Digital Receiver Decision Techniques
for Certain Fixed Length Binary Block Codes
Transmitted Through the Gaussian Channel

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ABSTRACT

Various types of fixed length binary block encodings of binary data are presented. Binary signaling is used to transmit the binary encoded data through an additive white gaussian noise channel. Certain block decision techniques are investigated and compared. For certain block encoding schemes, the notion of an intelligent receiver is developed.

This study is considered basic in analyzing the use of redundant encoding in digital communications systems.

PROBLEM STATUS

This is an interim report; work on the problem is continuing.

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INTRODUCTION

A study was undertaken to provide some of the background basis to the understanding and intelligent pursuit of problems involving the use of redundant encoding schemes in digital communication systems.

Binary data emitted from a source is encoded into fixed length (N) blocks of bits according to various types of block encoding schemes. By using each of the N bits in the block to be transmitted to binary modulate a carrier in phase or frequency, N bit signals are produced and these are transmitted as the signal for that block. Depending on whether the binary signaling is coherent (C2FSK, C2PSK) or noncoherent (NC2FSK), i.e., depending on the bit coherency, and the availability of coherency over the block, the receiver processes the received bit signals according to various block decision techniques. Diagrams of receiver implementations for these block decision techniques are given. The probability of a block error (P_{eB}) is derived for the various block decision techniques. Using various values of N and E_b/N_0 - the average signal energy per bit divided by the average noise power per unit bandwidth, the P_{eB} curves are plotted for each decision technique as a function of N and E_b/N_0. The block decision techniques are compared in terms of processing gain per bit for given values of N. For certain types of encodings the notion of an intelligent receiver is developed and some of its uses are discussed. Areas for further investigation are outlined.

ENCODING AND TRANSMISSION

Block Encoding Schemes

We assume that the source emits sequences of binary data elements where these may, for example, represent teletype characters or quantized analog signal samples.

Each data element emitted by the source is encoded into a block of N bits (N even, positive). Due to the binary nature of the data, for each data element the encoder must form a block B_0 which is to be transmitted if the data element is a zero and a block B_1 which is to be transmitted if the data element is a one. We let the sequences \( \{b_{0i}\} \) and \( \{b_{1i}\} \) represent the B_0 and B_1 blocks, respectively where \( b_{0j}, b_{1j} \in \{0, 1\} \) for all \( i = 1, \ldots, N \) and \( j = 0, 1 \).

Next we characterize the important properties of the B_0 and B_1 blocks. In this connection we let \( \rho(B_0, B_1) \) denote the un-normalized correlation between the B_0 and B_1 blocks, i.e., \( \rho(B_0, B_1) \) equals the number (A) of bits \( i \), such that \( b_{0i} = b_{1i} \) minus the number (D) of bits \( i \), such that \( b_{0i} \oplus b_{1i} \). Clearly \( A + D = N \).

We say that the B_0 and B_1 blocks belong to a complementary block encoding scheme if \( b_{01} = b_{11} \oplus 1 \) (where \( \oplus \) denotes addition modulo two) or equivalently if \( \rho(B_0, B_1) = -N \). For this encoding scheme there are
\[ \sum_{K=1}^{N} \binom{N}{K} = 2^N - 1 \] distinguishable ways to form \( B_0 \) and \( B_1 \). When \( N = 8 \), \( B_0 = 01001011 \) and \( B_1 = 10110100 \) is such a distinguishable complementary encoding.

If the \( B_0 \) and \( B_1 \) blocks belong to a complementary block encoding scheme and either \( b_{i1} = 0 \) and \( b_{i1} = 1 \) or \( b_{i1} = 1 \) and \( b_{i1} = 0 \) for all \( i = 1, \ldots, N \), then we say that the \( B_0 \) and \( B_1 \) blocks belong to a simple complementary block encoding scheme. For this encoding scheme there is only one distinguishable way to form \( B_0 \) and \( B_1 \). When \( N = 8 \), then \( B_0 = 00000000 \) and \( B_1 = 11111111 \) is such a simple complementary encoding.

Let \( N_0(B) \) and \( N_1(B) \) denote the number of zeros and ones, respectively in the block \( B \).

If the \( B_0 \) and \( B_1 \) blocks belong to a complementary block encoding scheme and we have \( N_0(B_0) = N_0(B_1) = N_1(B_0) = N_1(B_1) = N/2 \), then we say that the \( B_0 \) and \( B_1 \) blocks belong to an equi-distributed complementary block encoding scheme. For this scheme there are \( \binom{N}{N/2} = N! / \left[ \left( N/2 \right)! \left( N/2 \right)! \right] \) distinguishable ways to form \( B_0 \) and \( B_1 \). When \( N = 8 \), \( B_0 = 01101010 \) and \( B_1 = 10010101 \) is such an equi-distributed complementary encoding.

Let \( N_{b1} \) denote the number of bits \( i \) for which \( b_{i1} = j \) and \( b_{i1} = k \) where \( j = 0, 1 \) and \( k = 0, 1 \) and \( i = 1, \ldots, N \).

Suppose that \( N \) is such that \( N/4 \) is an integer. We say that the \( B_0 \) and \( B_1 \) blocks belong to an equi-distributed random orthogonal block encoding scheme if:

(i) \( N_0(B_0) = N_0(B_1) = N_1(B_0) = N_1(B_1) = N/2 \) and
(ii) \( \rho(B_0, B_1) = 0 \).

Clearly for this scheme \( A = D = N/2 \) and \( N_{b1} = N_{b2} = N_{b3} = N_{b4} = N/4 \).

Also, for this scheme there are \( \binom{N}{N/2} \cdot \binom{N/2}{N/4} \cdot \binom{N/4}{N/4} = N! / \left[ \left( N/4 \right)! \right]^3 \) distinguishable ways to form \( B_0 \) and \( B_1 \). When \( N = 8 \), \( B_0 = 01101010 \) and \( B_1 = 10010101 \) is such an equi-distributed random orthogonal encoding.

If the \( B_0 \) and \( B_1 \) blocks belong to an equi-distributed random orthogonal block encoding scheme and we form blocks \( B_0 \) and \( B_1 \) from \( B_0 \) and \( B_1 \), respectively by using only the \( D = N_{b1} + N_{b2} = N/2 \) bits where \( B_0 \) and \( B_1 \) disagree, then \( B_0 \) and \( B_1 \) will belong to an equi-distributed complementary block encoding scheme of length \( N = N/2 \).
Given a particular block encoding scheme we allow that the \( B_0 \) and \( B_1 \) blocks chosen from this scheme can be the same for all binary data elements, vary for each data bit according to a deterministic rule, or vary for each data bit according to some discrete probability distribution.

Modulation and the Channel

Now suppose the source emits a data bit which in turn determines through a particular block encoding scheme the block \( B_i = \{b_{ij}\} \) = 1, ..., \( N \) to be transmitted. The bits \( b_{ij} \) of \( B_i \) binary modulate a signal \( S(t) \) in frequency or phase producing bit signals \( X_{ij}(t) \) where for all \( i \), \( X_{ij}(t) \) belongs to a binary PSK or FSK signal set and each \( X_{ij}(t) \) has average energy \( E_b \).

Hence, the block \( B_i \) determines a block signal \( X(t) = \{X_{ij}(t)\} \) \( i = 1, ..., N \) which is transmitted bit signal by bit signal through a channel, which we assume adds white Gaussian noise of one-sided spectral density \( N_0 \) to each bit signal. If \( T_b \) is the bit time and \( W \) is the effective receiver bandwidth, then we will assume throughout it follows that \( T_b W = 1 \).

RECEPTION AND DECISION TECHNIQUES

Now we consider certain block decision techniques for various block encoding schemes and binary signal sets. We assume that the \( B_0 \) and \( B_1 \) blocks are available at the receiver.

The Bit by Bit Block Decision Technique

In this decision technique we employ for each bit signal \( X_{ij}(t) \) a binary signal detection method the form of which depends on the modulation used and the available bit phase coherency at the receiver. After the detection procedure we make a bit decision on which signal of the binary signal set was transmitted. This decision corresponds to a binary decision \( d_i \) on \( b_{ij} \). After completing this for all \( N \) bit signals, we have formed a block \( D_i = \{d_i\} i = 1, ..., N \) which is the receiver's estimate of the transmitted block \( B_i = \{b_{ij}\} i = 1, ..., N \). We make a block decision by computing \( \rho(D, B_i) \) and \( \rho(D, B_{i'}) \) and then deciding that \( B_0 \) was transmitted if \( \rho(D, B_0) > \rho(D, B_{i'}) \) and that \( B_1 \) was transmitted if \( \rho(D, B_1) > \rho(D, B_{i'}) \). If \( \rho(D, B_0) = \rho(D, B_1) \), then we flip an unbiased coin to reach a decision.

Diagram 1 gives an implementation of the bit by bit block decision technique receiver when NC2FSK signaling is employed with logic corresponding to frequency.

Now \( \text{probability of a block error, } P_{EB} \), for a given \( N \) and \( E_b/N_0 \), using the bit by bit decision technique is:
\[ P_{\text{eb}} = \begin{cases} \sum_{K=0}^{\lfloor \frac{N+1}{2} \rfloor} \binom{N}{K} P_{\text{eb}} \left(1 - P_{\text{eb}}\right)^{N-K}, & N \text{ odd} \\ \sum_{K=0}^{\frac{N}{2}} \binom{N}{K} P_{\text{eb}} \left(1 - P_{\text{eb}}\right)^{N-K} + \frac{1}{2} \binom{N}{\frac{N}{2}} P_{\text{eb}} \left(1 - P_{\text{eb}}\right), & N \text{ even} \end{cases} \]

\[ \text{Pe}_b \] is the probability of a bit error, i.e., \( P(d_1 \neq b_1) = P_{\text{eb}} \) and thus clearly, \( P_{\text{eb}} \) depends on the type of signaling employed. It is well known that for white gaussian noise of one-sided spectral density \( N_0 \), \( P_{\text{eb}} \) equals \( \frac{1}{2} \text{erf}(\sqrt{E_b/2N_0}) \) for C2PSK signaling, \( \frac{1}{2} \text{erf}(\sqrt{E_b/2N_0}) \) for C2FSK signaling and \( \frac{1}{2} \exp(-E_b/2N_0) \) for NC2FSK signaling where

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-y^2) \, dy. \]

For C2FSK and NC2FSK signaling we assume that the modulation index is \( m = \frac{1}{2} \) or \( m = 1 \).

The \( P_{\text{eb}} \) curves for C2PSK, C2FSK, and NC2FSK signaling are given in Figures 1, 2, and 3, respectively. These curves were computed on NRL's CDC 3300 computer using the fact that for a given signaling type, \( P_{\text{eb}} = P_{\text{eb}}(N) \) for \( N \) odd, \( P_{\text{eb}}(N) = P_{\text{eb}}(N+1) \).

Next we consider for which block encoding schemes these \( P_{\text{eb}} \) curves hold. Suppose that the \( B_0 \) and \( B_1 \) blocks are such that \( -N < p(B_0, B_1) < N \). Clearly a bit error \( (d_1 \neq b_1) \) will influence \( P_{\text{eb}} \) only when \( b_{i1} \neq b_{11} \), since if \( b_{i1} = b_{11} \), then both \( p(D_0, B_0) \) and \( p(D_0, B_1) \) have a minus one contribution for that bit. Hence, the \( P_{\text{eb}} \) curves using the formula given above hold only for complementary, simple complementary, or equi-distributed complementary block encoding schemes. If the \( B_0 \) and \( B_1 \) blocks (of length \( N \)) belong to an equi-distributed random orthogonal block encoding scheme, then we see from the considerations above that the \( P_{\text{eb}} \) curves for this scheme are given by the curves of Figures 1, 2, and 3 where the redundancy is \( N = N/2 \).

The NC2FSK-Sum Block Decision Technique

For this decision technique we assume that binary FSK modulation is employed with modulation index \( m = \frac{1}{2} \) or \( m = 1 \) such that frequency corresponds to log2 and due to a lack of bit phase knowledge, we must for each bit signal \( X_{i1}(t) \) detect the envelopes \( E_i(t) \) and \( E_i(t) \) at the binary signaling frequencies. Then using \( B_0 \) and \( B_1 \) blocks we form for \( X_0(t) \) and \( X_1(t) \) \( E(B_0) \) and \( E(B_1) \), respectively where \( E(B_0) \) is a sum of bit envelopes \( X_{i1}(t) \) such that the envelope \( E_i(t) \) is that which would contain the signal had the \( B_i \) block been transmitted. This summing of appropriate bit envelopes is a method of post-detection integration. We make a block decision by
deciding $B_0$ was transmitted if $E(B_0) > E(B_1)$ and that $B_1$ was transmitted if $E(B_0) < E(B_1)$. If $E(B_0) = E(B_1)$, then we flip an unbiased coin to reach a decision. From references 1 and 2, we see that this technique as described above is optimum for high $E_b/N_0$ ratios and the optimum form of this technique for low $E_b/N_0$ ratios is obtained by replacing the bit envelopes $E_1$ and $E_2$ by $E_3$ and $E_4$.

Diagram 2 gives an implementation of the NC2FSK-Sum Block Decision Technique receiver for low $E_b/N_0$.

Given $N$ and $E_b/N_0$ (low $E_b/N_0$ ratio) and assuming a complementary, simple complementary or equi-distributed complementary block encoding scheme, we have, from Appendix 1 assuming the $B_0$ block was transmitted, that $E(B_0)$ is determined to within a constant $C$ by a non-central Chi-Squared distribution with $2N$ degrees of freedom and non-central parameter $2N E_b/N_0$ and $E(B_1)$ is determined to within the same constant $C$ by a Chi-Squared distribution with $2N$ degrees of freedom. Using these distributions and the approximations found in reference 3, the $P_{eb}$ curves, given in Figure 4, were computed on NRL's CDC 3300 computer using function subprograms for erf(X) and the inverse cumulative distribution function of the Chi-Squared distribution with $2N$ degrees of freedom.

Now suppose the $B_0$ and $B_1$ blocks (of length $N$) belong to an equi-distributed random orthogonal block encoding scheme. Since in this case $Z_0 = Z_1$ for the bits $i$ where the blocks agree, the bits where the blocks agree do not affect the block decision and hence the $P_{eb}$ curves for this scheme are given by the curves of Figure 4 where the redundancy is $\tilde{R} = N/2$.

The NC2FSK-Add Block Decision Technique

For this decision technique we assume that binary FSK modulation is employed with modulation index $m = 1$, frequency corresponds to logic, and there is phase coherency over the block such that for either frequency the initial phase of all $N$ possible bit signals at that frequency is the same but this value is unknown. Let $f_s$ and $f_c$ denote the binary signaling frequencies.

The decision technique forms for each bit, the quadrature components at the binary signaling frequencies; i.e., for the $i^{th}$ bit the receiver forms $X(i, s, f_s), X(i, c, f_c), X(i, s, f_c)$ and $X(i, c, f_c)$. Due to the phase coherency over the block with respect to $f_s$ and $f_c$ and since frequency corresponds to logic; i.e., $b_{i,j} = 0$ corresponds to $f_s$ and $b_{i,j} = 1$ corresponds to $f_c$ ($j = 0, 1$), we can employ predetection integration and form for block $B_j$ ($j = 0, 1$):

\[ Y(B_j, s, f_s) = \sum X(i, s, f_s), \quad Y(B_j, c, f_c) = \sum X(i, c, f_s) \]

where the summation is over $i$ s.t. $b_{i,j} = 0$ with $i = 1, \ldots, N$; and

\[ Y(B_j, s, f_c) = \sum X(i, s, f_s), \quad Y(B_j, c, f_s) = \sum X(i, c, f_s) \]

where the summation is over $i$ s.t. $b_{i,j} = 1$ with $i = 1, \ldots, N$. 

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Now since the initial phase of any bit signal at either frequency is unknown, we form for block $B_j: (j = 0, 1)$, the block type envelope at each frequency; i.e., we form

$$E(B_j, f_0) = \sqrt{\gamma^2(B_j, z, f_0) + \gamma^2(B_j, c, f_0)}$$

and

$$E(B_j, f_1) = \sqrt{\gamma^2(B_j, z, f_1) + \gamma^2(B_j, c, f_1)}.$$

Then we form for $B_j: (j = 0, 1)$, $E(B_j) = E(B_j, f_0) + E(B_j, f_1)$ and we make a block decision by deciding that $B_0$ was transmitted if $E(B_0) > E(B_1)$ and that $B_1$ was transmitted if $E(B_0) < E(B_1)$. If $E(B_0) = E(B_1)$ then we flip an unbiased coin to reach a decision. From References 1 and 2 we see that this technique as described above is optimum for high $E_b/N_0$ ratios. For low $E_b/N_0$ ratios, the optimum form of this technique is obtained by replacing the block type envelopes $E(B_j, f_0)$ and $E(B_j, f_1)$ by $F^2(B_j, f_0)$ and $F^2(B_j, f_1)$ ($j = 0, 1$).

Diagram 3 gives an implementation of the HCO2FSK-Add Block Decision Technique receiver for low $E_b/N_0$ ratios.

Next we consider the $P_{eb}$ curves resulting for the various block encoding schemes assuming low $E_b/N_0$ ratios. Suppose that the $B_0$ and $B_1$ blocks (of length $N$) belong to a simple complementary block encoding scheme. Let us suppose that for all $i = 1, \ldots, N$, $b_{0i} = 0$ and $b_{1i} = 1$. In this case $Y(B_j, z, f_i) = Y(B_j, c, f_i) = Y(B_j, z, f_0) = Y(B_j, c, f_0) = 0$ and hence $E(B_0) = F^2(B_j, f_0)$ and $E(B_1) = F^2(B_j, f_1)$. We see that for simple complementary block encoding schemes the decision for high or low $E_b/N_0$ ratios results in the same $P_{eb}$ curves. From Reference 3 we see that for a given $N$ and $E_b/N_0$, the probability of a block error is given by:

$$P_{eb} = \exp[-N^2b/N_0]$$

These curves are given in Figure 1 and were computed on a Wang 400 Series calculator.

Suppose now that the $B_0$ and $B_1$ blocks (of length $N$) belong to an equi-distributed complementary block encoding scheme. From Appendix A, assuming the $B_0$ block was transmitted, we have that $E(B_0)$ is determined to within a constant $C$ by a non-central Chi-Squared distribution with $N$ degrees of freedom and non-central parameter $N^2b/N_0$ and $E(B_1)$ is determined to within the same constant $C$ by a Chi-Squared distribution with $N$ degrees of freedom. Using these distributions and the approximations found in Reference 4, the $P_{eb}$ curves, given in Figure 1, were computed on NRL's CDC 6600 computer using function subroutines for the Chi-Squared and the inverse cumulative distribution function of the Chi-Squared distribution, with $N$ degrees of freedom.

Suppose next that the $B_0$ and $B_1$ blocks have length $N$ and belong to a complementary block encoding scheme. Comparing the two curves assuming simple complementary encoding to the $P_{eb}$ curves assuming equi-distributed complementary encoding and noting the nature of complementary encoding in terms of these limiting cases we see that for a given $N$. 
the PeB curve for complementary encoding lies between the PeB curve for simple complementary encoding at N and the PeB curve for equi-distributed complementary encoding at N. Exactly where the PeB curve for complementary encoding lies depends on the distribution of ones and zeros in either the B_0 or B_1 block. The more nearly uniform this distribution is the closer the PeB curve for complementary encoding approaches to the PeB curve for equi-distributed complementary encoding and the more nearly this distribution resembles the degenerate distribution the closer the PeB curve for complementary encoding approaches to the PeB curve for simple complementary encoding.

Suppose finally that the B_0 and B_1 blocks (of length N) belong to an equi-distributed, random orthogonal block encoding scheme. In this case, we see from the properties of this type of encoding scheme that Y(B_0, s, f_s) and Y(B_0, c, f_c) will have N/4 terms in common with Y(B_1, s, f_s) and Y(B_1, c, f_c), respectively and Y(B_0, s, f_s) and Y(B_0, c, f_c) will have N/4 terms in common with Y(B_1, s, f_s) and Y(B_1, c, f_c), respectively. Hence, E^0(B_0, f_s) and E^0(B_1, f_s) will each consist of a block type squared envelope term where the quadrature components appearing therein are taken with respect to the complementary bit portions of the blocks plus a term which is the same for both E^0(B_0, f_s) and E^0(B_1, f_s) plus a term consisting of a product of two sums of quadrature components where one sum is the same for both E^0(B_0, f_s) and E^0(B_1, f_s). The same relation exists between E^0(B_0, f_s) and E^0(B_1, f_s). If we regard the block decision as consisting only of a comparison of the sums of the block type squared envelopes, then, since these are formed only with respect to the complementary bit portions of the blocks, the resulting PeB curves would be the same as those for equi-distributed complementary block encoding where the redundancy is R = N/2. But because of the extra terms, which are just normal random variables appearing in E^0(B_0, f_s) and E^0(B_1, f_s) and E^0(B_0, f_s) and E^0(B_1, f_s), we see that the PeB curves for equi-distributed random orthogonal block encoding schemes (of length N) are slightly to the left of the PeB curves (of length N/2) for equi-distributed complementary block encoding schemes.

The NCBB-FSK-Add, Subtract Block Decision Technique

For this decision technique we assume that binary FSK modulation is employed with modulation index m = 1, frequency corresponds to logic, and there is phase coherency over the block such that for either frequency the initial phase of all N possible bit signals at that frequency is the same but this value is unknown. Let f_s and f_c denote the binary signaling frequencies.

The decision technique forms for each bit, the quadrature components at the binary signaling frequencies; i.e., for the i'th bit the receiver forms X(i, s, f_s), X(i, c, f_c), X(i, s, f_c), and X(i, c, f_s).
Due to the phase coherency over the block and since frequency corresponds to logic, we can employ predetection integration and form for \( B_j \) \((j = 0, 1)\):

\[
Z(B_j, s, f_0) = \sum_{i=1}^{N} b_i \cdot X(i, s, f_0)
\]

\[
Z(B_j, c, f_0) = \sum_{i=1}^{N} b_i \cdot X(i, c, f_0)
\]

\[
Z(B_j, s, f_1) = \sum_{i=1}^{N} [b_i + 1] \cdot X(i, s, f_1)
\]

\[
Z(B_j, c, f_1) = \sum_{i=1}^{N} [b_i + 1] \cdot X(i, c, f_1)
\]

Now since the initial phase of any bit signal at either frequency is unknown, we form for block \( B_j \) \((j = 0, 1)\) the block type envelopes at each frequency; i.e., we form \( E(B_j, s, f_0) = E_{L}(B_j, s, f_0) + E_{R}(B_j, c, f_0) \) and \( E(B_j, c, f_0) = E_{L}(B_j, s, f_0) + E_{R}(B_j, c, f_0) \). Then we form for \( B_j \) \((j = 0, 1)\), \( E(B_j) = E_{L}(B_j, s, f_0) + E_{R}(B_j, s, f_0) \) and we make a block decision by deciding that \( B_j \) was transmitted if \( E_{L}(B_j) > E_{R}(B_j) \) and that \( B_j \) was transmitted if \( E_{R}(B_j) > E_{L}(B_j) \). Hence, \( E_{L}(B_j) = E_{R}(B_j) \), then we flip an unbiased coin to reach a decision. From References 1 and 2, we see that this technique as described above is optimum for high \( E_b/N_0 \) ratios. For low \( E_b/N_0 \) ratios, the optimum form of this technique is obtained by replacing the block type envelopes \( E(B_j, s, f_0) \) and \( E(B_j, c, f_0) \) by \( E_{L}(B_j, f_0) \) and \( E_{R}(B_j, f_0) \) \((j = 0, 1)\).

Diagram 3 gives an implementation of the NCB2FSK-Add, Subtract Block Decision Technique receiver for low \( E_b/N_0 \) with \( Y(B_j, s, f_0) \) and \( Y(B_j, c, f_0) \) replaced by \( Z(B_j, s, f_0) \) and \( Z(B_j, c, f_0) \) for \( j = 0, 1 \) and \( k = 0, 1 \). Given \( N \) and \( E_b/N_0 \) (low \( E_b/N_0 \) ratio) and assuming an equi-distributed random orthogonal block encoding scheme, we have from Appendix 3, assuming the \( B_j \) block was transmitted, that \( E(B_j) \) is determined to within a constant \( C \) by a non-central Chi-Squared Distribution with \( 2N \) degrees of freedom and non-central parameter \( N E_b/N_0 \) and that \( E(B_j) \) is determined to within the same constant \( C \) by a Chi-Squared Distribution with \( 2N \) degrees of freedom. Comparing these distributions for \( E(B_j) \) and \( E(B_j) \) with those for the equi-distributed complementary case of the NCB2FSK-Add Block Decision Technique given above we find that for this decision technique the \( P_eB \) curves for \( B_j \) and \( B_j \) blocks (of length \( N \)) belonging to an equi-distributed random orthogonal block encoding scheme are given by the \( P_eB \) curves for the equi-distributed complementary block encoding case of the NCB2FSK-Add Block Decision Technique at a redundancy \( N = N/2 \).
The CB2FSK Block Decision Technique

For this decision technique we assume that binary FSK modulation is employed with modulation index \( m = 1 \), frequency corresponds to logic, and there is phase coherency over the block such that for either frequency the initial phase of all possible bit signals at that frequency is the same and this value is known.

If we assume that the \( B_n \) and \( B_{n+1} \) blocks (of length \( N \)) belong to a simple complementary block encoding scheme, then from Reference 6 we see that we can optimally matched filter process the received block signal with the resulting \( P_{EB} \) for a given \( N \) and \( E_b/N_0 \) being given by:

\[
P_{EB} = \frac{1}{2} \left[ 1 - \text{erf} \left( \sqrt{N E_b/2N_0} \right) \right].
\]

These curves are given in Figure 7 and were computed on a 300 Series Wang calculator using Reference 5.

The CB2FSK Block Decision Technique

For this decision technique we assume that binary FSK modulation is employed and there is phase coherency over the block such that for either bit signal the initial phase is the same and this value is known for \( N \) repetitions of the bit signal.

If we assume that the \( B_n \) and \( B_{n+1} \) blocks of length \( N \) belong to a simple complementary block encoding scheme, then we can optimally process the received block signal with the resulting \( P_{EB} \) for a given \( N \) and \( E_b/N_0 \) being given by:

\[
P_{EB} = \frac{1}{2} \left[ 1 - \text{erf} \left( \sqrt{N E_b/2N_0} \right) \right].
\]

These curves are given in Figure 8 and were computed on a 300 Series Wang calculator using Reference 5.

A Comparison of Decision Techniques

Suppose binary FSK modulation is employed with modulation index \( m = \frac{1}{2} \) or 1, frequency corresponds to logic and the initial phase of each of the \( N \) possible bit signals at either frequency is different and this value is unknown. Clearly one can employ only the NC2FSK signaling case of the Bit by Bit Decision Technique or the NC2FSK-Sum Block Decision Technique in making a block decision. If we compare these alternatives, then from Figures 3 and 4 we see that for any given \( N \), the \( P_{EB} \) curve for the NC2FSK-Sum Block Decision Technique show a processing gain advantage of approximately 2 dB per bit with respect to the \( P_{EB} \) curves for the NC2FSK signaling case of the Bit by Bit Decision Technique. We note that this comparison holds for all types of block encoding schemes.

Now suppose that \( m = 1 \) and the initial phase of each of the \( N \) possible bit signals at either frequency is the same but this value is unknown. This allows us to employ the NC2FSK-Add Block Decision Technique in making a block decision. If we recall the relation between the \( P_{EB} \) curves for the different types of block encoding schemes for this
If we compare, from a processing gain advantage per bit standpoint, the NCB2FSK-Add Block Decision Technique with the NC2FSK-Sum Block Decision Technique using Figures 4, 5, and 6, then we see that this advantage will be a function of N, the block encoding scheme employed and the value of $P_{EB}$ chosen as a reference. For any block encoding scheme and any value of $P_{EB}$, we see that the NCB2PSK-Add Block Decision Technique has a processing gain advantage over the NC2FSK-Sum Block Decision Technique for any $N \geq 4$ and this advantage clearly increases as $N$ increases. For example, if we employ equi-distributed complementary encoding and set $P_{EB} = 10^{-3}$, then for $N = 4$ the advantage is about 1 dB per bit whereas for $N = 2048$ the advantage is about 12 dB per bit. Using these comparisons of the NCB2FSK-Add technique with the NC2FSK-Sum technique we can, using the results previously obtained, easily extend these results to comparisons of the NC2FSK Bit by Bit technique with the NCB2FSK-Add technique.

Next let us suppose that we can employ coherent signaling per bit and there is phase coherency over the block with respect to each of the bit signals in the binary signaling set. If the block encoding scheme can be any of the various types mentioned, we must employ the Bit by Bit Decision Technique. Due to the fact that we can employ coherent signaling per bit, we can utilize either the C2PSK case or the C2FSK case of the Bit by Bit Decision Technique. If we compare these alternatives, then from Figures 1 and 2 we see that for any $N$ and any type of block encoding scheme, the $P_{EB}$ curves for the C2PSK case show a processing gain advantage of approximately 3 dB per bit with respect to the $P_{EB}$ curves for the C2FSK case.

Now if we restrict the block encoding schemes to the simple complementary block encoding scheme, then we can employ the CB2PSK and CB2FSK Block Decision Techniques. If we compare these alternatives, then from Figures 7 and 8 we see that for any $N$ the $P_{EB}$ curves for the CB2PSK technique show a processing gain advantage of 3 dB per bit with respect to the $P_{EB}$ curves for the CB2FSK technique. If we compare the CB2FSK technique and the C2FSK Bit by Bit technique or the CB2PSK technique and the C2FSK Bit by Bit technique, then we see from Figures 1 and 3 and 2 and 7 that the Bit by Bit technique shows approximately a 2 dB loss in processing gain per bit versus the corresponding technique for any N and when simple complementary block encoding is employed.
THE INTELLIGENT RECEIVER

Let $N_k$ denote the number of bits $i$ for which $b_i = j$ and $b_{i+1} = k$ where $j = 0$, $1$ and $k = 0$, $1$ and $i = 1$, ..., $N$. Suppose we employ a block encoding scheme which has the property that, for any pair of $B_0$ and $B_1$ blocks belonging to this scheme, $N_{11} + 0$ and $N_{00} + 0$. Clearly an equi-distributed random orthogonal block encoding scheme has this property since $N_{00} = N_{11} = N/4$, but a simple complementary, complementary, or equi-distributed complementary block encoding scheme does not have this property. Now suppose that, in addition to implementing a particular block decision technique, for each of the $N_{00} + N_{11}$ bits where the blocks agree, the receiver makes a hard decision on which of the bit signals was transmitted. Thus, the receiver makes a hard bit decision $d_i$ on the transmitted bit $b_i$ for these $N_{00} + N_{11}$ bits $i$. Due to the fact that the $B_0$ and $B_1$ blocks are available at the receiver, the receiver knows precisely the value of $b_{i+1}$ for these $N_{00} + N_{11}$ bits $i$. Hence, the receiver can compute $r_o = \sum d_i$, where the summation is over $i$ s.t. $b_i = b_{i+1} = 0$ with $i = 1$, ..., $N$; and $r_1 = \sum (d_i \oplus 1)$ where the summation is over $i$ s.t. $b_i = b_{i+1} = 1$ with $i = 1$, ..., $N$, which represent the number of obvious bit errors with respect to the $N_{00}$ and $N_{11}$ bits, respectively. Assuming a block encoding scheme is employed where $N_{00} \neq 0$ and $N_{11} \neq 0$, a receiver which, in addition to implementing a particular block decision technique, operates in a manner described above to compute the numbers $r_o$, $r_1$ and $N_{00}$, $N_{11}$ will be called an intelligent receiver. An implementation of the intelligent portion of the receiver for NC2PSK signaling is given in Diagram 4.

Now let us examine the utilizations of the intelligent receiver. Basically we can distinguish two types of utilizations of the intelligent receiver. These involve either direct or indirect use of the measured values $r_o$, $r_1$ and $N_{00}$, $N_{11}$.

An example of a direct use of the measured values would be using these values to measure the actual probability of a bit error. $P_{eb}$ where $P_{eb}$ is simply the relative frequency of all obvious bit errors, i.e., for a particular data bit transmission $P_{eb} = \frac{r_o + r_1}{N_{00} + N_{11}}$. From Reference 7, pages 191-209, we see that for $N_{00} + N_{11}$ sufficiently large it follows from the Laws of Large Numbers that $P_{eb}$ converges to $P_{eb}$ assuming the conditions used to determine $P_{eb}$ are valid. If $N_{00} + N_{11}$ is not sufficiently large so that $P_{eb}$ converges to $P_{eb}$ we can form $P_{eb}$ with respect to a number of bit transmissions; i.e., we can add the $r_o$ and $r_1$ values for these transmissions together and add the $N_{00}$ and $N_{11}$ for these transmissions together and then determine the relative frequency $P_{eb}$ with respect to these sums.

An example of an indirect use of the measured values would be using these values in testing a hypothesis concerning the nature of the noise in
the communication system. Suppose we let $H_0$ be the hypothesis that the noise is white Gaussian noise. In the case when $H_0$ is true and given a binary signaling technique (C2PSK, C2FSK, or NC2FSK) we can determine $P_{eb}$. In order to test the hypothesis $H_0$ we choose an error probability $\alpha$ such that we want our test to decide correctly that $H_0$ is true with probability $1 - \alpha$. We use the probability $\alpha$ to determine a threshold $R$ such that if $r_0 + r_1 \leq R$, then by choosing $H_0$ to be true we can be correct with probability $1 - \alpha$. Thus, when we reject the hypothesis $H_0$ for $r_0 + r_1 > R$, we are incorrect only with probability $\alpha$. Clearly $R$ is a function of $P_{eb}$, $\alpha$, and $N_0 + N_1$ and $R$ is determined by the equation:

$$R = \left( \frac{N_0 + N_1}{\sum_{K=0}^{K} P_{eb} (1 - P_{eb})^K} \right)^{1 - \alpha}$$

AREAS FOR FURTHER INVESTIGATION

In the area of redundant encoding techniques, concatenated or nested codes will be investigated since they show promise of high noise resistance. In the area of modulation and signaling techniques, multi-level or M-ary techniques such as MPSK, MFSK and combined modulation techniques will be investigated. A previous report (see Reference 9) will provide the background basic to this investigation. In the area of detection techniques, non-parametric detectors will be investigated due to their promise of a lower probability of error when the noise is not additive white Gaussian noise.

It is also planned that physical implementation of these techniques will be pursued in the form of a real time digital receiver with ample consideration being given to software as well as hardware oriented digital receiver design.

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APPENDIX 1

For the low $E_b/N_0$ ratio case of the NC2FSK-Sum Block Decision Technique we investigate the distribution of $E(B_i)$ assuming that the $B_i$ block was transmitted and that the $B_j$ block was not transmitted. In what follows, we will use the symbols given in Diagram 2 and let $s$ be the signal power, $\sigma^2$ be the average noise power and $\varphi_i$ be the phase distortion of the signal for the $i$th bit.

First let us suppose that the $B_i$ block was not transmitted. It is well known (see Reference 1, page 168) that, for all $i = 1, \ldots, N$, $\sqrt{z_{ij}}$ is a Rayleigh distributed random variable with density function $P(z_{ij};X)$ given by:

$$P(z_{ij};X) = \begin{cases} \frac{X}{\sigma^2} \exp\left[-\frac{X^2}{2\sigma^2}\right], & X > 0 \\ 0, & X \leq 0 \end{cases}$$

From Reference 8, page 79 it follows that the density function of $Z_{ij}$ is:

$$P(Z_{ij};Y) = \begin{cases} \frac{1}{\sigma^2} \exp\left[-Y/\sigma^2\right], & Y > 0 \\ 0, & Y \leq 0 \end{cases}$$

Thus from Reference 8, page 173 we see that for all $i = 1, \ldots, N$, $Z_{ij}$ is an exponential random variable with parameter $1/\sigma^2$. Now $E(B_i) = \sum_{j=1}^{N} Z_{ij}$ is the sum of $N$ independently and identically distributed exponential random variables with parameter $1/\sigma^2$. From Reference 8, pages 194-5 we have that the moment generating function of each $Z_{ij}$ is given by:

$$M(Z_{ij};t) = \frac{1}{\frac{1}{\sigma^2} - t} \quad \text{for} \; t < \frac{1}{\sigma^2}$$

From Reference 8, page 203 we see that the moment generating function of $E(B_i)$ is given by:

$$M(E(B_i); t) = \left(\frac{1}{\frac{1}{\sigma^2} - t}\right)^N \quad \text{for} \; t < \frac{1}{\sigma^2}$$
Now $M(\mathbf{B}_i; t)$ is the moment generating function of the random variable $\mathbf{E}(\mathbf{B}_i)$ where $\mathbf{E}(\mathbf{B}_i)/\sigma^2$ has a Chi-Squared Distribution with $2N$ degrees of freedom. Thus $\mathbf{E}(\mathbf{B}_i)$ is determined to within the constant $C = 1/\sigma^2$ by a Chi-Squared Distribution with $2N$ degrees of freedom.

Next let us suppose that the $\mathbf{B}_i$ block was transmitted. We consider first the distribution of $Z_{31}$. Clearly $Z_{31} = X^2(j, c, i) + X^2(j, s, i)$ where $X(j, c, i)$ and $X(j, s, i)$ are the quadrature components associated with $\mathbf{B}_i$ for the $i^{th}$ bit. From Reference 1, page 168 we see that, since $\mathbf{B}_i$ was transmitted, $X(j, c, i)$ has a normal distribution with mean $\sqrt{s} \cos \varphi_i$ and variance $\sigma^2$ and $X(j, s, i)$ has a normal distribution with mean $\sqrt{s} \sin \varphi_i$ and variance $\sigma^2$. From Reference 3, page 169 and Reference 4, pages 48-52 we see that $X^2(j, c, i)/\sigma^2$ and $X^2(j, s, i)/\sigma^2$ have a non-central Chi-Squared Distribution with one degree of freedom and non-central parameters $s \cos \varphi_i/\sigma^2$ and $s \sin \varphi_i/\sigma^2$, respectively; hence from the additive property $Z_{31}/\sigma^2 = X^2(j, c, i)/\sigma^2 + X^2(j, s, i)/\sigma^2$, has a non-central Chi-Squared Distribution with two degrees of freedom and non-central parameter $s/\sigma^2 = 2E_b/N_0$. Using the additive property and since the $Z_{31}$ are independently and identically distributed, we see that $\mathbf{E}(\mathbf{B}_i)/\sigma^2 = \sum_{i=1}^{N} Z_{31}/\sigma^2$ has a non-central Chi-Squared distribution with $2N$ degrees of freedom and non-central parameter $2N E_b/N_0$. Thus $\mathbf{E}(\mathbf{B}_i)$ is determined to within the constant $C = 1/\sigma^2$ by a non-central Chi-Squared distribution with $2N$ degrees of freedom and non-central parameter $2N E_b/N_0$. 

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For the low Eb/N0 ratio case of the NCB2FSK-Add Block Decision Technique, with equi-distributed complementary block encoding, we investigate the distribution of $\mathbf{E}(B_3)$ assuming that the $B_3$ block was transmitted and that the $B_1$ block was not transmitted. In what follows we will use the symbols given in Diagram 3 and let $s$ be the signal power, $\sigma^2$ be the average noise power and $\varphi_0$ and $\varphi_1$ be the initial, unknown signal phases at $f_0$ and $f_1$, respectively.

First let us suppose that the $B_3$ block was not transmitted. From Reference 1, page 168 we see that the quadrature components appearing in $Y(B_3, s, f_0)$, $Y(B_3, c, f_0)$, $Y(B_3, s, f_1)$ and $Y(B_3, c, f_1)$ have normal distributions with mean $= 0$ and variance $= \sigma^2$. From Reference 8, page 200 it follows that $Y(B_3, s, f_0)$, $Y(B_3, c, f_0)$, $Y(B_3, s, f_1)$ and $Y(B_3, c, f_1)$ have normal distributions with mean $= 0$ and variance $N_0^2/2$. Thus from Reference 8, pages 169, 180 it follows that $\mathbf{E}(B_3, s, f_0)/(N_0^2/2)$, $\mathbf{E}(B_3, c, f_0)/(N_0^2/2)$ and $\mathbf{E}(B_3, s, f_1)/(N_0^2/2)$ have a Chi-Squared Distribution with one degree of freedom. Hence $\mathbf{E}(B_3, f_0)/(N_0^2/2) = \mathbf{E}(B_3, s, f_0)/(N_0^2/2) + \mathbf{E}(B_3, c, f_0)/(N_0^2/2)$ and $\mathbf{E}(B_3, f_1)/(N_0^2/2) = \mathbf{E}(B_3, s, f_1)/(N_0^2/2) + \mathbf{E}(B_3, c, f_1)/(N_0^2/2)$ have, from Reference 8, page 201, a Chi-Squared Distribution with two degrees of freedom.

Next let us suppose that the $B_3$ block was transmitted. From Reference 1, page 168, we see that the quadrature components appearing in $Y(B_3, s, f_0)$, $Y(B_3, s, f_0)$, $Y(B_3, c, f_0)$ and $Y(B_3, c, f_0)$ have a normal distribution with means $= s \sin \varphi_0$, $s \cos \varphi_0$, $s \sin \varphi_1$ and $s \cos \varphi_1$, respectively and variance $= \sigma^2$. From Reference 8, page 200 it follows that $Y(B_3, s, f_0)$, $Y(B_3, s, f_0)$ and $Y(B_3, c, f_0)$ have a normal distribution with mean $= N_0^2 \sin \varphi_0/2$, $N_0^2 \cos \varphi_0/2$, $N_0^2 \sin \varphi_1/2$ and $N_0^2 \cos \varphi_1/2$, respectively and variance $N_0^2/2$. Thus from Reference 8, page 169 and Reference 4, pages 45-52 it follows that $\mathbf{E}(B_3, s, f_0)/(N_0^2/2)$, $\mathbf{E}(B_3, c, f_0)/(N_0^2/2)$, $\mathbf{E}(B_3, s, f_0)/(N_0^2/2)$ and $\mathbf{E}(B_3, c, f_0)/(N_0^2/2)$ have a non-central Chi-Squared Distribution with one degree of freedom and non-central parameters $N_0 \sin \varphi_0/2N_0$, $N_0 \cos \varphi_0/\sigma^2$, $N_0 \sin \varphi_1/2N_0$ and $N_0 \cos \varphi_1/\sigma^2$, respectively. Using the additive property we see that $\mathbf{E}(B_3, f_0)/(N_0^2/2) = Y^2(B_3, s, f_0)/(N_0^2/2) + Y^2(B_3, c, f_0)/(N_0^2/2)$ and $\mathbf{E}(B_3, f_1)/(N_0^2/2) = Y^2(B_3, s, f_1)/(N_0^2/2) + Y^2(B_3, c, f_1)/(N_0^2/2)$ have a non-central Chi-Squared Distribution with two degrees of freedom and non-central parameters $N_0 \sin \varphi_0/2N_0$, $N_0 \cos \varphi_0/\sigma^2$, $N_0 \sin \varphi_1/2N_0$ and $N_0 \cos \varphi_1/\sigma^2$, respectively. Using the additive property once again we see that $\mathbf{E}(B_3)/(N_0^2/2) = \mathbf{E}(B_3, f_0)/(N_0^2/2) + \mathbf{E}(B_3, f_1)/(N_0^2/2)$ has a non-central Chi-Squared Distribution with four degrees of freedom and non-central parameter $2N_0 \sigma^2/N_0$. Thus $\mathbf{E}(B_3)$ is determined to within the constant $C = 2/N_0^2$ by a non-central Chi-Squared Distribution with four degrees of freedom and non-central parameter $2N_0 \sigma^2/N_0$. 

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APPENDIX 3

For the low $E_b/N_0$ ratio case of the NCE2FSK-Add, Subtract Block Decision Technique with equi-distributed random orthogonal block encoding we investigate the distribution of $S(B_j)$ assuming that the $B_j$ block was transmitted and that the $B_i$ block was not transmitted. In what follows we will use the symbols given in Diagram 3 with $Y(B_j, s, f_k)$ and $Y(B_i, c, f_k)$ replaced by $Z(B_j, s, f_k)$ and $Z(B_i, c, f_k)$ for $j = 0, 1$ and $k = 0, 1$ and let $s$ be the signal power, $\sigma^2$ be the average noise power and $\phi_s$ and $\phi_c$ be the initial, unknown signal phases at $f_s$ and $f_c$, respectively.

First let us suppose that the $B_i$ block was not transmitted. Due to the encoding employed and from Reference 1, page 168 we see that, for each of the sums $Z(B_j, s, f_0)$, $Z(B_j, c, f_0)$, $Z(B_j, s, f_1)$ and $Z(B_j, c, f_1)$, $N/4$ of the $N/2$ positive and $N/4$ of the $N/2$ negative quadrature components appearing in the sum have a normal distribution with means $= s \sin \phi_{s, j}$, $\sqrt{s} \cos \phi_{s, j}$, $\sqrt{s} \sin \phi_{s, j}$ and $\sqrt{s} \cos \phi_{s, j}$ respectively and variance $= \sigma^2$ and the other $N/4$ positive and $N/4$ negative quadrature components appearing in the sum have a normal distribution with mean = 0 and variance $= \sigma^2$. Hence the sum of the $N/2$ quadrature components with respect to either the negative or the positive portion of each sum has from Reference 8, page 200 a normal distribution with means $= N/2 \sin \phi_{s, j}/N$, $N/2 \cos \phi_{s, j}/N$, $N/2 \sin \phi_{s, j}/N$ and $N/2 \cos \phi_{s, j}/N$ respectively and variance $= \sigma^2 N/2$. Hence taking into account the signs associated with each portion we see from Reference 9, pages 199-200 that $Z(B_j, s, f_0)/N_{2, j}$, $Z(B_j, c, f_0)/N_{2, j}$, $Z(B_j, s, f_1)/N_{2, j}$ and $Z(B_j, c, f_1)/N_{2, j}$ each have a Chi-Squared Distribution with one degree of freedom. From Reference 8, page 201 it follows that $E(S(B_j, s, f_0)/N_{2, j}) = E(S(B_j, c, f_0)/N_{2, j}) + E(S(B_j, s, f_1)/N_{2, j})/N_{2, j}$ has a Chi-Squared Distribution with four degrees of freedom. Thus $E(S_B)$ is determined to within a constant $C = 1/N_0$ by a Chi-Squared Distribution with four degrees of freedom.

Next let us suppose that the $B_i$ block was transmitted. In this case each of the $N/2$ positive quadrature component terms appearing in the sums $Z(B_j, s, f_0)$, $Z(B_j, c, f_0)$, $Z(B_j, s, f_1)$ and $Z(B_j, c, f_1)$ has a normal distribution with means $= s \sin \phi_{s, j}$, $\sqrt{s} \cos \phi_{s, j}$, $\sqrt{s} \sin \phi_{s, j}$ and $s \cos \phi_{s, j}$ respectively and variance $= \sigma^2$ and each of the $N/2$ negative quadrature component terms appearing in these sums has a normal distribution with mean $= 0$ and variance $= \sigma^2$. Thus from Reference 8, page 200 it follows that the sum of $N/2$ positive quadrature component terms in these sums have a normal distribution with means $= N/2 \sin \phi_{s, j}/N$, $N/2 \cos \phi_{s, j}/N$, $N/2 \sin \phi_{s, j}/N$ and $N/2 \cos \phi_{s, j}/N$ respectively and variance $= \sigma^2 N/2$ and the sum of the $N/2$ negative quadrature component terms in these sums have a normal distribution with mean $= 0$ and variance $= \sigma^2 N/2$. Hence taking into account the signs associated with each type of sum we see from Reference 8, pages 199-200 that $Z(B_j, s, f_0)$,
Z(B, c, f_{0}), Z(B, s, f_{0}) and Z(B, c, f_{1}) have a normal distribution with means $= \sqrt{3} \sin \varphi_{0}/2, \sqrt{3} \cos \varphi_{0}/2$, $\sqrt{3} \sin \varphi_{1}/2$ and $\sqrt{3} \cos \varphi_{1}/2$ respectively and variance $= N\sigma^{2}$. Thus from Reference 2, page 169 and Reference 4, pages 4-52 it follows that $Z(B, s, f_{0})/N\sigma^{2}$, $Z(B, c, f_{0})/N\sigma^{2}$, $Z(B, s, f_{1})/N\sigma^{2}$ and $Z(B, c, f_{1})/N\sigma^{2}$ have a non-central Chi-Squared Distribution with one degree of freedom and non-central parameters $N\sin^{2} \varphi_{0}/4$, $N\cos^{2} \varphi_{0}/4$, $N\sin^{2} \varphi_{1}/4$ and $N\cos^{2} \varphi_{1}/4$ respectively. Using the additivity property we see that $Z^{2}(B, s, f_{0})/N\sigma^{2} = Z^{2}(B, s, f_{0})/N\sigma^{2} + Z^{2}(B, c, f_{0})/N\sigma^{2}$ and $Z^{2}(B, s, f_{1})/N\sigma^{2} = Z^{2}(B, s, f_{1})/N\sigma^{2} + Z^{2}(B, c, f_{1})/N\sigma^{2}$ have a non-central Chi-Squared Distribution with two degrees of freedom and non-central parameter $N\sin^{2} \varphi_{0}/4$. Using the additivity property once again we see that $Z^{2}(B, s, f_{0})/N\sigma^{2} = Z^{2}(B, s, f_{0})/N\sigma^{2} + Z^{2}(B, c, f_{1})/N\sigma^{2}$ has a non-central Chi-Squared Distribution with four degrees of freedom and non-central parameter $N\sin^{2} \varphi_{0}/4$. Thus $Z^{2}(B, s, f_{0})/N\sigma^{2}$ is determined to within the constant $C = 1/4\sigma^{2}$ by a non-central Chi-Squared Distribution with four degrees of freedom and non-central parameter $N\sin^{2} \varphi_{0}/4$.
REFERENCES


BIT BY BIT DECISION TECHNIQUE C2PSK SIGNALING

Figure 1

N = 1024, N = 512, N = 256, N = 128, N = 64, N = 32, N = 16, N = 8, N = 4
NC2FSK-SUM BLOCK DECISION TECHNIQUE

Figure 4
Figure 5
NCB2FSK-ADD BLOCK DECISION TECHNIQUE
EQUI-DISTRIBUTED COMPLEMENTARY ENCODING

Figure 6
Figure 7
CB2PSK BLOCK DECISION TECHNIQUE

Figure 8

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BIT BY BIT DECISION TECHNIQUE RECEIVER
(2FSK SIGNALING)
NCB2FSK-ADD BLOCK DECISION TECHNIQUE RECEIVER

Diagram 3
INTELLIGENT PORTION OF A RECEIVER
(NC2FSK SIGNALING)

Diagram 4
DIGITAL RECEIVER DECISION TECHNIQUES FOR CERTAIN FIXED LENGTH
BINARY BLOCK CODES TRANSMITTED THROUGH THE GAUSSIAN CHANNEL

An interim report on the problem.

Noel C. Balthasar and Hugo M. Beck

November 20, 1969

This document has been approved for public release and sale; its
distribution is unlimited.

Various types of fixed length binary block encodings of binary data are
presented. Binary signaling is used to transmit the binary encoded data through
an additive white gaussian noise channel. Certain block decision techniques
are investigated and compared. For certain block encoding schemes, the notion
of an intelligent receiver is developed.

This study is considered basic in analyzing the use of redundant encoding
in digital communications systems.
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