ON A MODEL FOR COMPUTING ROUND-OFF ERROR OF A SUM

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GEORGE B. DANTZIG

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COMPUTER SCIENCE DEPARTMENT
School of Humanities and Sciences
STANFORD UNIVERSITY
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Computer Science Department
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Given real numbers \( a_1, a_2, \ldots, a_n \), we are interested in the classic problem of the error in computing \( S = \sum_{i=1}^{n} a_i \) when the sum is computed by \( \tilde{S}_0 = \sum_{i=1}^{n} \tilde{a}_i \) where \( \tilde{a}_i \) is the nearest integer to \( a_i \). We shall first study this error as a function of a \( \Delta \) shift, i.e., when all numbers \( a_i \) are each shifted \( \Delta \) and then rounded;

\[
(1) \quad S - n\Delta = \sum_{i=1}^{n} (a_i - \Delta)
\]

\[
(2) \quad \tilde{S}_\Delta - n\Delta = \sum_{i=1}^{n} (a_i - \Delta)^\star
\]

We will then let \( \Delta \) become a random variable that can take on uniformly any value in the interval \( -\frac{1}{2} \leq \Delta \leq \frac{1}{2} \). Different choices of \( \Delta \) give rise to different rounding errors \( \tilde{S}_\Delta - S \) and the variance of the distribution of \( \tilde{S}_\Delta - S \) can be used to measure the variability of the rounding error due to the random selection of the origin of the real numbers \( a_i \) with respect to that of the computer.

The cumulative error from (1) and (2) is

\[
(3) \quad \tilde{S}_\Delta - S = \sum_{i=1}^{n} [(a_i - \Delta)^\star - (a_i - \Delta)]
\]

Let \( f_i \) be the positive fractional part of \( a_i \) and let \( a_i \) be the largest integer not exceeding \( a_i \), i.e.,

\[
(4) \quad a_i = a_i + f_i
\]
Denoting by $r_i$ the error of the $i^{th}$ term, we have

$$r_i = \left[(a_i - \Delta)^* - (a_i - \Delta)\right] = \begin{cases} 1 - (f_i - \Delta) & \text{if } -\frac{1}{2} \leq \Delta \leq -\frac{1}{2} + f_i \\ -(f_i - \Delta) & \text{if } -\frac{1}{2} + f_i \leq \Delta \leq +\frac{1}{2} \end{cases}$$

To prove the above, we note that $f_i - \Delta = (a_i - \Delta) + a_i$. If $-\frac{1}{2} \leq f_i - \Delta \leq +\frac{1}{2}$ then $(a_i - \Delta)$ is rounded to $a_i$. Hence $a_i - \Delta$ is rounded down if $-\frac{1}{2} + f_i \leq \Delta$ otherwise rounded up.

Denoting expected value by $E$, we have by direct evaluation

$$E(r_i) = \int_{-\frac{1}{2}}^{\frac{1}{2}} r_i \, d\Delta = 0$$

Assume $f_i < f_j$, then

$$E(r_i r_j) = \int_{-\frac{1}{2}}^{\frac{1}{2}} r_i r_j \, d\Delta = \int_{-\frac{1}{2}}^{\frac{1}{2}} r_i r_j \, d\Delta + \int_{-\frac{1}{2}}^{\frac{1}{2}} r_i r_j \, d\Delta + \int_{-\frac{1}{2}}^{\frac{1}{2}} r_i r_j \, d\Delta$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} (f_i f_j - \Delta(f_i + f_j) + \Delta^2) \, d\Delta$$

$$+ \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[(1-f_i - f_j) + 2\Delta\right] \, d\Delta$$

$$+ \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[-f_i + \Delta\right] \, d\Delta$$

Performing indicated integration yields:

$$E(r_i r_j) = \frac{1}{2}[|f_j - f_i|^2 - |f_j - f_i| + \frac{1}{6}]$$
which is one-half the 2\textsuperscript{nd} order Bernoulli Polynomial in \(|f_j - f_i|\). For \(f_j \leq f_i\) we also get (7). Note that the individual errors \(r_i\) and \(r_j\) are not independent of one another.

It now follows that

\[
(9) \quad E(S) = S
\]

\[
(10) \quad E(S-S)^2 = E\left( \sum_{i=1}^{n} \sum_{j=1}^{n} r_i r_j \right) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ |f_i - f_j|^2 - |f_i - f_j| + \frac{1}{6} \right]
\]

The usual value of variance, \(E(S-S)^2 = n/12\), will result if we further assume \(f_i\) are independently drawn from uniform distributions on \([0 < f_i < 1]\).

**Theorem:** If the fractional parts of all \(a_i\) are equal to each other, then each term of (10) is maximum for \(0 \leq f_i \leq 1\) and

\[
(11) \quad \text{Max } E(S-S)^2 = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{1}{6} \right) = \frac{n^2}{12}.
\]

From (10) we have an interesting inequality, namely for all \(f_i\)

\[
(12) \quad V(f) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( |f_i - f_j|^2 - |f_i - f_j| + \frac{1}{6} \right) \geq 0
\]

This function is not convex even for \(n=2\), since \(f^{(1)} = (\frac{1}{2}, 0)\) and \(f^{(2)} = (-\frac{1}{2}, 0)\) yields \(V(f^0) = V(f^1) = \frac{1}{12} + \frac{1}{12} - \frac{1}{12} = \frac{1}{12}\) but
There appears to be no obvious direct way to establish that $V(f) \geq 0$ for all $0 \leq f \leq 1$. Our development shows $V(f)$ to be a variance and this, of course, constitutes an indirect proof.

We can replace (12) by a convex realization: Assume $f_i \geq f_{i+1}$ for all $i$, then the problem of finding $\min V(f)$ can be rewritten:

$$V(\frac{f^1 + f^2}{2}) = V(0) = \frac{3}{12}.$$  

subject to

$$f_1 \geq f_2 \ldots \geq f_n$$

$$0 \leq f_n \leq 1$$

Formally (13), (14), (15), is a positive definite quadratic program. Fortunately, as we shall see this can be solved by classical calculus by ignoring inequalities (14) and (15).

Theorem: Equally spaced $f_i = (n - i)/n$, $(i = 1, \ldots, n)$ yields

$$\min V(f) = \frac{1}{12}$$

independent of $n$, i.e., the variance of the sum in this case is minimum and is the same as the variance of the individual terms forming the sum.

Proof: Setting partials $= 0$ in (13) yields:
\[
\begin{align*}
2(n-1)f_1 & - 2f_2 \ldots - 2f_n = (n-1) \\
- 2f_1 + 2(n-1)f_2 & - 2f_n = (n-3) \\
- 2f_1 & - 2f_2 \ldots 2(n-1)f_{n-1} - 2f_n = -(n-3) \\
- 2f_1 & - 2f_2 \ldots 2(n-1)f_{n-1} = -(n-1)
\end{align*}
\]

Adding shows the equations to be dependent. Hence we may drop the last equation as redundant. Moreover, we can always translate the \( f_i \) so that the smallest \( f_i \), namely \( f_n = 0 \).

Re-adding yields:

\[
2f_1 + 2f_2 + \ldots + 2f_{n-1} + 0 = (n-1) \quad , \quad f_n = 0.
\]

Adding this last equation to each of the others gives

\[
2nf_1 = (n-2i+1) + (n-1) = 2(n-1)
\]

(17) \( f_i = (n-1)/n \)

Evidently the conditions \( 0 \leq f_i \leq 1 \) and \( f_i \geq f_{i+1} \) are (by good luck) also satisfied so that (17) yields the minimum, namely

\[
\text{Min } V(f) = \frac{n^2}{12} - \frac{1}{2} \sum_{i=1}^{n} (n-2i+1)f_i = \frac{1}{12}
\]

(18)