IMPACT LOADING OF SUBMARINE HULLS

By
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Contract No. ONR-N00014-68-A-0146-8
Report No. THEMIS-UM-70-2

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February 1970
Approved for Release


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ABSTRACT

Following Flugge's exact derivation for the buckling of cylindrical shells, the equations of motion for dynamic loading of cylindrical shells subjected to hydrostatic and axial pressure have been formulated.

The equations of motion are applicable for long, short, or thick shells, and are very useful in calculating deflections and stresses when the impact loads are applied to comparatively small regions of the shell. The normal mode theory was utilized to provide dynamic solutions for the equations of motion.

Solutions are also provided for the Timoshenko-type theory, and comparisons are made between the two theories by considering and neglecting in-plane inertia forces.

Comparison of results is exemplified by a numerical example which considers the effect of hydrostatic pressure on the dynamic response of a shell simply supported by a thin diaphragm and subjected to a localized unit radial impulse.
ACKNOWLEDGEMENT

This research was accomplished with the support of the Office of Naval Research. Computations were done at the University of Massachusetts Research Computer Center, Amherst, Massachusetts. The author wishes to record his thanks to Messrs. Aarij and Minhaj Kirmani for their assistance in computer programming and calculations.
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Introduction

The purpose of this investigation is to determine the effect of impact loads on submarine hulls. For ductile materials current design methods utilize static loads for design with a performance criterion that the hull behave in a ductile manner when subjected to an impact load that may occur during ground collision or depth charges. Since glass, considered as a possible material for design has brittle properties the present design practices must be re-evaluated. Hence, more rigorous analyses must be made to determine the dynamic stress and deformation characteristics of glass hulls subjected to impact loads, and deep hydrostatic pressures. The present investigation will consider only the response of the shell. The coupled response of the shell and ring frames will be considered in a future investigation.

The model to be investigated will essentially be a cylindrical shell under deep hydrostatic and axial pressure, and subjected to an impact load.

Following Flugge's exact derivation for the buckling of cylindrical shells, the equations of motion are formulated. The equations are applicable for long, short or thick shells, and are very useful in calculating deflections and stresses when the impact loads are applied to comparatively small regions of the shell. The normal mode theory is utilized to provide dynamic solutions for the equations of motion.
Solutions are also provided for the Timoshenko-type theory, and comparisons are made between the two theories by considering and neglecting in-plane inertia forces.

Equations of Motion

Following Flugge's [1] exact derivation for the buckling of cylindrical shells, the differential equations of motion for impact loading of cylindrical shells under hydrostatic pressure become:

\[
\begin{align*}
\alpha v''_x + \alpha x''_x &= p_a(u'' - u') - P'u'' + a^2p_x = \chi a^2 \frac{\partial u}{\partial t} \\
\alpha v''_y + \alpha y''_y &= p_a(v'' + v') - P'y'' + a^2p_y = \chi a^2 \frac{\partial v}{\partial t} \\
-\alpha x''_x - \alpha y''_x &= p_a(u'' - v'' + w'') - P'u'' + a^2p_r \\
&= \chi a^2 \frac{\partial w}{\partial t} 
\end{align*}
\]

where

\[
\begin{align*}
Q_x &= \frac{\partial u}{\partial a} \\
Q_y &= \frac{\partial v}{\partial a} \\
Q_z &= \frac{\partial w}{\partial a} \\
\alpha v'' &= \frac{u'' + v''}{a} + \frac{K}{a^4} (u'' + v'') \\
\alpha x'' &= \frac{u'' + v'' + w''}{a} - \frac{K}{a^4} w'' \\
\alpha x'' &= \frac{1}{\alpha} \left( u'' + v'' + w'' \right) - \frac{1}{\alpha^2} w'' \\
\alpha x'' &= \frac{1}{\alpha^2} \left( u'' + v'' + w'' \right) - \frac{1}{\alpha^4} \left( u'' + v'' + w'' \right)
\end{align*}
\]
\[ H_{x\phi} = \frac{u}{a} \frac{1 - \nu}{2} (u' + v') + \frac{K}{a^2} \frac{1 - \nu}{2} (v' - w') \]  \hspace{1cm} (9)

\[ H_{\phi} = \frac{K}{a^2} (w + w' + vw') \]  \hspace{1cm} (10)

\[ H_{x} = \frac{K}{a^2} (w'' + vw'' - u' - uv') \]  \hspace{1cm} (11)

\[ H_{\phi x} = \frac{K}{a^2} (1 - \nu)(w'' + \frac{1}{2} u' - \frac{1}{2} v') \]  \hspace{1cm} (12)

\[ H_{x\phi} = \frac{K}{a^2} (1 - \nu)(w'' - v') \]  \hspace{1cm} (13)

Substitution of equations (4) through (13) into equations (1) through (3) yields:

\[ u'' + \left( \frac{1 - \nu}{2} \right) u'' + \frac{1 + \nu}{2} v'' + \nu w' + k \left( \frac{1 - \nu}{2} u'' - w'' + \frac{1 - \nu}{2} w'' \right) \]

\[ - q_1 (u'' - w') - q_2 u'' + \frac{p(x, t)a^2}{D} = \frac{\rho a^2}{3} \frac{2 \nu}{3 t^2} \]  \hspace{1cm} (14)

\[ \frac{1 + \nu}{2} u'' + v'' + \frac{1 - \nu}{2} v'' + w'' + k \left( \frac{3}{2} (1 - \nu)v'' - \frac{3 - \nu}{2} w'' \right) \]

\[ - q_1 (v'' + w'') - q_2 v'' + \frac{p(x, t)a^2}{D} = \frac{\rho a^2}{3} \frac{2 \nu}{3 t^2} \]  \hspace{1cm} (15)

\[ \nu u'' + v' + w + k \left( \frac{1 - \nu}{2} u'' + u'' - \frac{3 - \nu}{2} v'' + w'' \right) \]

\[ + 2w'' + w'' + 2w'' + w' + q_1 (u'' + v'' + w') \]

\[ + q_2 w'' - \frac{p_r(x, t)a^2}{D} = -\frac{\rho a^2}{3} \frac{2 \nu}{3 t^2} \]  \hspace{1cm} (16)

where

\[ k = \frac{k^2}{12 a^2} \]

\[ q_1 = \frac{p a}{D} \]

\[ q_2 = \frac{p}{D} \]
Equations (14) through (16) may be written:
\[
\alpha_3 u'' + \alpha_4 v'' + \frac{1 + \nu}{2} v'' + \alpha_3 w'' + k\left(\frac{1 - \nu}{2} w'' - w''\right) + \frac{p_x(x, t)}{D} a^2 = \frac{\rho \alpha^2}{\frac{\partial^2 u}{\partial t^2}}
\]
\[
\beta_1 + \nu + \frac{\alpha_4}{2} (v'' + w'') \beta_1 + \alpha_3 v'' - \frac{k}{2} (3 - \nu) \beta_2 + \frac{p_\beta(x, t)a^2}{D} = \frac{\rho \alpha^2}{\frac{\partial^2 v}{\partial t^2}}
\]
\[
\alpha_3 u'' + \alpha_4 v'' + (2k + 1 - \alpha_4) w'' + k\left(\frac{1 - \nu}{2} u'' - u''\right) - \frac{3 - \nu}{2} v'' + w'' + 2w'' + w'' + \beta_1 + \left(\frac{1 + k}{k}\right) \beta_2 + (1 - \alpha_5) w'' - \frac{p_r(x, t)a^2}{D} = -\frac{\rho \alpha^2}{\frac{\partial^2 w}{\partial t^2}}
\]

where
\[
\alpha_1 = \frac{1 - \nu}{2} (1 + 3k) - q_2
\]
\[
\alpha_2 = \frac{1 - \nu}{2} (1 + k) - q_1
\]
\[
\alpha_3 = (\nu + q_1)
\]
\[
\alpha_4 = 1 - q_1
\]
\[
\alpha_5 = 1 - q_2
\]

Orthogonality and Modal Vibrations

For free vibrations, equations (1) through (3) become
\[
\alpha_1' + \alpha_1' - p(a u'' - w') - p u'' = \frac{\rho \alpha^2}{\frac{\partial^2 u}{\partial t^2}}
\]
\[
\alpha_3' + \alpha_3' - aQ_0 - p(a v'' + w') - p v'' = \frac{\rho \alpha^2}{\frac{\partial^2 v}{\partial t^2}}
\]
Equations (20) through (22) yield the free vibration frequencies and mode shapes. The orthogonality condition is derived by assuming that the displacements \( u \), \( v \), and \( w \) have the form

\[
u = u_n(x, \phi) e^{i\omega t}, \quad v = v_n(x, \phi) e^{i\omega t}, \quad w = w_n(x, \phi) e^{i\omega t}
\]

Finding the orthogonality condition involves the following steps:

1. The \( m \)th terms of expressions (23) are inserted into equations (20) through (22), and the resulting equations are multiplied by \( u_m(x) \), \( v_m(x) \), and \( w_m(x) \), respectively, integrated over the domain, and added;
2. The \( m \)th terms of expressions (23) are inserted into equations (20) through (22), and the resulting equations are multiplied by \( u_n(x) \), \( v_n(x) \), and \( w_n(x) \), respectively, integrated over the domain, and added;
3. The two equations resulting from Step 2 are subtracted from those resulting from Step 1; they are integrated by parts, and use is made of equations (4) through (13) to obtain the final orthogonality relation. The orthogonality condition may be written as follows:

\[
\left(\omega_n^2 - \omega_m^2\right) \int_0^L \rho \left(u_n u_m + v_n v_m + w_n w_m\right) \, dx =
\]

\[
\begin{align*}
&u_n \left(\frac{H_{xmn}}{a} + p \frac{3u_m}{3x} - \frac{3u_n}{3x}\right) + v_n \left(\frac{H_{xnm}}{a} - p \frac{3v_m}{3x}\right) - v_n \left(\frac{H_{xnm}}{a} + p \frac{3w_m}{3x}\right) + w_n \left(\frac{H_{xnm}}{a} - p \frac{3v_n}{3x}\right) - v_n \left(\frac{H_{xnm}}{a} + p \frac{3u_n}{3x}\right) - w_n \left(\frac{H_{xnm}}{a} - p \frac{3w_n}{3x}\right) = 0, \quad m \neq n
\end{align*}
\]
where the natural boundary conditions for the fixed, simply supported and free condition are given as:

Fixed  

at \( x = 0 \),

\[
\begin{align*}
    u &= v = w = \frac{\partial w}{\partial x} = 0 \\
    H_x &= C 
\end{align*}
\]  

(25)

Hinge  

at \( x = 0 \),

\[
\begin{align*}
    u &= v = w = 0. \\
    H_x &= C 
\end{align*}
\]  

(26)

Simply Supported  

at \( x = 0 \),

\[
\begin{align*}
    w &= 0 \\
    H_x &= 0 \\
    H_x - \frac{H_x}{a} - p \frac{\partial v}{\partial x} &= 0 \\
    H_x + pw - P \frac{\partial u}{\partial x} &= 0 
\end{align*}
\]  

(27)

Free  

at \( x = 0 \),

\[
\begin{align*}
    H_x &= 0 \\
    Q_x + p \frac{\partial w}{\partial x} &= 0 \\
    H_x - \frac{H_x}{a} - p \frac{\partial v}{\partial x} &= 0 \\
    H_x + pw - P \frac{\partial u}{\partial x} &= 0 
\end{align*}
\]  

(28)

The differential equations (17) - (19) may be solved by assuming

\[
\begin{align*}
    u &= Ae^{i\lambda x/a} \cos(m\phi) e^{i\omega t} \\
    v &= Be^{i\lambda x/a} \sin(m\phi) e^{i\omega t} \\
    w &= Ce^{i\lambda x/a} \cos(m\phi) e^{i\omega t} 
\end{align*}
\]  

(29)
Inserting equation (29) into equations (17) - (19) yields equation (30).

\[
\begin{bmatrix}
\alpha^2 - \frac{1}{2} \omega^2 + (\alpha \phi \omega^2 \mu/v) & \left(\frac{1}{2} \omega^2\right) m \lambda \\
-(\frac{1}{2} \omega^2) m \lambda & -\alpha^2 \omega^2 + \alpha \lambda^2 + (\alpha \phi \omega^2 \mu/v) & -\alpha \phi k (\frac{3}{2} \omega^2) m^2 \\
\alpha \omega k \left(\frac{1}{2} \omega^2 \mu - k \lambda^3 \right) & \alpha \omega k \left(\frac{3}{2} \omega^2 \mu \right) - (2k + 1) \alpha \phi m^2 + 1 & -(\alpha \phi k \omega^2) + (1 - \alpha)^2 \phi \mu \\
&(\alpha^2 - \frac{1}{2} \omega^2)^2 \mu + \phi \mu^2 \nu/f & \left(\frac{2}{\phi} \omega^2 \mu^2 / \nu \right) & 0
\end{bmatrix} \begin{bmatrix}
A_i \\
B_i \\
C_i
\end{bmatrix} = 0
\]

(30)

The characteristic equation is found by setting the determinant of equation (30) equal to zero. To determine the eigenvalues, \( \omega^2 \), the following method is utilized: A value of \( \omega^2 \) is guessed and inserted into the characteristic equation. The characteristic equation will yield eight roots. For unequal roots, equations (29) may be written as follows:

\[
u = \sum_{i=1}^{8} A_i e^{i \lambda_i x/a} (\cos m \phi e^{i \omega t}), \quad v = \sum_{i=1}^{8} B_i e^{i \lambda_i x/a} (\sin m \phi e^{i \omega t})
\]

\[
w = \sum_{i=1}^{8} C_i e^{i \lambda_i x/a} (\cos m \phi) e^{i \omega t}
\]

(31)

where for each \( \lambda_i \) there exists a relationship between the amplitudes \( A_i, B_i \) and \( C_i \) from the determinant of equation (30).

Equations (31) with the necessary boundary conditions will lead to a determinant \( |a_{ij}| \). A plot is then made of the determinant \( |a_{ij}| \) versus \( \omega^2 \). The eigenvalues, \( \omega^2 \), are those for which \( |a_{ij}| = 0 \). At a point, \( \omega^2 \), when \( |a_{ij}| = 0 \), the ratio of the amplitudes \( A_i, B_i \) and \( C_i \) can be calculated from the determinant of equation (30).
For impact loads, local bending action will predominate, and the principal mode of response will be in the radial direction. Neglecting inertia forces in the longitudinal and circumferential directions, equation (32) reduces to equation (33).

\[ u_{5}\cdot^{2} - \omega_{5}^{2} \phi_{5} = 0 \]

\[ v_{5}^{2} + \frac{3\omega_{5}^{2}}{2} \phi_{5} = 0 \]

\[ \omega_{3} = 1 \left( \frac{5\omega_{5}}{2} \right)^{2} - 4\omega_{5}^{2} \]

\[ \phi_{3} = \left( \frac{5\omega_{5}}{2} \right)^{2} - 4\omega_{5}^{2} \]

\[ \phi_{1} = \left( \frac{5\omega_{5}}{2} \right)^{2} - 4\omega_{5}^{2} \]

Solutions for Forced Vibrations

Equations (17) through (19) may be solved by assuming

\[ u = \sum_{n=0}^{\infty} u_{n}(x, \phi) q_{n}(t) \]

\[ v = \sum_{n=0}^{\infty} v_{n}(x, \phi) q_{n}(t) \]

\[ w = \sum_{n=0}^{\infty} w_{n}(x, \phi) q_{n}(t) \]

Substituting the above equations into equations (17) through (19), and utilizing the orthogonality condition (24) yields the following:

\[ q_{n}(t) = \frac{2\pi}{r^{2}\pi} \int_{0}^{2\pi} \int_{0}^{r} \left[ P_{x}(x, \phi, \lambda) u_{n}^{2} + P_{r}(x, \phi, \lambda) v_{n}^{2} + P_{\phi}(x, \phi, \lambda) w_{n}^{2} \right] \sin \omega_{n}(t - \omega_{n} \lambda) \, dx \, d\phi \]

\[ q_{n}(t) = \frac{2\pi}{r^{2}\pi} \int_{0}^{2\pi} \int_{0}^{r} \left[ c_{n}^{2} + v_{n}^{2} + w_{n}^{2} \right] \sin \omega_{n}(t - \omega_{n} \lambda) \, dx \, d\phi \]

(34)
For an impact loading as shown in Figure 1, equation (34) becomes

\[
q_n(t) = \frac{r_{(+t)} - r_{(-t)}}{a} \int_{-\infty}^{t} \int_{0}^{2\pi} \left[ p_x(x,\phi,\tau)u_n + p_\phi(x,\phi,\tau)v_n + p_r(x,\phi,\tau)w_n \right] \sin \omega_m (t-\tau) \; d\tau \; d\phi.
\]

For a concentrated impact loading, equation (34) becomes

\[
q_1(t) = \frac{1}{\sin \omega_m t} \int_{0}^{2\pi} \int_{0}^{\tau} \left[ p_x(x,\phi,\tau)u_n + p_\phi(x,\phi,\tau)v_n + p_r(x,\phi,\tau)w_n \right] \sin \omega_m (t-\tau) \; d\tau \; d\phi
\]

where

\[
p_x = \frac{p}{4b_1 b_2}, \quad p_\phi = \frac{p}{4b_1 b_2}, \quad p_r = \frac{p}{4b_1 b_2}.
\]

Solution for Impulse

Consider an impulse per unit area, \( i_x(x, \phi), i_\phi(x, \phi) \) and \( i_r(x, \phi) \) acting on the cylinder for an infinitely short time. The cylinder may now be considered to be vibrating freely with the following initial conditions:

At \( t = 0 \)

\[
u = v = w = 0
\]
Figure 1-a

Cylindrical Shell Subjected to Dynamic and Hydrostatic Loading
The displacements for free vibrations are given as

\[ u = \sum_{m=0}^{\infty} u_m (A_m \cos \omega_m t + B_m \sin \omega_m t) \]

\[ v = \sum_{m=0}^{\infty} v_m (A_m \cos \omega_m t + B_m \sin \omega_m t) \] (39)

\[ w = \sum_{m=0}^{\infty} w_m (A_m \cos \omega_m t + B_m \sin \omega_m t) \]

Substituting the initial conditions (37) and (38) into equation (39) and making use of orthogonality yields the following

\[ u = \sum_{m=0}^{\infty} u_m \lambda_m \sin \omega_m t \]

\[ v = \sum_{m=0}^{\infty} v_m \lambda_m \sin \omega_m t \] (40)

\[ w = \sum_{m=0}^{\infty} w_m \lambda_m \sin \omega_m t \]

where

\[ \lambda_m = \frac{1}{\omega_m} \int_0^{2\pi} \int_0^{2\pi} \left( i u_m + i v_m + i w_m \right) dx d\phi \]

\[ \int_0^{2\pi} \int_0^{2\pi} \left( \omega_m^2 + \rho_m(u_m^2 + v_m^2 + w_m^2) \right) dx d\phi \] (41)

For a distributed impulse as shown in Figure 1, equation (41) becomes
For a concentrated impulse, equation (41) becomes:

\[ \lim_{\epsilon_1 \rightarrow 0} \int_{r_1 - \epsilon_2}^{r_1 + \epsilon_2} \int_{\gamma}^{\xi_2} \left[ i_m^x u_m + i_m^\phi v_m + i_m^r w_m \right] dx \, d\gamma \]

\[ \frac{b_m}{w_m} \left[ \int_0^{2\pi} \int_0^\gamma \rho h(u_m^2 + v_m^2 + w_m^2) \, dx \, d\gamma \right] \]

where

\[ i_x = \frac{I_x}{4\epsilon_1 \epsilon_2} \]

\[ i_\phi = \frac{I_\phi}{4\epsilon_1 \epsilon_2} \]

\[ i_r = \frac{I_r}{4\epsilon_1 \epsilon_2} \]

Solutions for \( m = 0 \)

For \( n = 0 \), equations (1) through (3) and (14) through (16) degenerate to the following equations:

\[ a_{11}^u u'' + a_{19}^u P u'' = \rho \phi a^2 \frac{\partial^2 u}{\partial \xi^2} \]

\[ a_{11}^\phi \phi'' + a_{19}^\phi \phi P \phi'' = \rho \phi a^2 \frac{\partial^2 \phi}{\partial \xi^2} \]

\[ -a_{11}^\phi \phi'' - a_{19}^\phi \phi P \phi'' = \rho \phi a^2 \frac{\partial^2 \phi}{\partial \xi^2} \]

\[ -a_{33}^u u''' + k u''' + (p_x a^2 u') = \rho \phi a^2 \frac{\partial^2 u}{\partial \xi^2} \]

\[ u_1 v'' + (p_x a^2 u') = \rho \phi a^2 \frac{\partial^2 v}{\partial \xi^2} \]

\[ -a_{33}^u u''' + k u''' + \left( \frac{k}{k+1} \right) u'' - (1 - a_y) w'' \]

\[ + (p_r a^2 w') = \rho \phi a^2 \frac{\partial^2 w}{\partial \xi^2} \]
Solutions for equations (47) and (49) can be determined from two solutions given for unsymmetrical loading. For \( m = 0 \), equation (30) becomes

\[
\begin{bmatrix}
\alpha_5 \lambda + \frac{\phi a^2 \omega^2}{D} & 0 & \alpha_3 \lambda - k \lambda^3 \\
0 & \alpha_1 \lambda^2 + \frac{\phi a^2 \omega^2}{D} & 0 \\
\lambda^3 - k \lambda^2 & 0 & k \lambda^4 + (1 - \alpha_5) \lambda^2 \\
\end{bmatrix}

\begin{bmatrix} A \\ B \end{bmatrix} = 0 
\]

or

\[
\lambda^6 + g_1 \lambda^4 - g_2 \lambda^2 - g_3 = 0
\]

where

\[
g_1 = [\alpha_5 (1 - \alpha_5) + k (\frac{\phi a^2 \omega^2}{D} + 2 \alpha_3)] / (\alpha_5 - k) \\
g_2 = [\alpha_3^2 - \alpha_5 (1 + k) - (1 - 2 \alpha_5) (\frac{\phi a^2 \omega^2}{D})] / (\alpha_5 - k) \\
g_3 = \frac{\phi a^2 \omega^2}{D} (\frac{\phi a^2 \omega^2}{D} - k - 1) / (\alpha_5 - k)
\]

Equations (47) through (49) are now uncoupled and may be solved independently.

The solution for equation (48) is as follows:

For free vibrations

\[
v = n_1 \cos \frac{\phi a}{D} \omega \frac{x}{a} + n_2 \sin \frac{\phi a}{D} \omega \frac{x}{a}
\]

The orthogonality conditions are

\[
(\omega_n^2 - \omega_m^2) \int_0^L \frac{\phi a}{D} v_n v_m dx = \left| v_n \frac{\partial v_n}{\partial x} - v_m \frac{\partial v_m}{\partial x} \right|_0
\]

The forced vibration solution becomes
The orthogonality conditions from equation (24) become:

\[
(\omega_n^2 - \omega_m^2) \int_0^t \sin(\omega_n t) (u_n u_m + w_n w_m) \, dt = 0
\]

\[
= u_{nn} (u_{nn} + pu_{nn} - \frac{\partial u_n}{\partial x}) - u_n (Q_{nn} + P \frac{\partial u_n}{\partial x})
\]

\[
- \int \left[ u_{nn} + p u_{nn} - \frac{\partial u_n}{\partial x} \right] (Q_{nn} + P \frac{\partial u_n}{\partial x})
\]

\[
0 = 0
\]

The dynamic solutions from equations (33) and (34) are:

\[
u = \sum_{n=1}^\infty u_{on}(x) q_{on}(t)
\]

\[
w = \sum_{n=1}^\infty w_{on}(x) q_{on}(t)
\]

where:

\[
q_{on}(t) = \frac{\int_0^t \left[ p_x(x, t) u_{no} + p_r(x, t) u_{no} \right] \sin \omega_{on} (t - \cdot) \, dx \, dt}{\int_0^t \phi (u_{no}^2 + w_{no}^2) \, dx}
\]

**Integral of the Square of Eigenfunctions**

The integral of the square of the eigenfunctions is evaluated from equation (24) by a limiting process. For any prescribed boundary condition, the evaluation of the integral may be determined as follows:
\[ \int_{-l/2}^{l/2} \sin[u_n^2(x) + v_n^2(x) + w_n^2(x)] \, dx \]

\[ = \sin \left( \frac{-l}{2} \int_{0}^{l} \left( u_{n,m}^2 + \frac{\partial^2 u_n}{\partial x^2} - p \frac{\partial^2 u_n}{\partial x^2} \right) + \frac{m}{a} \frac{\partial v_n}{\partial x} - v_{n,m}^2 \left( q \frac{\partial x}{\partial x} + p \frac{\partial^2 v_n}{\partial x^2} \right) + \frac{n}{a} \frac{\partial w_n}{\partial x} \right) \int_{0}^{l} \left( u_{n,m}^2 + \frac{\partial^2 u_n}{\partial x^2} - p \frac{\partial^2 u_n}{\partial x^2} \right) + \frac{m}{a} \frac{\partial v_n}{\partial x} - v_{n,m}^2 \left( q \frac{\partial x}{\partial x} + p \frac{\partial^2 v_n}{\partial x^2} \right) + \frac{n}{a} \frac{\partial w_n}{\partial x} \right) \]

Illustrative Example for Cylinder Supported by Thin Diaphragm

For a cylinder supported by a thin diaphragm, the following displacements satisfy the natural boundary conditions as derived from the orthogonality conditions:

\[ u = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} U_{n,m} \cos \frac{\pi n x}{a} \cos \frac{\pi m y}{b} \]

\[ v = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} V_{n,m} \sin \frac{\pi n x}{a} \sin \frac{\pi m y}{b} \]

\[ w = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} W_{n,m} \cos \frac{\pi n x}{a} \sin \frac{\pi m y}{b} \]

To determine the natural frequencies and mode shapes, the determinant for the frequency equation becomes
Solutions for Forced Vibrations

From equations (33) and (34) the dynamic displacements in, com.

\[
\begin{align*}
    u &= \sum_{m_1} \sum_{n_1} \cos m_1 \cos n_1 \cos \left(\frac{n_1 x}{r}\right) q_{m_1 n_1} (t) \\
    v &= \sum_{m_1} \sum_{n_1} \sin m_1 \sin n_1 \sin \left(\frac{n_1 x}{r}\right) q_{m_1 n_1} (t) \\
    w &= \sum_{m_1} \sum_{n_1} \sin m_1 \cos n_1 \cos \left(\frac{n_1 x}{r}\right) q_{m_1 n_1} (t)
\end{align*}
\]
\[ u_{\text{on}}(t) = \int_0^{2\pi} \int_0^1 \int_0^t \left( p_X(x, \varphi, \lambda) u_{\text{on}} \cos \frac{nx}{\xi} + \sum_{m=1}^\infty \int_0^{2\pi} \int_0^1 \int_0^t \left( p_X(x, \varphi, \lambda) u_{\text{on}} \cos \frac{nx}{\xi} + \sum_{m=1}^\infty \left( \cos \frac{mx}{\xi} \right) \right) \sin \frac{n \sin t}{\xi} \right) \frac{\sin \frac{n \sin t}{\xi} \, dt \, dx \, d\varphi}{\omega_{\text{on}} \sin x_{\text{on}}(t) \, dx \, d\delta} \]

The ratio of the mode shape coefficients are

\[ \rho = \frac{U_{\text{on}}}{U_{\text{mn}}} = \frac{-CD + BE}{AD - B^2} \]

\[ \delta = \frac{V_{\text{on}}}{V_{\text{mn}}} = \frac{-AE + BC}{AD - B^2} \]

where:

\[ \beta = -\sqrt{\frac{m^2}{\xi} - m^2 \frac{\pi^2}{\xi^2} + \frac{\pi^2}{\xi^2}} \]

\[ \alpha = \frac{1}{\xi} \left( \frac{m^2}{\xi} \right) \]

\[ \gamma = \frac{\pi^2}{\xi^2} \]

\[ \delta = -\frac{\pi^2}{\xi^2} \]

\[ \epsilon = -\frac{\pi^2}{\xi^2} \]

For \( m = 1 \)

\[ u_{\text{on}}(t) = \int_0^{2\pi} \int_0^1 \int_0^t \left( p_X(x, \varphi, \lambda) u_{\text{on}} \cos \frac{nx}{\xi} + \sum_{m=1}^\infty \left( \cos \frac{mx}{\xi} \right) \right) \sin \frac{n \sin t}{\xi} \frac{\sin \frac{n \sin t}{\xi} \, dt \, dx \, d\varphi}{\omega_{\text{on}} \sin x_{\text{on}}(t) \, dx \, d\delta} \]

(61)
For $m \neq 0$

\[ q_{nm}(t) = \int_0^r \int_0^t \left[ p_x(x, \phi, \lambda)u_{nm} \cos \frac{nx}{r} \cos \frac{nm\phi}{a} ight. \]
\[ + p_y(x, \phi, \lambda)v_{nm} \sin \frac{nx}{r} \sin \frac{nm\phi}{a} \]
\[ + p_z(x, \phi, \lambda)w_{nm} \cos \frac{nx}{r} \sin \frac{nm\phi}{a} \]
\[ \times \frac{\sin \omega_{n,m}(t - \lambda) \, d\lambda \, dx \, d\phi}{\omega_{nm} \rho \frac{\pi}{2} \left( U_{nm}^2 + V_{nm}^2 + W_{nm}^2 \right)} \]

(62)

**Solution for Radial Impact**

1. **Unit Step Loaded Distributed over Finite Area ($\lambda_1$, $\lambda_2$)**

\[ u = \frac{4}{\rho m^2} \left( \frac{c_1}{a} \right) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[ \frac{\alpha_{nm}}{\beta_{nm}} \left( \frac{1}{\alpha_{nm}} + \frac{1}{\beta_{nm}} \right) \right] \left( \frac{\sin \frac{nx}{r}}{\sin \frac{nx}{a}} \right) \left( \sin \frac{nm\phi}{a} \right) \left( \frac{1 - \cos \omega_{nm} t}{\omega_{nm}} \right) \]
\[ + \left( \frac{8}{\rho m^2} \right) \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[ \frac{\alpha_{nm}}{\beta_{nm}} \left( \frac{1}{\alpha_{nm}} + \frac{1}{\beta_{nm}} \right) \right] \left( \frac{\sin \frac{nx}{r}}{\sin \frac{nx}{a}} \right) \left( \sin \frac{nm\phi}{a} \right) \left( \frac{1 - \cos \omega_{nm} t}{\omega_{nm}} \right) \]

(63)
u. Concentrated Unit Step Load

\[
\begin{align*}
    u &= \frac{1}{\pi \alpha_0} \sum_{n=1}^{\infty} \left( \frac{a_{mn}}{\omega_{m0} + \gamma} \right) \left[ \sin \frac{n \pi x}{L} \right] \left[ \cos \frac{n \pi x}{L} \right] \left( \frac{1 - \cos \omega_{m0} t}{\omega_{m0}^2} \right) \\
    &\quad + \frac{2}{\pi \alpha_0} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \frac{a_{mn}}{\omega_{m0} + \gamma} \right) \left[ \cos \frac{m \pi n}{a} \right] \left[ \sin \frac{n \pi x}{L} \right] \left[ \cos \omega_{m0} t \right] \\
    &\quad \times \left( \cos \frac{n \pi x}{L} \right) \left( \frac{1 - \cos \omega_{m0} t}{\omega_{m0}^2} \right) \\
    v &= \frac{2}{\pi \alpha_0} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \frac{a_{mn}}{\omega_{m0} + \gamma} \right) \left[ \cos \frac{m \pi n}{a} \right] \left[ \sin \frac{n \pi x}{L} \right] \left[ \sin \omega_{m0} t \right] \\
    &\quad \times \left( \sin \frac{n \pi x}{L} \right) \left( \frac{1 - \cos \omega_{m0} t}{\omega_{m0}^2} \right) \\
    w &= \frac{1}{\pi \alpha_0} \sum_{n=1}^{\infty} \left( \frac{1}{\omega_{m0} + \gamma} \right) \left[ \sin \frac{n \pi x}{L} \right] \left[ \sin \frac{n \pi x}{L} \right] \left( \frac{1 - \cos \omega_{m0} t}{\omega_{m0}^2} \right) \\
    &\quad + \frac{2}{\pi \alpha_0} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \frac{a_{mn}}{\omega_{m0} + \gamma} \right) \left[ \cos \frac{m \pi n}{a} \right] \left[ \sin \frac{n \pi x}{L} \right] \left[ \cos \omega_{m0} t \right] \\
    &\quad \times \left( \sin \frac{n \pi x}{L} \right) \left( \frac{1 - \cos \omega_{m0} t}{\omega_{m0}^2} \right)
\end{align*}
\]

(64)

c. Unit Impulse

Solutions for a unit impulse can be found by differentiating with respect to time the solutions for a unit step function. A typical displacement relationship for a concentrated unit impulse is as follows:

\[
\begin{align*}
    u &= \frac{1}{\pi \alpha_0} \sum_{n=1}^{\infty} \left( \frac{a_{mn}}{\omega_{m0} + \gamma} \right) \left[ \sin \frac{n \pi x}{L} \right] \left[ \cos \frac{n \pi x}{L} \right] \left( \frac{1 - \cos \omega_{m0} t}{\omega_{m0}^2} \right) \\
    &\quad + \frac{2}{\pi \alpha_0} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \frac{a_{mn}}{\omega_{m0} + \gamma} \right) \left[ \cos \frac{m \pi n}{a} \right] \left[ \sin \frac{n \pi x}{L} \right] \left( \frac{1 - \cos \omega_{m0} t}{\omega_{m0}^2} \right) \\
    &\quad \times \left( \cos \frac{n \pi x}{L} \right) \left( \frac{1 - \cos \omega_{m0} t}{\omega_{m0}^2} \right)
\end{align*}
\]

(65)
Triangular Loading with Suddenly applied Value of Unity, and decreasing Linearly to Zero at Time, $t_d$

$$u = \frac{4}{\rho h^2} \left[ \frac{\nu^2}{n^2} \right] \left( \frac{1}{t} \right) \left( \sin \frac{n \pi x}{l} \right) \left( \sin \frac{n \pi y}{l} \right) \left( \cos \frac{n \pi x}{l} \right) (F_{no}(t))$$

$$+ \frac{8}{\rho h^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left( \frac{\sin \frac{n \pi x}{l}}{n \pi} \right) \left( \sin \frac{n \pi y}{l} \right) \left( \sin \frac{m \pi l}{a} \right) \left( \sin \frac{n \pi y}{l} \right) \left( \cos \frac{m \pi l}{a} \right) (F_{nm}(t))$$

$$+ \left( \cos \omega_o \right) \left( \cos \frac{n \pi y}{l} \right) \left( \sin \frac{n \pi x}{l} \right) \left( \sin \frac{n \pi y}{l} \right) \left( \cos \frac{n \pi x}{l} \right) (F_{no}(t))$$

$$v = \frac{8}{\rho h^2} \left[ \frac{1}{\alpha_{nm}^2} \frac{1}{a^2} \right] \left( \frac{1}{\alpha_{nm}^2} \frac{1}{a^2} \right) \left( \sin \frac{n \pi x}{l} \right) \left( \sin \frac{n \pi y}{l} \right) \left( \sin \frac{m \pi l}{a} \right) \left( \sin \frac{n \pi y}{l} \right) \left( \cos \frac{m \pi l}{a} \right) (F_{nm}(t))$$

$$+ \left( \sin \omega_o \right) \left( \sin \frac{n \pi x}{l} \right) \left( \sin \frac{n \pi y}{l} \right) \left( \cos \frac{n \pi x}{l} \right) \left( \cos \frac{n \pi y}{l} \right) \left( \sin \frac{n \pi x}{l} \right) (F_{no}(t))$$

$$w = \frac{4}{\rho h^2} \left[ \frac{1}{\alpha_{nm}^2} \frac{1}{a^2} \right] \left( \frac{1}{\alpha_{nm}^2} \frac{1}{a^2} \right) \left( \sin \frac{n \pi x}{l} \right) \left( \sin \frac{n \pi y}{l} \right) \left( \sin \frac{m \pi l}{a} \right) \left( \sin \frac{n \pi y}{l} \right) \left( \cos \frac{m \pi l}{a} \right) (F_{nm}(t))$$

where

$$F_{no}(t) = \frac{1}{\omega_{no}^2} \left( 1 - \cos \omega_{no} t + \frac{\sin \omega_{no} t}{\omega_{no} t} - \frac{t}{t_d} \right)$$

$$F_{nm}(t) = \frac{1}{\omega_{nm}^2} \left( 1 - \cos \omega_{nm} t + \frac{\sin \omega_{nm} t}{\omega_{nm} t} - \frac{t}{t_d} \right) \quad t < t_d$$

$$F_{no}(t) = \frac{1}{\omega_{no}^2} \left( \sin \omega_{no} t - \sin \omega_{no} (t - t_d) - \frac{1}{\omega_{no}} \cos \omega_{no} t \right)$$

$$F_{nm}(t) = \frac{1}{\omega_{nm}^2} \left( \sin \omega_{nm} t - \sin \omega_{nm} (t - t_d) - \frac{1}{\omega_{nm}} \cos \omega_{nm} t \right) \quad t > t_d$$
e. Rectangular Pulse Loading with Suddenly Applied Value of Unity

and Duration, $t_d$

Expressions for $u$, $v$, and $w$ are identical to those corresponding to a triangular loading with the exception that $F_{no}(t)$ and $F_{nm}(t)$ be defined as follows:

\[
F_{no}(t) = \frac{1}{\omega_{no}} (1 - \cos \omega_{no} t) \quad t < t_d
\]

\[
F_{nm}(t) = \frac{1}{\omega_{nm}} (1 - \cos \omega_{nm} t) \quad t \leq t_d
\]

\[
F_{no}(t) = \frac{1}{\omega_{no}^2} \left\{ \cos \omega_{no} (t - t_d) - \cos \omega_{no} t \right\}
\]

\[
F_{nm}(t) = \frac{1}{\omega_{nm}^2} \left\{ \cos \omega_{nm} (t - t_d) - \cos \omega_{nm} t \right\} \quad t > t_d
\]
Equations of Motion for Timoshenko Theory

Equations (1) through (3) can be reduced to those presented by Timoshenko and Gere by assuming the following conditions:

a. The circumferential strain $\gamma_\phi$, $\gamma_x$, and $\gamma_x$ are equal to zero in calculation of $\chi_\phi$ and $\chi_x$.

b. Membrane forces are not affected by bending stresses, nor bending moments by membrane stresses.

Assumptions (a) and (b) yield $\eta_x = \eta_{xx}$ and $\eta_{xx} = \eta_{xx}$, assuming

\[ \gamma_x = (v' + w)/a, \quad u', u'' = 0, \quad \text{and} \quad \gamma_x = (u' + v')/a = 0, \]

equations (1) through (3) become:

\[ \eta_t + \eta_t + pa(v'' + w') + a^2 p_x = \rho a^2 \frac{\partial^2 u}{\partial t^2} \] \hspace{1cm} (68)

\[ \eta_t + \eta_t + aQ_x - Pw'' + a^2 p_x = \rho a^2 \frac{\partial^2 v}{\partial t^2} \] \hspace{1cm} (69)

\[ -aQ_x + aQ_x + aQ_x - Pw'' - a^2 p_x = \rho a^2 \frac{\partial^2 w}{\partial t^2} \] \hspace{1cm} (70)

The membrane forces and moments from equations (6) through (13) become:

\[ ii_x = \frac{v}{a} (v' + w + vv') \]

\[ ii_x = \frac{w}{a} (u' + vv' + uu') \]

\[ ii_{xx} = \frac{v}{a} \left( \frac{1}{2} - \frac{v}{2} \right) (u' + v') \]

\[ ii_{xx} = \frac{w}{a} \left( \frac{1}{2} - \frac{v}{2} \right) (u' + v') \]

\[ ii_x = \frac{v}{a} (v'' + w'' + vv'') \]

\[ ii_x = \frac{w}{a} (w'' + vv'' - vv') \]
Substitution of equations (71) into equations (68) - (70) yields the following:

\[ u'' + \left(\frac{1 + \nu}{2}\right) v'' + \frac{1 - \nu}{2} u'' + w'' + q_1 (v'' + w') \]
\[ + \frac{a^2 p_x}{\nu} = \frac{\phi a^2 \nu^2 u}{\partial t^2} \]  
\[ (\frac{1 + \nu}{2}) u'' + v'' + \left(\frac{1 - \nu}{2}\right)v'' + u'' - k\nu''' + w''' \]
\[ + k \left[(1 - \nu)v' + v''\right] - q_2 v'' + \frac{a^2 p_y}{\nu} = \frac{\phi a^2 \nu^2 v}{\partial t^2} \]  
\[ wu' + v' + u + k\nu'' + 2w'' + w'''' - k\nu'''' + (2 - \nu)v''' \]
\[ + q_2 w'' + q_1 (w'' + w) - \frac{a^2 p_r}{\nu} = -\phi a^2 \nu^2 \]  

Equations (72) - (74) may be written

\[ u'' + \frac{1 - \nu}{2} u'' + \beta_1 v'' + \beta_2 v'' \]
\[ + \frac{a^2 p_x}{\nu} = \frac{\phi a^2 \nu^2 u}{\partial t^2} \]  
\[ (\frac{1 + \nu}{2}) u'' + \beta_3 v'' + \beta_4 v'' + w'' - k\nu'' + w'''' \]
\[ + \frac{a^2 p_y}{\nu} = \frac{\phi a^2 \nu^2 v}{\partial t^2} \]  
\[ wu' + v' + \beta_5 w + k\nu'' + 2w'' + w'''' - k\nu'''' + (2 - \nu)v''' \]
\[ + q_2 w'' + q_1 w'' - \frac{a^2 p_r}{\nu} = -\phi a^2 \frac{\nu^2}{\partial t^2} \]  

where

\[ \frac{1 + \nu}{2} + q_1 = \beta_1 \]
\[ v + q_1 = \beta_2 \]
\[ k + 1 = \beta_3 \]
\[ (1 - v)(k + \frac{1}{2}) - q_2 = \beta_4 \]
\[ 1 + q_1 = \beta_5 \]
Data for Illustrative Example for Unit Radial Impulse

\[ n = 1.2 \text{ inches} \quad a = 60 \text{ inches} \quad r = 24 \text{ inches} \]

\[ r = 12 \text{ inches} \quad \epsilon_1 = 2 \text{ inches} \quad \epsilon_2 = 2 \text{ inches} \]

\[ r = 0 \text{ radians} \quad \gamma = 0.3333 \quad n = 1 - 30 \quad m = 0 - 29 \]

\[ P_c(\text{Flugge}) = 5003.855 \text{ psi} \quad P_c(\text{Timoshenko}) = 5071.3 \text{ psi} \]
### Table 1

Effect of Maximum Pressure on Frequencies in Buckling Mode

Material: Steel

Parameters: \( \eta/2a = 0.01 \)  \( \xi/2a = 0.2 \)

Buckling mode: \( n = 1 \)  \( m = 9 \)

Buckling Pressure: \( P_c = 5003.555 \)

### Flugge's Theory

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Table II
EFFECT OF HYDROSTATIC PRESSURE ON FREQUENCIES IN BUCKLING TUBE

Material: steel
Parameters: h/2a = 0.01  \( t/2a = 0.2 \)
Buckling Mode: \( n = 1, \nu = 9 \)
Buckling Pressure: 3071.793

Timoshenko's Theory

<table>
<thead>
<tr>
<th>P/P_c</th>
<th>( f_1 )</th>
<th>( f_2 )</th>
<th>( f_3 )</th>
<th>( f_1 )</th>
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</thead>
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<td>464.94</td>
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<td>6809.14</td>
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</tr>
</tbody>
</table>
TABLE III

EFFECT OF HYDOSTATIC PRESSURE ON FREQUENCIES IN BUCKLING MODE

Material: steel
Parameters: h/2a = 0.1 \( t/2a = 3 \)
Buckling Mode: \( n = 1, m = 2 \)
Buckling Pressure, \( P_c = 63211.4 \)

Flugge's Theory

<table>
<thead>
<tr>
<th>( P/P_c )</th>
<th>( f_1 )</th>
<th>( f_2 )</th>
<th>( f_3 )</th>
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TABLE IV

EFFECT OF HYDROSTATIC PRESSURE ON FREQUENCIES IN LOUANG HOLE

Material: steel

Parameters: \( h/2a = 0.1 \), \( t/2a = 0 \)

Buckling modes: \( n = 1, \ m = 2 \)

Buckling pressure, \( P_c = 63356.4 \)

Timoshenko's Theory

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<tr>
<th>( P/P_c )</th>
<th>( f_1 )</th>
<th>( f_2 )</th>
<th>( f_3 )</th>
<th>( f_1 )</th>
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### Table V

**Comparison of Frequencies for Various Final Shapes**

**Flugge's Theory**

Parameters: \( n/2a = 0.01 \quad \varepsilon/2a = 0.2 \)

\( n = 1 \)

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<th>( P/P_c = 0.5 )</th>
</tr>
</thead>
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<td>566.22</td>
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<td>545.79</td>
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<td>492.44</td>
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<td>461.46</td>
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<td>8</td>
<td>443.95</td>
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<td>9</td>
<td>437.04</td>
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### Table VI

Comparison of Frequencies for Various Tool Shapes

**Fugge’s Theory**

Parameters: \( h/2a = 0.01 \) \( t/2a = 0.2 \)

\( n = 3 \)

<table>
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<th>( P/P_c )</th>
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<th>( f_3 )</th>
<th>( P/P_c )</th>
<th>( f_1 )</th>
<th>( f_2 )</th>
<th>( f_3 )</th>
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<td>13404.75</td>
<td>1087.77</td>
<td>7716.36</td>
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<td>7457.86</td>
<td>13416.82</td>
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<td>7723.81</td>
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<td>13452.99</td>
<td>1018.46</td>
<td>7744.65</td>
<td>13440.26</td>
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</tr>
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<td>13513.04</td>
<td>1031.89</td>
<td>7779.25</td>
<td>13500.14</td>
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<td>13832.83</td>
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<td>7963.35</td>
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<td>8410.40</td>
<td>14562.79</td>
<td>2086.67</td>
<td>8382.92</td>
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### Table VII

Comparison of Frequencies for Various Hole Shapes

**Flugge's Theory**

Parameters: \( h/2a = 0.01 \)  \( t/2a = 0.2 \)

\( n = 5 \)

<table>
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<tr>
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<th>( f_2 )</th>
<th>( f_3 )</th>
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<td>22339.84</td>
<td>4993.41</td>
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<td>22347.03</td>
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<td>12927.79</td>
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<td>22498.72</td>
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<td>22599.16</td>
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<td>13010.27</td>
<td>22577.42</td>
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<td>13101.79</td>
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<tr>
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<td>22793.32</td>
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<td>22919.20</td>
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<td>22896.56</td>
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<tr>
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<td>5414.29</td>
<td>13310.62</td>
<td>23052.99</td>
<td>5315.34</td>
<td>13270.66</td>
<td>23029.97</td>
</tr>
</tbody>
</table>
TABLE VIII
EFFECT OF HYDROSTATIC PRESSURE ON FUNDAMENTAL FREQUENCY

Parameters: \( h/2a = 0.01 \) \( \varepsilon/2a = 0.2 \) \( P_c = 5063.655 \)

<table>
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<tr>
<th>( P/P_c )</th>
<th>( n, m )</th>
<th>( f )</th>
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<td>0</td>
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<td>1, 7</td>
<td>374.72</td>
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<tr>
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<td>1, 8</td>
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<tr>
<td>0.8</td>
<td>1, 8</td>
<td>230.72</td>
</tr>
<tr>
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<td>1, 9</td>
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<tr>
<td>0.98</td>
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<tr>
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### Table IX

**Effect of Hydrostatic Pressure on Higher Frequencies**

Flugge's Theory

Parameters: \( \pi/2a = 0.01 \)  \( t/2a = 0.2 \)

\( m = 10 \)

<table>
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<th>( f_3 )</th>
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<td>10564.18</td>
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<tr>
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<td>8410.40</td>
<td>14562.79</td>
<td>2088.67</td>
<td>8382.92</td>
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<td>5414.29</td>
<td>13310.62</td>
<td>23052.99</td>
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<td>13270.66</td>
<td>23029.97</td>
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<td>$\frac{P}{P_c}$</td>
<td>$u \times 10^{12}$</td>
<td>$v \times 10^5$</td>
<td>$w \times 10^2$</td>
<td>$\varepsilon_x \times 10^5$</td>
<td>$\varepsilon_\phi \times 10^4$</td>
<td>$\gamma_{x\phi} \times 10^{13}$</td>
</tr>
<tr>
<td>-------------</td>
<td>------------------</td>
<td>----------------</td>
<td>----------------</td>
<td>----------------</td>
<td>----------------</td>
<td>----------------</td>
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<td>-0.0562</td>
<td>-2.5181</td>
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</table>

**TABLE X**

**EFFECT OF HYDROSTATIC PRESSURE ON DYNAMIC RESPONSE FOR UNIT IMPULSE**

**Flugge's Theory**

*Time = 0.0006 sec    $P_c = 5071.793$*
**TABLE XI**

**DYNAMIC RESPONSE FOR UNIT IMPULSE WITH AND WITHOUT IN-PLANE INERTIA**

Flugge's Theory  
Time = 0.0006 sec  \( P/P_c = 0.5 \)

<table>
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<tr>
<th>( \phi )</th>
<th>( u \times 10^{12} )</th>
<th>( v \times 10^5 )</th>
<th>( w \times 10^2 )</th>
<th>( \epsilon_x \times 10^5 )</th>
<th>( \epsilon_\phi \times 10^4 )</th>
<th>( \gamma_{x\phi} \times 10^{12} )</th>
<th>( \sigma_x )</th>
<th>( \sigma_\phi )</th>
<th>( \tau_{x\phi} \times 10^5 )</th>
</tr>
</thead>
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<td>0.0000</td>
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<td>10053.00</td>
<td>0.0000</td>
</tr>
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<td>-0.0021</td>
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<td>-120.53</td>
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<td>42.27</td>
<td>-94.49</td>
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<table>
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<th>( \phi )</th>
<th>( u \times 10^{12} )</th>
<th>( v \times 10^5 )</th>
<th>( w \times 10^2 )</th>
<th>( \epsilon_x \times 10^5 )</th>
<th>( \epsilon_\phi \times 10^4 )</th>
<th>( \gamma_{x\phi} \times 10^{12} )</th>
<th>( \sigma_x )</th>
<th>( \sigma_\phi )</th>
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<td>4.5379</td>
<td>0.2693</td>
<td>1.2245</td>
<td>-0.00451</td>
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<td>408.19</td>
<td>122.53</td>
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<tr>
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<td>1.4828</td>
<td>-0.0398</td>
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<td>-0.00328</td>
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<td>198.19</td>
<td>56.22</td>
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### TABLE XII

**EFFECT OF HYDROSTATIC PRESSURE ON DYNAMIC RESPONSE FOR UNIT IMPULSE**

Comparison of Theories with In-Plane Inertia Included

Time = 0.0006 sec

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<th>$P/P_c$</th>
<th>$u \times 10^{12}$</th>
<th>$v \times 10^5$</th>
<th>$w \times 10^2$</th>
<th>$\epsilon_x \times 10^5$</th>
<th>$\epsilon_y \times 10^4$</th>
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<td>5.0308</td>
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<td>3.0685</td>
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<td>10597.0</td>
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**TABLE XIII**

DYNAMIC RESPONSE FOR UNIT IMPULSE WITH IN-PLANE INERTIA

Comparison of Theories

Time = 0.0006 sec  \( P/P_c = 0.5 \)

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<tr>
<th>( \phi )</th>
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<th>( v \times 10^5 )</th>
<th>( w \times 10^2 )</th>
<th>( \varepsilon_1 \times 10^5 )</th>
<th>( \varepsilon_2 \times 10^4 )</th>
<th>( \gamma_{x\theta} \times 10^{12} )</th>
<th>( \sigma_x )</th>
<th>( \sigma_\theta )</th>
<th>( \tau_{x\theta} \times 10^5 )</th>
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<td>-0.1895</td>
<td>-0.0569</td>
<td>-0.2341</td>
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<tr>
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<td>0.2459</td>
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<td>0.0000</td>
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<td>-94.49</td>
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<table>
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<th>( \phi )</th>
<th>( u \times 10^{12} )</th>
<th>( v \times 10^5 )</th>
<th>( w \times 10^2 )</th>
<th>( \varepsilon_1 \times 10^5 )</th>
<th>( \varepsilon_2 \times 10^4 )</th>
<th>( \gamma_{x\theta} \times 10^{12} )</th>
<th>( \sigma_x )</th>
<th>( \sigma_\theta )</th>
<th>( \tau_{x\theta} \times 10^5 )</th>
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### Table A.14

**Effect of Hydrostatic Pressure and Loading Area on Dynamic Response for Unit Impulse**

<table>
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<th>( u \times 10^5 )</th>
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<th>( x 3 \times 10^4 )</th>
<th>( x \times 10^3 )</th>
<th>( x 3 \times 10^4 )</th>
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<td>( \frac{Z}{Z_2} = \frac{1}{2} )</td>
<td>( \frac{Z}{Z_2} = \frac{1}{2} )</td>
<td>( \frac{Z}{Z_2} = \frac{1}{2} )</td>
<td>( \frac{Z}{Z_2} = \frac{1}{2} )</td>
<td>( \frac{Z}{Z_2} = \frac{1}{2} )</td>
<td>( \frac{Z}{Z_2} = \frac{1}{2} )</td>
<td>( \frac{Z}{Z_2} = \frac{1}{2} )</td>
<td>( \frac{Z}{Z_2} = \frac{1}{2} )</td>
<td>( \frac{Z}{Z_2} = \frac{1}{2} )</td>
</tr>
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<td>( 0 )</td>
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<td>( 0.40 )</td>
<td>( 0.60 )</td>
<td>( 0.80 )</td>
<td>( 0.40 )</td>
<td>( 0.40 )</td>
<td>( 0.60 )</td>
<td>( 0.80 )</td>
</tr>
<tr>
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<td>( 3.9164 \times 10^{-4} )</td>
<td>( 3.5165 \times 10^{-4} )</td>
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<td>( 3.3732 \times 10^{-4} )</td>
<td>( 3.3732 \times 10^{-4} )</td>
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<td>( 2.937 \times 10^{-4} )</td>
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<td>( 2.937 \times 10^{-4} )</td>
<td>( 2.937 \times 10^{-4} )</td>
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<tr>
<td>( \phi )</td>
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<td>( \phi )</td>
<td>( \phi )</td>
<td>( \phi )</td>
<td>( \phi )</td>
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</table>
Figure 2. Radial Displacement versus Pressure for $\phi = 0$, $x = \varepsilon/2$ at Various Values of Time
Figure 3. Radial Displacement versus Time for $P/P_c = 0$, $\phi = 0$, $x = \frac{r}{2}$ at Various Values of Time.
Figure 4. Longitudinal Strain versus Time for $P/P_c = 0$, $\phi = 0$, $x = 1/2$
Figure 1. Longitudinal Strain versus Time for \( P/P_c = 0.5, \phi = 0, x = \frac{1}{2} \)
Figure 6. Longitudinal strain versus time for
\[ P/P_c = 0, \; t = 0, \; x = t/\theta \]
Figure 7. Circumferential Strain versus Time for \( P/P_c = 0, \phi = 0, x = t/2 \).
Conclusions

Large hydrostatic pressures and small variations of impact area greatly affect the dynamic response of deep submersible hulls subjected to a localized impact loading.

For free vibrations deep hydrostatic pressures reduce the lower frequencies substantially while the higher frequencies are not appreciably affected. Hydrostatic pressures in the neighborhood of 50 percent of the buckling pressure can reduce the fundamental frequencies by 30 percent, while the higher frequencies, especially the second and third frequencies of the $n, m$ mode will have no appreciable change.

Comparison of frequencies with the Flugge and Timoshenko theories show good agreement as illustrated in Tables I and II.

For forced vibrations as illustrated by a localized unit impulse, the following conclusions can be made:

a. Deep hydrostatic pressures have predominantly large effects on longitudinal displacements and strains. Consequently the longitudinal stresses, $\sigma_x$, will be more sensitive to change while the circumferential strains and stresses will increase moderately.

b. Shearing stresses experience moderate increases and are very small in magnitude.

c. Radial displacements and response times will have considerable increases as shown in Figure 3.

d. Small changes in the area of loading have tremendous influence on displacements and stresses as shown in Table XIV.
e. Comparison of theories indicates the following:

(1) The greatest discrepancy occurs in longitudinal displacements and strains.

(2) Within the area of impact, stresses, radial and circumferential displacements have good agreement, while those outside the area of impact can have large discrepancies.

(3) A good estimate of stresses, radial and circumferential displacements within the area of impact can be found by neglecting in-plane inertias.

References


Following Flugge's exact derivation for the buckling of cylindrical shells, the equations of motion for dynamic loading of cylindrical shells subjected to hydrostatic and axial pressure have been formulated.

The equations of motion are applicable for long, short, or thick shells, and are very useful in calculating deflections and stresses when the impact loads are applied to comparatively small regions of the shell. The normal mode theory was utilized to provide dynamic solutions for the equations of motion.

Solutions are also provided for the Timoshenko-type theory, and comparisons are made between the two theories by considering and neglecting in-plane inertia forces.

Comparison of results is exemplified by a numerical example which considers the effect of hydrostatic pressure on the dynamic response of a shell simply supported by a thin diaphragm and subjected to a localized unit radial impulse.