BOOLEAN MATRICES AND GRAPH THEORY
by
R. S. Ledley

NBR Report No. 68032/3265
March 1968

Contract N000-3265(00)

National Biomedical Research Foundation
11200 Lockwood Drive
Silver Spring, Maryland 20901

Available for
Unlimited Distribution
Reproduced by
NATIONAL TECHNICAL INFORMATION SERVICE
Springfield, Va. 22151
Introduction. A net is a set of points between any two of which may be a connecting line. We will consider directed nets, in which the connecting lines have a direction: if the line $v_{12}$ connecting points $p_1$ and $p_2$ is directed from $p_1$ toward $p_2$, then $p_1$ is called the origin of the line, and $p_2$ is called the insertion of the line. In particular, we will restrict ourselves to nets in which there cannot be two or more lines in the same direction between the same pair of points, and in which a line can only connect two distinct points: such a net is called a directed graph. Figures 1a and 1b are not valid directed graphs; Fig. 1c is a valid directed graph.

The study of directed graphs has many applications in information and computer science. The trees of the previous section are directed graphs; the skeletons of flow charts are graphs; the state diagrams to be studied in Part V below are related to directed graphs; and so forth. Information flow in communication network can be analyzed in terms of directed graphs, as can social structures and many mathematical relationships. In this short section we can do no more than present a bare introduction to this subject; nevertheless, its importance in information and computer science necessitates inclusion, however briefly, in this survey.

The Incidence Matrix $G$. Associated with a directed graph is a Boolean matrix $G$, called the incidence matrix, in which $g_{ij} = 1$ if there is a line in the graph with origin $p_i$ and insertion $p_j$, and $g_{ij} = 0$ otherwise. Thus for the directed graph of Fig. 1c we have

$$G = \begin{pmatrix}
0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
2 & 0 & 1 & 0 & 0 \\
3 & 0 & 1 & 0 & 0
\end{pmatrix}$$
Note that if we renumbered the points as in Fig. 2, then a different matrix would result, even though the graph would have the same basic structure. For the numbering of Fig. 2 we have

\[
G' = \begin{pmatrix}
0 & 1 & 2 & 3 \\
0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 \\
2 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 \\
\end{pmatrix}
\]

If we wanted to show that the two matrices indeed represented the same basic graph structure, then we would have to demonstrate that there is a renumbering, or permutation, of the rows and columns of \( G \) which would make it identical to \( G' \). Recall that when a column-unitary matrix multiplies a matrix on its left, the result is to permute the columns of that matrix. Similarly when a row-unitary matrix (with a single unit in each row) multiplies a matrix on its right, it permutes the rows of that matrix. Since we must permute both the columns and rows of \( G \) to get \( G' \), we must multiply \( G \) both on the left and right. To get the same permutation of rows and columns, the row- and column-unitary matrices must be transposes of each other. Thus if \( P \) is the desired permutation, then

\[
P^\top \otimes G \otimes P = G'
\]

For our case we see from Fig. 2 that

\[
P = \begin{pmatrix}
0 & 1 & 2 & 3 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 \\
3 & 1 & 0 & 0 \\
\end{pmatrix}
\]

whence from Eq. (1) we find

\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
\end{pmatrix} \otimes \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
\end{pmatrix}
\]

which is \( G' \), as desired.

\*\*A true permutation matrix \( P \) is a (row or column) unitary matrix with a single unit in each row and each column. If point \( i \) \( \rightarrow \) point \( j \), then \( P_{ij} = 1 \) in \( P \).
For our case of directed graphs (i.e. where $g_{ii} = 0$), it can be shown that, in general, given two incidence matrices $G$ and $G'$, all permutations $P$, if any exist at all, that satisfy Eq. (1) can be generated by the following process:

**Step 1.** For each row $j$ of $G$ (written as a column) and each row $i$ of $G'$ (written as a row) form†

$$S_{ij} = \begin{pmatrix} \text{\text{ith row of } G} \\ \text{written as a column} \end{pmatrix} \ast \begin{pmatrix} \text{\text{i}th row of } G' \\ \text{written as a row} \end{pmatrix}$$

$$= \begin{pmatrix} g_{j1} \\ g_{j2} \\ \vdots \\ g_{jj} \end{pmatrix} \ast (e'_{11} \ e'_{12} \ \ldots \ e'_{jj})$$

**Step 2.** Form every product

$$T_k = S_{a_1} \cdot S_{a_2} \cdot \ldots \cdot S_{a_J}$$

where the $a_n$ are chosen, in some order, from 1, 2, ..., $J$, so that $a_n \neq a_m$ (i.e. no two alphas are the same in the same product). Retain only those $T_k$ that do not have an all-zero column or row.

**Step 3.** For each such product $T_k$ form the permutation matrix $P_k$ with elements $P_{k, a_i} = 1$, zeros otherwise (i.e. with a unit element corresponding to each term in the product). If

$$P_k \ast T_k$$

then $P = \frac{1}{k}$ is a permutation that satisfies Eq. (1).

†If $A$ is a matrix with a single column of elements $a_j$, and $B$ is a matrix with a single row of elements $b_j$, then the elements of the $0$ (theta) product of $A$ and $B$, namely $C = A \ast B$, are

$$c_{ij} = a_i b_j + \bar{a}_i \cdot \bar{b}_j$$
As an example, consider $G$ and $G'$ as given above, namely

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad G' = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then for the $S^{ij}$ matrices we have

$$S^{00} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad S^{01} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} \quad S^{02} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad S^{03} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$S^{10} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad S^{11} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad S^{12} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad S^{13} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$S^{20} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad S^{21} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad S^{22} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad S^{23} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$S^{30} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad S^{31} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad S^{32} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad S^{33} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Note that $S^{00}, S^{01}, S^{02}, S^{13}, S^{23},$ and $S^{33}$ were crossed off because they each had a zero column. For the first two rows of matrices, we can form the products

$$S^{03} \cdot S^{10} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad S^{03} \cdot S^{11} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \quad S^{03} \cdot S^{12} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Including the third of $S^{ij}$ matrices, only

$$S^{03} \cdot S^{10} \cdot S^{22} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad S^{03} \cdot S^{11} \cdot S^{22} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad S^{03} \cdot S^{12} \cdot S^{20} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

and $S^{03} \cdot S^{12} \cdot S^{21} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

need be retained as having no all-zero columns (or rows).
Finally, including the fourth row of \( S_{ij} \) matrices, we find

\[
S_{03}S_{12}S_{20}S_{31} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix} = T_k
\]

as the product with no all-zero columns or rows. Thus

\[
P_k = \begin{pmatrix}
0 & 1 & 2 & 3 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 10 \\
2 & 1 & 0 & 0 \\
3 & 0 & 1 & 0
\end{pmatrix}
\]

and certainly \( P_k + T_k \). Hence

\[
P = P_k^t = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

is the desired permutation.

The Reachability Matrix \( R \). A path from \( P_i \) to \( P_j \) is a collection of points \( P_i, P_k, P_n, P_m, \ldots P_r, P_j \) and the lines \( v_{ik}, v_{kn}, v_{nm}, \ldots, v_{rj} \), where the insertion of each line is the origin of the next, except for the last. If such a path from \( P_i \) to \( P_j \) actually exists in a graph, then point \( P_j \) is said to be reachable from point \( P_i \) in the graph. For instance, in Fig. 3, we see that \( P_3 \) is reachable from \( P_1 \), but \( P_0 \) is not reachable from any other point on the graph. The reachability matrix \( R \) of a graph has elements \( r_{ij} = 1 \) if \( P_j \) is reachable from \( P_i \), and zero otherwise. Note: that every point is trivially reachable from itself, and hence \( r_{ii} = 1 \) for all points \( P_i \). For Fig. 3, the reachability matrix is

\[
R = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 \\
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
2 & 0 & 0 & 1 & 1 \\
3 & 0 & 0 & 0 & 1 \\
4 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

with \( G = \)

\[
G = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
2 & 0 & 0 & 0 & 1 \\
3 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

clearly \( G \rightarrow R \), for if \( c_{ij} = 1 \), then \( P_j \) is reachable from \( P_i \) and hence \( r_{ij} = 1 \). For any point \( P_i \), we can determine the set of points that can be reached from \( P_i \).
by inspecting the $i^{th}$ row of $R$: the set corresponds to those columns with units in that row. Similarly we can find the set of points from which $p_i$ is reachable by inspecting the $j^{th}$ column of $R$: the set corresponds to those rows with units in that column.

The matrix $R$ can be obtained from the matrix $G$ of a graph as follows:

Note that $G$ gives the "one step" reachability of the graph. If we form

$$G \otimes G = G^2$$

we get the "two step" reachability of the graph, i.e. all those points that can be reached from another point in two lines. For instance, for Fig. 3,

$$G = \begin{bmatrix}
0 & 1 & 2 & 3 & 4 \\
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 \\
2 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0
\end{bmatrix}$$

and

$$G \otimes G = \begin{bmatrix}
0 & 1 & 2 & 3 & 4 \\
0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 \\
2 & 0 & 0 & 0 & 1 \\
3 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Similarly, a "three step" reachable graph can be constructed by forming

$$G \otimes G \otimes G = G^3$$

For our illustration, this becomes

$$(G \otimes G) \otimes G = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

This process can be continued to find an "n step" reachability matrix $G^n$. Now the reachability matrix $R$ includes all reachable steps, and the unit diagonal matrix $I$ as well. That is,

$$R = I + G + G^2 + G^3 + \ldots + G^n$$

where $I$ has the elements $\delta_{ii} = 1$, $\delta_{ij} = 0$ for $i \neq j$. Therefore, to construct the reachability matrix $R$ from the incidence matrix $G$, (logically) add to $I$ the
matrix $G$ and its successively higher powers until no new units can be included in $R$. For our case we have

$$R = I + G + G^2 + G^3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Finally, note that since $I \otimes A = A \otimes I$ we have

$$(I + G)^2 = (I + G) \otimes (I + G) = I + G + G^2$$

or in general

$$(I + G)^n = I + G + G^2 + ... + G^n$$

Then, if no new units can be included after the $n^{th}$ step,

$$R = (I + G)^n$$

For our example, we have

$$(I+G) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(I+G)^2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(I+G)^3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

whence $R = (I+G)^3$

Connectedness and Components. A directed graph is called strong, or strongly connected, when every point is reachable from every other point of the graph. A directed graph is weakly connected when for every two points, at least one point is reachable from the other (but not necessarily the reverse). A directed graph is unconnected when there exist at least two points neither of which is reachable from the other, but there is at least a sequence of lines, disregarding direction, between every two points. A directed graph is disconnected when it is not connected in any of the just-mentioned ways. Figure 4 illustrates the types of connectedness of graphs.
If a graph is strongly connected, then $P_i$ must be reachable from $P_j$ and $P_j$ must be reachable from $P_i$, for every pair of points $P_i$ and $P_j$ of the graph. Hence, if $J$ represents the matrix all of whose elements are units, then

$$R = J$$

if and only if the graph is strongly connected. If the graph is weakly connected, then $P_i$ must be reachable from $P_j$ or $P_j$ must be reachable from $P_i$. Hence if $R^t$ is the transpose of the reachability matrix $R$ then

$$R + R^t = J$$

if and only if the graph is weakly connected. Of course strong connectedness implies weak connectedness. For our illustrations of Fig. 4, we have:

(a) \[ R = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \]

(b) \[ R = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad R^t = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad R + R^t = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \]

(c) \[ R = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad R^t = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad R + R^t = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \]

(d) \[ R = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad R^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad R + R^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \]

Strong subcomponents of a graph can be recognized. For instance, consider Fig. 5 with the matrices

$$G = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Now if $P_i$ is reachable from $P_j$, then $r_{ij} = 1$; if $P_j$ is reachable from $P_i$, then $r_{ij}^t = r_{ji} = 1$. Hence the unit elements of the product $R \cdot R^t$ represent mutually
reachable pairs of points which are therefore members of some strong subcomponent of the graph. For our illustration of Fig. 5, we have

\[
R \cdot R^t = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1
\end{pmatrix} \cdot \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1
\end{pmatrix} = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
2 & 0 & 1 & 1 & 1 & 1 \\
3 & 0 & 0 & 1 & 1 & 0 \\
5 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Here the dashed lines represent a decomposition of the matrix into the strong subcomponents of the graph, namely point sets \( \{P_0, P_1\} \), \( \{P_2, P_3, P_4\} \), and \( \{P_5\} \).

Note that a single point is always a strong subcomponent of itself (why?). The complete decomposition would be characterized by

\[
\{P_0, P_1\} : \quad G = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \quad R = \begin{pmatrix}
1 & 1
\end{pmatrix}
\]

\[
\{P_2, P_3, P_4\} : \quad G = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix} \quad R = \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

\[
\{P_5\} : \quad G = \begin{pmatrix}
1
\end{pmatrix} \quad R = \begin{pmatrix}
1
\end{pmatrix}
\]

We can construct a new directed graph, called the condensed graph, that displays the relationships of the strong components (see Fig. 6). The condensed graph will have points corresponding to the strong components, say \( Q_1 \), \( Q_2 \), and \( Q_3 \) for \( \{P_0, P_1\} \), \( \{P_2, P_3, P_4\} \), and \( \{P_5\} \). If at least one line connects a point of one component with a point of another, then there will be a line in the same direction connecting the corresponding points in the condensed graph.

For our illustration, from the full-sized \( G \) above (or see Fig. 5) we will have a line \( v_0 v_1 \) connecting \( Q_0 \) and \( Q_1 \) and a line \( v_2 v_3 \) connecting \( Q_2 \) and \( Q_3 \) (see Fig. 6). For our condensed graph, we now have

\[
G = \begin{pmatrix}
0 & 1 & 2 \\
0 & 1 & 0 \\
2 & 0 & 1
\end{pmatrix} \quad R = \begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 0 \\
2 & 0 & 1
\end{pmatrix} \quad R \cdot R^t = \begin{pmatrix}
1 & 1 & 0 \\
1 & 1 & 1 \\
0 & 1 & 1
\end{pmatrix}
\]

- 9 -
Of course, the condensed graph is not strongly connected (why?) and for our example is not weakly connected. Another very important question can be answered with the condensed graph, namely, which is the minimum set of points of the original graph from which all other points can be reached. Clearly, if we find such a minimum set for the condensed graph, then we have answered the question for the original graph, since a point chosen for the condensed-graph case can be replaced by any one of the points of the strong component it represents. To find the minimum set for the condensed graph, we chose those points which correspond to columns with a single unit in the R matrix. We know, of course, that such columns (with a single unit) must exist in the condensed graph (why?). This single unit simply means that the point is reachable from itself. All remaining columns (with more than one unit) will then correspond to points reachable from the other points. For instance, for our example column zero and column two of the condensed R have single units, corresponding to points $Q_0$ and $Q_2$ (see Fig. 6). Hence a minimum set of points for the original graph (see Fig. 5) could be $P_1$ (from $Q_0$) and $P_5$ (from $Q_2$) and all points of the graph can be reached from one of these. Such a minimum set of points is called a point basis for the graph.

**Application of Arithmetic Matrices to Graph Theory.** Up to now in our treatment of graph theory, we have been utilizing only Boolean matrices with Boolean, or logical, operations. In this paragraph, we will turn to the use of arithmetic matrices using ordinary arithmetic operations in matrix multiplication. We will, however, again start with the incidence matrix. Our first observation is that the value of an element $E_{ij}^{(n)}$ of $G^n$, the $n$th arithmetic power of the incidence matrix $G$, is the number of paths from $P_i$ to $P_j$ of length $n$ (where by the length of a path we mean the number of lines or steps required). For instance, for the graph of Fig. 7

\[
G = \begin{pmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad G^2 = \begin{pmatrix}
1 & 1 & 0 & 2 & 2 \\
0 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 2 & 1 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}, \quad G^3 = \begin{pmatrix}
0 & 3 & 1 & 2 & 2 \\
0 & 1 & 0 & 1 & 1 \\
1 & 3 & 0 & 2 & 3 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}, \quad G^4 = \begin{pmatrix}
1 & 4 & 0 & 4 & 5 \\
0 & 1 & 0 & 1 & 2 \\
0 & 4 & 1 & 4 & 4 \\
0 & 1 & 0 & 0 & 1
\end{pmatrix}
\]
For $G^2$, note, for example, that there are two paths of length 2 from $F_0$ to $F_3$, one through $F_1$ and the other through $F_2$. For $G^3$, note that there are three paths of length 3 from $F_0$ to $F_1$ (namely $F_0F_2F_3F_4$, $F_0F_1F_4F_5$, and $F_0F_2F_0F_1$), and so forth.

For $G^4$ there are five paths of length 4 from $F_1$ to $F_3$ (namely $F_0F_2F_0F_3F_4$, $F_0F_1F_2F_3F_4$, $F_0F_1F_4F_5F_6$, $F_0F_2F_0F_1F_2$, and $F_0F_2F_0F_1F_4$).

If we form the matrix $T = G \times G^t$ then

$$t_{ij} = e_{i1} e_{j1} + e_{i2} e_{j2} + \cdots + e_{in} e_{jn}$$

Hence if both $e_{ik}$ and $e_{jk}$ are units, a unit will be contributed to the sum composing $t_{ij}$. Elements $e_{ik}$ and $e_{jk}$ both being units means that lines go from both $P_i$ and $P_j$ to $P_k$. Hence the value of $t_{ij}$ is the number of points that are insertions for lines having origins at both $P_i$ and $P_j$. For diagonal elements of $T$, namely $t_{ii}$, the value is the number of points that are insertions for lines having $P_i$ as origin, i.e. the number of lines emanating from $P_i$. For the illustration of Fig. 7, we have

$$T = G \times G^t = \begin{pmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{pmatrix} \times \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0
\end{pmatrix} = \begin{pmatrix}
3 & 1 & 2 & 0 & 1 \\
1 & 2 & 1 & 1 & 0 \\
2 & 1 & 3 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1
\end{pmatrix}$$

Thus, for example, there are three lines emanating from (having their origins at) $F_0$, two from $F_1$, three from $F_2$, and one each from $F_3$ and $F_4$. By similar reasoning, we can see that for matrix $H = G^t \times G$, the value of $h_{ij}$ is the number of points that are origins for lines having insertions at both $P_i$ and $P_j$, and that value of the diagonal elements $h_{ii}$ is the number of lines converging on $P_i$. For our illustration,

$$H = G^t \times G = \begin{pmatrix}
1 & 1 & 0 & 1 & 0 \\
1 & 3 & 1 & 2 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 2 & 1 & 3 & 1 \\
0 & 0 & 0 & 1 & 2
\end{pmatrix}$$

Thus $F_3$, for instance, is the point of insertion of three lines, and so forth.
Finally, let us observe that we can easily construct a distance matrix $D$ for a graph by observation of the sequence of Boolean matrices $G, G', G^2,\ldots,G^n$. The value of $d_{ij}$ is found by examining the sequence of $i_{th}$ elements of the matrix sequence; the exponent of the matrix in which this element first becomes a unit is the value of $d_{ij}$. We must, however, use the conventions that a point and itself the distance between is 0, and between two points for which there is no connecting path the distance is $\infty$. For instance, for Fig. 8 we have:

$$G = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad G^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad G^3 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad G^n = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The distance matrix becomes:

$$D = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 1 \\ \infty & \infty & \infty & \infty \end{pmatrix}$$
Fig. 2. A renumbering of the points of the graph of Fig. 1c, where 0-2, 1-3, and 2-0.

Fig. 1. (a) Not a valid graph because the two lines v12 and v21 are in the same direction between the same points. (b) Not a valid graph because loops at P1 and P2 do not connect distinct points. (c) Valid directed graph.

Fig. 3. Graph illustrating recontability.
Fig. 5. Illustration of decomposition of a graph into strong subcomponents \{p_0,p_1\}, \{p_2,p_3,p_4\}, and \{p_5\}.

Fig. 6. The condensed graph corresponding to the strong components of Fig. 5, with the new points called Q_0, Q_1, and Q_2.

Fig. 7. Illustration for the application of arithmetic matrices.

Fig. 8. Illustration for the distance matrix.