ANOTHER ESTIMATE OF EXPECTED CRITICAL PATH LENGTH IN PERT NETWORKS

John Lindsey, II

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John Lindsey, II*
Northern Illinois University
DeKalb, Illinois

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Professor Lindsey is a consultant to the Mathematics Department of The RAND Corporation.
SUMMARY

The Program Evaluation and Review Technique is a very well-known and widely-applied method for estimating the amount of time required to complete a complex project. However, if the time required to complete each subtask is a random variable, the computation of the expected time to complete the total project may be infeasible (for reasons of dimensionality). The simplest way to estimate the expected time to complete the project is to assume each subtask takes exactly its expected time to complete; and then find a critical path. This procedure, except in very simple cases, gives an underestimate of the true expected time required. In 1962 D. R. Fulkerson published a technique for estimating the expected time required which gives values intermediate between the simplest estimate and the true value. This paper suggests another technique which, while more complex than Fulkerson's method, is more accurate in that it gives estimates which fall between Fulkerson's estimates and the true expected value.
ANOTHER ESTIMATE OF EXPECTED CRITICAL PATH LENGTH
IN PERT NETWORKS

As in [1], we assume that we have a PERT network with nonnegative work times assigned along the various arcs depending on a discrete random variable $u$, where the probability of the value $u$ occurring is $\psi(u)$. Number the nodes so that $j < i$ if there is an arc from node $j$ to node $i$. We also assume that work times on arcs coming into any node $i$ are independent from work times on arcs coming into a different node $j$. In other words, $u$ may be considered to be a Cartesian product of random variables $u_i$ on which work times on arcs coming into the node $i$ depend, with $\psi(u)$ being a product of functions $\psi_i(u_i)$.

We wish to approximate the expected longest path. The simplest approach is to simply assign each arc its expected work time and calculate the longest path $P_o$ and its work time for this assignment. As the expectation of the sum of random variables is the sum of the expectations of the random variables, this equals the expected length of a critical (longest) path [3] if and only if for no value of $u$ is some path longer than the path $P_o$. The method employed in [1] has greater flexibility in that it does not force us to use the same path for all events. In fact, the next to last node of the path is allowed to depend on the work lengths which $u$ assigns to arcs to the final node. However, for each $u$, the next to
last node of a critical path also depends on the work
lengths which u assigns to earlier arcs. This dependence
is the reason that Fulkerson's estimate [1] may fall short
of the actual expected length of a critical path. The
following example illustrates this.

The arcs from 1 are allowed to take on the values 0, 1 inde-
dependently with probability \( \frac{1}{2} \). As there is only one assign-
ment of work lengths along arcs to node 4, Fulkerson's method
forces us to go through node 2 all the time, or through node
3 all the time. This results in the estimate \( \frac{3}{4} \) while the
actual expected critical length is \( \frac{2}{3} \). The problem occurs when
for a fixed assignment of work times on arcs to the final
node and variable assignment of work times to earlier arcs,
there is a close decision as to which node, j or k, should
be the next to last node. In general, random fluctuation
will make a path through j longer than a path through k
for some \( u \) and vice versa for other \( u \). However, if a
critical path to j tends to fluctuate the same way as a
critical path to k (a critical path to j is long when a
critical path to k is long) then still, little is lost in
always choosing a path through u for a fixed assignment of
work times along arcs to i. In summary, the Fulkerson
estimate is weakest when there is a large variance in
critical path lengths to these nodes i and k, and there
is little or negative correlation between these critical
path lengths. It is in precisely this situation that
the following modification aims to improve Fulkerson's
estimate.

Let \( x_j(u) \) be the length of a critical path from node
1 to node j. We define the following quantities by recursion
on i or \( \max(i, j) \): \( k_i, h_i, n_i, N_i, M_{ij} \) for \( i \neq j \), and
\( C_{ij} \) for \( i > j \). Define \( k_1 = h_1 = n_1 = N_1 = C_{ij} = M_{jj} = 0 \).
Fix i and let S be the set of nodes from which there are
arcs to i. Let work times along arcs to i depend on the
random variable t and work times along arcs to nodes j,
j < i, depend on the random variable s where s and t are
independent, the probability of s occurring being \( \psi(s) \), and
the probability of t being \( \rho(t) \). If S has one element, let
\( a(t) = b(t) \) be this element. Otherwise, define \( a(t), b(t) \in S
\) so that for \( j \in S - \{a(t), b(t)\} \),
\[ h_a(t) + t_A(t)(t) \geq h_b(t) + t_B(t)(t) \geq h_j + t_j(t) \]
where \( t_j(t) \) is the work time along the
arc from j to i for the random variable t. In the following, if
a term has 0 in its denominator, set the term equal to 0. Let

\[ k_i = \sum_t \rho(t) (h_a(t) + t_a(t)(t)), \quad \rho_i = \sum_t \rho(t) \max\{h_a(t) + t_a(t)(t), h_b(t) + t_b(t)(t)\} + k(t) \]

where \( k(t) = [C_b(t)b(t) - 2C_a(t)b(t) + C_a(t)a(t) + (h_a(t) + t_a(t)(t) - h_b(t) + t_b(t)(t)))^2]^{1/2} \max(M_b(t)a(t) + t_b(t)(t) - t_a(t)(t), M_a(t)b(t) + t_a(t)(t) - t_b(t)(t)) \),
n_i = \min_{t} (n_a(t) + t_a(t)(t))$, $N_i = \max_{t} [\max(N_a(t) + t_a(t)(t), N_b(t) + t_b(t)(t))$, for $i > j, M_{ji} = \max_{t} (Mja(t) - t_a(t)(t))$ and $M_{ij} = \max_{t} [\max(Ma(t)j + t_a(t)(t), Mb(t)j + t_b(t)(t))$, 
\[ C_{ii} = \exp(t)[C_a(t)a(t) + (h_a(t) + t_a(t) - h_1^2)] - 2(h_i - n_i) (h_i - k_i), \]
and for $i > j, C_{ij} = \exp(t)C_a(t) + (h_i - k_i)(N_j - h_j)$.

Define $C_{ji} = C_{ij}$ for $j < i$. We shall show by induction on $i$, that for each node $i$, there is a function $y_i(u)$ of the random variable $u$, satisfying the following:

I. $y_i(u) \leq x_i(u), n_i \leq y_i(u) \leq N_i$, $y_j(u) - y_k(u) \leq M_{jk},$

\[ y_a(t)(s) + t_a(t)(t) \leq y_i(s, t) = y_i(u) \leq \max (y_a(t)(s)) + t_a(t)(t), y_b(t)(s) + t_b(t)(t)); \]

II. $h_i = \bar{y}_i = \exp(u)y_i(u);$

III. $\exp(u)(y_i(u) - h_i)^2 \geq C_{ii};$ and

IV. $\exp(u)(y_j(u) - h_j)(y_k(u) - h_k) \leq C_{jk}.$

Assume that the above statements are true when $i, j$, and $k$ are replaced by integers less than $i$. For each $s, t$ the longest path from node 1 to node $i$ has length $x_j(s, t)(s) + t_j(s, t)(t)$ for some node $j(s, t)$ with an arc from node $j(s, t)$ to node $i$. Then $x_j(s, t)(s) + t_j(s, t)(t) \geq x_a(t)(s) + t_a(t)(t).$ Similarly, $x_j(s, t)(t) + t_j(s, t)(t) \geq \max (y_a(t)(s) + t_a(t)(t), y_b(t)(s) + t_b(t)(t)) = \exp(y_a(t)(s) +$
\[\begin{align*}
\tau_a(t)(t) + Y_b(t)(s) + t_b(t)(t)) + \frac{1}{2} |Y_b(t)(s) + t_b(t)(t) - Y_a(t)(s) - \tau_a(t)(t)|. \quad \text{Then } h_a(t) + t_a(t)(t) &= \sum \phi(s) (y_a(t)(s) + \tau_a(t)(t), y_b(t)(s) + t_b(t)(t)) \\
+ \sum \phi(s) \max [y_a(t)(s) + t_a(t)(t), y_b(t)(s) + t_b(t)(t)] \\
= \frac{1}{2} \sum \phi(s) [y_a(t)(s) + t_a(t)(t) + y_b(t)(s) + t_b(t)(t)] + \\
+ \frac{1}{2} \sum \phi(s) |y_a(t)(s) + t_a(t)(t) - y_b(t)(s) - t_b(t)(t)|. \\
\text{Also, } \frac{1}{2} \sum \phi(s) [y_b(t)(s) + t_b(t)(t) - y_a(t)(s) - \tau_a(t)(t)| \\
\geq \sum \phi(s) [y_b(t)(s) + t_b(t)(t) - y_a(t)(s) - \tau_a(t)(t)]^2/2 \max \\
[M_a(t)b(t) + t_b(t)(t) - t_a(t)(t), M_a(t)b(t) + t_a(t)(t) - t_b(t)(t)] \\
\text{and } \sum \phi(s) [y_b(t)(s) + t_b(t)(t) - y_a(t)(s) - t_a(t)(t)]^2 \\
= \sum \phi(s) [(y_b(t)(s) - h_b(t)) - (y_a(t)(s) - h_a(t))] + [(h_b(t) + t_b(t)(t)) - (h_a(t) + t_a(t)(t))]^2 \\
= \sum \phi(s) [y_b(t)(s) - h_b(t)]^2 - 2 \sum \phi(s) (y_b(t)(s) - h_b(t)) \\
(y_a(t)(s) - h_a(t)) + \sum \phi(s) (y_a(t)(s) - h_a(t))^2 + [h_b(t) + t_b(t)(t)] \\
- (h_a(t) + t_a(t)(t))]^2 \\
\geq c_b(t)b(t) - 2c_a(t)b(t) + c_a(t)a(t) + [h_b(t) + t_b(t) - (h_a(t) + t_a(t)(t))]^2.
\end{align*}\]

This shows that we may define \( y_i(s, t) \) to satisfy the following:

\[\begin{align*}
\tau_a(t)(t) + y_a(t)(s) + t_a(t)(t) \leq y_i(s, t) \leq \max (y_a(t)(s) + t_a(t)(t), \\
y_b(t)(s) + t_b(t)(t)) \leq \max [x_a(t)(s) + \tau_a(t)(t), x_b(t)(s) + t_b(t)(t)] \leq x_i(s, t), \quad \text{and } h_i = \frac{1}{s} \rho(s) y_i(s, t) \\
\sum_{s,t} \rho(t) \check{x}_1(s, t) = \check{x}_1. \quad \text{This shows by induction that } h_i \text{ lies between the actual expected length of a critical path and} \\
\text{Fulkerson's estimate for it.}\]
Furthermore, \( n_i = \min_{s,t} [n_a(t) + t_a(t)(t)] \leq \min_{s,t} [y_a(t)(s) + t_a(t)(t)] \leq \min_{s,t} [y_i(s, t)] \leq \max_{s,t} y_i(s, t) \leq \max_{s,t} \max \{y_a(t)(s) + t_a(t)(t), y_b(t)(s) + t_b(t)(t)\} \]

\[
\leq \max_{s,t} \max \{N_a(t) + t_a(t)(t), N_b(t) + t_b(t)(t)\} = N_i.
\]

For \( i > j \), \( M_{ji} = \max_{t} (M_{ja}(t) - t_a(t)(t)) \)

\[
\geq \max_{s,t} \{y_j(s) - y_a(t)(s) - t_a(t)(t)\} = \max_{s,t} \{y_j(s) - y_i(s, t)\}
\]

and \( M_{ij} = \max_{t} \max \{M_{aj}(t) + t_a(t)(t), M_{bj}(t) + t_b(t)(t)\} \)

\[
\geq \max_{s,t} \max \{y_a(t)(s) - y_j(s) + t_a(t)(t), y_b(t)(s) - y_j(s) + t_b(t)(t)\}
\]

\[
= \max_{s,t} \max \{y_a(t)(s) + t_a(t)(t), y_b(t)(s) + t_b(t)(t)\} - y_j(s)
\]

Also, \( C_{ii} = \sum_{t} \rho(t) [C_a(t)a(t) + (h_a(t) + t_a(t)(t) - h_i)^2] - 2(h_i - n_i)(h_i - k_i) \]

\[
\leq \sum_{s,t} \rho(t)\varphi(s) [y_a(t)(s) - h_a(t) + (h_a(t) + t_a(t)(t) - h_i)]^2
\]

\[
- 2(h_i - n_i) [h_i - \sum_{s,t} \rho(t)\varphi(s)(y_a(t)(s) + t_a(t)(t))]
\]

\[
= \sum_{s,t} \rho(t)\varphi(s)(y_a(t)(s) + t_a(t)(t) - h_i)^2
\]

\[
- \sum_{s,t} \rho(t)\varphi(s)(y_a(t)(s) - y_a(t)(s) - t_a(t)(t))(2h_i - 2n_i)
\]

\[
- \sum_{s,t} \rho(t)\varphi(s)(y_a(t)(s) + t_a(t)(t) - h_i)^2
\]

\[
- \sum_{s,t} \rho(t)\varphi(s)(y_i(s, t) - y_a(t)(s) + t_a(t)(t))(2h_i - y_i(s, t))
\]

\[
- \sum_{s,t} \rho(t)\varphi(s)(y_a(t)(s) - t_a(t)(t))\]

\[
= \sum_{s,t} \rho(t)\varphi(s)(y_i(s, t) - h_i)^2. \quad \text{Finally, for} \ i > j, \ C_{ij}
\]

\[
= \sum_{t} \rho(t)C_a(t)j + (h_i - k_i)(N_j - h_j)
\]

\[
\geq \sum_{s,t} [\rho(t) - \varphi(s)(y_a(t)(s) - h_a(t))(y_j(s) - h_j)]
\]

\[
+ \sum_{s,t} [\rho(t) - \varphi(s)(y_i(s, t) - y_a(t)(s) - t_a(t)(t))(N_j - h_j)]
\]

\[
\geq \sum_{s,t} [\rho(t) - \varphi(s)(y_a(t)(s) - h_a(t))(y_j(s) - h_j)]
\]
\[ + \sum_{s,t} p(t) \phi(s) (y_i(s, t) - y_a(t)(s) - r_a(t)(t))(y_j(s) - h_j) + \sum_{t} p(t) (h_a(t) + r_a(t)(t) - h_i) \sum_{t} \phi(s) (y_j(s) - h_j) \]
\[ = \sum_{s,t} p(t) \phi(s) (y_i(s, t) - h_i)(y_j(s) - h_j). \]

For the example:

For the example:

\[ h_2 = k_2 = h_3 = k_3 = 1, \quad n_2 = n_3 = 0, \quad N_2 = N_3 = 1, \quad M_{12} = M_{13} = 0, \quad M_{21} = M_{31} = 1, \quad M_{23} = M_{32} = 1, \quad C_{32} = 0, \quad C_{22} = C_{33} = i, \quad \text{and} \]
\[ h_4 = \frac{1}{2} (\frac{1}{2} + 0 + \frac{1}{2} + 0) + (\frac{1}{2} - 0 + \frac{1}{2} + (\frac{1}{2} + 0 - \frac{1}{2} - 0)^2)/2 = 3/4, \]
the actual expected critical length. Naturally, the example exaggerates the accuracy of our estimate. One problem is that for some assignment of work times to arcs into a node i, there may be a close decision between three or more nodes as to which should immediately precede i on a critical path. In the example

our estimate is still 3/4, but the actual expected critical path length is \(1 - 2^{-n}\). Without using the assumption that work times on arcs in bundles leading into nodes j, j < i, are independent, it is not possible to use the correlations between critical path lengths coming in two nodes to calculate how much critical paths through c(t) to i increase.
max \( (y_a(t)(s) + t_a(t)(t), y_b(t)(s) + t_b(t)(t); \) that is, by how much \( \sum_s \sigma(s) \max (y_a(t)(s) + t_a(t)(t), y_b(t)(s) + t_b(t)(t), y_c(t)(s) + t_c(t)(t)) \) exceeds \( \sum_s \sigma(s) \max (y_a(t)(s) + t_a(t)(t), y_b(t)(s) + t_b(t)(t)) \). In fact, suppose that \( t_a(1)(1) = t_b(1)(1) = t_c(1)(1) = 0 \) and \( s \) takes on four values with probability \( \frac{1}{4} \).

\[
\begin{array}{cccc}
  s & I & II & III & IV \\
  x_a(1)(s) = y_a(1)(s) & 0 & 0 & 1 & 1 \\
  x_b(1)(s) = y_b(1)(s) & 0 & 1 & 0 & 1 \\
  x_c(1)(s) = y_c(1)(s) & 0 & 1 & 1 & 0 \\
\end{array}
\]

Then there is no correlation between any two of the following: \( y_a(t)(s) \), \( y_b(1)(s) \), and \( y_c(1)(s) \). However, it is never necessary to use \( c(1) \) in a critical path; regardless of \( s \), there is a critical path through \( a \) or \( b \). However, in some sense there is correlation, as \( C_{a(1)b(1)c(1)} = \sum_s \varphi(s)(y_a(1)(s) - \overline{y_a(1)})(y_b(1)(s) - \overline{y_b(1)})(y_c(1)(s) - \overline{y_c(1)}) = \frac{1}{4}(-\frac{1}{2})(-\frac{1}{2})(\frac{1}{2}) + 3(\frac{1}{4})(\frac{1}{4})(\frac{1}{4})(-\frac{1}{4}) = -\frac{1}{2} \). There are two problems in trying to use this. First, we would have to calculate an expression \( C_{ijk} \) for each triple of nodes. Secondly, we would have the problem of using a third degree term \( C_{ijk} \) with the second degree terms \( C_{ij} \).

We have already experienced these difficulties to some extent. We have to calculate expressions \( C_{ij} \) and \( M_{ij} \) for each pair of nodes, whereas, in Fulkerson's estimate, it was only necessary to calculate \( f_i \) for each node. Another
The problem was that we were able to calculate
\[ \sum_{s} \varphi(s)(y_b(t)(s) + \tau_b(t)(t) - y_a(t)(s) - \tau_a(t)(t))^2 \]
but we needed
\[ \sum_{s} \varphi(s)|y_b(t)(s) + \tau_b(t)(t) - y_a(t)(s) - \tau_a(t)(t)| \]
instead. Our solution was to bound the ratio by the
\[ \max_{s,t} |y_b(t)(s) + \tau_b(t)(t) - y_a(t)(s) - \tau_a(t)(t)|. \]
With more simplicity and less sharpness, we could have used
\[ \max (N_b(t) + \tau_b(t)(t) - n_a(t) - \tau_a(t)(t), N_a(t) + \tau_a(t)(t) - n_b(t) - \tau_b(t)(t)) \]
instead of
\[ \max (M_b(t)a(t) + \tau_b(t)(t) - \tau_a(t)(t), M_a(t)b(t) + \tau_a(t)(t) - \tau_b(t)(t)) \]
as a bound for this ratio. Either of these bounds does not lose very much
sharpness if
\[ y_b(t)(s) + \tau_b(t)(t) - y_a(t)(s) - \tau_a(t)(t) \]
as a function of the random variable \( s \) has a high probability
of taking on a value near the endpoints of its range, as in
our example. However, this will not usually be the case,
particularly if we have a continuous random variable or
if the node \( i \) is far from node 1 so that lots of random
fluctuations make
\[ y_b(t)(s) + \tau_b(t)(t) - y_a(t)(s) - \tau_a(t)(t) \]
behave nearly like a continuous distribution. One solution
is to assume that
\[ y_b(t)(s) + \tau_b(t)(t) - y_a(t)(s) - \tau_a(t)(t) \]
is distributed for fixed \( t \) as a normal distribution. If
the random fluctuations add, then this is a very plausible
assumption, if the node \( i \) is far from node 1, by the central-
limit theorem. Now
\[ y_b(t)(s) + \tau_b(t)(t) - y_a(t)(s) - \tau_a(t)(t) \]
has mean
\[ \bar{x} = h_b(t) + \tau_b(t)(t) - h_a(t)(t), - \tau_a(t)(t) \]
and
\[ \sigma^2 = c_b(t)b(t) - 2c_a(t)b(t) + c_a(t)a(t). \]
Let
\[ r = \frac{h_a(t) + \tau_a(t)(t) - h_b(t) - \tau_b(t)(t)}{\sigma}. \]
Then
\[ \sum_{s} \varphi(s)|y_b(t)(s) + \tau_b(t)(t) - y_a(t)(s) - \tau_a(t)(t)|
approximately equals

\[
\int_{-\infty}^{\infty} |y_b(t)(s) + t_b(t)(t) - y_a(t)(s) - t_a(t)(t)| \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{r^2}{2\sigma^2} + \frac{r\sigma}{\sigma^2} \int \frac{r}{\sigma} e^{-x^2/2} dx}
\]

where the last term can be expressed in terms of the error function:

\[
\frac{1}{\sqrt{2\pi}} \int_{0}^{r/\sigma} e^{-x^2/2} dx = \text{erf}(r/\sigma)
\]

The expression
\[ f(r, \sigma) = \frac{1}{\sqrt{2\pi}} e^{-r^2/2\sigma^2} \int_0^{r/\sigma} e^{-x^2/2} \, dx \] if \( \sigma^2 > 0, \)

\[ f(r, \sigma) = \frac{r}{2} \geq 0 \] if \( \sigma^2 \leq 0, \) where \( r = h_a(t) + t_a(t)(t) - h_b(t) - t_b(t)(t) \) and \( \sigma^2 = C_b(t)b(t) - 2C_a(t)b(t) + C_a(t)a(t) \)

may be used to replace \( K(t) \) in recursively defining \( h_i. \) Note that

\[ \frac{3f}{\partial \sigma} = \frac{1}{\sqrt{2\pi}} e^{-r^2/2\sigma^2} + \frac{r^2}{\sigma^2/2\pi} e^{-r^2/2\sigma^2} - \frac{r^2}{\sigma^2/2\pi} e^{-r^2/2\sigma^2} \]

\[ = \frac{1}{\sqrt{2\pi}} e^{-r^2/2\sigma^2} > 0. \]

Therefore, \( f(r, \sigma) \) increases as \( \sigma \) increases, and \( f(r, \sigma) \geq \lim_{\sigma \to 0} f(r, \sigma) = \frac{r}{2} = f(r, 0). \) Therefore, we may define

\[ h_i = \mathbb{E} [\rho(t)(h_a(t) + t_a(t) + h_b(t) + t_b(t)(t)) + f(r, \sigma)]. \]

Also, our conservative estimates of \( \sigma \) give a conservative estimate for \( f(r, \sigma). \) Advantages of the use of \( f(r, \sigma) \) are closer estimates of the expected critical path lengths and avoidance of the use of the expressions \( M_{i,j} \) and \( M_{j,i}. \) The major disadvantage is that we no longer have a guarantee that our estimate for the expected critical path length will be optimistic, although it usually will be. In fact, consider the example...
where the arc from 1 to 3 takes on the work time $0, p$ of the time; 1, $p$ of the time; and $\frac{1}{2}$ the rest of the time. Then $C_{22} = 0, C_{33} = \frac{1}{2}(2p)$. The actual expected critical path length is $\frac{1}{2} + p/2$. The new estimate for $h_4$ is $\frac{1}{2}(\frac{1}{2} + 0 + \frac{1}{2} + 0) + \sqrt{\frac{p/2}{\sqrt{2\pi}}}$.

As $p$ gets small, sooner or later this exceeds the expected critical path length.

Even when there are too many nodes to keep track of the $C_{i,j}$ for each pair of nodes, it is still possible to carry through this procedure for the first hundred or so nodes, label the $h_i$ as $f_i$, and continue with Fulkerson's recursive definition [1] of the $f_i$. When our original method of defining the $h_i$ is used, we still obtain an estimate which is no larger than the expected critical path length.
REFERENCES

