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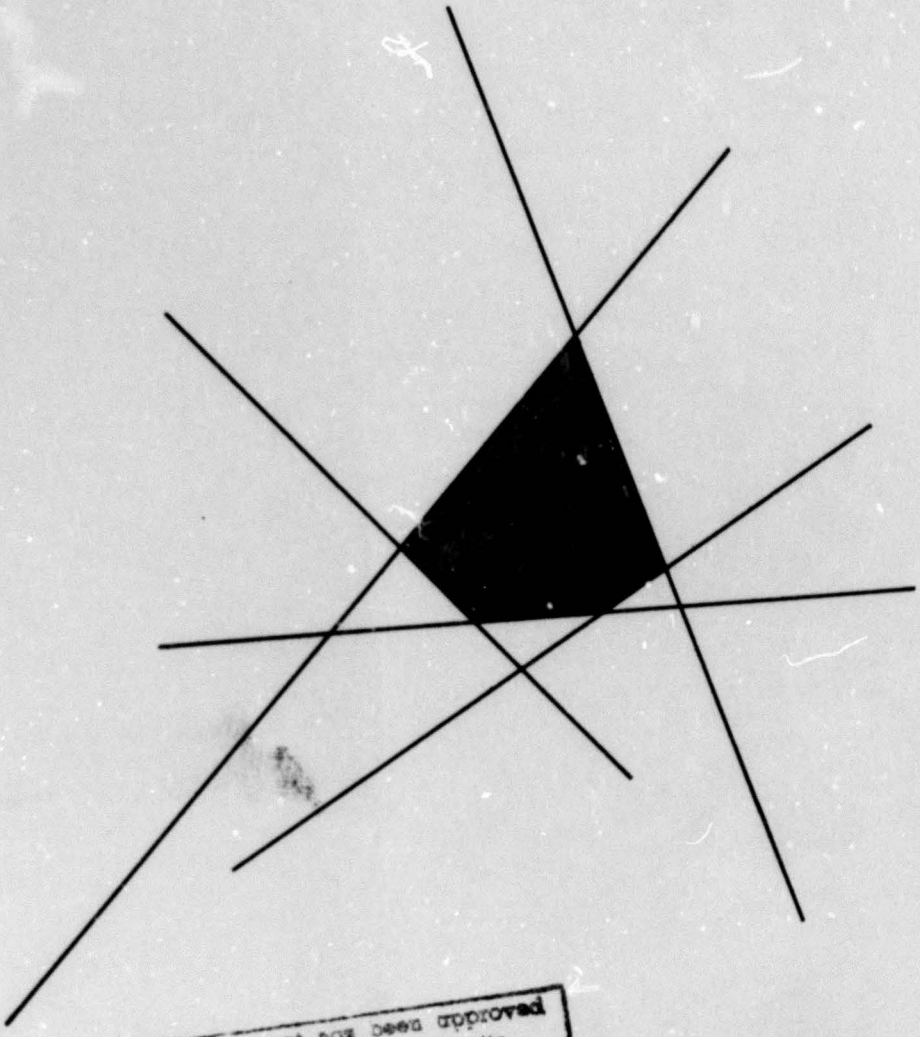

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by

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1. INTRODUCTION

The game of pure strategy, sometimes called Goofspiel or Gops (see [2] and [3]), is played by two players, using a normal deck of cards, as follows. The 13 clubs are first taken out of the deck and of the remaining 39 cards the 13 hearts are given to Player I, the 13 diamonds to Player II, and the 13 spades are placed face down in the center. The spades are shuffled and one is turned face up. At this point, the two players choose one of their cards and then simultaneously discard it. The one who discards the higher card (ace being low, king high) wins from the other an amount equal to the value of the upturned spade (ace = 1 , king = 13). If both players discard the same card, then neither wins. The three cards are then thrown away, a new spade upturned and the game continues. After 13 plays, there are no remaining cards and the game ends.

In Section 2, we consider this game under the assumption that Player II discards his cards in a completely random manner. Given this information, we show that the best thing for Player I to do is to always match the upturned spade, i.e., if the upturned card is an ace then I should play his ace, etc. The expected winnings of Player I is shown to equal 28 .

In Section 3, we show how Goofspiel may be treated as a stochastic game. This special structure is then utilized to determine a dynamic programming type recursion algorithm for solving it.

In Section 4, we consider the game of Hidden Card Goofspiel. In this variation, it is supposed that the players must discard before the middle card is turned face up. The randomizing strategy is then shown to be optimal for both players.

2. GOOFSPIEL AGAINST A RANDOMIZING OPPONENT

Let us first generalize our game. Suppose that Player I has N cards having values $V_1, V_2, \ldots, V_N$, where $V_1 \leq V_2 \leq \ldots \leq V_N$; Player II has N cards having values $Y_1, Y_2, \ldots, Y_N$, where $Y_1 \leq Y_2 \leq \ldots \leq Y_N$; and the N cards in the middle have values $P_1, P_2, \ldots, P_N$, where $P_1 \leq P_2 \leq \ldots \leq P_N$. The game is played as before: One of the center cards is turned face up. The players then simultaneously discard and whoever's card has the higher value wins from the other an amount equal to the value of the middle card. These three cards are then thrown away and the play continues until there are no cards left.

Theorem 1:

If Player II discards in a completely random manner, then the strategy maximizing Player I's expected winning is the one which discards the card having value V_i whenever the upturned middle card has value P_i, $i = 1,2, \ldots, N$.

Proof:

The proof is by induction on N. The theorem is trivially true for $N = 1$, so assume it for $N - 1$. Suppose now that for the N-card problem the initial upturned card has value P_j and consider any strategy which calls for Player I to play V_i where $i < j$. After this first discard, I has cards $1, \ldots, i-1, i+1, \ldots, j, \ldots, N$, while the center has cards $1, \ldots, i, \ldots, j-1, j+1, \ldots, N$. Hence, from the induction hypothesis, it follows that if the initial upturned card has value P_j then, among those strategies which call for I to play V_i, the best is the one which plays

$$
(1) \qquad \begin{array}{lll} V_k \text{ on } P_k, & & k = 1, \ldots, i-1 \\ V_i \text{ on } P_j & & \\ V_{k+1} \text{ on } P_k, & & k = i, \ldots, j-1 \\ V_k \text{ on } P_k, & & k = j+1, \ldots, N. \end{array}
$$

Compare this, however, with the strategy which is the same as (1) with the exception that it uses

$$(2) \qquad \begin{matrix} V_{i+1} & \text{on} & P_j \\ V_i & \text{on} & P_i \end{matrix} .$$

That is, strategies (1) and (2) are identical except that (1) uses

$$\begin{matrix} V_i & \text{on} & P_j \\ V_{i+1} & \text{on} & P_i \end{matrix}$$

and the second uses (2). The expected payoff to Player I for these two plays is, under strategy (1)

$$1/N\ P_j[(\text{Number}\ \ k : Y_k < V_i) - (\text{Number}\ \ k : Y_k > V_i)]$$
$$+\ 1/N\ P_i[(\text{Number}\ \ k : Y_k < V_{i+1}) - (\text{Number}\ \ k : Y_k > V_{i+1})]$$

while under strategy (2) it is

$$1/N\ P_i[(\text{Number}\ \ k : Y_k < V_i) - (\text{Number}\ \ k : Y_k > V_i)]$$
$$+\ 1/N\ P_j[(\text{Number}\ \ k : Y_k < V_{i+1}) - (\text{Number}\ \ k : Y_k > V_{i+1})] .$$

Hence, strategy (2) is at least as good as strategy (1). Therefore, for any $i < j$, whenever the initial upturned card is P_j, there is a strategy which plays V_{i+1}, that is, at least as good as any which plays V_i. By repeating this argument, it follows that there is a strategy which initially plays V_j, that is, at least as good as any playing V_i. Similar results may be shown for $i > j$ and hence by the induction hypothesis the strategy which always matches the upturned card is optimal.

Q.E.D.

Corollary 1:

If Player II plays randomly, then

(i) for any value x, the probability that Player I's winnings exceeds x is maximized by the matching strategy, and

(ii) the expected winnings of Player I is

$$1/N \sum_{i=1}^{N} P_i[(\text{Number } j : Y_j < V_i) - (\text{Number } j : Y_j > V_i)] .$$

(iii) If $V_i = P_i = Y_i = i$, then (ii) equals

$$\frac{(N - 1)(N + 1)}{6} .$$

Proof:

Part (i) is proved by showing that Player I's winnings is stochastically larger under strategy (1) than it is under strategy (2). This is shown by considering all possible outcomes of the two plays P_i and P_j.

Parts (ii) and (iii) are obvious.

3. GOOFSPIEL AS A SUPER-GAME

We first note that the number of pure strategies for each player is

$$N^N \prod_{k=1}^{N-1} k^{k(k+1)} . \tag{3}$$

To see why (3) is true, reason as follows. For each initial upturned middle card, Player I has a choice of N cards; hence, the first term N^N. Now, conditional on the first upturned card and the first card played by I, the choice of I on the second play is determined by the second upturned card and the first card played by Player II; hence, the second term $(N-1)^{(N-1)N}$. The reasoning progresses similarly.

From (3), it is clear that it is not possible to write down all the pure strategies and calculate the payoff matrix.[†] Rather, we shall attempt to treat N-card Goofspiel as a supergame consisting of N subgames, and develop a dynamic programming type recursion relation. Towards this end, let $f(V_1, \ldots, V_N, Y_1, \ldots, Y_N, P_1, \ldots, P_N, P_k)$ be the value of the game to I if I initially has values $V_1, \ldots, V_N$, II initially has values $Y_1, \ldots, Y_N$, the middle initially has values $P_1, \ldots, P_N$, and the initial upturned card is P_k. Then

$$f(V_1, \ldots, V_N, Y_1, \ldots, Y_N, P_1, \ldots, P_N, P_k) = \text{value of the } N \times N \text{ game with payoff matrix } [X_{ij}] \tag{4}$$

where

[†] For $N = 4$, Equation (3) tells us that there are more than 8.4 billion pure strategies.

$$X_{ij} = P_k \bar{\delta}_{ij}$$

$$+ \frac{1}{N-1} \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N} f(V_1 \cdots V_{i-1} V_{i+1} \cdots V_N, Y_1 \cdots Y_{j-1} Y_{j+1} \cdots Y_N, P_1 \cdots P_{k-1} P_{k+1} \cdots P_N, P_\ell)$$

$$\bar{\delta}_{ij} = \begin{cases} 1 & V_i > Y_j \\ 0 & V_i = Y_j \\ -1 & V_i < Y_j \end{cases} . \tag{5}$$

Equation (4) is true because after the initial play the situation is the same as if the players had started with $N - 1$ cards. Thus the N card problem may be solved by first solving all $N - 1$ card problems, which may be solved in terms of all $N - 2$ card problems, etc. Hence, solving recursively (or backwards), we would need to solve

$$j\binom{N}{j}^3 \quad j \text{ by } j \text{ games for } j = 1,2, \ldots, N .$$

For instance, when $N = 4$, rather than having to solve one 8.4 billion by 8.4 billion game, we would need to solve 4 four-by-four games, 192 three-by-three games, and 432 two-by-two games.

The necessary computation simplifies considerably if we suppose that the middle cards are turned over in some fixed order.† In this case, we would need to recursively solve

$$\binom{N}{j}^2 \quad j \text{ by } j \text{ games for } j = 1,2, \ldots, N .$$

†In this case, the number of pure strategies available for each player is $N \prod_{k=1}^{N-1} k^{k+1}$.

4. HIDDEN CARD GOOFSPIEL

The game of Hidden Card Goofspiel is played as before with the exception that the players are required to discard their cards before the point value of the play is revealed to them. That is, the middle cards are shuffled and one is placed face down and then the players simultaneously discard a card. The three cards are then turned face up and the game continues.

Theorem 2:

For Hidden Card Goofspiel randomizing is optimal for each player and the value of the game to Player 1 is

$$1/N^2 \sum_{k=1}^{N} P_k \sum_{i,j=1}^{N} \bar{\delta}_{ij} \tag{6}$$

where $\bar{\delta}_{ij}$ is given by (5).

Proof:

We prove this by showing that if I randomizes then his expected return is given by (5) irregardless of II's strategy. This is proven by induction on N. It is obvious for $N = 1$, hence assume it for $N - 1$. Suppose now that for the N-card problem II initially plays Y_j. Then, by the induction hypothesis, it follows that I's expected payoff given that he randomizes is exactly

$$1/N^2 \sum_{i,k=1}^{N} \left[P_k \bar{\delta}_{ij} + \frac{1}{(N-1)^2} \sum_{\ell \neq k} P_\ell \sum_{\substack{\ell \neq i \\ k \neq j}} \bar{\delta}_{\ell k} \right] \tag{7}$$

and the induction will be completed if we can show that (7) equals (6). This, however, follows by first noting that (6) is just the expected payoff to I given that I and II both randomize. However, by writing the payoff to I as the payoff to I on the play for which II uses Y_j plus the payoff to I on the remaining plays of the game, it follows by conditioning on the middle value and I's card on the

play that II uses Y_j that (7) also represents the expected payoff to I given that I and II both randomize. Hence, (7) equals (6) and the induction is complete. This, however, implies that the randomized strategy guarantees I the value (6) irregardless of II's strategy. Also, by reversing the roles of I and II, it follows that if II randomizes then he can lose no more than (6) and the result follows.

Remark:

Theorem 2 is somewhat similar to a result proven by Gale [1].

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