THE DANZIG-WOLFE DECOMPOSITION PRINCIPLE AND
MINIMUM COST MULTICOMMODITY NETWORK FLOWS

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September 1969
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ABSTRACT

J. A. Tomlin published a paper [1] on meeting required multi-
commodity network flows at minimum cost. He formulated this problem
in both node-arc and arc-chain form. The node-arc linear program was
attacked by the Dantzig-Wolfe decomposition principle by expressing
the derived master program as convex combinations of the extreme points
of the derived subprograms. In this note, it is shown that this prob-
lem is really a special case of the problem where one is attempting to
meet minimum cost multicommodity flows without flow requirements on the
individual commodities. Tomlin's algorithm is then modified to solve
this more general problem. When this is done, the subprograms are
homogeneous and the master program is a nonnegative combination of
their independent solutions.

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INTRODUCTION

Consider a network \([N,A]\) with \(n\) nodes and \(m\) directed arcs joining pairs of nodes. Assign \((i,j)\), the arc directed from \(i\) to \(j\), a capacity \(b_{ij} \geq 0\) and a cost \(c_{ij}\) per unit of flow. The \(c_{ij}'s\) are such that the total cost on any directed cycle is nonnegative. Commodity \(k\), \(k = 1, \ldots, q\), is identified by its source \(s_k\) and its sink \(t_k\). It is required to find a multicommodity flow that satisfies the capacity constraints at minimum cost.

Tomlin [1] published a paper in which \(c_{ij} \geq 0\) and a flow \(r_k\) of commodity \(k\) was required. It will be shown later that this problem is a special case of the one treated here. Other special cases treated by this more general formulation are maximizing a linear combination of the \(k\) distinct commodity flows and finding an efficient routing when the value of a routing is proportional to individual commodity flows but this value can be offset by transportation costs.
DECOMPOSITION

Let \( y_{ij}^k \) be the flow of commodity \( k \) on arc \((i,j)\) and \( v_k \) the total flow of commodity \( k \). Then it is required to find \( y_{ij}^k, v_k, \) and \( \min Z \) such that

\[
Z = \sum_{(i,j)} c_{ij} \left( \sum_{k=1}^{q} y_{ij}^k \right)
\]

\[
\sum_{k=1}^{q} y_{ij}^k \leq b_{ij} \quad \text{all } (i,j)
\]

\[
u = \begin{cases} -v_k & \text{for } i = s_k \\ v_k & \text{for } i = t_k \\ 0 & \text{otherwise} \end{cases}
\]

\( k = 1, \ldots, q \)

Letting \( A_k \) be the node-arc incidence matrix of the network; \( d_k \) an \( n \) vector containing \(-1\) in the \( s_k \) position, \(+1\) in the \( t_k \) position, and \(0\) elsewhere; \( y_k \) the vector of arc flows \(< y_{ij}^k >\) for commodity \( k \); \( b \) the vector of arc capacities; \( c' \) the vector \(< c_{ij} >\) of arc costs; and \( s \) a vector of slacks, (1-3) may be written as

\[
c'y_1 + c'y_2 + \ldots + c'y_q = Z \min
\]

\[
Iy_1 + Iy_2 + \ldots + Iy_q + Is = b
\]

\[
A_1 y_1 + d_1 v_1 = 0
\]

\[
A_2 y_2 + d_2 v_2 = 0
\]

\[
\ldots
\]

\[
A_q y_q + d_q v_q = 0
\]
Note that solutions to \( A_ky_k + d_kv_k = 0 \) may be decomposed into units of flow along paths from \( s_k \) to \( t_k \) and along cycles. However, since all cycles have nonnegative cost and since elimination of flow on cycles from a feasible solution cannot yield an infeasible solution, cycle flows need not appear in an optimal solution and can be eliminated from consideration. Thus, let \( K_k = \{ W_{k1}, \ldots, W_{kN_k} \} \) be the set of solutions corresponding to one unit of flow on a directed path from \( s_k \) to \( t_k \). \( K_k \) is then a set of points that span the set of solutions to \( A_ky_k + d_kv_k = 0 \) which contain no cycles. Then, applying the Dantzig-Wolfe decomposition principle [2] one may write \( y_k \) as a nonnegative combination of the elements of \( K_k \) and (4) becomes

\[
(5) \quad \sum_{j=1}^{N_1} \lambda_{1j} (c'w_{ij}) + \cdots + \sum_{j=1}^{N_q} \lambda_{qj} (c'w_{qj}) = Z \min
\]

\[
\sum_{j=1}^{N_1} \lambda_{1j} (Iw_{ij}) + \cdots + \sum_{j=1}^{N_q} \lambda_{qj} (Iw_{qj}) + Is = b
\]

\[\lambda_{kj} \geq 0\]

The number of variables in (5) is of course too large to enumerate. However, suppose we have a basic feasible solution and let \( \pi_{ij} \) be the corresponding simplex multiplier for the row containing \( b_{ij} \) on the right-hand side and \( \pi' = \langle \pi_{ij} \rangle \). If \( \pi_{ij} > 0 \), then \( s_{ij} \) may be introduced into the basis to give an improved basic solution. If all \( \pi_{ij} \leq 0 \), then any solution with

\[
(6) \quad c'w_{kj} - \pi' Iw_{kj} < 0
\]
will yield an improvement. If no \( w_{k_j} \) satisfies (6), then the current solution is optimal. If arc \((i,j)\) is assigned a cost of \( c_{ij} - \pi_{ij} \), then the left-hand side of (6) is equal to the total cost of the path represented by \( w_{k_j} \). Thus the search for a solution satisfying (6) may be found by finding the shortest path between \( s_k \) and \( t_k \) for \( k = 1, \ldots, q \).

Efficient methods for finding shortest chains are given in [3-5]. Many of them require that the network contain no negative cycles. This is assured when all \( \pi_{ij} \leq 0 \) due to our initial restriction of the \( c_{ij} \)'s.

The phase I procedure for finding a starting feasible basis to initiate the algorithm is accomplished by setting \( I s = b \).
FLOW REQUIREMENTS

Tomlin [1] treats the problem where it was required to minimize total network cost subject to the restriction that the total flow of commodity $k$ be equal to $r_k$. Furthermore, it was assumed that all $c_{ij} \geq 0$. This can be treated as a special case of the problem treated in this paper. For each $k$ one merely attaches an artificial node and directs an artificial arc from it to $s_k$ (or alternatively to it from $t_k$). This artificial arc is assigned a capacity of $r_k$ and a very large negative cost. Since none of these artificial arcs belong to cycles, our initial restriction on the $c_{ij}$'s will hold.

The node-arc formulation in (1) differs only slightly from ours. Specifically, (4) is replaced by

\begin{align*}
&c_1y_1 + c_2y_2 + \ldots + c_qy_q = Z \min \\
&I y_1 + I y_2 + \ldots + I y_q + Is = b \\
&A_1y_1 = d_1 \\
&A_2y_2 = d_2 \\
&\ldots \\
&A_qy_q = d_q \\
&y_k \geq 0 \text{ all } k
\end{align*}

Here $d_k$ is the vector with an $r_k$ in the $s_k$ position, a $-r_k$ in the $t_k$ position, and zero everywhere else; all other variables are as defined before.

This problem is also treated by decomposition. However, the sub-programs, $A_qy_q = d_q$, are no longer homogeneous and consequently, instead
of looking for a nonnegative combination of independent solutions to
the subprograms, we look for convex combinations of their extreme
points. Thus letting \( W_k = \{ w_{k1}, \ldots, w_{kN_k} \} \) be the extreme points of
\( A_k y_k = d_k \), (5) is replaced by

\[
\begin{align*}
(8) \quad \sum_{j=1}^{N_1} \lambda_{kj} (c^I w_{lj}) + \cdots + \sum_{j=1}^{N_q} \lambda_{kj} (c^I w_{lj}) &= Z \min \\
\sum_{j=1}^{N_1} \lambda_{kj} (I w_{lj}) + \cdots + \sum_{j=1}^{N_q} \lambda_{kj} (I w_{lj}) + Is &= b \\
\sum_{j=1}^{n_q} \lambda_{kj} &= 1 \\
\lambda_{kj} \geq 0
\end{align*}
\]

The extreme points of these subprograms are also paths and hence the
\( W_k \) defined here is identical to that defined in the last section. In
form, (8) differs from (5) only in that one must add a convexity con-
straint on the \( \lambda_{kj} \) for each \( k \). However, if (5) is used, one must also
add constraints reflecting capacities on the added artificial arcs
(i.e. the \( b \) vector in (5) is larger than that in (8) by the number of
commodities) which differ only slightly from the convexity constraints
in (8). Thus for this special case the formulations are almost identical.
ARC-CHAIN FORMULATION

In [1] the minimum cost multicommodity problem was also formulated as a linear program in terms of its arc chain incidence matrix. Specifically, rows correspond to arcs and columns to chains. The program was then solved by a simple extension of a method due to Ford and Fulkerson [6] which has since been extended by Wollmer [7]. It was shown that this method turned out to be equivalent to applying decomposition to the node-arc program.

In this paper, we will not go into the details of this method other than to comment that if the more general problem of this paper is formulated in terms of its arc-chain incidence matrix, it may also be solved by a similar extension to the method proposed in [6] and that this method is also equivalent to applying decomposition to the node-arc linear program.
CONCLUDING REMARKS

It has been shown that the problem treated in [1] is a special case of that treated here. As other special cases, the problem treated here includes (i) maximizing a linear combination of the individual commodity flows and (ii) finding an efficient routing that takes into consideration the value of both the individual commodity flows and the transportation costs involved. The former of these is accomplished by attaching, for each commodity, an artificial node and an artificial arc directed from it to the source. The artificial arcs are given infinite capacities and costs whose negatives are proportional to the linear coefficients of the commodity flows in the linear function that is to be maximized. For the latter problem, one also adds these same artificial nodes and arcs, the new arcs having infinite capacity. The cost on the artificial arc for commodity k is the negative of the value of a unit of flow of commodity k. Thus the scope of problems treated in [1] may be significantly increased by relatively small changes in the problem formulation and algorithms.
REFERENCES


