.9 \text{ K} \)
\( R_6 = \frac{30 \text{ K}}{13.3} \cdot \frac{151}{51} = 6.7 \text{ K} \)
\( R_1 = (6.6 \text{ K}) \cdot \frac{13.3}{14.3} = 6.1 \text{ K} \)

5. \( C = .01 \mu \text{fd} \)
\( R(96 \text{ Hz}) = 166 \text{ K} \)
\( R(187 \text{ Hz}) = 85.2 \text{ K} \).

B. Experimental Results

1. Figure 3 shows the final circuit; Figure 4 shows the gain vs the frequency.

2. For \( f_o = 96 \text{ Hz} \)
   
   For gain \( = 1 @ 10 & 100 \text{ Hz} \), \( R_1 = 6.13 \text{ K} \)
   
   For notch attenuation \( > 44 \text{ db} \), \( R_3 = 16.5 \text{ K} \)
   
   Gain = \( \frac{1}{2} @ f_- = 90.5 \text{ Hz} \) and \( f_+ = 101.1 \text{ Hz} \), \( R_6 = 6.2 \text{ K} \), \( Q = 9.06 \)
   
   For \( f_o = 187 \text{ Hz} \)
   
   Gain = \( 1 @ 10 & 100 \text{ Hz} \), \( R_1 = 5.8 \text{ K} \)
   
   Notch attenuation \( > 43 \text{ db} \) \( R_3 = 16.5 \text{ K} \)
   
   Gain = \( \frac{1}{2} @ f_- = 176 \text{ Hz} \) and \( f_+ = 199.5 \text{ Hz} \), \( R_6 = 7.2 \text{ K} \), \( Q = 7.96 \)

3. Figure 5 shows the maximum stable \( Q \) vs the \( Q \) for \( R_6 = \infty \).

4. Figure 6 shows the range of \( Q \) available using a 20 K ohm potentiometer for \( R_6 \).

5. Figure 7 shows picture, completed dual filter.
REFERENCES

Fig. 1 - Schematic diagram of inverting notch filter.
Fig. 2 - Schematic diagram of non-inverting notch filter.
Fig. 3 - Completed dual-notch filter with notch frequencies at 96 and 187 Hz respectively.
Fig. 4 - Typical gain curve of dual notch filter.
Fig. 5: Maximum $Q$ of design example vs bridged-T with no feedback loop.

- $Q = 22.2$ (Curve for minimum stable $R_b$)
- $Q = 0.58$ (Theoretical curve for $R_b = \infty$)

Gain, dB
Fig. 6 - Typical $Q$ values available from design example.
Fig. 7 - Picture, completed dual filter.
APPENDIX

I. Output of a bridged T-Network (see Figure A) for low driving impedance and high load impedance:

1. Assuming \( Z_L = \infty \)

\[ Z_S = 0 \]

and let \( u_b = 1/RC \)

\[ A = -i \frac{u_b}{w} \]

Then it can be shown that

\[ \frac{e_2}{e_{in}} = \frac{A^2 + 2A}{A^2 + 3A + 1} \]

\[ \frac{e_1}{e_{in}} = \frac{1}{A^2 + 3A + 1} \]

\[ \frac{e_{out}}{e_{in}} = \frac{e_1 + e_2}{e_{in}} = \frac{A^2 + 2A}{A^2 + 3A + 1} \]

2. Thus

\[ \frac{e_{out}}{e_{in}} = \frac{(\frac{u_b}{w})^2 - 2i(\frac{u_b}{w}) + 1}{(\frac{u_b}{w})^2 - 3i(\frac{u_b}{w}) + 1} \]

\[ = \frac{[w - \frac{u_b}{w} - 2i][w - \frac{u_b}{w} + 3i]}{(w - \frac{u_b}{w})^2 + 9} \]

\[ = \frac{(w - \frac{u_b}{w})^2 + 6}{(w - \frac{u_b}{w})^2 + 9} + i \frac{(w - \frac{u_b}{w})}{(w + \frac{u_b}{w})^2 + 9} \]
3. Letting \( X = \left( \frac{w}{w_b} - \frac{w}{w} \right) \)

\[
\frac{e_{\text{out}}}{e_{\text{in}}} = \frac{x^2 + 6}{x^2 + 9} + i \frac{x}{x^2 + 9}
\]

\[
= \frac{5}{6} + \frac{1}{6} \left[ \frac{x^2 - 9}{x^2 + 9} + i \frac{6x}{x^2 + 9} \right]
\]

or

\[
\frac{e_{\text{out}}}{e_{\text{in}}} = \frac{5}{6} + \frac{1}{6} e^{i \alpha}
\]

where

\[
\alpha = \alpha + i \sin \alpha = \frac{x^2 - 9}{x^2 + 9} + i \frac{6x}{x^2 + 9}
\]
Fig. A – Bridged T-network.