ANGLE OF ARRIVAL CALCULATIONS AT 10.6μ

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Stuart A. Collins, Jr.

The Ohio State University
ElectroScience Laboratory

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Raymond P. Uetz
RAEC Project Engineer
ABSTRACT

This report deals with the specification of the angular position of objects by measurement of the angles of arrival of light beams; and with the measurement uncertainties due to atmospheric turbulence. A survey of the literature is presented showing two approaches to angle of arrival, one for large aperture receivers and one for a single small aperture or a pair of pinholes. Pertinent defining equations are presented for both cases and values are calculated for arrival angle mean square measured with a large aperture and correlation of arrival angles for small apertures. A synonymity between the small aperture correlation function and large aperture mean square angle of arrival is presented and used to present further large aperture mean square values. This provides a significant simplification in calculation and measurement capabilities. Finally the regions of accuracy are presented for the Rytov approximation which forms the basis for the numeric calculations.
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I. INTRODUCTION

When optical methods are used to observe an object through the atmosphere, the turbulence in the atmosphere may cause the apparent position of the object to fluctuate. This motion is caused by turbulence-induced fluctuations in the electromagnetic phase of the received wave. The received rays appear to change direction, so that the term angle of arrival fluctuations is used to describe this phenomenon. Knowledge of the angle of arrival fluctuations or its statistical properties is obviously necessary for analysis of systems specifying the accurate location of terrestrial or extraterrestrial object by optical means. It is also conceivable that, since angle of arrival fluctuations are induced by atmospheric turbulence, arrival angle statistics could be used as a convenient tool for study of turbulence characteristics.

The literature on angle of arrival is scattered and offers a wide range of views, approaches, and restrictions. For example there are two general approaches to arrival angle. The first considers the light detected by one or several minute apertures where phase is linear and amplitude is constant across the aperture. The quantities of interest for this approach are the arrival angle variance and correlation function, the mean being easily demonstrated to be zero. Alternatively there is consideration given to a single large aperture where the phase front is crinkled and the amplitude is not at all constant across the aperture. Generally the literature considers only the arrival angle variance for such a situation. For either approach the derivations based on the Rytov, the geometrical optics or other approximation may be given.

It is desirable to be able to predict the angle of arrival observed by both large and small aperture methods in terms of the propagation range, receiver and aperture characteristics, wavelength, and turbulence parameters; outer and inner scale and structure parameter. Some cases have been considered, but most are valid only for restricted ranges of the variables described earlier and all use several approximations.

The original object of this report was to examine the literature, to categorize these different situations and approximations, and to extend the results where feasible. This has been done. Basic extensions involve the computation of arrival angle functions using an index spectrum which accounts for the turbulence inner and outer scales.

In the process of performing these calculations, a synonymity between a small aperture correlation function and large aperture variance function was discovered. This synonymity provides great simplification in the calculation and measurement of these two quantities. The synonymity is presented and derived.
In the balance of the report there is first a general discussion of angle of arrival in Chapter II. This is followed by analytic expressions for angle of arrival statistical functions for first the small aperture approach (Chapter III) and then the large aperture approach (Chapter IV). Numerical results are presented in Chapter V in graphical form for a wide variety of cases along with a discussion of the limits of applicability of the Rytov approximation. Chapter VI deals with the synonymity of the large and small aperture approaches. Finally, the summary appears in Chapter VII.

II. GENERAL DISCUSSION OF ANGLE OF ARRIVAL

In this section we introduce in detail the various concepts and quantities associated with the two major approaches to angle of arrival. First the general measurement situation is considered. The small and large aperture situations are discussed to the extent that typical formulae are presented.

As generally used, the term angle of arrival is concerned with light emitted from an object and detected by a receiver. Pragmatically what is desired is a listing of the angular coordinates of the object with respect to some fixed reference system. In simple minded terms this can be obtained by imaging the object either with a lens (finite aperture) or a pinhole camera (very small aperture) and taking a line from the object through the center of the input aperture. The angular coordinates of the line then define the direction of the object. If there are no intervening refractive index fluctuations then the image of a point object may be a diffraction limited spot with a well defined center. The angular coordinates determined by the spot center can be very precisely chosen in such a case.

Complications arise because of random index fluctuations which cause the image to move around and/or to blur, depending on the physical details of the receiver. Random image motion is then interpreted as random fluctuations in the observed angle of arrival. Further fluctuations in the shape of the received image of a point object also cause fluctuations in the "center of gravity" of the image and may be interpreted as fluctuations of the angle of arrival. It is the statistical description of these random fluctuations in arrival angle caused by refractive index fluctuations that form the basic subject of this report.

It might be noted that the arrival angles measured with an optical instrument might, even in the absence of refractive index fluctuations, not be the true angular coordinates of the object. For example, if there were a refractive index gradient perpendicular to the light path, the light rays would be bent causing an anomalous indication of the true angles. This effect causes the phenomenon of "looming" and causes the sun to be visible after it has in actuality dropped below the horizon. Such effects will not be considered in the present report.
There are various approaches to the measurement and prediction of angle of arrival fluctuations. The measurements are generally categorized by the size of the aperture used for the receiver. Perhaps the cleanest approach involves the use of a very small pinhole sized aperture, as shown in Fig. 1. The pinhole is chosen sufficiently small so as to be smaller than the spatial extent of any amplitude variation, and so as to contain only a flat wavefront. The displacement, $d$, of the center of the diffraction pattern in a given direction, then, is related to the angle of arrival, $\alpha$, by the expression

$$d = R \alpha$$

where $R$ is the distance from pinhole to diffraction pattern. In this situation, motion is caused only by phase fluctuations and not by amplitude fluctuations of the incoming light.

Fig. 1. Definition of small aperture angle of arrival.
As shown in Fig. 1, two angles of arrival are defined; the elevation angle \( \alpha \), and the azimuth angle, \( \beta \). One can consider various statistical functions of these angles. The simplest are the variances,* such as \( \langle (\alpha - \bar{\alpha})^2 \rangle \). (The angular brackets indicate ensemble average.)

There are also the two point arrival angle correlation functions for angles of arrival defined at two separate pinholes as shown in Fig. 2. Two such correlation functions exist, differentiated by the relative positions of the pinholes with respect to the direction of motion of the spots, as shown in Fig. 2. These will be discussed in more detail in the next section.

To calculate angle of arrival for a pinhole aperture the wavefront tilt is determined using the optical distance, \( L \), or the equivalent electromagnetic phase shift, \( S = k \frac{L}{\lambda} \), along two rays through the turbulent atmosphere from the source to opposite sides of the receiver aperture. As shown in Fig. 3, we have for

\[
(2) \quad \alpha = \lim_{n \to 0} \frac{L(y_n) - L(y)}{n} = \frac{\partial L}{\partial y} = \frac{1}{k} \frac{\partial S}{\partial y}
\]

The type of wave, i.e., plane, spherical, etc., will determine the functional form of \( L \). The arrival angle variance \( \langle \alpha^2 \rangle \) is then expressed as the ensemble average of the square of the average product of two optical phases.

\[
(2a) \quad \langle \alpha^2 \rangle = \frac{1}{k^2} \frac{\partial^2}{\partial y_1 \partial y_2} \langle S(y_1) S(y_2) \rangle \bigg|_{y_1 = y_2}
\]

\[
(2b) \quad = \frac{1}{k^2} \frac{\partial^2}{\partial y_1 \partial y_2} B_S(y_2 - y_1) \bigg|_{y_1 = y_2}
\]

\[
(2c) \quad = -\frac{1}{2k^2} \frac{\partial^2}{\partial y_1 \partial y_2} D_S(y_2 - y_1) \bigg|_{y_1 = y_2}
\]

*The terms angle of arrival variance and mean square value will be used interchangeably. They are identical since the mean is chosen to be zero. Similarly, the terms covariance and correlation function will be used interchangeably.
Fig. 2. Definition of small aperture angle of arrival correlation functions.
Equation (2a) uses the standard form for the phase correlation function
\[ B_s(y_2-y_1) = <s(y_1)s(y_2)> \]
and the associated structure function,
\[ D_s(y_2-y_1) = 2 [B_s(0) - B_s(y_2-y_1)] \]
written for a homogeneous atmosphere. Equations similar to Eq. (2c) will be used subsequently in the report.

Various techniques appear in the literature for the extensions and evaluation of Eq. (2c). They use expressions for the phase correlation or structure functions based on either the eikonal equation approach\[1,2,3\] of ray optics, or based on the Rytov approximation\[4,5,6,7\] (which reduces to the eikonal equation approach for short ranges). They also use a variety of correlation functions, structure functions or spatial spectra to represent the atmosphere.
An apparatus using pinhole apertures has one possible drawback as a practical device for angular coordinate measurement. That is the high source signal level required because of the small receiver aperture size. However, it does provide a measure of arrival angle variance and correlation functions and, therefore, could be useful under controlled conditions where the source is available. Further, the associated analytical calculations of the statistical quantities are simplified by being amplitude independent.

The other approach to arrival angle measurement employs a large aperture receiver with a lens to image the object as shown in Fig. 4.

This situation is more practical because of its light-gathering ability and subsequent increased sensitivity. However, it is more difficult to work with conceptually because of random spatial wavefront and amplitude fluctuations superimposed upon the no-turbulence fields. Thus, while the wavefront normal provided a reasonable indication of arrival angle for a pinhole aperture, there is very apt to be no unique wavefront normal for a large aperture. One can instead use the focused image, but amplitude and phase fluctuations about the mean wavefront of the incoming light cause motion of the image and deterioration of its shape, thereby increasing measurement uncertainty.
The quantity generally calculated for the large aperture approach is the arrival angle variance. To do this the instantaneous arrival angle is defined using some sort of optimization technique involving either fitting a smooth wavefront to the input fields[8,9] or defining the arrival angle to be that which has the maximum power.[10] These calculations are generally more complex than pinhole aperture calculations because of amplitude fluctuations over the input aperture, and because of the multiple integrations required.

As an indication of the approach to arrival angle calculations for large aperture situations, consider the following approximate calculation from Hufnagel.[8] Assume that to a good degree of approximation the wavefront at any instant can be described by a plane which matches the actual wavefront at four points, the intersection of the transverse coordinate axis with the aperture edge. Then the phase deviation along the x axis will be

\[ \phi(D, 0) - \phi(-D, 0), \]

the linear variation of the wavefront from the x axis will be

\[ [\phi(D, 0) - \phi(-D, 0)]/k \]

and the angle of tilt in the x direction will be

\[ \alpha_x = [\phi(D, 0) - \phi(-D, 0)]/kD. \]

Assuming that \( \alpha_x \) has a zero mean, then its variance will be

\[ \langle \alpha_x^2 \rangle = \langle (\phi(D, 0) - \phi(-D, 0))^2 \rangle/k^2 D^2 \]

\[ = D_s(D)/k^2 D^2 \]

\[ = 6.88(D/r_0)^{5/3}/k^2 D^2 \]

\[ = 6.88/k^2 r_0^{5/3} D^{1/3} \]

where the phase structure function, \( D_s(p) \), has been approximated by
\[
D_s(\rho) = 6.88(\rho/r_0)^{5/3}
\]

where \( r_0 \) is Fried's transverse coherence length given by

\[
r_0 = (6.88/2.91 k^2 L C_n^2)^{3/5}.
\]

There are also other approaches to angle of arrival. One is an examination of arrival angle of light from a line source,[11,12] representing an extension of the small aperture approach. Another is a calculation of arrival angle variance using the Fokker-Plank equation[13] which represents a unique approach. These techniques will not be mentioned further. A few experimental measurements have been described by Coulman.[14,15]

To summarize, this section has presented a general discussion of various aspects of angle of arrival work, indicating the sources of uncertainty considered, and discussion of the two models for approaching the subject.

III. SMALL APERTURE ANGLE OF ARRIVAL CALCULATIONS

In this section the exact expression for the small aperture angle of arrival statistical quantities are derived and examined. They include two types of correlation functions and the mean square angle of arrival. The formalism to be presented follows closely that presented by Tatarski [5, section 40]*. As mentioned earlier, the small aperture technique assumes that the aperture is small enough that the aperture size has no effect on the angle of arrival indication. Thus, the phase difference across the aperture is assumed linear and the instantaneous amplitude is assumed to be independent of position in the aperture. Under these conditions the wavefront normal will uniquely define the angle of arrival at any instant of time.

Since the aperture size will not affect the final result, the optical path lengths to pairs of points are considered and the aperture sizes shrink to zero. The situation is shown in Fig. 5a. Using the definition in Eq. (2) we have

*For equations taken from [5], the section number and equation number will be given in parentheses.
Fig. 5. Geometry used to calculate small aperture arrival angle correlation functions.
\[ \alpha_1 = \lim_{n_1 \to 0} \frac{L(z,0,0) - L(z,0,n_1)}{n_1} \]

\[ = \lim_{n_1 \to 0} \frac{S_1 - S_2}{kn_1} \]

\[ \alpha_2 = \lim_{n_2 \to 0} \frac{[L(z,0,y_1-n_2/2) - L(z,0,y_1 + n_2/2)]}{n_2} \]

\[ = \lim_{n_2 \to 0} \frac{S_3 - S_4}{kn_2} \]

where

- \( S_1 \) = phase at the point \((z,0,0)\)
- \( S_2 \) = phase at the point \((z,0,n_1)\)
- \( S_3 \) = phase at the point \((z,0,y_1-n_2/2)\)
- \( S_4 \) = phase at the point \((z,0,y_1+n_2/2)\).

These points are shown in Fig. 5a.

The elevation angle of arrival correlation is defined as

\[ B_a(y_1) = \langle a(z,0,0) \ a(z,0,y_1) \rangle. \]

To simplify we substitute Eq. (4a) and Eq. (4b) into Eq. (5a) giving
\[ B_a(y) = \lim_{\eta_1 \to 0} \frac{<(S_1 - S_2)(S_3 - S_4)>}{k^2 \eta_1^2} \]

where \( \eta_1 = \eta_2 \).

Using the identity that
\[ (S_1 - S_2)(S_3 - S_4) = \frac{1}{2}[(S_1 - S_4)^2 + (S_2 - S_3)^2 - (S_1 - S_3)^2 - (S_2 - S_4)^2]. \]

Equation (5b) can be written in terms of the phase structure function, \( D_s(\rho) \).

\[ B_a(y) = \lim_{\eta_1 \to 0} \frac{1}{2k^2} \left[ D_s(y_1 + \eta_1) + D_s(y_1 - \eta_1) - 2D_s(y_1) \right]. \]

Expanding \( D_s(y_1 \pm \eta_1) \) in a Taylor series we have
\[ D_s(y_1 \pm \eta_1) = D_s(y_1) \pm \eta_1 D_s'(y_1) + \frac{1}{2} \eta_1^2 D_s''(y_1) + \cdots \]

which, when substituted into Eq. (5d), yields \([5, (40.35)]\) written as Eq. (5f)

\[ B_a(y_1) = \frac{1}{2k^2} D_s''(y_1) \]

where \( D_s(y) \) is the phase structure function and the primes denote derivatives with respect to the argument.

The azimuth angle of arrival, \( \beta \), shown in Fig. (5b) is calculated in a similar manner. Using the points shown in Fig. (5b) we have

\[ \beta_1 = \lim_{\zeta_1 \to 0} \frac{(S_1 - S_2)}{k\zeta_1} \]
(6b) \[ B_2 = \lim_{\zeta_2 \to 0} \frac{(S_3 - S_4)}{k\zeta_2} \]

where

\[ S_1 = \text{phase at the point } (z, -\zeta_1/2, 0) \]
\[ S_2 = \text{phase at the point } (z, +\zeta_1/2, 0) \]
\[ S_3 = \text{phase at the point } (z, -\zeta_2/2, y_1) \]
\[ S_4 = \text{phase at the point } (z, +\zeta_2/2, y_1) \]

and with \( \zeta_1 = \zeta_2 \)

(6c) \[ B(y_1) = \lim_{\zeta_1 \to 0} \frac{1}{k^2\zeta_1^2} \langle (S_2 - S_1)(S_4 - S_3) \rangle. \]

Using the identity in Eq. (5c) we again have the correlation in terms of the phase structure function. The only difference is that

(6d) \[ \langle (S_1 - S_4)^2 \rangle = D_s \left( \frac{\sqrt{y_1^2 + \zeta_1^2}}{y_1} \right). \]

Then Eq. (6c) becomes

(6e) \[ B(y_1) = \lim_{\zeta_1 \to 0} \frac{1}{k^2\zeta_1^2} \left[ D_s \left( \sqrt{y_1^2 + \zeta_1^2} \right) - D_s(y_1) \right]. \]

Since the limit will force \( \zeta \) to be arbitrarily small the square root in Eq. (6e) can be expanded in a power series.

(6f) \[ \left( \sqrt{y_1^2 + \zeta_1^2} \right) = y_1 + \frac{\zeta_1^2}{2y_1} + \cdots. \]
Using the Taylor's series expansion in Eq. (5e) we have

\[ D_s \left( \sqrt{y_1^2 + z_1^2} \right) = D_s(y_1) + \frac{z_1^2}{2y_1} D_s'(y_1) + \cdots \]  

When substituted into Eq. (6e) the resulting expression is \([5,(40.38)]\), written as Eq. (6h)

\[ B_\theta(y_1) = \frac{1}{2k^2} \frac{D_s'(y_1)}{y_1} \]

Comparing Eqs. (5f) and (6h) we see that the interpair displacement with respect to the pair separation determines the final form for \( B_\alpha(y_1) \) and \( B_\beta(y_1) \). To calculate \( B_\alpha(y_1) \) the displacement was parallel to the separation. On the other hand, the displacement was transverse to the separation when \( B_\beta(y_1) \) was calculated.

The dependence on the interpair displacement when the interpair direction is neither parallel or perpendicular to the interpoint direction has been found explicitly by Strobehn and Clifford.[7] For sample points oriented as in Fig. 6 we have

![Fig. 6. Generalized geometry to calculate small aperture arrival angle correlation functions.](image-url)
\[ B_e(\rho) = \frac{1}{2k^2} D_s''(\rho) \cos^2\gamma + \frac{1}{2k^2} \frac{D_s'(\rho)}{\rho} \sin^2\gamma \]

where

\( e = \) angle of arrival of interest
\( \gamma = \) angle between interpair displacement and pair separation.

For \( \gamma = 0 \) and \( \pi/2 \) we have the expressions for \( B_a(\rho) \) and \( B_B(\rho) \) respectively.

The smallness criterion for the aperture size can now be considered. Recall that it was assumed that the aperture size had no effect on the angle of arrival. From Eqs. (5f) and (6h) we see that for \( D_s(\rho) \alpha \rho^2 \), the angle of arrival correlation is independent of \( \rho \). It is well known that for the Kolmogorov refractive index fluctuation spectrum, \( D_s(\rho) \) has this form for \( \rho \) less than the turbulence inner scale. Thus, these results apply to apertures smaller than the inner scale. For apertures larger than this, the large aperture method should be used.

In this section the basic equations for small aperture angle of arrival correlation functions have been derived. These expressions will be used in sections V and VI.

IV. LARGE APERTURE ANGLE OF ARRIVAL CALCULATIONS

In this section pertinent formulas for the large aperture angle of arrival mean square value are derived and discussed. The linear phase and constant amplitude assumptions made for the small aperture angle of arrival calculations begin to break down as the aperture becomes larger than the inner scale. The phase front becomes more distorted than can be accurately described with a linear wave tilt. Secondly, amplitude distortions become significant. These problems have been considered in part by various workers.

Two major approaches are used, both involving an extremum technique. The first approach to define the instantaneous arrival angle as the direction for which the instantaneous power is the maximum. The second is to fit expression for the instantaneous wave-front to a series of polynomial surfaces using a least squares fit.
One of the polynomials is a flat plane, so that the normal to the plane then is taken as the direction of the wave. Both approaches give the same basic expressions for the instantaneous angle of arrival direction. The derivations will now be outlined.

The first approach to be considered is that which uses the direction of maximum instantaneous power to define angle of arrival[10]. The technique is to split the mathematical expression for the electromagnetic phase into two parts, one representing a plane wave and the other representing the deviations from the plane wave. The direction of the plane wave normal is chosen to maximize the expression for the power. It is assumed that the deviations of the electromagnetic phase from the plane wave value are small and that amplitude fluctuations over the input aperture are negligible. Thus, let the field arriving at the receiving aperture be

\[ E e^{i\phi(\vec{r})}, \]

where \( \vec{r} = (x,y) \) denotes the transverse position of the input aperture. The term \( \phi(\vec{r}) \) is equivalent to \( S(x,y,z) \) in the previous section. Then the component of the field in the direction with transverse spatial frequency \( \vec{\kappa} \) is

\[ e(\vec{\kappa}) = \int d\vec{r} E e^{i(\phi(\vec{r}) - \vec{\kappa} \cdot \vec{r})} \]

and the power in the same direction is

\[ P(\vec{\kappa}) = \frac{1}{4\pi} \left| \int d\vec{r} E e^{i(\phi(\vec{r}) - \vec{\kappa} \cdot \vec{r})} \right|^2. \]

Following Heidbredder,[10] we define \( \Delta(\vec{r}) \) the phase difference from a flat plane,

\[ \Delta(\vec{r}) = \phi(\vec{r}) - \vec{\kappa} \cdot \vec{r}. \]

Our first object is to find an expression for the value of \( \vec{\kappa} \) for which \( P(\vec{\kappa}) \) will be maximum. We assume that the phase difference \( \Delta(\vec{r}) \) from that optimum value will be small. Then, rewriting Eq. (9),
where the \( r \) dependence of \( \Delta \) is implied. Differentiating Eq. (10c) with respect to \( \kappa_x \) to find the optimum value we find

\[
\frac{\partial P}{\partial \kappa_x} = 0 = 2 \int d\vec{r}_1 \left( 1 - \frac{\Delta^2}{2} \right) E \int d\vec{r}_2 \frac{\partial \Delta}{\partial \kappa_x} E + \int d\vec{r} \Delta E \int d\vec{r}_2 \frac{\partial \Delta}{\partial \kappa_x} E.
\]

\[
\frac{\partial P}{\partial \kappa_x} = 2 \int d\vec{r}_1 E \int d\vec{r}_2 (\phi_2 - \kappa \cdot \vec{r}_2) x_2 E
\]

\[
- \int d\vec{r} E \int d\vec{r}_2 E x_2.
\]

The aperture is assumed to be symmetric so that the integrals over odd powers of \( x \) and \( y \) are zero. Thus the integral in the second term and the integral over \( x_2 y_2 \) in the first term vanish. Solving for \( \kappa_x \) then gives

\[
\kappa_x = \frac{\int d\vec{r} \phi(\vec{r}) x E}{\int d\vec{r} x^2 E}.
\]
The mean value of $\kappa_X$ is taken to be zero. This is based on the assumption that the atmospheric turbulence is isotropic so that the mean value of $\phi(r)$ is zero, i.e., fluctuations of phase are equally likely.

The variance of $\kappa_X$ and the associated component of the angle of arrival, $\alpha_X$

\begin{equation}
\alpha_X = \frac{\kappa_X}{k}
\end{equation}

are given by

\begin{equation}
<\alpha^2_X> = \frac{<\kappa^2_X>}{k^2} = \frac{\int \int d\vec{r_1} d\vec{r_2} E_1 E_2 \langle \phi(\vec{r_1}) \phi(\vec{r_2}) \rangle x_1 x_2}{k^2 \int d\vec{r} x^2 E^2}.
\end{equation}

Identifying the phase correlation function

\begin{equation}
B_s(r_2-r_1) = \langle \phi(r_1) \phi(r_2) \rangle
\end{equation}

relating it to the phase structure function

\begin{equation}
B_s(r_2-r_1) = \frac{1}{2} [D_s(\infty) - D_s(r_2-r_1)]
\end{equation}

and again using the fact that the aperture is symmetric gives

\begin{equation}
<\alpha^2_X> = \frac{1}{2k^2} \frac{\int \int d\vec{r_1} d\vec{r_2} E_1 E_2 D_s(r_2-r_1) x_1 x_2}{\int d\vec{r} x^2 E^2}.
\end{equation}

Equation (16) is generally evaluated for circular apertures. To obtain the appropriate expression for that case first define
\[
\begin{align*}
    r_1 &= |\overline{r}_1| & r_2 &= |\overline{r}_2| \\
    x_1 &= r_1 \cos \theta_1 & x_2 &= r_2 \cos \theta_2
\end{align*}
\]

so that the denominator of Eq. (16) is (taking the amplitude \( E \) to be constant),

\[
(17) \quad 2k^2 \left[ \int_0^{2\pi} \int_0^R \overline{r}_1 \, d\theta \, r_1^3 \cos^2 \theta \, E \right]^2 = 2k^2 \left( \frac{R^4}{4} \right)^2 E^2.
\]

The expression for the arrival angle variance is then

\[
(18) \quad \langle a_x^2 \rangle = - \frac{2k^2}{2k^2 \left( \frac{R^4}{4} \right)^2 E^2} \left[ \int_0^{2\pi} \int_0^R \overline{r}_1 \, d\theta \, r_1^2 \right] \left[ \int_0^{2\pi} \int_0^R \overline{r}_2 \, d\theta \, r_2^2 \right] E_1 E_2 \cos(\theta_1 - \theta_2) \mathcal{D}_s((r_1^2 + r_2^2) - 2r_1 r_2 \cos(\theta_2 - \theta_1))
\]

After further defining

\[
\gamma = \theta_2 - \theta_1 \\
\beta = 1/2(\theta_2 + \theta_1)
\]

and performing the \( \beta \) integral, the result is,

\[
(19) \quad \langle a_x^2 \rangle = - \frac{\int_0^R \int_0^R \overline{r}_1^2 \, d\gamma \, r_1^2 \int_0^\pi \cos \gamma \, D_s((r_1^2 + r_2^2) - 2r_1 r_2 \cos \gamma)}{\pi \kappa^2 (R^4/4)^2}. 
\]
Equation (19) will be used in the next section for numerical evaluations. Heidbreder evaluates Eq. (10) with various symmetric aperture weighting functions $W(r_1)W(r_2)$ multiplying the phase structure function finding relatively small changes for gaussian weighting functions with half-widths the order of the radius. He also finds some small correlation between the plane wave fluctuations of the phase front and the residual phase for a one dimensional aperture (infinitesimal in the other dimension).

The second approach, due to Fried,[9] in which the expression for the instantaneous wavefront is fitted to a series of polynomials will now be considered. The polynomials are similar to the Zernike polynomials, orthogonal over a circular area. The first six are

\[
\begin{align*}
F_1(r) &= \frac{1}{\sqrt{\pi R^2}}, & F_4(r) &= \frac{(x^2 - y^2)}{\sqrt{\pi R^6/12}} \\
F_2(r) &= \frac{x}{\sqrt{\pi R^4/4}}, & F_5(r) &= \frac{(x^2 - y^2)}{\sqrt{\pi R^6/6}} \\
F_3(r) &= \frac{y}{\sqrt{\pi R^4/4}}, & F_6(r) &= \frac{xy}{\sqrt{\pi R^6/24}}
\end{align*}
\]

they are orthonormal so that

\[
\int d\vec{r} W(\vec{r}) F_n(\vec{r}) F_m(\vec{r}) = \delta_{mn}
\]

where

\[
W(\vec{r}) = \begin{cases} 
1 & \vec{r} \leq R \\
0 & \vec{r} > R
\end{cases}
\]
The wavefront \( \phi(\mathbf{r}) \) is then expressed as an infinite series of these polynomials:

\[
\phi(\mathbf{r}) = \sum_{n=1}^{\infty} a_n F_n(\mathbf{r}).
\]

The second and third polynomials represent the flat of the wavefront. Thus the spatial frequency and arrival angle associated with the flat component are given by

\[
(23a) \quad \vec{\alpha} = \frac{\vec{k}}{k} = \overline{\text{grad}} (a_2 F_2(\mathbf{r}) + a_3 F_3(\mathbf{r}))
\]

\[
= \frac{1}{\pi R^4/4} (a_2 \hat{x} + a_3 \hat{y})
\]

(\( \hat{x} \) and \( \hat{y} \) are unit vectors.) Further, the variance of the angle of arrival in the x direction is

\[
(24) \quad \langle a_x^2 \rangle = \frac{1}{\pi R^4/4} \langle a_2^2 \rangle.
\]

The polynomial series is fitted to the wavefront at any instant by choosing the coefficients so as to minimize the mean square deviation between the series and the wavefront proper:

\[
(25) \quad \frac{d}{da_2} \int d\mathbf{r} \, W(\mathbf{r}) \left[ \phi(\mathbf{r}) - \sum_{n=1}^{\infty} a_n F_n(\mathbf{r}) \right]^2 = 0.
\]

Squaring the term in the brackets, performing the indicated differentiation, and solving for \( a_2 \) gives

\[
(26) \quad a_2 = \int d\mathbf{r} \, W(\mathbf{r}) \, \phi(\mathbf{r}) \, F_2(\mathbf{r}).
\]
Equation (26) can be shown to be comparable with the expression by Heidbreder. To do that, use the expression for $F_2(r)$ in Eq. (20) to give $a_2$ in Eq. (26) and substitute into Eq. (24). After using Eq. (15) to simplify, the result is

$$<a^2_x> = \frac{1}{2k^2(\pi R^2/4)^2} \iint d\vec{r}_1 d\vec{r}_2 W(\vec{r}_1) W(\vec{r}_2) x_1 x_2 D_s(|\vec{r}_2-\vec{r}_1|)$$

which coincides very nicely with Eq. (18).

Fried and Heidbreder both evaluate the angle of arrival variance assuming that the wave structure function is approximated by the phase structure function, i.e., the range is sufficiently short that amplitude effects are negligible:

$$D_w(r) = D_s(r) = 6.88(\frac{r}{r_o})^{5/3}$$

where $r_o$ is Fried's transverse coherence length given by

$$r_o = (6.88/2.91 k L_{\text{C}}^2 N^{3/5}).$$

The result is

$$<a^2_x> = \frac{7.064}{k^2} r_o^{5/3} D^{1/3}$$

$$= 1.026 \times 6.88/k^2 r_o^{5/3} D^{1/3}.$$

Equation (30b) agrees closely with the approximate expression derived by Hufnagel[8] (Eq. (3e)).

To account for amplitude effects the phase structure function can be replaced with the wave structure function and the polynomial series now has complex coefficients[9]. One drawback as explained by Fried is that the results are difficult to interpret.
Concluding, we see that the angle of arrival for a large aperture depends on the phase structure function also. Thus, these results are as general as the approximations used to calculate the phase structure function. A specific phase structure function more general than Eq. (28) will now be used to calculate extended numeric results.

V. NUMERIC CALCULATIONS OF ANGLE OF ARRIVAL

In this section the general expressions given previously for arrival angle mean square and correlation functions are evaluated. The effects of such parameters as outer and inner scale, range and type of wave are demonstrated for both large and small aperture cases. The range of parameters covered is such as to give a reasonably complete set of curves for comparison with experiments 10.6B performed at the RADC Laser Propagation Range.

To proceed with the evaluation, we note that the angle of arrival statistical functions in Eqs. (2c), (5f), and (6h) are all in terms of the phase structure function. To evaluate these expressions an integral form of $D_s(p)$ is used,

$$D_s(p) = 4\pi^2 k^2 L \int_0^\infty \left[ 1 - J_0(\kappa p) \right] \left[ 1 + \frac{k}{\kappa L} \sin \frac{\kappa^2 L}{k} \right]^2 \Phi_N(\kappa) \kappa \, d\kappa$$

where

- $k = \frac{2\pi}{\lambda}$
- $p$ = separation
- $L$ = propagation range
- $\Phi_N(\kappa)$ = three dimensional refractive index fluctuation spectrum.

For a spherical wave, a similar expression was derived by Carlson and Ishimaru.[16]
Both of these expressions are the result of using the Rytov approximation and are valid where the Rytov approximation is valid. This validity will be discussed later.

Both expressions (31) and (32), use the refractive index fluctuation spectrum $\Phi_N(\kappa)$. Several models for $\Phi_N$ have been proposed and these are discussed in Appendix A. For the calculations in this report, the model used is

\begin{equation}
\Phi_N(\kappa) = 0.033 C_n^2 \frac{e^{-\kappa^2/\kappa_m^2}}{(\kappa^2 + \kappa_0^2)^{11/6}}
\end{equation}

where

\begin{align*}
\kappa_m &= 5.92/\kappa_0 \\
\kappa_0 &= \text{inner scale of the turbulence} \\
\kappa_0 &= 1/L_0 \\
L_0 &= \text{outer scale of the turbulence}.
\end{align*}

Using Eq. (31) in the expressions derived earlier in Eqs. (2c), (5f), and (6h) we have for the small aperture angle of arrival expressions

\begin{equation}
<\alpha^2> = \pi^2 L \int_0^\infty \left[ 1 + \frac{k}{\kappa L} \sin \left( \frac{\kappa L}{k} \right) \right] \kappa^3 \Phi_N(\kappa) \, d\kappa
\end{equation}
\begin{align}
B_\beta (\rho) &= 2\pi^2 L \int_0^\infty \left[ 1 + \frac{k}{k^2 - L} \sin \frac{\kappa^2 L}{k} \right] \frac{J_0(\kappa \rho)}{\kappa^3} \phi_N(\kappa) \, d\kappa \\
B_\alpha (\rho) &= 2\pi^2 L \int_0^\infty \left[ 1 + \frac{k}{k^2 - L} \sin \frac{\kappa^2 L}{k} \right] J_0(\kappa \rho) \kappa^3 \phi_N(\kappa) \, d\kappa - B_\beta (\rho).
\end{align}

Similar expressions were also obtained by Stroehn. The corresponding equations for the spherical wave case can be found by making the substitution

\begin{align}
L \left[ 1 + \frac{k}{k^2 - L} \sin \frac{\kappa^2 L}{k} \right] \phi_N(\kappa) &\rightarrow 2 \int_0^L \left[ \frac{L}{\kappa} \cos \left( \frac{L(\kappa - n^2)^2}{2\kappa n} \right) \right]^2 \\
&\cdot \phi_N(n) \, dn
\end{align}

in Eqs. (34)-(36). These expressions are suitable for numeric evaluation since \( \phi_N(\kappa) \) approaches zero rapidly for a finite value of \( \kappa \) so that the upper limit of the integrals can be made finite.

As noted in Appendix B, when the outer scale, \( L_0 \) becomes infinite, Eq. (33) reduces to the Kolmogorov spectrum with inner scale, \( \varepsilon_0 \), and Eqs. (32) and (34)-(36) can be integrated analytically. Thus, defining

\[ D = \frac{k m^2 L}{2} \quad \text{and} \quad g = \frac{k m^2}{4}, \]

we have
\begin{align*}
B_{B}(\rho) &= 1.028 L C_{N}^{2} \kappa_{m}^{1/3} \left\{ \frac{5}{6} F_{1}(\frac{1}{6}, 2, -g) \\
&\quad + D^{-1/6} \left[ \cos \frac{\pi}{12} \Re \left\{ F_{1}(\frac{-5}{6}, 2, -g) \right\} + \sin \frac{\pi}{12} \Im \left\{ F_{1}(\frac{-5}{6}, 2, -g) \right\} \right] \right\} \\
B_{a}(\rho) &= B_{B}(\rho) - 0.226 L C_{N}^{2} \kappa_{m}^{7/3} \left\{ \frac{1}{6} F_{1}(\frac{7}{6}, 3, -g) \\
&\quad + D^{-7/6} \left[ \sin \frac{\pi}{12} \Re \left\{ F_{1}(\frac{1}{6}, 3, -g) \right\} - \cos \frac{\pi}{12} \Im \left\{ F_{1}(\frac{1}{6}, 3, -g) \right\} \right] \right\} \\
\langle \alpha^{2} \rangle &= \langle \beta^{2} \rangle = L C_{N}^{2} \kappa_{m}^{1/3} (0.904 + 1.088 D^{-1/6} \cos \frac{\pi}{12}).
\end{align*}

For large \( \rho \), the asymptotic forms are

\begin{align*}
B_{B}(\rho) &\approx 2.43 L C_{N}^{2} \rho^{-1/3} \\
B_{a}(\rho) &\approx 1.62 L C_{N}^{2} \rho^{-1/3}.
\end{align*}

The small aperture angle of arrival mean square and correlation function were calculated for many cases. These will now be presented. Equations (38), (39), and (40) were evaluated for several typical values of range and inner scale. The results are shown in Figs. 7-34. The parameters used were chosen to agree with those that might be present at the RADC Laser Propagation Range. For example the ranges of 304.8 m, 914.4 m and 1524 m correspond respectively to 1000 ft, 3000 ft and 5000 ft, all ranges used at the RADC Range.
First the mean square angle of arrival is considered. Figures 7 and 8 show it as a function of range, first for two inner scales and infinite outer scale as given in Eq. (40), and second for three different outer scales taking the inner scale as 1 mm and numerically integrating Eqs. (33) and (34). For simplicity the mean square is normalized to the refractive index structure parameter, $C_n^2$. It is noted that the mean square increases as the outer scale increases and as the inner scale decreases. In either case the range of turbulent spatial frequencies interacting with the light beam has increased, thus increasing the effect. Of the two scales the inner scale has the larger effect on the arrival angle fluctuations, because, while the smaller scale eddies are less prevalent, they scatter light at the greatest angles.

Typical angle of arrival correlation curves are shown in Figs. 9, 10, and 11, evaluated from Eqs. (34), (35), and (36). They show the effect of inner scale on the correlation functions. Figures 9 and 10 show the autocorrelation functions $B_{\alpha}(p)/<\alpha^2>$ and $B_{\theta}(p)/<\alpha^2>$ for a range of 304.8 m, infinite outer scale and inner scales of 1 cm and 1 mm. Figure 11 shows the left hand portion of Fig. 10 in detail to display the fine structure of the initial roll-offs. The rapid roll-offs of the autocorrelations, as the separation is increased slightly from zero, is evident. However, as the separation exceeds the inner scale, the curves level off. The roll-off and break point are strongly dependent on the inner scale.

The inner scale dependence can be explained. As noted earlier for $p < s_o$, the phase structure function obeys a square power law and the correlations $B_{\phi}(\rho)$ and $B_{\alpha}(\rho)$ are equal to the mean square angle of arrival. From a physical standpoint we note that the inner scale is a characteristic dimension of the smallest blob in a turbulence cell. Thus, we expect the angle of arrival correlation for $p < s_o$ to be nearly the same as the mean square value. It is very likely, that the two regions considered on the wavefront were perturbed by the same blob or a closely correlated blob. As the separation is increased and approaches the inner scale the number of blobs increases rapidly in comparison to the increase in separation. However, as the separation becomes sufficiently large the total number of blobs masks the increase in blobs. Thus, the roll-off of the correlation slows down and levels off.

The inner scale dependence can be derived from the expressions used earlier. Since we are dealing with an inner scale phenomenon the outer scale can be ignored and the analytic expressions are used. Thus, for $p << s_o$ we have

$$\frac{B_{\theta}(\rho)}{<\theta^2>} = 1 - 0.0206 \cdot \frac{p^2}{m^2}.$$
Fig. 7. Small aperture mean square angle of arrival versus range for infinite outer scale and two inner scales.
Fig. 8. Small aperture mean square angle of arrival versus range for two inner scales and three outer scales.
Fig. 9. Small aperture elevation angle of arrival autocorrelation function for two inner scales.
Fig. 10. Small aperture azimuth angle of arrival autocorrelation function for two inner scales.
Fig. 11. Comparison of exact and approximate equations for the small aperture azimuth angle of arrival autocorrelation function.
within a correction of $D^{-1/6} \cos \pi/12$. For $\rho \ll L_0$, we see that $K \approx \rho$ will be less than one and from Eq. (43), $B_\theta(\rho) = \langle \delta \theta^2 \rangle$ as expected.

A comparison of the exact result and the approximate result is shown in Figs. 11 and 12. Figure 11 shows both cases of $\ell_0 = 1$ mm and 1 cm and Fig. 12 is a magnified view of the case of $\ell_0 = 1$ cm. The structure function, $D_N(\rho)/C_N^2$, is also shown in Fig. 12. We see that the approximation given in Eq. (43) breaks down around the point where $D_N(\rho)$ changes from a $r^2$ dependence to $r^{2/3}$ dependence as expected.

The information in Figs. 9, 10, and 11 is repeated in Figs. 13-16, along with other curves to show the effect of varying the range. Instead of plotting the curves on linear scale which has a rapid variation near zero separation, the curves are plotted on log-log paper. Also, instead of plotting the autocorrelation which suppresses the range information, the correlation function is plotted, normalized for convenience to the turbulent structure parameter. The fact that the curves are identical except for a vertical displacement indicates that the range information is primarily in the mean square for infinite outer scale.

The effects of the outer scale obtained from computer evaluation of Eqs. (33)-(36) are shown in Figs. 17-20. The azimuth and elevation angle correlation are shown in Figs. 17 and 18 for a range of 304.8 m, inner scale of 1 mm and three values of the outer scale. The information is repeated in Figs. 19 and 20 as log-log plots of the two arrival angle correlation functions normalized to the turbulence structure parameter, $C_N^2$. The effect of the outer scale is shown quite dramatically. The elevation angle of arrival, $B_\phi(\rho)/\langle \alpha^2 \rangle$ in Fig. 17 shows a slightly negative correlation for $\rho = L_0$. This negative correlation is sufficiently large that it is not believed to be an artifact of the computation. Further showing the effect of outer scale, the azimuth angle of arrival correlation $B_\theta(\rho)/\langle \alpha^2 \rangle$ shown in Fig. 18 falls below 5% of the mean value for $\rho = L_0$.

The effects of range with a finite outer scale are shown in Figs. 21 and 22 where we see the normalized correlation functions plotted for an outer scale of 1 m, and inner scale of 1 mm and ranges of 304.8 m ft, 914.4 m ft and 1524 m ft. Figures 23 and 24 contain the same information except with an inner scale of 1 cm. It is noted that in these cases the outer scale dependence is almost entirely determined by the variation of the mean square arrival angle caused by the outer scale. Thus, the correlation function is almost directly proportional to range.

The preceding graphs have all been for a plane wave. The equivalent graphs are also presented for a spherical wave in Figs. 25-34. Thus Figs. 25 and 26 give the mean square arrival angle for two outer scales for 1 mm and 1 cm inner scales respectively. Figures 27 and 28 give the normalized arrival angle correlation functions for three ranges for an infinite outer scale and inner scale of 1 mm.
Fig. 12. Magnified comparison of exact and approximate equations for the small aperture azimuth angle of arrival auto-correlation function and the refractive index fluctuation structure function.
Fig. 13. Small aperture elevation angle of arrival correlation versus separation for plane wave input, $L_0 = \infty$, $t_0 = 0.01$ m.
$l_0 = \infty$

$l_0 = 0.01 \text{m}$

$\lambda = 10.6 \mu$

PLANE WAVE

Fig. 14. Small aperture azimuth angle of arrival correlation versus separation and large aperture mean square angle of arrival versus aperture radius for plane wave input. $l_0 = \infty$, $s_0 = 0.01 \text{ m.}$
Fig. 15. Small aperture elevation angle of arrival correlation versus separation for plane wave input, \( L_0 = \infty, \ell_0 = 0.001 \, \text{m} \).
Fig. 16. Small aperture azimuth angle of arrival correlation versus separation and large aperture mean square angle of arrival versus aperture radius for plane wave input, $L_0 = \infty$, $\lambda_0 = 0.001$ m.
Fig. 17. Small aperture elevation angle of arrival auto-correlation versus separation for plane wave input and three outer scales.

Fig. 18. Small aperture azimuth angle of arrival auto-correlation versus separation for plane wave input and three outer scales.
Fig. 15. Small aperture elevation angle of arrival correlation versus separation for plane wave input and three outer scales.
Fig. 20. Small aperture azimuth angle of arrival correlation versus separation and large aperture mean square angle of arrival versus aperture radius for plane wave input and three outer scales.
Fig. 21. Small aperture elevation angle of arrival correlation versus separation for plane wave input $L_0 = 1 \text{ m}$, $\lambda_0 = .001 \text{ m}$. 

$B_a (\rho) / C_n^2$ 

$B_a (\rho) / C_n^2 = \left( \frac{\rho^2}{\lambda_0^2} \right)^{2/3}$ 

$L = 914.4 \text{ m}$ 
$L = 1524 \text{ m}$ 
$L = 304.8 \text{ m}$ 

$\rho (\text{m})$ 

$\rho (\text{m}) = 10^{-4}$, $10^{-3}$, $10^{-2}$, $10^{-1}$, 1.0, 10 

Scale 

NOTE: DASHED LINE INDICATES A NEGATIVE CORRELATION.

$L_0 = 1 \text{ m}$ 
$\lambda_0 = 0.001 \text{ m}$ 
$\lambda = 10.6 \mu$

PLANE WAVE
Fig. 22. Small aperture azimuth angle of arrival correlation versus separation and large aperture mean square angle of arrival versus aperture radius for plane wave input, $L_0 = 1\, \text{m}$, $\rho_0 = 0.001\, \text{m}$. 
Fig. 24. Small aperture azimuth angle of arrival correlation versus separation and large aperture mean square angle of arrival versus aperture radius for plane wave input, $L_0 = 1\text{ m}$, $\lambda_0 = 0.01\text{ m}$. 

- $L_0 = 1\text{ m}$
- $I_0 = 0.01\text{ m}$
- $\lambda = 10.6\text{ }\mu$ 

**PLANE WAVE**
Fig. 25. Small aperture mean square angle of arrival for spherical wave input, finite outer scale and two inner scales.
Fig. 26. Small aperture mean square angle of arrival for spherical wave input, infinite outer scale, and two inner scales.
Fig. 27. Small aperture elevation angle of arrival correlation versus separation for spherical wave input, $L_0 = \infty$, $\lambda_0 = 0.001$ m.
Fig. 28. Small aperture azimuth angle of arrival correlation and large aperture mean square angle of arrival versus aperture radius for spherical wave input, $L_0 = \infty$, $\ell_0 = 0.001$ m.
Figures 29 and 30 give the same data except that the inner scale is 1 cm. Figures 31 and 32 give the normalized arrival angle correlations for a 1 m outer scale and 1 mm inner scale for the three different ranges and Figs. 33 and 34 hold for 1 m outer scale, 1 cm inner scale and the three different ranges. These were all computed the same as the plane wave data, except with Eq. (32) replacing (31).

The accuracy of the computer evaluations of Eqs. (33)-(36) was checked by evaluating them for the case where \( L_0 = \infty \) and comparing with the closed form analytic expressions in Eqs. (38)-(40). Several orders of Gaussian quadrature integration were tried and compared to the closed form results as shown in Fig. 35. We see that the 96-point Gaussian quadrature integration used in the previous calculations provides good agreement.

Extensive results are also available for the large aperture arrival angle variance. Figures 36 and 37 are plots of mean square arrival angle normalized to turbulence structure parameter, \( C_n^2 \), plotted as a function of aperture radius. The numbers were obtained by numerically integrating Eq. (19) with the expression in Eq. (33) used for \( \phi_N(\kappa) \). In Fig. 36 curves are shown for plane waves for an inner scale of 1 mm, a range of 304.8 m ft and three values of outer scale. Figure 37 shows the effect of range for an outer scale of 1 m and an inner scale of 1 mm. These curves show changes with outer scale and range similar to the small aperture arrival angle correlation functions.

Verifying the accuracy for these results is much more difficult and time consuming since four numeric integrations are used. However, increasing the order of integrations used yielded results which differed in the third and higher order digits. Thus, these results are believed to be accurate to 3 digits. Higher accuracy can be obtained by increasing the Gaussian integration orders.

Further data is also available on large aperture mean square arrival angle, because, to a very good degree of approximation, the large aperture mean square arrival angle taken as a function of aperture radius, is equal to the small aperture azimuth arrival angle correlation function, \( B_0(\rho) \) taken as a function of point separation, \( \rho \). This synonymity is discussed in detail in the next section. However the results are used here to indicate the further data on large aperture arrival angle mean square available. Thus the even numbered figures between Fig. 14 and Fig. 24 can all be retitled to give large aperture mean square angle of arrival versus radius for the particular case for plane waves, and the even numbered figures between Fig. 27 and Fig. 34 can all be retitled to give the mean square arrival angle versus radius for the particular cases for spherical waves. Thus data is available for a large number of cases for the large aperture situation also. The discussion of the curves parallels that for the small aperture case.
Fig. 29. Small aperture elevation angle of arrival correlation versus separation for spherical wave input, $L_0 = \infty$, $\varepsilon_0 = 0.01$ m.
Fig. 30. Small aperture azimuth angle of arrival correlation and large aperture mean square angle of arrival versus aperture radius for spherical wave input, $L_0 = \infty$, $\lambda_0 = 0.01$ m.
Fig. 31. Small aperture elevation angle of arrival correlation versus separation for spherical wave input, $L_0 = 1 \text{ m}$, $\xi_0 = 0.001 \text{ m}$. 

$L_0 = 1 \text{ m}$  
$f_0 = 0.001 \text{ m}$  
$\lambda = 10.6 \mu$  
SPHERICAL WAVE
Fig. 32. Small aperture azimuth angle of arrival correlation versus separation and large aperture mean square angle of arrival versus aperture radius for spherical wave input, $L_0 = 1$ m, $d_0 = 0.001$ m.
Fig. 33. Small aperture elevation angle of arrival correlation versus separation for spherical wave input, $L_0 = 1 \text{ m}$, $\varepsilon_0 = 0.01 \text{ m}$. 
Fig. 34. Small aperture azimuth angle of arrival correlation versus separation and large aperture mean square angle of arrival versus aperture radius for spherical wave input, $L_0 = 1 \text{ m}$, $\ell_0 = 0.01 \text{ m}$.
Fig. 36. Large aperture mean square angle of arrival versus aperture radius for three outer scales.
Fig. 37. Large aperture mean square angle of arrival versus aperture radius for three ranges.
This concludes the presentation of data. Large and small aperture angle of arrival data predicting experimental results have been calculated for a range of typical cases. The effects of range, inner and outer scales, separation and aperture size have all been demonstrated.

Since all of the equations derived in this report are functions of \( D_s(p) \), the validity of these results depends on the validity of the Rytov approximation. This topic will now be briefly considered. The question has been the subject of many discussions [18-26] and the criterion which seems appropriate at this time to the calculations in this report is \( D_s(p) \ll \pi \). This result was reached by Tatarski from calculations of the second order term of \( D_s(p) \). [18] A similar result was also reached by DeWoif. [19] To quantify this result we consider \( \rho \) much greater than \( \lambda_0 \) and set [5, (47.37)]

\[
D_s(p) = 2.92 C_n^2 k^2 \rho^{5/3} L \ll \pi.
\]

For convenience, replace the "much less than \( \pi \)" with \( \pi/10 \). Thus, we can write Eq. (44) as

\[
\rho = \left( \frac{\pi}{29.2k^2} \right)^{3/5} (C_n^2 L)^{-3/5}.
\]

This is shown plotted in Fig. 38.

In Fig. 38 several range, \( C_n^2 L \), combinations are shown. We see that for the longest range used earlier, \( 1524 \text{ m} \) and for strong turbulence \( C_n^2 = 10^{-14} \text{ m}^{-2/3} \), the maximum permissible separation, \( \rho \), is approximately 9.2 cm. This means that the angle of arrival two-point correlation is valid for \( \rho < 9.2 \text{ cm} \) and the large aperture results are valid for \( R < 9.2 \text{ cm} \). As the range and turbulence levels decrease we note that the restrictions are relaxed.

It should also be noted that the form used for \( D_s(p) \) does not include saturation effects due to the outer scale. Since the validity criterion was derived independently of the exact form of the refractive index fluctuations saturation effects may extend the region of validity. Furthermore, it has been suggested the region of validity may be larger than calculated. [5], [27] Measurements by Bouricius and Clifford at \( .632 \mu \) [27] suggest this result and indicate that more study is required.
Fig. 38. Areas of validity for Rylov approximation.
To summarize the work presented in this section, data were presented predicting results of various angle of arrival experiments for various conditions. First the formulae to be used for the phase structure function and refractive index turbulence spectra were presented. Then expressions given in previous sections for small aperture arrival correlation functions were evaluated, and plotted. For infinite outer scale the expressions could be written as closed form expressions, for finite outer scale they had to be evaluated numerically. Both elevation and azimuth angle correlation functions, respectively \( B_e(\rho) \) and \( B_a(\rho) \), were calculated and graphed for plane waves for ranges 304.8 m, 914.4 m and 1524 m, outer scales of infinity, 1 m and 0.159 m, and inner scales of 0.001 m and 0.01 m. This data was then repeated for spherical waves. The large aperture mean square arrival angle was also evaluated from a previously presented formula. Only a few cases were evaluated because of a synonymity which allowed the small aperture azimuth arrival angle correlation curves as a function of separation, \( \rho \), to be relabeled with large aperture mean square arrival angle as a function of aperture radius, \( R \), thus allowing many of the results to serve double duty. The section was concluded with a discussion of the Rytov approximation on which the phase structure function was based.

VI. ANGLE OF ARRIVAL SYNONYMITY

This section deals explicitly with a relationship between the angle of arrival variance for large aperture receiver as a function of aperture radius, \( R \), and the small aperture azimuth correlation function, \( B_a(\rho) \) taken as a function of \( \rho \). The particular relationship is that they are synonymous over a sizable range of values of the argument. These two situations are shown in Fig. 39 to facilitate the comparison between the two. Figure 39a shows a standard large aperture arrival angle measurement situation. An atmospherically degraded beam from a point source is focussed to a degraded spot. The information about angle of arrival is obtained from the transverse motion of the spot, the arrival angle being proportional to the spot displacement. Figure 39b shows a possible situation for the measurement of small aperture arrival angle correlation function. The angles of arrival at two pinholes are determined independently by monitoring the two pinhole diffraction patterns, the motion of the center of each pattern being proportional to the individual arrival angle. In the case at hand, only the x component of motion is of interest. The pinholes are sufficiently small that the amplitude is constant over the hole. Then the quantity of interest is the correlation of arrival angles between these two pinholes, the correlation being a function of \( \rho \), the separation of the pinholes. In the case of interest the pinholes are separated in the direction perpendicular to the direction in which the spot motion is being monitored, (the x direction in Fig. 39).
Fig. 39. Comparison of the definitions of the large aperture mean square angle of arrival and the small aperture angle of arrival correlation.
Fig. 40. Comparison of the large aperture mean square angle of arrival and the small aperture azimuth angle of arrival correlation function.

\( L = 304.8 \text{ m} \)
\( \lambda_0 = 0.001 \text{ m} \)
\( \lambda = 10.6 \mu \)
Comparing Eqs. (46) and (47) we see that both expressions can be written in the form

\[ 4\pi^2L \int T(\kappa) F(\kappa) \phi_N(\kappa) \kappa^3 d\kappa \]

where \( T(\kappa) \) is the filter function peculiar to the particular operation, \( F(\kappa) \) identifies the type of wave, and \( \phi_N(\kappa) \) is the refractive index turbulence spectrum. Denoting the particular small and large aperture filter functions by \( SA(\kappa) \) and \( LA(\kappa) \) respectively, we have

\[ SA(\kappa) = J_1(\kappa)/\kappa \]

\[ LA(\kappa) = -\left(16/\pi R^8\kappa^2\right) \int_0^R r^2 dr \int_0^R \rho^2 d\rho \int_0^\pi \cos \gamma [1-J_0(\kappa\sqrt{r^2+\rho^2-2\rho\cos \gamma})] d\gamma \]

Equations (49) and (50) describe the manner in which the receiving aperture interprets the angle of arrival. Graphs of these functions are shown in Fig. 41. For convenience in comparison we shall take them both to be functions of the same variable to be called \( R \). We see in this figure that the two filter functions are synonymous over a wide range of \( \kappa R \).

The synonymity is limited with respect to the range of the variable \( R \) over which it applies. This occurs because of the relative size of the filter function and the other functions, \( F(\kappa)\phi_N(\kappa) \kappa^3 \) in the frequency integral. If the other functions are of significant size only when the filter functions are almost identical, i.e., in the region, \( 0 < \kappa R < 4 \), then the identity will hold. If the other functions have large contributions for \( \kappa R > 4 \) then the synonymity will break down.

To more quantitatively define the regions of applicability for the angle of arrival synonymity we define \( \epsilon(R) \), the error between the two functions relative to the small aperture correlation function,

\[ \epsilon(R) = \frac{B_B(R) - \langle a_x^2 \rangle}{B_B(R)} \]
Fig. 41. Comparison of the large and small aperture angle of arrival filter functions.
The numerator of Eq. (51) is given by an integral, $I_D(\kappa R)$ of the difference between Eqs. (46) and (47). Using the difference function $\Delta(\kappa R)$ given in Eq. (52) and shown in Fig. 42,

\begin{equation}
\Delta(\kappa R) = SA(\kappa R) - LA(\kappa R)
\end{equation}

the numerator integral is

\begin{equation}
I_D(\kappa R) = 4\pi^2 L \int_0^\infty \Delta(\kappa R) F_S(\kappa) \phi_N(\kappa) \kappa^3 \, d\kappa.
\end{equation}

For a plane wave, $F_S(\kappa)$ varies between 1 and 2 and has little effect, thus, making $\Delta(\kappa R)$ and $\kappa^3 \phi_N(\kappa)$ the two functions of most interest.

We now proceed to examine $\varepsilon(R)$ qualitatively, but in some detail to determine the approximate functional dependence. This is to be done by examining the curves for $SA(\kappa R)$, $LA(\kappa R)$, $\Delta(\kappa R)$, and $\kappa^3 \phi_N(\kappa)$ and the various ways they can combine for various values of $R$. These curves are plotted quite precisely in Figs. 41, 42, and 43, using for $\phi_N(\kappa)$ the Von Karman spectrum given in Eq. (33). The various functions are compared in Fig. 44 for several values of $R$. The effect of choosing various values of $R$ is to select particular horizontal scales for the filter functions $SA(\kappa R)$, $LA(\kappa R)$ and $\Delta(\kappa R)$. Thus, as seen in Fig. 41, $LA(\kappa R)$ goes essentially to zero at $\kappa = 4.4/R$ and $SA(\kappa R)$ has its first zero at $\kappa = 3.9/R$. Fig. 42 shows that the difference function, $\Delta(\kappa R)$ has its first zero at $\kappa = 3.4/R$, pretty much in the same region. Examining curves in Fig. 44a and we see that for $R << L_0$ the small and large aperture filter functions are nearly constant over the complete range of $\kappa^3 \phi_N(\kappa)$ so that $B_R(\rho)$ and $<\phi^2(\kappa)>$ are identical, each being given by that integral over $\kappa^3 \phi_N(\kappa)$. For $R = 4.4/z_0$, comparing Figs. 44a and 44c, we see that both $LA(\kappa R)$ and $SA(\kappa R)$ go to zero at about the same place that $\kappa^3 \phi_N(\kappa)$ becomes very small. For that case both curves are finite over the range $0-1/L_0$, the product at any point being less than the value of either curve. The integral over the product is then starting to decrease. This is the region of $R$ where the curves of $B_R(\rho)$ and $<\phi^2(\kappa)>$ start to decrease, as shown on graphs of these functions. Further the value of the error $\int \Delta(\kappa R) \kappa^3 \phi_N(\kappa) \, d\kappa$ has been increasing with $R$ and has possibly a local maximum because $\Delta(\kappa R)$ has a negative portion where $\kappa^3 \phi_N(\kappa)$ is still finite.
Fig. 42. Large and small aperture angle of arrival difference function versus spatial frequency.
Fig. 43. Modified refractive index fluctuation spectrum for finite outer scale versus spatial frequency.
Fig. 44. Comparison of large and small aperture filter functions with the modified refractive index fluctuation spectrum.
For $R$ in the region of $L_0$, comparing Figs. 44a and 44d shows that $\Delta(\kappa R)$ and $\kappa^3 \phi_\infty(\kappa)$ both have large positive portions in the region of $\kappa = 1/L_0$. One would expect a local maximum in the error around this region. As $R$ increases slightly past $L_0$, the first zero of the $\Delta(\kappa R)$ curve approximately coincides with the peak of the $\kappa^3 \phi_\infty(\kappa)$ curve so that the product is positive for $\kappa < 1/L_0$ and negative for $\kappa > 1/L_0$. One would expect the error to decrease for this region of $R$. Finally as $R$ becomes still larger, the error curve will oscillate. This expected qualitative behavior is sketched in Fig. 45.

Fig. 45. Integrated large and small aperture angle of arrival difference function.

One can further examine the relative error, $\varepsilon$, defined in Eq. (51). In particular for the case where $R < L_0$, the product of $L_A(\kappa R)$ and $\kappa^3 \phi_\infty(\kappa)$ becomes quite small, being limited for small $\kappa$ by $\kappa^3 \phi_\infty(\kappa)$ and for large $\kappa$ by $L_A(\kappa R)$. Then $<\alpha_x^2(\kappa)> \text{ in Eq. (51)}$ is relatively small and $\varepsilon(R)$ approaches unity. Thus the asymptotic value of $|\varepsilon(R)|$ is 1.00, i.e., 100% error. The expected qualitative behavior for $\varepsilon(R)$ is then plotted using the data on $\Delta(\kappa R)$, $SA(\kappa R)$ and the asymptotic value. The plot is given in Fig. 46.
These qualitative results are confirmed by the quantitative numeric evaluation in Eq. (53), (including $F_S(\kappa)$) shown in Fig. 47. These numeric calculations required considerable computation time so only enough points were calculated to show the nature of the curves. Thus, some oscillations have more than likely been suppressed and the final value of $\epsilon(R)$ was not reached. Accuracy in these calculations was difficult to obtain, however, the results are certainly correct to at least order of magnitude and demonstrate the behavior predicted earlier.

Three cases are shown in Fig. 47 corresponding to earlier calculations. The first characteristic we note is that $\epsilon(R)$ is fairly small for $R < L_0$. This implies that for aperture diameters less than twice the outer scale, the large aperture mean square angle of arrival can be approximated by the small aperture azimuth correlation function. From Fig. 47 we see that this error will be on the order of 10%. Second, we note the range has very little effect as is to be expected. What effect the range, $L$, does have is incorporated in $F_S(\kappa)$.

To summarize, in this section, we have stated that the large aperture angle of arrival mean square value and the small aperture azimuthal angle of arrival correlation function are synonymous. The functional dependence was demonstrated to be very nearly equal for typical cases. The relationship was examined in the spectral domain both qualitatively and quantitatively showing that the results should be expected to be well within ten percent for values of the argument up to the outer scale.
Fig. 47. Angle of arrival synonymity error function for two outer scales and two ranges.
VII. SUMMARY

In this report the subject of angle of arrival of light beams degraded by propagation through atmospheric turbulence has been considered. Data on arrival angle experiments was calculated for two types of experiments for a wide variety of cases and an equivalence between two types of angle of arrival experiments was demonstrated. In Chapter II the subject was introduced and reviewed, and the literature enumerated. It was pointed out that there were two approaches to angle of arrival considerations, one using a pair of pinhole apertures and the other using a single large aperture. In Chapter III the area of small aperture angle of arrival calculations was reviewed in detail, the basic theoretical expressions being derived. The calculation of large aperture angle of arrival was discussed in Chapter IV. In Chapter V the analytic expressions derived in Chapters III and IV were evaluated using a particular expression for the refractive index spatial spectrum, namely the Von Karman spectrum. The results were presented graphically for the two angle of arrival situations for variety of cases including various inner and outer scales. The Rytov approximation used in the derivation of the phase structure function used was considered and a graph given showing the expected range of validity the calculations. In Chapter VI a synonymity was presented between the large aperture angle of arrival mean square value expressed as a function of aperture radius and the small aperture azimuth angle of arrival correlation function taken as a function of aperture separation. This synonymity was used in Chapter V to simplify calculations and also offers the possibility of simplification of experimental measurements. The limitations on the synonymity were demonstrated.
APPENDIX A
DISCUSSION OF REFRACTIVE INDEX FLUCTUATION SPECTRA

Various forms of the refractive index fluctuation spectra have been used to find solutions of the wave equation for an inhomogeneous random media. Two forms which have been used recently are compared in this appendix. The choice of one of the constants is explained and several interesting characteristics noted. The derivations of these spectra are not presented since they are outside the scope of this report.

Kolmogorov theory predicts that the refractive index structure functions should have the following asymptotic behavior

\[ D_N(r) = \begin{cases} 
C_2 \varepsilon_0^{2/3} \left( \frac{r}{\varepsilon_0} \right)^2 & 0 < r < \varepsilon_0 \\
C_2 r^{2/3} & \varepsilon_0 < r < L_0 
\end{cases} \]

where

\( \varepsilon_0 = \text{inner scale of turbulence} \)
\( L_0 = \text{outer scale of turbulence} \)
\( C_2 = \text{structure constant of the turbulence}. \)

It can be shown that the Kolmogorov spectrum as used by Tatarski in [5] and given in Eq. (A2)

\[ \Phi_N(\kappa) = 0.033 C_N^2 e^{-\kappa^2/\kappa_m^2} \kappa^{-11/3} \]

\[ \kappa_m = \frac{5.92}{\varepsilon_0} \]

has a structure function of the form
\( D_N(r) = 1.67 c_N^2 \kappa_m^{-2/3} \left[ \, _1 F_1 \left( -\frac{1}{3}, \frac{3}{2}, -\frac{\kappa_m^2 r^2}{4} \right) \right] - 1 \)

where \(_1 F_1\) signifies a confluent hypergeometric function. Further, it can be shown the confluent hypergeometric function has the proper asymptotic behavior such that Eq. (A1) is satisfied.

Another form of \( \Phi_N(\kappa) \) is the Von Karman spectrum which includes outer and inner scale effects. Several algebraic forms have been used but they can be generalized to

\[
\Phi_N(\kappa) = \frac{0.033 c_N^2 e^{-\kappa^2/\kappa_m^2}}{(\kappa^2 + \kappa_m^2)^{1/6}}
\]

with

\[
\kappa_0 = \frac{A}{L_0}
\]

where \( A \) is a normalization constant. The value of \( A \) has been chosen as 1 in [17] and 2\( \pi \) in [28]. For \( L_0 = \infty \), Eq. (A4) reduces to Eq. (A3) as expected.

The value of \( A \) used in this report is 1. The reason for this choice will now be explained. We note that when \( A = 2\pi \), the effect is to normalize \( L_0 \) to a new value \( L_0' \) such that

\[
L_0' = \frac{L_0}{2\pi}.
\]

The physical interpretation of \( L_0 \) is that it is a characteristic dimension of saturation effects. To examine this situation the refractive index spectrum and structure function calculated from Eq. (A3) and Eq. (A4) and the resulting phase structure function are shown in Figs. A1-A3 respectively. Dashed lines have been added at \( \rho = 1/\kappa = L_0 \) for \( L_0 = 1 \) and \( L_0' = 1/2\pi \).

Two interesting characteristics are noted. First, all of the curves tend to saturate at a value predicted approximately by
Fig. A1. Refractive index fluctuation spectrum versus spatial frequency.
Fig. A2. Refractive index fluctuation structure function versus separation.
Fig. A3. Phase structure function versus separation.
the Kolmogorov spectrum at $p = 1/\kappa = L_0$. Secondly, the break point of the saturation occurs at the same point. These two characteristics are consistent with the original definition of the outer scale, $L_0$, and the use of $A = 2\pi$ merely clouds the issue. Thus, for this report $A$ is chosen to be one.

Another interesting characteristic is seen in Fig. A3. The slope of the phase structure function in the inertial subrange is not unique. Since the effect of the outer scale is to cause saturation the slope must change continuously from that of square law to zero. Only for the special case of $L_0 = \infty$ is the slope a constant. Therefore, the outer scale makes it difficult to predict a unique slope of the phase structure function in the inertial subrange.

Concluding, two refractive index fluctuation spectra have been described and the criteria used to determine one of the constants explained. Several interesting characteristics were demonstrated.
APPENDIX B
COMPUTER PROGRAM USED FOR NUMERIC CALCULATIONS

The computer programs used for numeric calculations in this report are presented in this appendix. Each program will be described briefly and data input explained. A listing of the source deck is also presented. All programs were written for use on an IBM 360/75 computer in FORTRAN IV(G). Some subroutines are contained in the IBM Scientific Subroutine Package (360-CM-03X) Version III.

1. Small Aperture Angle of Arrival Calculations

Three programs were written to make the small aperture calculations. The first program evaluates the analytic functions for the angle of arrival mean square and 2 pt. correlation function, using the Kolmogorov spectrum and plane wave input. The second program evaluates the integral expressions using the Von Karmann spectrum and plane wave input. The third program evaluates the integral expressions for spherical wave input. These programs also calculate the refractive index fluctuation spectrum and structure function, the phase structure function and the wave structure function. Each program has numerous comment cards to describe the program flow and the variable names correspond closely to variables used in the report.

(a). The computer program used to calculate the angle of arrival correlation functions for the Kolmogorov spectrum is shown in listing (B1). It evaluates the confluent hypergeometric functions for pure real or pure imaginary arguments using formulae taken from Tables of Functions, Jahnke and Emde.

The first card in the input data specifies the parameters, \( \epsilon_0 \), \( L \), and \( \lambda \). The variables are punched into ten column fields in decimal form. They are:

Columns

<table>
<thead>
<tr>
<th>Columns</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-10</td>
<td>( XLO = \varepsilon ), the inner scale of the turbulence in meters</td>
</tr>
<tr>
<td>11-20</td>
<td>( XL = L ), the range in meters</td>
</tr>
<tr>
<td>21-30</td>
<td>( XLAMB = \lambda ), the wavelength in meters</td>
</tr>
</tbody>
</table>

If \( XL = 0 \), execution is terminated. The following card specifies the starting point, increment, and number of increments for the separation distance. These variables are:
LISTING B-1

C ***** 10.6 PROPAGATION PHENOMENON PROGRAM *****
C ***** SMALL APERTURE ANGLE OF ARRIVAL CALCULATION FOR
C THE KOLMOGOROV SPECTRUM *****
C
C COMPLEX G,GD
REAL*8 RDYP2, XDYP2, RDYP3, XDYP3, B1, B2
C INITIALIZE NORMALIZATION CONSTANTS
BANOMK=1.
BBNORM=1.
BAGORM=1.
BBGORM=1.
C READ DATA
25 READ(5,900)XLO,XL,XLAB
IF(XLO.EQ.0.)CALL EXIT
50 READ(5,900)RHOS,DLRH,XNR
IF(XNR.EQ.0.)GO TO 25
900 FORMAT(3E10.3)
WRITE(6,1001)
1001 FORMAT('1')
C CALCULATE CONSTANTS
XX=6.2821/XLAB
XKM=5.94/XLO
D=XX**2*XL/XX
NR=XNR+1
A1=1.088*XL*XKM**.333333
A2=0.**(-.1666667)
A3=D**(-1.1666667)
B1=DCOS(3.1415927/12.)
B2=DSIN(3.1415927/12.)
WRITE(6,1002)IXX,IXKM,IX
1002 FORMAT('XX= ',E10.3,'XX=',E10.3,'DD=',E10.3)
C BEGIN ITERATION ON RH0
DO 100 I=1,NR
RH0=RHOS+DLRH*FLOAT(I-1)
G=XX**2*RH0**2/4.
XXD=(RH0**2*XX)/(4.*XL)
GD=(4.*XL)*XXD
C BRANCH TO ASYMMETRIC CALCULATION
IF(XXD.GT.100.)GO TO 75
C
C SMALL RHO FORMULATION
C
C CALCULATE CONFLUENT HYPERGEOEMTRIC FUNCTIONS
CALL HYPERG(1.6666,2.,-G,RHP1,XHP1)
CALL DHYPER(-.8333,2.,GD,RDYP2,XDYP2)
C PRINT VALUES OF CONFLUENT HYPERGEOMTRIC
WRITE(6,1003)RHP1,XHP1, RDYP2, XDYP2
1003 FORMAT(4(E10.3,5X))
C CALCULATE BETA CORRELATION
   \[ \text{BBETA} = A1 \times (B34 \times \text{RHYP}1 + (A2 \times (B1 \times \text{RDYP}2 + B2 \times \text{XDYP}2))) \]

C CALCULATE CONFLUENT HYPERGEOMETRIC FUNCTIONS
   CALL HYPERG(1.1666,2.,-G,RHYP2,XHYP2)
   CALL DHYPER(1.1667,3.,-G,RDYP3,XDYP3)

C PRINT VALUES OF CONFLUENT HYPERGEOMETRIC
   WRITE(6,1003) RHYP2, XHYP2, RDYP3, XDYP3

C CALCULATE ALPHA CORRELATION
   \[ \text{BALPHA} = \text{BBETA} - 0.2260 \times \text{RH0}^2 \times \text{XL} \times \text{XKM}^{2.333} \times (0.166667 \times \text{RHYP}2 \times A3 \times \text{BP3} + B2 \times \text{RDYP}3) \]
   GO TO 60

75 CONTINUE

C ASYMPTOTIC FORMULATION FOR LARGE RH0
   CALL HYPERG(1.1666,2.,-G,RHYP1,XHYP1)
   \[ \text{BBETA} = 1.81 \times \text{XL} \times \text{XKM}^{2.333} \times \text{RHYP1} \]
   CALL HYPERG(1.1666,3.,-G,RHYP2,XHYP2)
   \[ \text{BALPHA} = \text{BBETA} - 1.81 \times \text{XL} \times \text{XKM}^{2.333} \times \text{RH0}^2 / 24. \times \text{RHYP2} \]
   GO TO 60

60 CONTINUE

C NORMALIZE TO MEAN SQUARE ANGLE OF ARRIVAL
   \[ \text{BBN} = \text{BBETA} / \text{BBNORM} \]
   \[ \text{BAN} = \text{BALPHA} / \text{BANORM} \]

C PRINT RESULTS
   WRITE(6,1000) RH0, BBETA, BALPHA
   WRITE(6,1004)

   1004 FORMAT(T65,'CORRELATION NORMALIZED TO MEAN SQUARE VALUE ')
   WRITE(6,1000) RH0, BBN, BAN
   IF(RH0.NE.0.)GO TO 100
   BBNORM=BBETA
   BANORM=BALPHA
   BBGORM=BBETAG
   BAGORM=BALPHG

100 CONTINUE

1000 FORMAT(T65,'RH0=',E10.3,E10.3,'BBETA=',E10.3,E10.3,'BALPHA=',E10.3)
   GO TO 50

END
SUBROUTINE DHYPER(ALPHA,GAMMA,CZ,RHYP,IHYP)
C THIS SUBROUTINE CALCULATES THE CONFLUENT HYPERGEOMETRIC FUNCTION
C ARGUMENTS ARE ALPHA,GAMMA, AND CZ. RESULT IS (RHYP,IHYP) (COMPLEX)
C CZ IS ASSUMED TO BE PURE IMAGINARY
C
IMPLICIT COMPLEX(C),REAL*(A-B,D-O-Z)
REAL*4 ALPHA,GAMMA,ARGZ,ARGG,A,B,DMA,SIGN,Z
COMPLEX*16 CDZ,CRSUM,CSSUM,CRTERM,CSTERM,CF,DCONJG
INTEGER I
ARG(CZZ)=ATAN2(AIMAG(CZZ),REAL(CZ2))
C
1000 FORMAT('ON CONVERGENCE'/ 'CHYP=',2(1PE10.3,2X),5X,'IHYP=',2(1PE10.3,5X))
1001 FORMAT(* I=*,15,5X,* ALPHA=*,F10.5,5X,* GAMMA=*,F10.5,3X,* CZ=*,2(1PE10.3,2X),3X,* RHYP=*,E10.3,5X,* IHYP=*,E10.3)
C
C SMALL ARGUMENT EXPANSION
C
C SET UPPER LIMIT ON THE NUMBER OF TERMS
LMT=200
100 I=2
C
C CALCULATE REAL PART
C
RHYP=1.
IHYP=0.
Z=AIMAG(CZ)
C TEST FOR ZERO
IF(ABS(Z).LT.1.E-50)RETURN
C TEST FOR LARGE ARGUMENT
IF(ABS(AIMAG(CZ)).GT.3.)GO TO 500
RHYPD=ALPHA/GAMMA*Z*(ALPHA+1.)/(GAMMA+1.)*Z/2.*(-1.)
SRHYP=RHYPD
120 I=I+2
C CHECK NUMBER OF TERMS
IF(I.LT.LMT)GO TO 140
K=I-1
RHYP=-RHYPD*(ALPHA+K-1.)/(GAMMA+K-1.)*Z/FLOAT(I-1)*(ALPHA+K)/GAMMA
SRHYP=SRHYP+RHYP
C TEST FOR CONVERGENCE
IF(DABS(RHYP-RHYPD).LT.1.E-05)GO TO 130
RHYPD=RHYP
GO TO 120
C WRITE ERROR MESSAGE
140 WRITE(6,1000)RHYP,RHYPD,CHYPD
WRITE(6,1001)I,ALPHA,GAMMA,CZ,RHYP,IHYP
C
IHYP = 0.

C CALCULATE IMAGINARY PART

RHYP = SRHYP + 1.
I = 1
IHYP = ALPHA / GAMMA * Z
SIHYP = IHYP

220 I = I + 2

C CHECK NUMBER OF TERMS

IF (I .GT. LMT) GO TO 240
IHYP = - IHYP * (ALPHA + K - 1.) / (GAMMA + K - 1.) * Z / FLOAT (I - 1) * (ALPHA + K) / (GAMMA - ALPHA - K) * Z / FLOAT (I)
SIHYP = IHYP + SIHYP

C TEST FOR CONVERGENCE

IF (ABS (IHYP - IHYP) .LT. 1.E-05) GO TO 230
IHYP = IHYP
GO TO 220

C WRITE ERROR MESSAGE

240 WRITE (6, 1000) IHYP, IHYP, CHYPO
WRITE (6, 1001) I, ALPHA, GAMMA, CZ, RHYP, IHYP
IHYP = SIHYP
RETURN

500 CONTINUE

C ASYMPTOTIC EXPANSION FOR LARGE ARGUMENT

CDZ = CZ
CRSUM = I.
CSSUM = I.
CRTERM = I.
CSTERM = I.

C SET UPPER LIMIT ON THE NUMBER OF TERMS

LMT = 5
IF (ABS (Z) .GT. 20.) LMT = 1
DO 600 I = 1, LMT
K = I - 1
CRTERM = (ALPHA + K) * (ALPHA - GAMMA + 1.) * K / (FLOAT (I) * (-CDZ)) * CRTERM
CRSUM = CRSUM + CRTERM
CSTERM = (GAMMA - ALPHA + K) * (1. - ALPHA + K) / (FLOAT (I) * CDZ) * CSTERM
CSSUM = CSSUM + CSTERM

600 CONTINUE
SIGN = - 1.
IF (AIMAG (CZ) .LE. 0.) SIGN = - 1.
CALL GAMMB (ALPHA, GAMMA, A, B, BMA)
ARGZ = 3.141592 / Z.
CF = CDABS (CDZ) ** (ALPHA) * CEXP ((0., 1.) * ALPHA * SIGN * ARGZ) / BMA * CRSUM
IF (ABS (Z) .GT. 20.) GO TO 650
CF = CF + CABS(CZ)**(ALPHA-GAMMA)*CEXP((0., 1.)*(ALPHA-GAMMA)*ARGZ+CZ) 
1/A*CSSUM

650 CONTINUE

CF = CF*B
RHYP = .5*(DCONJG(CF)+CF)
IHYP = (0., 5.)*(DCONJG(CF)-CF)
RETURN
END
SUBROUTINE HYPERGCALPHA,GAMMA,CZ,THYP,THYP)

C THIS SUBROUTINE CALCULATES THE CONFLUENT HYPERGEOMETRIC FUNCTION
C ARGUMENTS ARE ALPHA, GAMMA, AND CZ. RESULT IS (THYP,THYP) (COMPLEX)
C CZ IS ASSUMED TO BE PURE REAL

C IMPLICIT COMPLEX (C)
REAL THYP,THYP
ARG(CZ)=ATAN2(AIMAG(CZ),REAL(CZ))

C SET UPPER LIMIT ON NUMBER OF TERMS
LMT=500
ARG=-ARG(-CZ)
ARGZ=ARG(CZ)

C TEST FOR LARGE ARGUMENT
IF(CABS(CZ).GT.3.)GO TO 500

C SMALL ARGUMENT EXPANSION
IHYP=0.
RHYP=ALPHA/GAMMA*Z
SRHYP=RHYP

C CHECK NUMBER OF TERMS
IF(I.GT.LMT)GO TO 340
K=I-1
RHYP=RHYP*(ALPHA+K)/(GAMMA+K)*Z/FLOAT(I)
SRHYP=SRHYP+RHYP

C TEST FOR CONVERGENCE
IF(ABS(RHYP-THYP).LT.1.E-05)GO TO 330
RHYP=RHYP
GO TO 320

C WRITE ERROR MESSAGE
WRITE(6,1000)RHYP,THYP,CHYPO
WRITE(6,1001)ALPHA,GAMMA,CZ,RHYP,THYP
RETURN

C ASYMPTOTIC EXPANSION FOR LARGE ARGUMENT
RSUM=1.
SSUM=1.
RTERMO=1.
STERMO=1.
CALL GAMMB(ALPHA,GAMMA,A,B,BMA)

C SET UPPER LIMIT ON NUMBER OF TERMS
LMT=5
DO 600 I=1,LMT
K=I-1
RTERM=(ALPHA+K)*(ALPHA-GAMMA+1+K)/(FLOAT(I)*(-Z))*RTERMO
RSUM=RSUM+RTERM
RTERMO=RTERM
STERMO=(GAMMA-ALPHA+K)*(1.-ALPHA+K)/(FLOAT(I)*Z)*STERMO
SSUM=SSUM+STERM
STERMO=STERM
600 CONTINUE
CF=CABS(CZ)**(-ALPHA)*CEXP(0.*1.)*(-ALPHA)*ARZG)/BMA*RSUM
IF(CABS(CZ).GT.99.)GO TO 700
CF=CF+CABS(CZ)**(ALPHA-GAMMA)*CEXP((0.*1.)*(ALPHA-GAMMA)*ARGZ+CZ)/I*SSUM
700 CONTINUE
RHYP=REAL(CF)*B
IHYP=AIMAG(CF)*B
RETURN
END

SUBROUTINE GAMMB(A,B,A1,B1,BMA)
C
C THIS SUBROUTINE CALCULATES GAMMA FUNCTIONS FOR CONFLUENT HYPERGEOMETRIC
C FUNCTION'S LARGE EXPANSIONS
C
B1=GAMMA(B)
IF(A.GT.B)GO TO 200
BMA=GAMMA(B-A)
150 CONTINUE
IF(A.LT.0.)GO TO 100
A1=GAMMA(A)
RETURN
100 A1=-3.141592/(GAMMA(-A+1.)*SIN(-3.141592*A))
RETURN
200 BMA=-3.141592/(GAMMA(A-B+1.)*SIN(3.141592*(A-B)))
GO TO 150
END
Again the variables are punched in ten column fields in decimal form. If XNR = 0, then the program reads a new parameter card.

(b). The Von Karmann spectrum calculations of the mean square angle of arrival, two point correlation functions, and structure functions mentioned earlier are performed by numerical integration. The programs for both plane and spherical wave input are shown in listings (B-2) and (B-3), respectively. The integrations are performed by using a 96 point Gaussian quadrature algorithm. The Bessel functions were evaluated using a polynomial expansion from the Handbook of Mathematical Functions by Abramowitz and Stegun.

The weighting factors and arguments for the Gaussian integration are read as input data and are shown in listing (B-4). All user input data should follow these cards. Since both programs use the same input data only one data deck will be explained.

The first user input data card specifies the parameters for the angle of arrival correlation function calculation. The data is again punched in ten column fields in decimal form. The input variables are:

Columns
1-10 DHOS = the first value of \( \rho \)
11-20 DELRH = the increment of \( \rho \)
21-30 XNR = the total number of increments.

If XL = 0, the correlation function calculations are terminated and the program begins the structure function calculations.

Otherwise, the program reads the separation parameters: starting value, increment, and total number of increments. These variables are punched in ten column fields in decimal form. They are:

Columns
1-10 XLO = \( L_0 \), the inner scale in meters
11-20 XL = \( L \), the range in meters
21-30 XLAMB = \( \lambda \), the wavelength in meters
31-40 XCAPLO = \( L_0 \), the outer scale in meters
41-50 A = \( A \), the outer scale normalizing constant.

If XL = 0, the correlation function calculations are terminated and the program begins the structure function calculations.

Otherwise, the program reads the separation parameters: starting value, increment, and total number of increments. These variables are punched in ten column fields in decimal form. They are:
LISTING B-2

C ***** 10.6 MICKEN PROPAGATION PHENOMENA PROGRAM *****
C ***** ANGLE OF ARRIVAL AND RELATED STRUCTURE FUNCTION CALCULATION
C FOR THE VON KARMANN SPECTRUM, AND PLANE INPUT *****
EXTERNAL FCT, FC10, FC11, FC12, FC13
REAL*6 YINT, YINTG, FCT, XL, XK, XKM, XK0
COMMON RHO, XL, XK, XKM, XK0
COMMON /INTG/TEST
CALL GAUS2
TEST=0.
C READ INPUT DATA
100 READ(5,900)XLG, XL, XLAMB, XCAPLO, A
900 FORMAT(5E10.3)
   IF(XL.EQ.0.)GO TO 201
C CALCULATE CONSTANTS
   XKM=5.94/XL0
   XK=XK0/XCAPLO
   PI=3.1415927
   XK=6.283184/XLAMB
C READ ARGUMENT VALUES
150 READ(5,900)RHOS, CELRHO, XNR
   IF(XNR.EQ.0.)GO TO 100
C CALCULATE ANGLE OF ARRIVAL CORRELATION
   NR=XNR
   WRITE(6,1002)
1002 FORMAT(1X,'ANGLE OF ARRIVAL CORRELATION FUNCTION')
   WRITE(6,1005)XL, XLAMB, XCAPLO, A
1005 FORMAT(1X,'RANGE=' ,1PE11.3,' M',5X,'INNER SCALE=' ,1PE10.3,' M',5X,'OUTER SCALE=' ,1PE10.3,' M',5X,'A=' ,1PE10.3,5X,'LAMBDA=' ,1PE10.3,' M')
DO 200 I=1,NR
   RHO=RHOS+DELRHO*FLGAT(I-1)
C ELEVATION ANGLE OF ARRIVAL CORRELATION
   CALL INTGRL(0.,1.E-3,8,FCT,YINT)
   BALPHA=2.*PI**2*XL*YINT
   IF(RHO.EQ.0.)BAN=BALPHA
   BALPHN=BALPHA/BAN
   WRITE(6,1000)RHO, BALPHA, BALPHN
1000 FORMAT(1X,'RHO=' ,1PE10.3,5X,'BALPHA=' ,1PE10.3,5X,'BALPHN=' ,1PE10.3)
C AZIMUTH ANGLE OF ARRIVAL CORRELATION
   CALL INTGRL(0.,1.E-3,8,FCT0,YINT)
   BBETA=2.*PI**2*XL*YINT
   IF(RHO.EQ.0.)BBN=BBETA
   BBETAN=BBETA/BBN
   WRITE(6,1001)BBETA, BBETAN
1001 FORMAT(2X,'BBETA=' ,1PE10.3,5X,'BBETAN=' ,1PE10.3)
200 WRITE(6,1001)BBETA, BBETAN
GO TO 150
C CALCULATE STRUCTURE FUNCTIONS
C READ INPUT DATA
DOUBLE PRECISION FUNCTION FCT(XKAP)
C INTEGRAND FOR CALCULATING ELEVATION ANGLE OF ARRIVAL CORRELATION
REAL*8 ARG1,FCT,XKAP
COMMON RHO,XL,XK,XKM,XKO
ARG3=(XKAP/XKM)**2
IF(ARG3.GT.144.)GOTO 100
ARG2=XKAP*RHO
ARG1=XKAP**2*XL/XK
IF(ARG1.GT.0.)GOTO 100
FCT=(1.+USIN(ARG1)/ARG1)*(BES0(ARG2)-BES1X(ARG2))*XKAP**3*PHIEPS(XKAP,XKM,XKO)
RETURN
100 FCT=0.
RETURN
END
DOUBLE PRECISION FUNCTION FCT0(XKAP)
C INTEGRAND FOR CALCULATING AZIMUTH ANGLE OF ARRIVAL
REAL*8 ARG1,FCT0,XKAP
COMMON RHO,XL,XK,XKM,XKO
ARG3=(XKAP/XKM)**2
IF(ARG3.GT.1T74.)G0T0 100
ARG2=XKAP*RHO
ARG1=XKAP**2*XL/XK
IF(ARG1.EQ.0.)G0T0 100
FCT0=(1.+DSIN(ARG1)/ARG1)*BES1X(ARG2)*XKAP**3*PHIEPS(XKAP,XKM,XKO)
RETURN
100 FCT0=0.
RETURN
END

DOUBLE PRECISION FUNCTION FCT1(XKAP)
C INTEGRAND FOR CALCULATING INDEX STRUCTURE FUNCTION
REAL*8 ARG2,FCT1,XKAP
COMMON RHO,XL,XK,XKM,XKO
ARG1=XKAP**2*XL/XK
ARG2=XKAP*RHO
ARG3=(XKAP/XKM)**2
IF(ARG3.GT.1T74.)G0T0 100
IF(ARG1.EQ.0.)G0T0 100
FCT1=(1.-BSQ(ARG2)/ARG2)*(1.+DSIN(ARG1)/ARG1)*PHIEPS(XKAP,XKM,XKO)*XKAP**2
RETURN
100 FCT1=0.
RETURN
END

DOUBLE PRECISION FUNCTION FCT2(XKAP)
C INTEGRAND FOR CALCULATING PHASE STRUCTURE FUNCTION
REAL*8 ARG1,FCT2,XKAP
COMMON RHO,XL,XK,XKM,XKO
ARG3=(XKAP/XKM)**2
IF(ARG3.GT.1T74.)G0T0 100
ARG2=XKAP**2*XL/XK
ARG1=XKAP*RHO
IF(ARG1.EQ.0.)G0T0 100
FCT2=(1.-BSQ(ARG2))*(1.+DSIN(ARG1)/ARG1)*PHIEPS(XKAP,XKM,XKO)*XKAP**2
RETURN
100 FCT2=0.
RETURN
END
DOUBLE PRECISION FUNCTION FCT3(XKAP)

C INTEGRAND FOR CALCULATING THE WAVE STRUCTURE FUNCTION

REAL*8 XKAP, FCT3
COMMON RHU, XL, XK, XKM, XKO
ARG2=XKAP*RHO
ARG3=(XKAP/XKM)**2
IF(ARG3.GT.174.)GOTO 100
FCT3=(1.-BESU(ARG2))*PHIEPS(XKAP, XKM, XKO)*XKAP
RETURN
100 FCT3=0.
RETURN
END

FUNCTION PHIEPS(XK, XKM, XKO)

C REFRACTIVE INDEX FLUCTUATION SPECTRUM

REAL*8 XK
TEST1=(XK/XKM)**2
IF(TEST1.GT.174.)GO TO 100
A=EXP(-TEST1)
B=(XK**2+XKO**2)**(11./6.)
TEST2=ALCG10(A)-ALCG10(B)
IF(ABS(TEST2).GT.60.)GO TO 100
PHIEPS=.033*A/B
RETURN
100 PHIEPS=0.
RETURN
END
SUBROUTINE INTGRL(XLS, XUS, NINT, FCT, YINT)

COMMON /INTG/TEST
REAL*8 YINT, YINTG, FCT, XL, XU
EXTERNAL FCT

1003 FORMAT(' INTEGRAL= ', 1PE10.3, 5X, 'LOWER LIMIT= ', E10.3, 5X,
          'UPPER LIMIT= ', E10.3)

1002 FORMAT(/)
  IF (TEST .GT. 0.) WRITE(6, 1002)
  XL = XLS
  XU = XUS
  CALL DG96(XL, XU, FCT, YINT)
  IF (TEST .GT. 0.) WRITE(6, 1003) YINT, XL, XU
  DO 300 J = 1, NINT
    XL = XU
    XU = XU * 10.
    CALL DG96(XL, XU, FCT, YINTG)
    YINT = YINT + YINTG
    IF (TEST .GT. 0.) WRITE(6, 1003) YINTG, XL, XU
  CONTINUE
  RETURN
END

SUBROUTINE DG96(XL, XU, FCT, YINT)

C 96 POINT GAUSSIAN QUADRATURE INTEGRATION ROUTINE

IMPLICIT RFAL*8 (A-H, O-Z)
COMMON /GAUSS/X(48), W(48)
SUM = (XU + XL) / 2.
DIF = (XU - XL) / 2.
YINT = 0.
DO 100 I = 1, 48
  ARG1 = DIF * X(I) + SUM
  ARG2 = DIF * X(I) + SUM
  100 YINT = W(I) * (FCT(ARG1) + FCT(ARG2)) + YINT
YINT = YINT + DIF
RETURN
END
SUBROUTINE GAUSZ
C READ AND STORE GAUSSIAN INTEGRATION ARGUMENTS AND WEIGHTS
IMPLICIT REAL*8 (A-H,0-C-Z)
COMMON /GAUSS/X(16),W(48)
REAL(*,999)*
REAL(*,999)
999 FORMAT(5F16.15)
WRITE(6,1111)X
WRITE(6,1111)W
1111 FORMAT(1X,6(F17.15,3X/))
RETURN
END

FUNCTION BESU(X)
REAL*8 THETA
C JO(X)
IF(X.GT.3.)GO TO 100
Y=X/3.
BESC=1.0-2.24999*Y**2+1.26562*Y**4-.316387*Y**6+.044447*Y**8-3.94414E-3*Y**10+2.16000E-4*Y**12
RETURN
100 Y=3./X
THETA=X-.785398-.041663*Y-.934E-05*Y**2+2.62573E-3*Y**3-5.4125E-4*Y**4+1*Y**5-2.9333E-4*Y**6
FC=-7.918467-7.7E-7*Y-5.5274E-3*Y**2-9.512E-5*Y**3+1.37237E-3*Y**4-17.2805E-4*Y**5+1.4476E-4*Y**6
BESC=FC*DCCS(THETA)/SQRT(X)
RETURN
END

FUNCTION BESIX(X)
REAL*8 THETA
C J1(X)/X
IF(X.GT.3.)GO TO 100
Y=X/3.
BESIX=.50-.562499*Y**2+.210935*Y**4-3.95428E-2*Y**6+4.43319E-3*Y**8-3.1761E-2*Y**10+1.109E-5*Y**12
RETURN
100 Y=3./X
THETA=X-2.35619+.124996*Y+.650E-05*Y**2-6.37879E-3*Y**3+7.4348E-4*Y**4-1*Y**5+1.9824E-4*Y**6-2.9166E-4*Y**8
FC=-.797884+1.65E-6*Y+1.6596E-2*Y**2-1.7105E-4*Y**3-2.4951E-3*Y**5+1.1353E-3*Y**6-2.0033E-4*Y**8
BESIX=FC*DCCS(THETA)/(X*SQRT(X))
RETURN
END

98
C **** 10.6 KICKCN PROPAGATION PHENOMENA PROGRAM ****
C **** Angle of Arrival and Related Structure Function Calculation
C for the Von Karmann Spectrum, and Spherical Wave Input ****
EXTERNAL FCT,FC10,FC11,FC12,FCT3
REAL*8 YINT,YINTG,FCT,YL,XU,YINTL,YINT2,XXAP
COMMON RHO,XL,XK,XKM,XK0
COMMON /YINTG/TEST
CALL GAUSZ
TEST=0.
C READ INPUT DATA
100 READ(5,900)XL0,XL,XYCAPL0,AL
900 FORMAT(5E10.3)
   IF(XL.EQ.0.)GO TO 201
C CALCULATE CONSTANTS
XKM=5.94/XL0
XKL=A/XCAPL0
PI=3.1415927
XX=6.283184/XLAMB
C READ ARGUMENT VALUES
150 READ(5,900)RHUS,DL,LKH0,XNR
   IF(XNR.EQ.0.)GO TO 100
C CALCULATE ANGLE OF ARRIVAL CORRELATION
XNR=XNR-1.
NR=XNR
WRITE(6,1002)
1002 FORMAT('ANGLE OF ARRIVAL CORRELATION FUNCTION ')
   WRITE(6,1005)XL,XL0,XYCAPL0,AL,XLAMB
1005 FORMAT('RANGE=',1Pe1.3,' M',5X,'INNER SCALE=',E10.3,' M',5X,'A=',E10.3,5X,'LAMBDA=',E10.3,' M',2)
   DC 200 I=1,NR
   RHC=RHUS*DELRHC*FLCAT(I-1)
C ELEVATION ANGLE OF ARRIVAL CORRELATION
   CALL INTEGR(0.,1.,5.,E-3,8,FCT,YINT)
   BALPHA=.5*PI**2*8.*YINT
   IF(RH0.EQ.0.)BAN=BALPHA
   BALPHN=BALPHA/BAN
   WRITE(6,1000)RHO,BALPHA,BALPHN
1000 FORMAT('RHO=',1Pe1.3,' M',5X,'BALPHA=',E10.3,5X,'BALPHAN=',E10.3)
C AZIMUTH ANGLE OF ARRIVAL CORRELATION
   CALL INTEGR(0.,1.,5.,E-3,8,FCT0,YINT)
   BBETA=.5*PI**2*8.*YINT
   IF(RH0.EQ.0.)BBN=BBETA
   BBETAN=BBETA/BBN
1001 FORMAT(21X,'BBETA=',1Pe1.3,5X,'BBETAN=',E10.3)
   200 WRITE(6,1001)BBETA,BBETAN
   GO TO 150
C CALCULATE STRUCTURE FUNCTIONS
C READ INPUT DATA
201 READ(5,900)XL,XL,XLAM,XCAPLO,A
    IF(XL,EQ.0.)CALL EXIT
C CALCULATE CONSTANTS
    XK=6.283184/XLAM
    XK=5.94/XLO
    XK=A/XCAPLO
    WRITE(6,1003)
1003 FORMAT('STRUCTURE FUNCTIONS AND SPECTRA')
C CALCULATE ARGUMENT VALUES
230 READ(5,900)RHOS,DELRRH,XNR
    IF(XNR.EQ.0.)GO TO 201
    NR=XNR+1.
    DO 220 I=1,NR
        RH=RHOS+DELRRH*FLOAT(I-1)
        XKAP=RHO
    C REFRACTIVE INDEX FLUCTUATION SPECTRUM
    PHIK=PHIEPS(XKAP,XKM,XKO)
C REFRACTIVE INDEX FLUCTUATION STRUCTURE FUNCTION
    CALL INTEGR(1.E-03,1.E-02,8,FCT1,YINT1)
    DN=8.*PI*YINT1
C PHASE STRUCTURE FUNCTION
    CALL INTEGR(1.E-02,1.E-01,6,FCT2,YINT2)
    DS=YINT2*PI**2*XK**2*8.
220 WRITE(6,1004)RH,RH,KP,NR,DS
1004 FORMAT('ARGUMENT= ',1PE10.3,/, ' REFRACTIVE INDEX FLUCTUATION SPECTRUM
        ITRUM= ',1PE10.3,/, ' STRUCTURE FUNCTION= ',1PE10.3,
        ' PHASE STRUCTURE
        2FUNCTION= ',1PE10.3)
    GO TO 230
END

DOUBLE PRECISION FUNCTION FCT(XKAPPA)
C INTEGRAND FOR CALCULATING ELEVATION ANGLE OF ARRIVAL CORRELATION
EXTERNAL FCT4
REAL*8 ARG1,FCT,XKAP,CXL,DSPHR,XKAPPA,FCT4
COMMON RHOS,XL,XK,XKM,XKO
COMMON /SPHERE/XKAP
XKAP=XKAPPA
CXL=XL
ARG2=XKAP*RHO
CALL DGBs (0.,CXL,FCT4,DSPHR)
FCT=(BESC(ARG2)-BES1X(ARG2))*XKAP**3*DSPHR
RETURN
END
DOUBLE PRECISION FUNCTION FCIO(XKAPPA)
C INTEGRAND FOR CALCULATING AZIMUTH ANGLE OF ARRIVAL
EXTERNAL FC4
REAL*8 ARGI,FCT0,XKAP,DXL,DSPHR,XKAPPA,FCT4
COMMON RHO,XL,XK,XKM,XKO
COMMON /SPHERE/XKAP
XKAP=XKAPPA
ARG2=XKAP*RHO
DXL=XL
CALL DCG8 (0.,DXL,FCT4,DSPHR)
FCTC=BSLX(ARG2)*XKAP**3*DSPHR
RETURN
END

DOUBLE PRECISION FUNCTION FCII(XKAP)
C INTEGRAND FOR CALCULATING INDEX STRUCTURE FUNCTION
EXTERNAL FC4
REAL*8 ARG2,FCTI,XKAP
COMMON RHO,XL,XK,XKM,XKO
ARG1=XKAP**2*XL/XK
ARG2=XKAP*RHO
ARG3=(XKAP/XKM)**2
IF(ARG3.GT.174.)GO TO 100
IF(ARG1.GT.0.)GO TO 100
FCTI=(1.-DSIN(ARG2)/ARG2)*PHIEPS(XKAP,XKM,XKO)*XKAP**2
RETURN
100 FCTI=0.
RETURN
END

DOUBLE PRECISION FUNCTION FC2(XKAPPA)
C INTEGRAND FOR CALCULATING PHASE STRUCTURE FUNCTION
EXTERNAL FC4
REAL*8 ARGI,FCT2,XKAP,DSPHR,DXL,XKAPPA,FCT4
COMMON RHO,XL,XK,XKM,XKO
COMMON /SPHERE/XKAP
XKAP=XKAPPA
ARG2=XKAP*RHO
DXL=XL
CALL DCG8 (0.,DXL,FCT4,DSPHR)
FCT2=(1.-BES0(ARG2))*DSPHR*XKAP
RETURN
END
DOUBLE PRECISION FUNCTION FCT3(XKAP)
C INTEGRAND FOR CALCULATING THE WAVE STRUCTURE FUNCTION
REAL*8 XKAP,FCT3
COMMON RHO,XL,XK,XKM,XKU
ARG2=XKAP*RHO
ARG3=(XKAP/XKM)**2
IF(ARG3.GT.174.)GO TO 100
FCT3=(1.-BESG(ARG2))*PHIFPS(XKAP,XKM,XKO)*XKAP
RETURN
100 FCT3=0.
RETURN
END

DOUBLE PRECISION FUNCTION FCT4(DELTA)
REAL*8 XKAP,DELTA,FCT4
COMMON RHO,XL,XK,XKM,XKU
COMMON /SPHERF/XKAP
ARG1=XKAP*XK/DELTA
FCT4=(XL/DELTA*CCCS(XL*(XL-DELTA)**2)/(2.*XI*K*DELTA))**2*PHIFPS(IARG1,XKM,XKO)
RETURN
END

FUNCTION PHIFPS(XK,XKM,XKO)
C REFRACTIVE INDEX FLUCTUATION SPECTRUM
REAL*8 XK
TEST1=(XK/XKM)**2
IF(TEST1.GT.174.)GO TO 100
A=EXP(-TEST1)
B=(XK**2*XKO**2)**(11./6.)
TEST2=ALCG10(A)-ALCG10(B)
IF(ABS(TEST2).GT.60.)GO TO 100
PHIFPS=.033*A/B
RETURN
100 PHIFPS=0.
RETURN
END
SUBROUTINE INTGML(XLS,XUS,NINT,FCT,YINT)
C ORDER OF MAGNITUDE INTEGRATION ROUTINE
COMMON /INTG/TEST
REAL*8 YINT,YINTG,FCT,XL,XU
EXTERNAL FCT
1003 FORMAT(' INTEGRAL= ',1PE10.3,5X,'LOWER LIMIT= ',1PE10.3,5X,
1'UPPER LIMIT= ',1PE10.3)
1002 FORMAT(/)
   IF(TEST.GT.0.)WRITE(6,1002)
   XL=XLS
   XU=XUS
   CALL DGG96(XL,XU,FCT,YINT)
   IF(TEST.GT.0.)WRITE(6,1003)YINT,XL,XU
   DO 100 J=1,NINT
   XL=XU
   XU=XU*10.
   CALL DGG96(XL,XU,FCT,YINTG)
   YINT=YINT+YINTG
   IF(TEST.GT.0.)WRITE(6,1003)YINTG,XL,XU
300 CONTINUE
   RETURN
END

SUBROUTINE DGG96(XL,XU,FCT,YINT)
C 96 POINT GAUSSIAN QUADRATURE INTEGRATION ROUTINE
IMPLICIT REAL*8 (A-H,O-Z)
COMMON /GAUSS/X(48),W(48)
SUM=(XU+XL)/2.
DIF=(XU-XL)/2.
YINT=0.
DO 100 I=1,48
ARG1=DIF*X(I)+SUM
ARG2=-DIF*X(I)+SUM
100 YINT=W(I)*(FCT(ARG1)+FCT(ARG2))+YINT
YINT=YINT*DIF
RETURN
END

SUBROUTINE GAUSZ
C READ AND SICKE GAUSSIAN INTEGRATION ARGUMENTS AND WEIGHTS
IMPLICIT REAL*8 (A-H,C-Z)
COMMON /GAUSS/X(48),W(48)
READ(5,999)X
READ(5,999)W
999 FORMAT(5F16.15)
WRITE(6,1111)X
WRITE(6,1111)W
1111 FORMAT(1X,6(F17.15,3X))
RETURN
END
FUNCTION BESG(X)
C JU(X)
IF(X.GT.3.)GO TO 100
Y=X/3.
BESL=1.0-2.24999*Y**2+2.26562*Y**4-3.16387*Y**6+0.44447*Y**8-3.944E-3*Y**10+2.10000E-4*Y**12
RETURN
100 Y=3./X
THETA=X-.785398-.041663*Y-3.954E-05*Y**2+2.62573E-3*Y**3-5.4125E-4*Y**4+2.9333E-4*Y**5+1.3556E-4*Y**6
FO=-7.978864-7.7E-/Y-5.5274E-3*Y**2-9.512E-5*Y**3+1.37237E-3*Y**4-17.2805E-4*Y**5+1.4476E-4*Y**6
BESC=FCOS(THETA)/SQRT(X)
RETURN
END

FUNCTION BESIX(X)
REAL*H THEA
C JU(X)/X
IF(X.GT.3.)GO TO 100
Y=X/3.
BESIX=.50-.552499*Y**2+.210935*Y**4-3.95428E-2*Y**6+4.43319E-3*Y**8-3.1761E-4*Y**10+1.09E-5*Y**12
RETURN
100 Y=3./X
THETA=X-2.35619+.12496*Y+5.650E-05*Y**2-.637879E-3*Y**3+7.43348E-4*Y**4+7.9824E-4*Y**5-.2.9166E-4*Y**6
FO=-7.97884+.556E-6*Y+1.65966F-2*Y**2-1.7105E-4*Y**3+2.94511E-3*Y**4+1.3653E-3*Y**5-2.0033E-4*Y**6
BEX=F1*DCOS(THETA)/(X*SQRT(X))
RETURN
END
Listing B-4
Columns

<table>
<thead>
<tr>
<th>Columns</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-10</td>
<td>RHOS = starting value of $\rho$</td>
</tr>
<tr>
<td>11-20</td>
<td>DELRHO = increment</td>
</tr>
<tr>
<td>21-30</td>
<td>XNR = total number of increments.</td>
</tr>
</tbody>
</table>

If XNR = 0, a new parameter card is read, otherwise another set of separation parameters is expected.

The data deck for the structure function calculations is identical to the correlation data deck. The same variables are used and the same action is taken for XNR = 0. When XL = 0, the program is terminated.

2. **Large Aperture Calculations**

Two programs were written to perform the calculation of the large aperture results. One calculates the large aperture angle of arrival and linear phase error; the other calculates the error between the large and small aperture results. Both programs use the Von Karmann spectrum.

The large aperture angle of arrival program shown in listing (B5) performs four nested integrations to calculate $K_{OX}$, $<\Delta^2>$ and $<\Delta^2K>$. Gaussian quadrature integration of different orders is used in each case. The Gaussian subroutines are contained in the IBM Scientific Subroutine Package. They are essentially the same as DQG96 used earlier except the weighting factors and arguments, are contained in the subprograms. The large aperture error program evaluates the error function $\Delta(\kappa R)$ in a similar manner.

The input data for both programs is read with a namelist format. The variables are

- XCAPLO = $L_0$, the outer scale in meters
- XO = $\xi_0$, the inner scale in meters
- RCAP = $R$, the aperture radius in meters
- A = $A$, normalizing constant in Von Karmann spectrum
- XLAMB = $\lambda$, wavelength in meters
- XL = L, range in meters.

After each calculation a new data card is read. The program is terminated by setting XLAMB = 0.
LISTING B-5

C **** 10.6 MICRON PROPAGATION PHENOMENA PROGRAM ****
C **** ANGLE OF ARRIVAL AND ERROR CALCULATION
C USING THE PHASE STRUCTURE FUNCTION ****
IMPLICIT REAL*8 (D)
COMMON R,RHO,GAMMA,RCAP
COMMON /PHI/XKM,XKO
COMMON /PHASE/XK,XL
COMMON /APRAD/DDCAP
EXTERNAL DINTl
V
NAMELIST/INPUT/XCAPLO,XLO,RCAP,A,XLAMB,XL
PI=3.1415927
WRITE(6,1000)
1000 FORMAT('ANGLE OF ARRIVAL CALCULATION USING PHASE STRUCTURE FUNCTION
ITION*/* (RESULTS NORMALIZED TO CN**2).
1005 FORMAT(' RANGE=',1PE11.3,' M',5X,'INNER SCALE=',E10.3,' M',5X,
'OUTER SCALE=',E10.3,' M',5X,'A=',E10.3,5X,'LAMBDA=',E10.3,' M' 2)
100 READ(5,INPUT)
 WRITE(6,1005)XL,XLO, XCAPLO,A,XLAMB
 IF(XLAMB.EQ.0.)CALL EXIT
 XK=6.28318/XLAMB
 XKM=5.92/XL
 DRCAP=RCAP
 XKM=A/XCAPLO
 CALL DQG12(0.,DRCAP,DINT1,DRINT1)
 CALL DQG12(0.,DRCAP,DDEN,DRDEN)
 GXO=-4.*PI**2*XK**2*XL*DRINT1/(PI*DRDEN**2)
 AOA=G XO/XK**2
 DDCAP=2.*DRCAP
 CALL DQG12(0.,DDCAP,DERFCT,DRNT1)
 DERROR=4.*PI**2*XK**2*XL*DRNT1/DDCAP**2
 DPSERR=DERROR/XK**2
 WRITE(6,1001)DRCAP,G XO,DERRO R, AOA,DPSERR
1001 FORMAT(' APERTURE RADIUS=',1PE10.3,5X,'KAPPA X=',E10.3,5X,'ERROR
1=',E10.3/33X,'MEAN SQUARE ANGLE OF ARRIVAL=',E10.3,5X,'ERROR=',E10.3)
 GO TO 100
END
DOUBLE PRECISION FUNCTION OERFCT(DR)
C LINEAR PHASE ERROR FUNCTION
IMPLICIT REAL*8 (D)
EXTERNAL DFCT1
COMMON R,RHO,GAMMA,RCAP
COMMON /APRAD/RCAP
RHC=DR
DARG=DR/RCAP
CALL INTEGR(1.E-02,1.0E-01,6,DFCT1,DINT1)
DFL=6.*DARCOS(DARG)-(14.*DARG-8.*DARG**3)*DSQRT(1.-DARG**2)
DERFCT=OR*DFL*DINT1
RETURN
END

DOUBLE PRECISION FUNCTION DFCT1(DXKAP)
C INTEGRAND FOR CALCULATING PHASE STRUCTURE FUNCTION FOR OERFCT
IMPLICIT REAL*8 (D)
COMMON R,RHO,GAMMA,RCAP
COMMON /PHI/XK,XKO
COMMON /PHASE/XK,XL
ARG3=(DXKAP/XK)**2
IF(ARG3.GT.174.)GO TO 100
DARG1=DXKAP**2*XK/XK
ARG2=DXKAP*RHO
IF(DARG1.EQ.0.)GO TO 100
XKAP=DXKAP
DFCT1=(1.-BES0(ARG2))*(1.+DSIN(DARG1)/DARG1)*PHIN(XKAP,XK,XKO)*
DXKAP
RETURN
100 DFCT1=0.
RETURN
END

DOUBLE PRECISION FUNCTION DINT1(DR)
C R INTEGRAND
IMPLICIT REAL*8 (D)
COMMON R,RHO,GAMMA,RCAP
EXTERNAL DINT2
R=DR
DRCAP=RCAP
CALL DQGB (0.00,DRCAP,DINT2,DINT2)
DINT1=DINT2*R*R*W(R)
RETURN
END

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DOUBLE PRECISION FUNCTION DINT2(DRHO)
C RHO INTEGRAND
IMPLICIT REAL*8 (D)
COMMON R,RHO,GAMMA,RCAP
EXTERNAL DINT3
RHO=DRHO
CALL DQG4 (0.00,3.1415927D0,DINT3,DRINT3)
DINT2=DRINT3*RHO*RHO*W(RHO)
RETURN
END

DOUBLE PRECISION FUNCTION DINT3(DGAMMA)
C GAMMA INTEGRAND
IMPLICIT REAL*8 (D)
COMMON R,RHO,GAMMA,RCAP
EXTERNAL DFCT
GAMMA=DGAMMA
CALL INTGRL(1.0E-03,1.E-02,7,DFCT,DPHASE)
DINT3=DPHASE*COS(GAMMA)
RETURN
END

DOUBLE PRECISION FUNCTION DFCT(DKAP)
C PHASE STRUCTURE FUNCTION INTEGRAND
IMPLICIT REAL*8 (D)
COMMON R,RHO,GAMMA,RCAP
COMMON /PHASE/XK,XL
COMMON /PHI/XKM,XKO
ARG=DKAP*SQRT(R*R+RHO*RHO-2.*RHO*R*COS(GAMMA))
XKAP=DKAP
DARG1=XKAP**2*XK/XK
DFCT=(1.-BESO(ARG))*PHIN(XKAP,XKM,XKO)*XKAP*(1.+DSIN(DARG1)/DARG1)
RETURN
END
FUNCTION PHIN(XKAP,XKM,XKO)
C REFRACTIVE INDEX FLUCTUATION SPECTRUM
TEST=(XKAP/XKM)**2
IF(TEST.GT.174.)GO TO 100
A=EXP(-TEST)
B=(XKAP**2+XKO**2)**(11./6.)
TEST=ALOG10(A)-ALOG10(B)
IF(ABS(TEST).GT.30.)GO TO 100
PHIN=.033*A/B
RETURN
100 PHIN=0.
RETURN
END

FUNCTION W(R)
C APERTURE WEIGHTING FUNCTION
W=1.
RETURN
END

DOUBLE PRECISION FUNCTION DDEN(DR)
C DENOMINATOR INTEGRAND
IMPLICIT REAL*8 (D)
R=DR
DDEN=DR**3*W(R)
RETURN
END

SUBROUTINE INTEGR(XL,XU,NINT,FCT,YINT)
C ORDER OF MAGNITUDE INTEGRATION ROUTINE
EXTERNAL FCT
XL=XLS
XU=XUS
CALL DG32(XL,XU,FCT,YINT)
DO 300 J=1,NINT
XL=XU
XU=XU*10.
CALL DG32(XL,XU,FCT,YINT)
YINT=YINT+YINTG
300 CONTINUE
RETURN
END
FUNCTION BESO(X)
C JO(X)
  IF(X.GT.3.)GO TO 100
  Y=X/3.
  BESC=1.0-2.24999*Y**2+1.26562*Y**4-.316387*Y**6+.044447*Y**8-3.944
  14E-3*Y**10+2.10000E-4*Y**12
  RETURN
  100  Y=3./X
  THETA=X-.785398-.041663*Y-3.954E-05*Y**2+2.62573E-3*Y**3-5.4125E-4
  1*Y**4-2.9333E-4*Y**5+1.3558E-4*Y**6
  FO=.79788467.7E-7*Y-5.5274E-3*Y**2-9.512E-5*Y**3+1.37237E-3*Y**4-
  17.2805E-4*Y**5+1.4476E-4*Y**6
  BESO=FO*COS(THETA)/SQRT(X)
  RETURN
END
**LISTING B-6**

**C **** 10.6 MICRON PROPAGATION PHENOMENA PROGRAM ****
C **** ANGLE OF ARRIVAL ERROR CALCULATION
C USING THE PHASE STRUCTURE FUNCTION ****
IMPLICIT REAL*8 (D)
COMMON R,RHO,GAMMA,RCAP, XKAP
COMMON /PHI/XKM,XKO
COMMON /PHASE/XK,XL
EXTERNAL DFCT
NAMESLIST/INPUT/XCAPLO, XL, RCAP, A, XLAMB, XL
PI=3.1415927
WRITE(6,1000)
1000 FORMAT("ANGLE OF ARRIVAL CALCULATION USING PHASE STRUCTURE FUNCTION
10N"/" (RESULTS NORMALIZED TO CN**2)
1005 FORMAT("ORANGE=",1PE11.3, "M", 5X, "INNER SCALE=",E10.3, " M", 5X, 
100 READ(5, INPUT)
WRITE(6,1005) XL, XLO, XCAPLO, A, XLAMB
XX=6.28318/XLAMB
XKM=2.92/XL
XKO=A/XCAPLO
DRCAP=RCAP
CALL INTEGRAL(1.0E-03,1.E-02,7,DFCT,DELTAA)
DELTAA=4.*PI**2*XL*DELTAA
WRITE(6,1001) DRCAP, DELTAA
1001 FORMAT(" APERTURE RADIUS=",1PE10.3,5X,"DELTA A0A=",E10.3,/)
GO TO 100
END

DOUBLE PRECISION FUNCTION DFCT(DKAP)
**C PHASE STRUCTURE FUNCTION INTEGRAND
IMPLICIT REAL*8 (D)
REAL*8 AUASA, AOALA, BESIX, AOAL
COMMON R,RHO,GAMMA,RCAP, XKAP
COMMON /PHASE/XK,XL
COMMON /PHI/XKM,XKO
EXTERNAL DINT1
XKAP=DKAP
DARG1=XKAP**2*XL/XK
DRCAP=RCAP
CALL DGG12(0.0, DRCAP, DINT1, DINT1)
AOALA=-16.*DINT1/(3.141592*DRCAP**8)
AOAL=AOALA/DKAP**2
AOASA=BESIX(DKAP*DRCAP)/2.0000000000000DO
DIF=AOASA-AOAL
DFCT=DIF*UKAP**3*PHIN(XKAP, XKM, XKO)*(1.+D5IN(DARG1)/DARG1)
RETURN
END

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DOUBLE PRECISION FUNCTION DINT1(UR)
C K INTEGRAND
IMPLICIT REAL*8 (D)
COMMON R,RHO,GAMMA,RCAP,XKAP
EXTERNAL DINT2
R=UR
ORLAP=RCAP
CALL DQG6 (0.DO,ORLAP,DINT2,DINT2)
DINT1=DINT2*R*R
RETURN
END

DOUBLE PRECISION FUNCTION DINT2(URHO)
C RH0 INTEGRAND
IMPLICIT REAL*8 (D)
COMMON R,RHO,GAMMA,RCAP,XKAP
EXTERNAL DINT3
RHO=ORHO
CALL DQG6 (0.DO,3.1415927DO,DINT3,DINT3)
DINT2=DINT3*RHO*RHO
RETURN
END

DOUBLE PRECISION FUNCTION DINT3(DGAMMA)
C GAMMA INTEGRAND
IMPLICIT REAL*8 (D)
REAL*8 BESO
COMMON R,RHO,GAMMA,RCAP,XKAP
EXTERNAL DFCT
GAMMA=DGAMMA
DPhASE=1.0000000000000000-BESO(XKAP*DSQRT(R**2+RHO**2-2.*R*RHO*10COS(DGAMMA)))
DINT3=DPHASE*COS(GAMMA)
RETURN
END
FUNCTION PHIN(XKAP, XKM, XKO)
C REFRACTIVE INDEX FLUCTATION SPECTRUM
TEST = (XKAP/XKM)**2
IF (TEST .GT. 1.14) GO TO 100
A = EXP(-TEST)
B = (XKAP**2 + XKU**2)**(11./6.)
TEST = ALG10(A) - ALG10(B)
IF (ABS(TEST) .LT. 30.) GO TO 100
PHIN = .033*A/B
RETURN
100 PHIN = 0.
RETURN
END

SUBROUTINE INIGRUL(XLS, XUS, NINT, FCT, YINT)
C ORDER OF MAGNITUDE INTEGRATION ROUTINE
REAL*8 YINT, YINTG, FCT, XL, XU
EXTERNAL FCT
XL = XLS
XU = XUS
CALL UG16(XL, XU, FCT, YINT)
DO 300 J = 1, NINT
XL = XU
XU = XU*10.
CALL UG16(XL, XU, FCT, YINTG)
YINT = YINT + YINTG
300 CONTINUE
RETURN
END

FUNCTION BESO(X)
C JD(X)
REAL*8 BESO, X, Y, F0, THETA
C JD(X)
100 IF (X .GT. 1.) GO TO 100
Y = X/3.
BESO = 1.0 - 2.24999*Y**2 + 1.26562*Y**4 - 3.16387*Y**6 + .044447*Y**8 - 3.94*14E-3*Y**10 + 2.10000E-4*Y**12
RETURN
100 Y = 3./X
THETA = X -.785398 - .041663*Y - 3.954E-05*Y**2 + 2.62573E-3*Y**3 - 5.4125E-4*Y**4 + 2.9333E-4*Y**5 + 1.3558E-4*Y**6
F0 = -.7978846 - .77E-7*Y - 5.5274E-3*Y**2 - 9.512E-5*Y**3 + 1.37237E-3*Y**4 + 1.2805E-4*Y**5 + 1.4476E-4*Y**6
BESO = F0*DCOS(THETA)/DSQRT(X)
RETURN
END
FUNCTION BES1X(X)
  REAL*8 THETA, F1, Y, BES1X, X
  C J1(X)/X
  IF(X .GT. 3.) GO TO 100
  Y = X / 3.
  BES1X = .50 -.562499*Y**2 +.210935*Y**4 -3.95428*Y**6 +4.43319*Y**8
               18 -3.1761*Y**10 +1.109*5*Y**12
  RETURN
100     Y = 3./X
  THETA = X -2.35619 +.124996*Y +5.650E-05*Y**2 -6.37879*Y**3 +7.4348E-4
        1*Y**4 +7.9824E-4*Y**5 -2.9166E-4*Y**6
        F1 = .79788 +1.56E-6*Y +1.65966*2*Y**2 -1.7105E-4*Y**3 -2.49511E-3*Y**4
        14 +1.13653*3*Y**5 -2.0033E-4*Y**6
  BES1X = F1*DCOS(THETA) / (X*DSQRT(X))
  RETURN
END
REFERENCES


