STABILITY OF DIFFERENCE APPROXIMATIONS TO
DIFFERENTIAL EQUATIONS

by

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Consider the differential equation \((1) \dot{x} = f(x)\) in a Banach space and let \(x^*\) be an equilibrium. The basic question treated is the following: if \(x^*\) is asymptotically stable and if \((2) x_{k+1} = x_k + h \varphi(x_k, h)\) is a one-step method, with fixed step size \(h\), for integrating \((1)\), then does the sequence \(x_k\) converge to \(x^*\)? It is shown that uniform asymptotic stability of \((1)\) implies stability of \((2)\) and that exponential asymptotic stability of \((1)\) implies asymptotic stability of \((2)\).
Abstract

Consider the differential equation (1) $x = f(x)$ in a Banach space and let $x^*$ be an equilibrium. The basic question treated is the following: if $x^*$ is asymptotically stable and if (2) $x_{k+1} = x_k + h\Phi(x_k, h)$ is a one-step method, with fixed step size $h$, for integrating (1), then does the sequence $x_k$ converge to $x^*$? It is shown that uniform asymptotic stability of (1) implies stability of (2) and that exponential asymptotic stability of (1) implies asymptotic stability of (2).
1. Introduction.

Consider the differential equation

\[(1.1) \quad \dot{x} = f(x)\]

and let \(x^*\) be an equilibrium point. The basic question to be treated here is the following: if \(x^*\) is an asymptotically stable equilibrium and if

\[(1.2) \quad x_{k+1} = x_k + hf(x_k, h)\]

is a one step method, with fixed step size \(h\), for integrating (1.1), then does \(x_k\) converge to \(x^*\) as \(k\) tends to infinity? We shall show in our first main theorem that uniform asymptotic stability of (1.1) implies stability of (1.2) and in our second main theorem that exponential asymptotic stability of (1.1) implies asymptotic stability of (1.2) (improving a result of Skalkina [11]).

Our interest in the problem considered here stemmed from an investigation of iterative methods for solving the equation \(F(x) = 0\) in a Banach space. If \(f(x)\) is a function whose zeros include the zeros of \(F\) (for example, \(f(x) = -(F(x)^{-1})F(x)\)), then numerical integration of (1.1) will lead to iterates \(x_k\) corresponding to points \(x(t_k; x_0)\) on the solution curve. If the initial point \(x_0\) is in a region of attraction of the equilibrium \(x^*\), then under what condition does \(x_k\) converge to \(x^*\)? Various
authors have used similar ideas to develop algorithms for solving $F(x) = 0$ ([1,2,3,5,10]) in particular situations. For example, Boggs ([1]) has integrated the equation $\dot{x} = -(F'_x)^{-1} F(x)$ with the A-stable methods of Dahlquist to generate iterates $x_k$ which converge to a root of $F$. In [2,3,10], Euler and Runge-Kutta integration methods are used to generate iterates $x_k$ which eventually lie within the region of convergence of Newton's method. Here, results are developed for general one step methods.

2. Uniform Asymptotic Stability.

Let $X$ be a real Banach space with norm, $\|\cdot\|$, and let $S(r) = \{x : \|x\| < r\}$ be the closed ball of radius $r$ about $0$ in $X$. We let $f$ be a mapping of $X$ into itself and $x^*$ be a zero of $f$. We assume, without loss of generality, that $x^* = 0$. Now, suppose that $f$ is defined on the ball $S(R)$ and that $\varphi(x,h)$ is a mapping of $S(R) \times [0,h_0]$ into $X$. We assume throughout the sequel that the following conditions are satisfied:

(2.1) there are positive constants $L$ and $L'$ such that $\|f(x) - f(y)\| \leq L \|x-y\|$ and $\|\varphi(x,h) - \varphi(y,h)\| \leq L' \|x-y\|$ for all $x,y \in S(R)$ and $0 \leq h \leq h_0$;

(2.2) $\varphi(x,h)$ is uniformly continuous on $S(R) \times [0,h_0]$;

(2.3) $\varphi(x,0) = f(x)$ for all $x \in S(R)$; and

(2.4) $f(0) = \varphi(0,h) = 0$ for all $h \in [0,h_0]$. 
We consider the differential equation

\begin{equation}
\dot{x} = f(x)
\end{equation}

and the one step integration method

\begin{equation}
x_{k+1} = x_k + h f(x_k, h).
\end{equation}

[Note that (2.6) is consistent in view of the assumption (2.3)]. We now have

DEFINITION 2.7. The solution \( x = 0 \) of (2.5) is uniformly stable if, given \( e > 0 \), there is a \( \delta(e) > 0 \) such that \( \|x_0\| < \delta(e) \) implies that \( \|x(t; t_0, x_0)\| < e \) for \( t \geq t_0 \), where \( x(t; t_0, x_0) \) is the solution of (2.5) with \( x(t_0; x_0) = x_0 \). The solution \( x = 0 \) of (2.5) is uniformly asymptotically stable on a ball \( S(r) \) if it is uniformly stable and if, given \( e > 0 \), there is a \( T(e) > 0 \) such that \( \|x_0\| \leq r \) implies that \( \|x(t; t_0, x_0)\| < e \) for \( t > t_0 + T(e) \).

We note that since \( X \) may be infinite dimensional, uniform asymptotic stability and asymptotic stability are not equivalent ([9]).

We now assume that the solution \( x = 0 \) of (2.5) is uniformly asymptotically stable on the ball \( S(R) \) for some \( R > 0 \). If \( \delta(e) \) and \( T(e) \) are the functions characterizing the stability of (2.5) as in definition 2.7, then we may assume that \( \delta(\cdot) \) and \( T(\cdot) \) are strictly monotonic continuous functions (see [7, p. 309]). We also suppose for simplicity that \( t_0 = 0 \) and we let \( x(t; x_0) = x(t; 0, x_0) \). We then have:
LEMMA 2.8. Let \( r, b \) be real numbers such that \( 0 < r < b \leq 5(R) \). Then there is a \( t_1 > 0 \) such that \( \inf \{ \| x(t; x_0) \| \mid t \in [0, t_1], r \leq \| x_0 \| \leq b \} \) is strictly positive.

**Proof:** Since \( b \leq 5(R) \), \( \| x(t; x_0) \| \leq R \) and so
\[
\| x(t; x_0) - x_0 \| = \| \int_0^t \{ f(x(s)) - f(x_0) \} ds + tf(x_0) \| \leq L \int_0^t \| x(s) - x_0 \| ds + t \| f(x_0) \| \quad \text{(where } x(\cdot) = x(\cdot; x_0)) \).
\]
It follows from Gronwall's inequality and an integration by parts that

\[
\| x(t) - x_0 \| \leq \| x_0 \| t e^{Lt}
\]

Therefore, \( \| x(t_1; x_0) \| \geq \| x_0 \| \left( 1 - Le^{Lt} \right) \) and we may choose \( t_1 > 0 \) such that \( 1 - Lt_1 e^{Lt} > 0 \).

Following Massera ([9]), we let \( G(\cdot) \) be a continuous strictly increasing function with \( G(r) \leq 2r \), \( G(0) = 0 \) and we introduce the Lyapunov function \( V(\cdot) \) for (2.5) given by

\[
(2.10) \quad V(x_0) = \sup \{ G(\| x(t; x_0) \|) \frac{(1+2t)/(1+t)} \mid t \geq 0 \}
\]

for \( 0 \leq \| x_0 \| \leq \rho \) where \( \rho = \min(1, 5(R)) \).

LEMMA 2.11. ([9]). \( V(\cdot) \) has the following properties: (i) \( G(\| x_0 \|) \leq V(x_0) \leq 2\| x_0 \| \); (ii) \( \| V(x_0) - V(y_0) \| \leq M\| x_0 - y_0 \| \) for some \( M > 0 \); (iii)
\[
\dot{V}(x_0) = \limsup_{k \to 0^+} [V(x(k; x_0)) - V(x_0)] / k \leq -G(\| x_0 \|)(1 + 2\| x_0 \|/2)^{-2}; \quad \text{and,}
\]

(iv) \( V(x(k; x_0)) - V(x_0) \leq -G(\| x_0 \|)(1 + 2k + 2\| x(k,x_0) \|/2)^{-2} \) for
\[ \|x_0\|, \|y_0\| \leq \rho \quad \text{and} \quad k > 0. \]

Letting \( \psi(\|x_0\|) = G(\|x_0\|)(1+2T(\|x_0\|/2))^{-2} \), we have:

**Lemma 2.12.** If \( 0 < r < 5(\rho) \) and \( \epsilon > 0 \), then there is a \( k(r, \epsilon) > 0 \) such that

\[
V(x(k; x_0)) - V(x_0) < k( -\psi(\|x_0\|) + \epsilon G(\|x_0\|))
\]

for \( 0 \leq k < k(r, \epsilon) \), \( r \leq \|x_0\| \leq 8(\rho) \).

**Proof:** Choose \( t_1 > 0 \) by lemma 2.8 so that \( m = \inf(\|x(t; x_0)\|) \quad \text{for} \quad 0 \leq t \leq t_1 \), \( r \leq \|x_0\| \leq 8(\rho) \). Then \( 0 < m \leq \|x(t; x_0)\| \leq \rho \) for \( 0 \leq t \leq t_1 \) and \( r \leq \|x_0\| \leq 8(\rho) \).

Since \( A(k, \sigma) = (1 + 2k + 2T(\sigma))^{-2} \) is uniformly continuous on \([0, t_1] \times [m/2, \rho/2]\), there is an \( \eta = \eta(\epsilon) \) such that \( |A(k, \sigma') - A(0, \sigma)| < \epsilon \) if \( |k| + |\sigma' - \sigma| < \eta \). Let \( k(x, \epsilon) \) be the smaller of \( t_1 \) and the unique positive solution of \( k + \frac{1}{2}L(\rho)\epsilon k \|x\| = \eta \). Letting \( \sigma = \|x_0\|/2 \), it follows from (2.9) that \( |\sigma' - \sigma| \leq \|x(k; x_0)\| - \|x_0\|/2 \leq (L(\rho)\epsilon \kappa)^{-1} \) and hence, by virtue of lemma 2.11, that \( V(x(k; x_0)) - V(x_0) \leq -kG(\|x_0\|)A(k, \sigma') < k( -\psi(\|x_0\|) + \epsilon G(\|x_0\|)) \).

**Lemma 2.14.** There is an \( N > 0 \) such that, if \( x(t; x_0) \) and \( x_0 + hf(x_0) \) are elements of \( S(\mathbb{R}) \) for \( 0 \leq t \leq h \leq h_0 \), then \( \|x_0 + hf(x_0) - x(h; x_0)\| \leq Nh^2 \|x_0\| \).

**Proof:** Apply Gronwall's inequality.

**Lemma 2.15.** Let \( \rho = \min(1, \delta(\mathbb{R})) \) and suppose that \( \varphi(x, h) \) is uniformly...
continuous on \( S(\rho) \times [0, \rho_0] \). If \( 0 < r < \delta(\rho) \), then there is an \( h_1(r) > 0 \) such that

\[
(2.16) \quad V(x_0 + h\varphi(x_0, h)) - V(x_0) \leq - \frac{1}{2} h\psi(r) < 0
\]

whenever \( 0 < h < h_1(r), \ r \leq \|x_0\| \leq \delta(\rho) \).

**Proof:** Assume without loss of generality that \( \delta(\rho) < \rho \). Then, if \( h \leq (\rho - \delta(\rho))/[\delta(\rho)\max(L, L')] \), \( x_0 + h\varphi(x_0, h) \) and \( x_0 + h\varphi(x_0) \) are elements of \( S(\rho) \). Now, \( V(x_0 + h\varphi(x_0, h)) - V(x_0) \leq |V(x_0 + h\varphi(x_0, h)) - V(x_0 + h\varphi(x_0))| + |V(x_0 + h\varphi(x_0)) - V(h; x_0)| + V(x(h; x_0)) - V(x_0) \). It follows from the previous lemmas, that, for \( 0 < r < \delta(\rho) \) and \( \epsilon > 0 \), there is a \( k(r, \epsilon) > 0 \) such that if

\[
(2.17) \quad 0 < h < h^* = \min(k(r, \epsilon), \rho_0, (\rho - \delta(\rho))/[\delta(\rho)\max(L, L')])
\]

then

\[
(2.18) \quad V(x_0 + h\varphi(x_0, h)) - V(x_0) \leq \mathcal{M}_\rho \|\varphi(x_0, h) - f(x_0)\| + \mathcal{M}_\rho^2 \|x_0\|
\]

\[
\leq h\psi(\|x_0\|) + \mathcal{G}(\|x_0\|)h
\]

for \( r \leq \|x_0\| \leq \delta(\rho) \).

Let \( \alpha(h) = \sup(\|\varphi(x, h) - \varphi(x, 0)\| \leq \|x\| \leq \delta(\rho)) \) and take \( \epsilon \leq \psi(\|x_0\|)/\mathcal{G}(\delta(\rho)) \). [Note that \( \varphi(x, 0) = f(x) \).] Since \( \varphi(x, h) \) is uniformly
continuous, \( \alpha(h) \) is continuous. Moreover, \( \alpha(0) = 0 \). Thus, the equation
\[
\alpha(h) + Nh\beta(p) = \psi(r)/(4M)
\]
has a least positive root \( \hat{h} > 0 \). If \( 0 < h_1(r) \leq \min(h^*, \hat{h}) \), then it follows that
\[
(2.19) \quad V(x_0 + h\psi(x_0, h)) - V(x_0) \leq -\frac{1}{2} h\psi(r)
\]
for \( 0 < h < h_1(r) \) and \( r \leq \|x_0\| \leq \delta(p) \).

We can now prove the following:

**Theorem 2.20.** Suppose that the solution \( x = 0 \) of (2.5) is uniformly asymptotically stable on \( S(R) \). Then, for any \( \epsilon > 0 \), there are \( h(\epsilon) > 0 \) and \( K(\epsilon) > 0 \) such that if \( \|x_0\| \leq \delta(\delta(p))/2 \) and \( 0 < h < h(\epsilon) \), then the solution \( x_k \) of (2.6) starting from \( x_0 \) satisfies the inequalities (i) \( \|x_k\| \leq \rho \) for all \( k \geq 0 \); and (ii) \( \|x_k\| < \epsilon \) for all \( k \geq K(\epsilon)/h \).

**Proof:** We may assume that \( 0 < \epsilon < \delta(p) \). Let \( r = \delta(\delta(p))/4 \) and let \( h(\epsilon) = \min(h_1(r), 1/L') \) where \( h_1(r) \) is given by lemma 2.15. Also, let \( K(\epsilon) = \frac{2(\delta(\delta(p)) \delta(r/2))}{\delta(\delta(p))} \).

We consider three cases, namely: (i) \( 0 \leq \|x_0\| < r \), (ii) \( r \leq \|x_0\| < 2r \), and, (iii) \( 2r \leq \|x_0\| \leq \delta(\delta(p))/2 \).

**Case (i):** If \( \|x_k\| < r \) for all \( k \geq 0 \), then \( \|x_k\| < \delta(\delta(p))/4 \). On the other hand, if \( \|x_k\| < r \) for \( k = 0, 1, \ldots, n-1 \) and \( \|x_n\| \geq r \), then \( \|x_n\| = \|x_{n-1} + h\psi(x_{n-1}, h)\| \leq \|x_{n-1}\|(1 + hL') < 2r \) and we regard \( x_n \) as an initial point for case (ii).
Case (i): We claim that \( \|x_k\| < \epsilon \) for all \( k \geq 0 \). [Note that \( \epsilon < \delta(\rho) < \rho \).] Clearly \( \|x_0\| < 2r < \epsilon \). If \( r \leq \|x_k\| < \epsilon \) for \( 0 \leq k \leq n \),

then \( G(\|x_{n+1}\|) \leq V(x_{n+1}) - V(x_0) + \sum_{0}^{n} (V(x_{k+1}) - V(x_k)) \leq V(x_0) - (n+1)h\psi(r)/2 < 2\|x_0\| < 2r = \delta(\epsilon) \) by virtue of lemmas 2.11 and 2.15.

Since \( G \) is strictly monotone, \( \|x_{n+1}\| < \epsilon \) and the claim is established by induction.

Thus, combining cases (i) and (ii), we have shown that if \( \|x_0\| < 2r \),
then \( \|x_k\| < \epsilon \) for all \( k \geq 0 \).

Case (iii): Clearly \( \|x_0\| \leq G(\delta(\rho))/2 \leq \delta(\rho) \leq \rho \). Suppose that \( r \leq \|x_k\| \leq \delta(\rho) \) for \( k \leq n \). Then, \( G(\|x_{n+1}\|) \leq V(x_{n+1}) - V(x_0) + \sum_{0}^{n} (V(x_{k+1}) - V(x_k)) \leq V(x_0) - (n+1)h\psi(r)/2 < 2\|x_0\| \leq G(\delta(\rho)) \) by virtue of lemmas 2.11 and 2.15. Since \( G \) is strictly monotone, \( \|x_{n+1}\| < \delta(\rho) \leq \rho \) and so, \( \|x_k\| \leq \rho \) for all \( k \geq 0 \). Furthermore, if \( (n+1)h \geq K(\epsilon) \), then \( G(\|x_{n+1}\|) \leq V(x_0) - K(\epsilon)\psi(r)/2 < 2\|x_0\| - K(\epsilon)\psi(r)/2 \leq G(\delta(\rho)) - K(\epsilon)\psi(r)/2 = G(r/2) \). It follows that \( \|x_{n+1}\| \leq r/2 \). The theorem then follows from the first two cases.

We note that the theorem does not assert that the solution \( x_k = 0 \) of (2.6) is stable for fixed \( h \). In other words, we do not claim that for given \( h \) and any \( \epsilon > 0 \) there is \( \eta = \eta(\epsilon, h) \) such that if \( \|x_0\| < \eta \), then \( \|x_k\| < \epsilon \) for all \( k \). Bearing this in mind, we consider the following two-dimensional system:

\[
\begin{align*}
k &= y - x(x^2 + y^2) \\
y &= -x - y(x^2 + y^2).
\end{align*}
\]
Let $V(x, y) = x^n y^n$. Then $\dot{V}(x(t), y(t)) = \beta(x^n(t) + y^n(t))^2$ along solutions of (2.21) and so, the trivial solution is uniformly asymptotically stable. If Euler's method is applied to (2.21), then the difference system

$$
\begin{align*}
x_{n+1} &= x_n + h y_n - h x_n (x_n^2 + y_n^2) \\
y_{n+1} &= y_n + h x_n - h y_n (x_n^2 + y_n^2)
\end{align*}
$$

is obtained. Let $\Delta_n V(x, y)$ be given by

$$
\begin{align*}
\Delta_n V(x, y) &= h^2 (x^2 + y^2) (x^2 + y^2) \left( \frac{1 - (1 - h^2)^{1/2}}{h^2} \right) (x^2 + y^2) \left( 1 + (1 - h^2)^{1/2} \right)
\end{align*}
$$

so that $V(x_{n+1}, y_{n+1}) - V(x_n, y_n) = \Delta_n V(x_n, y_n)$. Using (2.23), it is easy to verify that the trivial solution of (2.22) is not stable and that all solutions with $0 < h(x_n^2 + y_n^2) < 1 + (1 - h^2)^{1/2}$ ($0 < h < 1$) are attracted to the invariant set $x^2 + y^2 = (1 - (1 - h^2)^{1/2})/h$. Although the trivial solution of (2.22) is not stable for fixed $h$, the solutions of (2.22) can be made to remain arbitrarily close to zero by initially choosing $h$ small enough.

In other words, the theorm asserts that for given $\epsilon > 0$, there is an $h(\epsilon)$ such that if $h < h(\epsilon)$, then the solutions of (2.22) will lie within the ball $B(\epsilon)$.

3. **Exponential Asymptotic Stability.**

We now consider the case of exponential asymptotic stability.

**Definition 3.1.** The solution $x = 0$ of (2.5) is exponentially asymptotically
stable on \( S(R) \) if there are positive constants \( \alpha, M \) such that
\[
\|x(t; t_0, x_0)\| \leq M\|x_0\|e^{-\alpha(t-t_0)} \quad \text{for} \quad \|x_0\| \leq r \quad \text{and} \quad t \geq t_0. 
\]
Similarly, the solution \( x_k = 0 \) of (2.6) is exponentially asymptotically stable if there are positive constants \( \alpha, h, M_2, M_1 \) such that \( \|x_k\| \leq M_1\|x_0\|e^{-\alpha h} \) for all \( k \geq 0 \) whenever \( 0 < h < h_1 \) and \( \|x_0\| \leq M_1 \).

Skalkina ([11]) has shown that if the zero solution of (2.5) is exponentially asymptotically stable, then so is the zero solution of (2.6). We shall shortly present an improved version of his result.

**Lemma 3.2.** If the solution \( x = 0 \) of (2.5) is exponentially asymptotically stable on \( S(R) \), then the function \( W(\cdot) \) defined by
\[
W(x_0) = \sup\{\|x(t; t_0, x_0)\|e^{\text{atan} \alpha t} | t > 0 \}
\]
for \( x_0 \in S(R) \) has the following properties: (i) \( \|x_0\| \leq W(x_0) \leq M\|x_0\| \); (ii) \( |W(x_0) - W(y_0)| \leq \alpha\|x_0 - y_0\| \); (iii) \( \dot{W}(x_0) \leq -aW(x_0) \); and, (iv) \( W(x(t; x_0)) - W(x_0) \leq -c\|x_0\|n \) for suitable positive constants \( \alpha, c, c' \).

(Where \( \alpha, M \) are the constants involved in the definition of exponential asymptotic stability).

**Proof:** Argue as in [7, pp. 309-311].

We now have

**Theorem 3.4.** Suppose that the solution \( x = 0 \) of (2.5) is exponentially asymptotically stable on \( S(R) \). Assume also that either (a) \( \psi(x, h) = f(x) \) or (b) \( \psi(x, h) \) is (Fréchet) differentiable in \( x \) and \( \varphi_x(x, h) \) is locally Lipschitz uniformly continuous on \( S(R) \times [0, P_0] \). Then the solution
$x_k = 0$ of (2.6) is exponentially asymptotically stable.

**Proof:** Let $b \in (0, R)$ and let $\delta = \min(\frac{1}{2} \log R/b, ((R/b) - 1)/\max(L, L'), h_0)$. If $0 < h < \delta$, then $x + h\varphi(x, h)$ and $x + hf(x)$ are in $S(R)$ for $\|x\| \leq b$ and $\|x(t, x_0)\| \leq R$ for all $t \geq 0$ if $\|x_0\| \leq b$ (as $\|x(t, x_0)\| \leq \|x_0\\| e^{Lt}$).

Now, let $h = 0$ or $\sup(\|\varphi(x, h) - \varphi(x, 0)\| \|x\| \leq b)$ according as hypothesis (a) or (b) holds. If hypothesis (a) holds, then $|W(x+h\varphi(x, h)) - W(x+hf(x))| = 0 \leq Kh_a(h)\|x\|$. On the other hand, if hypothesis (b) holds, then $|W(x+h\varphi(x, h)) - W(x+hf(x))| \leq Kh\|\varphi(x, h) - \varphi(x, 0)\| \leq Kh\|\varphi(x, h) - \varphi(0, h)\| \leq Kh\int_0^1 (\|\varphi_x(tx, h) - \varphi_x(t, 0)\| \|x\|) \, dt \leq Kh_a(h)\|x\|$. [Note that $\varphi(0, h) = \varphi(0, 0) = 0$.] In other words, we always have

$$
(W(x+h\varphi(x, h)) - W(x+hf(x))) \leq Kh_a(h)\|x\|
$$

for $\|x\| \leq b$.

Let $h'$ be the least positive root of $Kh_a(h) + Nh = c/2$ and let $h_1$ be any positive number with $h_1 < \min(\delta, h', 2M/c)$. If $h < h_1$ and $\|x_0\| \leq b$, then $W(x_0+h\varphi(x_0, h)) - W(x_0) \leq |W(x_0+h\varphi(x_0, h)) - W(x_0+hf(x_0))| + |W(x_0+hf(x_0)) - W(x(h, x_0))| + W(x(h, x_0)) - W(x_0) \leq h\|x_0\| (Kh_a(h)+Kh\cdot c) \leq -b\|x_0\| c/2$.

Now let $M = M$ and $\beta = c/(2M)$. We shall show by induction that if $\|x_0\| \leq b/M$, then

$$
\|x_k\| \leq M\|x_0\| e^{-\beta kh}
$$

for $\|x\| \leq b$.
for all $k$. Clearly (3.6) holds for $k = 0$ and so, we suppose it holds for $0 \leq k \leq n$. For any such $k$, 

$$
\|x_k\| \leq M\|x_0\|e^{-\beta k h} \leq M\|x_0\| \leq b \quad \text{and so,}
$$

$$
W(x_{k+1}) - W(x_k) \leq -\eta \|x_k\|c/2. \quad \text{Since } W(x_k) \leq M\|x_k\|, \quad W(x_{k+1}) \leq W(x_k)(1-\beta h).
$$

If $W(x_k) = 0$ for any $k \leq n$, then $W(x_{k+1}) = 0$ for all $t \geq 0$ and (3.6) is satisfied. Otherwise, 

$$
\|x_{n+1}\| \leq W(x_{n+1}) \leq W(x_n)(1-\beta h) \leq W(x_0)(1-\beta h)^{n+1} \leq M\|x_0\|e^{-\beta h(n+1)}. \quad \text{Thus, the theorem is established.}
$$
References


