COMPARISONS OF COORDINATE SYSTEMS AND
TRANSFORMATIONS FOR TRAJECTORY
SIMULATIONS

By
Bernard F. Engebos

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Four right-hand orthogonal coordinate systems are defined and the transformations between these systems are developed. They are: a local topocentric launcher system, an earth-centered inertial system, a body axis system, and an aeroballistic axis system.

The determination of time derivatives of direction cosines is used to transform information between the inertial and body systems. Euler angles are used to transform data between the inertial and aeroballistic axis systems. Strengths and weaknesses of these transforms are then compared. Finally, a comparison of results of trajectory simulations is presented.
NOTICES

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1. Coordinate Systems
2. Trajectory Simulation
3. Mathematical Models
4. Euler Rate Matrix
COMPARISONS OF COORDINATE SYSTEMS AND TRANSFORMATIONS FOR TRAJECTORY SIMULATIONS

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ABSTRACT

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INTRODUCTION

There has been and still is considerable interest in the simulation of unguided rocket trajectories. Several mathematical models [1,2] have been developed for this purpose. The Atmospheric Sciences Laboratory has developed a six-degree-of-freedom ballistic simulation model [3]. This model uses three right-hand orthogonal coordinate systems - the launcher, the inertial, and the "body" coordinate systems. The output from the simulations is referenced to the launcher coordinate system. Aerodynamic forces and moments are computed in the body system, and the inertial system is used for the development and integration of the equations of motion. Information is transformed from the inertial to body system via direction cosines. Due to computational inaccuracies, however, it is possible here to obtain a nonorthonormal transformation. Hence it was decided to determine what would happen if the body axis system were replaced by an aeroballistic axis system and if Euler angles were used in the transformation of information from the inertial to the aeroballistic axis system. This report compares the limitations and capabilities of these two systems and their results in theoretical trajectory simulations.

INERTIAL AND LAUNCHER COORDINATE SYSTEMS

Two right-hand orthogonal coordinate systems will be defined. (See Figure 1.) The launcher coordinate system (denoted \(X', Y', Z'\)) has its origin at the launcher and rotates with the earth. The positive \(X'\)-axis points east, the positive \(Y'\)-axis points north, and the positive \(Z'\)-axis points outward along the radius vector from the center of the earth.

The inertial coordinate system \((X, Y, Z)\) has its origin at the center of the earth and is oriented so that the X-Y plane lies in the earth's equatorial plane with the positive Y-axis pointing initially through the longitude of the launcher. The Z-axis is coincident with the earth's axis of rotation and positive toward the North Pole. This system does not rotate with the earth.

The mathematical simulation model considers the earth to be an oblate spheroid with an equatorial radius of 6,378,423 meters and an eccentricity of .00672267. The transformation from the launcher system to the inertial system is, in matrix form,

\[
\begin{bmatrix}
X' \\
Y' \\
Z'
\end{bmatrix}
= \begin{bmatrix}
-cos \omega t & sin \lambda sin \omega t & -cos \lambda sin \omega t \\
-sin \omega t & -sin \lambda cos \omega t & cos \lambda cos \omega t \\
0 & cos \lambda & sin \lambda
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
+ \mathbf{R}_0 \begin{bmatrix}
-sin \omega t cos \lambda \\
cos \omega t cos \lambda \\
sin \lambda
\end{bmatrix}
\]
where \( \lambda \) is the geodetic latitude of the launcher, \( \omega \) is the earth's rotational speed, \( R_Q \) is the distance from the center of the earth to the launcher, and \( t \) is time after launch. The preceding transformation and its inverse are derived in [2].

**THE BODY COORDINATE SYSTEM**

The "body" coordinate system \((x,y,z)\) has its origin at the center of gravity of the rocket. The \(x\)-axis coincides with the longitudinal axis of the rocket and is positive toward the nose. The positions of the \(y\) and \(z\)-axes are determined by the rocket's motion. The initial positions of these axes are defined as follows: Let \( \theta \) be the angle between the \(x\)-axis and the positive \(Z'-\)axis measured from the \(Z'-\)axis. The \(y\)-axis lies in the \(X-Y\) plane and is positive in the direction of positive \(\theta\). (See Figure 1.)

The transformation from the inertial to the body coordinate system via direction cosines is obtained as follows: Let \((\ell_1, \ell_2, \ell_3), (m_1, m_2, m_3),\) and \((n_1, n_2, n_3)\) be the respective direction cosines of the body system with respect to the inertial system [4]. Then

\[
\begin{bmatrix}
K_x \\
K_y \\
K_z
\end{bmatrix} =
\begin{bmatrix}
\ell_1 & \ell_2 & \ell_3 \\
m_1 & m_2 & m_3 \\
n_1 & n_2 & n_3
\end{bmatrix}
\begin{bmatrix}
K_x \\
K_y \\
K_z
\end{bmatrix}
\]

(1)

where \(K_i\) denotes a unit vector along the i axis. Since this is an orthogonal system, its inverse is its transpose. Thus,

\[
K_x = \ell_1 K_x + m_1 K_y + n_1 K_z.
\]

(2)

Differentiating (2) with respect to time yields

\[
0 = \ell_1 \dot{K}_x + \ell_1 K_x + m_1 \dot{K}_y + m_1 K_y + n_1 \dot{K}_z + n_1 K_z.
\]

(3)

Let \(\dot{w}\) define the angular velocity of the body system relative to the inertial system. Then

\[
\dot{w} = p K_x + q K_y + r K_z
\]

where \(p, q, r\) are the \(x, y, z\) components of \(\dot{w}\). Hence

\[
\dot{K}_x = K_y r - K_z q,
\]

(4)
\[ \dot{K}_y = K_x p - K_x r, \]

and

\[ \dot{K}_z = K_x q - K_y p. \]

Substituting (4) into (3) yields

\[ 0 = K_x (\ell_1 - rm_1 + qn_1) + K_y (m_1 - pn_1 + r\ell_1) + K_z (n_1 - q\ell_1 + pm_1). \]

Thus,

\[ \dot{\ell}_1 = rm_1 - qn_1 \]
\[ \dot{m}_1 = pn_1 - r\ell_1 \]
\[ \dot{n}_1 = q\ell_1 - pm_1. \]

Similarly, one obtains

\[ \ell_i = rm_i - qn_i \]
\[ \dot{m}_i = pn_i - r\ell_i \]
\[ \dot{n}_i = q\ell_i - pm_i. \]

Integration of the nine differential equations in (6) yields the desired direction cosine transformation.

It is easy to see that the aforementioned numerical integration can give rise to a nonorthonormal system due to computational inaccuracies. Hence the information transformed by this system will contain errors which will accumulate throughout the simulation and can cause inaccurate trajectory determinations. Hence it is necessary to somehow force this system to be orthonormal. This can be accomplished as follows.

First set

\[ \ell_i = \frac{\ell_i}{(\ell_1^2 + \ell_2^2 + \ell_3^2)^{1/2}}, \quad i = 1, 2, 3. \]

Then the dot product of

\[ \ell = (\ell_1, \ell_2, \ell_3) \]
and
\[ m = (m_1, m_2, m_3) \]
is taken, call it DOT. Then set
\[ \hat{m}_i = m_i - \hat{2}_i \times DOT, \quad i = 1, 2, 3. \]
Finally redefining
\[ m_i = \hat{m}_i / (\hat{m}_1^2 + \hat{m}_2^2 + \hat{m}_3^2)^{1/2}, \quad i = 1, 2, 3 \]
and taking the vector cross product of \( \ell \) and \( m \), yields
\[ n = (n_1, n_2, n_3). \]
This system is now orthonormal and is the system used to transform information between the inertial and body axis systems.

Several things should be noted. Only the differential equations for the \( \ell \) and \( m \) vectors need be used. Thus there are but six differential equations in (6) to be integrated. Further, the \( m \) vector is modified to force it to be orthogonal to \( \ell \) and \( n \) is completely determined by \( \ell \) and \( m \). It should be noted that it would have been just as easy to have started with the \( m \) or \( n \) vector instead of \( \ell \) and thus to obtain an analogous, but somewhat different system. The above procedure contains artificiality which leads one to suspect the result obtained through it. To alleviate this problem, it was decided to use Euler angles and an aeroballistic axis system.

EULER ANGLES

Let \( \phi \) be the angle of rotation about the \( Z \)-axis from the \( X \) to the \( Y \)-axis. Then from Figure 2 one sees that
\[ \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \]
(7)

Let \( \xi \) be the angle of rotation about the \( y_1 \)-axis from the \( x \) to the \( Z \)-axis. It follows from Figure 3 that
Rotation About Z-Axis

Figure 2
Rotation About $y_1$-Axis

Figure 3
Let ψ be the angle of rotation about the \( x_2 \)-axis from \( y_2 \) to \( z_2 \). From Figure 4 it follows that

\[
\begin{bmatrix}
  x_2 \\
  y_2 \\
  z_2 
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 0 \\
  0 & \cos \psi & -\sin \psi \\
  0 & \sin \psi & \cos \psi 
\end{bmatrix}
\begin{bmatrix}
  x_3 \\
  y_3 \\
  z_3 
\end{bmatrix}
\]

(9)

Combining (7), (8), and (9), one obtains

\[
\begin{bmatrix}
  x \\
  y \\
  z 
\end{bmatrix} =
\begin{bmatrix}
  \cos \xi \cos \psi & \cos \xi \sin \psi \sin \xi - \sin \xi \cos \psi \\
  \sin \xi \cos \psi & \sin \xi \sin \psi \sin \xi + \cos \psi \cos \Xi \\
  -\sin \xi & \cos \xi \sin \psi 
\end{bmatrix}
\begin{bmatrix}
  \cos \xi \sin \psi + \sin \phi \sin \Xi \\
  \sin \phi \sin \xi \cos \psi - \cos \phi \sin \psi \\
  \cos \psi \cos \Xi 
\end{bmatrix}
\]

(10)

This is the transformation from the body to the inertial system using Euler angles. Equation (10) contains many trigonometric functions and thus takes considerable time to evaluate on a computer. Unlike the direction cosine matrix, no artificiality is used.

Let \( \dot{\psi} \) be defined as before. Then, the body system Euler rate matrices become

\[
\begin{bmatrix}
  p \\
  q \\
  r 
\end{bmatrix} =
\begin{bmatrix}
  -\sin \xi & 0 & 1 \\
  \cos \xi \sin \psi & \cos \psi & \dot{\psi} \\
  \cos \xi \cos \psi - \sin \psi & 0 & \dot{\phi} 
\end{bmatrix}
\]

(11)

and
Rotation About $x_2$-Axis

Figure 4
\[
\begin{bmatrix}
\phi \\
\dot{\phi} \\
\dot{q} \\
\dot{r}
\end{bmatrix}
=egin{bmatrix}
0 & \sin \psi & \cos \psi \\
0 & \cos \psi & -\sin \psi \\
\sin \psi \sin \dot{\psi} & \sin \dot{\psi} \cos \psi & \cos \psi \\
\cos \psi \sin \dot{\psi} & \sin \dot{\psi} \cos \psi & -\sin \psi
\end{bmatrix}
\begin{bmatrix}
p \\
q \\
r
\end{bmatrix}
\]

This can be seen by examining Figure 5.

AEROBALLISTIC AXIS SYSTEM

An aeroballistic axis system is one such that the y and z axes are not allowed to rotate about x. Thus \( \dot{\psi} = 0 \), and \( p = \int \dot{\psi} \, dt \) where \( p \) is redefined as the physical rolling rate of the rocket. The transformation from the body to the inertial system by Euler angles is still valid although \( \dot{\psi} = \psi_0 \). The aeroballistic system Euler rate matrices become

\[
\begin{bmatrix}
p \\
q \\
r
\end{bmatrix}
= \begin{bmatrix}
\cos \xi & \sin \psi_0 & \cos \psi_0 \\
0 & 0 & 0 \\
\cos \xi & \cos \psi_0 & -\sin \psi_0
\end{bmatrix}
\begin{bmatrix}
\phi \\
\dot{\phi} \\
\dot{\psi}
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
p \\
q \\
r
\end{bmatrix}
= \begin{bmatrix}
0 & \sin \psi_0/\cos \xi & \cos \psi_0/\cos \xi \\
0 & \cos \psi_0 & -\sin \psi_0 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
p \\
q \\
r
\end{bmatrix}
\]

From equations (11) or (12) it is easily seen that the Euler rate matrices can become singular at \( \xi = \pi/2 \) or its inverse matrix, \( A(A) \), to be undefined at \( \xi = \pi/2 \). Hence one must be extremely careful in the use of such a system. Also, note that the choice of the aeroballistic, i.e., nonrolling, coordinate system reduces the number of equations to be integrated and the number of trigonometric functions to be evaluated, since \( \dot{\psi} = 0 \) for all \( t \). The previously discussed definitions for the Euler angles were chosen to avoid singularities of the Euler rate matrices for White Sands Missile Range usage.

ANALYSIS

The equations of motion of an unguided rocket as developed in [2] were used in each of the two following instances. First the inertial,
Euler Rates

Figure 5
launcher and body coordinate systems were used in one of the mathemati-
cal models in conjunction with the direction cosine transformation. In the second case the inertial, launcher, and aeroballistic axis sys-
tems were used with Euler angles. These two cases will be referred to as cases 1 and 2 in that which follows.

Several trajectories were simulated for both cases, and their results in computational time, burn-out positions and velocities, and impacts were compared. The only differences between these simulations were the coordinate systems used and the transformations between them.

The Aerobee 350 rocket was used for this comparison. This rocket is a two-stage unguided high-altitude probe which is fired near the vertical. Since this rocket burns out and then coasts out of the atmosphere, the equations of motion of a rocket were reduced to three degrees of freedom after the rocket leaves the atmosphere (61 km). This procedure results in no loss of accuracy and was followed in both cases. A constant launcher setting was assumed throughout which resulted in a peak altitude of \(\sim 487\) km and impact range of \(\sim 204\) km.

Rocket trajectories were simulated for both cases. Those simulations were made with and without wind and with and without the rocket rolling. All computations were performed on the Univac 1108 computer.

The burn-out positions, burn-out velocities, and impact points varied only slightly between the two cases. There was, however, a large difference in their computational time on the computer (Table I).

<table>
<thead>
<tr>
<th>I.D.</th>
<th>CASE 1</th>
<th>CASE 2</th>
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<tbody>
<tr>
<td>No wind, No roll</td>
<td>45.7</td>
<td>140.8</td>
</tr>
<tr>
<td>No wind, Roll</td>
<td>55.3</td>
<td>155.7</td>
</tr>
<tr>
<td>10 MPH East, No Roll</td>
<td>51.7</td>
<td>222.8</td>
</tr>
<tr>
<td>20 MPH East, Roll</td>
<td>55.4</td>
<td>248.9</td>
</tr>
</tbody>
</table>

These discrepancies in computer running time can be explained by the large number of evaluations of trigonometric functions needed for the desired transformation to and from the aeroballistic axis system in Case 2. On the average, Case 2 required four times as much computational time per trajectory simulation as Case 1.
It should be noted, however, that this was accomplished on a slowly rolling unguided rocket. For such rockets, Case I is vastly superior to Case 2 with respect to computational time and yet seems to maintain acceptable accuracy.

Suppose now, one wishes to simulate the trajectory of an artillery shell or a bullet (something with a rapid roll rate). A natural question arises. Are Case I type transformations still superior to those of Case 2? The author feels that, due to artificiality in the determination of the direction cosines, Case I methods can lead to problems and that the use of Euler angles will prove superior. This question is currently being investigated. Another method of transforming data between two coordinate systems is by the use of quaternions. This method also leads to an orthonormality problem and thus was not considered.

CONCLUSIONS

In comparing the two cases, several strong and weak points naturally appear. First, in Case I, due to the necessity of the numerical integration of the derivatives of the direction cosines, a nonorthonormal system can arise. The artificial forcing of the direction cosines to be orthonormal leads one to suspect any results obtained therefrom. However, for a slowly rolling rocket, the method of Case I results in a quite accurate determination (when compared to other methods) of the theoretical rocket flight path and a fast simulation time. For rapidly spinning shells, this induced artificiality can be a potential problem.

With regard to aeroballistic axis and Euler angles, one must contend with the evaluations of several trigonometric functions to obtain the described transformation. This case has none of the artificiality of the first case. Near the singularity of the Euler rate matrix, many problems can and do arise. Also, the method is slower than that of direction cosines and hence requires more computational time.
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