AN INVARIANCE PRINCIPLE FOR DYNAMICAL SYSTEMS ON BANACH SPACE:
APPLICATION TO THE GENERAL PROBLEM OF THERMOELASTIC STABILITY

by

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1. Introduction

Elastic stability is usually discussed from strictly mechanical considerations. Recently, however, attempts have been made to analyze the influence of the usually neglected thermodynamic properties of elastic materials. More specifically, one may ask what effects the second law of thermodynamics has on the asymptotic stability of equilibrium of thermoelastic materials.

KOITER [1] has studied the general nonlinear thermoelastic problem, and for materials with internal friction he obtains asymptotic stability of the equilibrium solutions. ERICKSEN [2] has posed the question as to the asymptotic stability of the equilibrium solutions of elastic materials without imposing the assumption of internal friction. DAFERMOS [3] answered this question to some degree by obtaining a description of the states that the material approaches as \( t \to \infty \).

This same question is studied here in a more general setting than was done by SLEMROD [4], but in the same spirit: it is shown that the results of [3] can be obtained as a simple application of an invariance principle for abstract dynamical systems [4,5].

2. Mathematical Preliminaries

The principal analytical tool to be used is a generalization due to HALE [5] for abstract dynamical systems of the well known invariance principle of LASALLE [6] for ordinary differential equations. The following
brief presentation of this tool emphasizes notation and concepts to be used in studying the problem of thermoelastic stability.

Let $\mathbb{R}^+ = [0, \infty)$ and let $\mathcal{B}$ be a Banach space with norm $\| \cdot \|_{\mathcal{B}}$ for $\phi \in \mathcal{B}$. Then,

**Definition 2.1.** $u$ is a dynamical system on a Banach space $\mathcal{B}$ if $u$ is a function $u: \mathbb{R}^+ \times \mathcal{B} \to \mathcal{B}$ such that $u$ is continuous, $u(0, \phi) = \phi$, $u(t + \tau, \phi) = u(t, u(\tau, \phi))$ for all $t, \tau \geq 0$ and all $\phi$ in $\mathcal{B}$. The positive orbit $O^+(\phi)$ through $\phi$ in $\mathcal{B}$ is defined as $O^+(\phi) = \bigcup_{t \geq 0} u(t, \phi)$. A point $\psi$ in $\mathcal{B}$ is an equilibrium point if $O^+(\psi) = \psi$.

This set of definitions simply generalize familiar notions from the theory of differential equations to dynamical systems.

**Definition 2.2.** A set $M$ in $\mathcal{B}$ is a positively invariant set of the dynamical system $u$ if for each $\phi$ in $M$, $O^+(\phi) \subseteq M$.

**Definition 2.3.** A set $M$ in $\mathcal{B}$ is an invariant set of the dynamical system $u$ if for each $\phi$ in $M$ there exists a function $U(s, \phi)$, $U(0, \phi) = \phi$ defined and in $M$ for $s \in (-\infty, \infty)$ and such that $u(t, U(s, \phi)) = U(t + s, \phi)$ for all $t \in \mathbb{R}^+$.

Definition 2.2 is well known. The second definition is used to extend backward in time those solutions of the dynamical system which lie in an invariant set. It is clear that if a set $M$ is invariant it is positively invariant but the converse is, in general, false.
Definition 2.4. If \( u \) is a dynamical system on \( \mathcal{B} \) and \( V \) is a continuous scalar functional on \( \mathcal{B} \), define the functional

\[
\dot{V}(\phi) = \lim_{t \to 0} \frac{1}{t} [V(u(t, \phi)) - V(\phi)].
\]

Then

Definition 2.5. \( V: \mathcal{B} \to \mathbb{R} \) is said to be a Liapunov functional on a set \( G \) in \( \mathcal{B} \) if \( V \) is continuous on \( \overline{G} \), the closure of \( G \), and if \( \dot{V}(\phi) \leq 0 \) for \( \phi \) in \( G \). Furthermore, denote by \( S \) the set \( S = \{ \phi \text{ in } \overline{G} | \dot{V}(\phi) = 0 \} \) and let \( M \) be the largest invariant set in \( S \) for the dynamical system \( u \). With these definitions it is then possible to prove:

Theorem 2.1 (HALE [5]). Let \( u \) be a dynamical system on \( \mathcal{B} \). If \( V \) is a Liapunov functional on \( G \) and a positive orbit \( O^+(\phi) \) belongs to \( G \) and is in a compact set of \( \mathcal{B} \) then \( u(t, \phi) \to M \) as \( t \to \infty \).

It is self-evident that in applications to the problem of asymptotic stability of an equilibrium point \( \psi \) it is necessary to show that \( M = \{ \psi \} \). Moreover, it should be emphasized that the usefulness of this theorem in applications depends on the very relaxed assumptions imposed on the Liapunov functional \( V \) and its derivative \( \dot{V} \). These conditions should be compared to the much stronger conditions imposed by standard asymptotic stability theorems (see, for example, PARKS [7]).

3. Constitutive Equations of Linear Thermoelasticity

A material point is identified by \( x = (x_1, x_2, x_3) \) in its state of
equilibrium (no stresses, constant temperature = \( r_0 \)). The displacement field at some time \( t \) following an initial disturbance at time \( t = 0 \) is given by \( u(x,t) \) and the temperature deviation by \( T(x,t) \); \( \rho(x) \) denotes the density at \( x \) in the equilibrium state.

Let \( \Omega \) be an open, bounded, connected set in \( \mathbb{R}^3 \) which is properly regular [8]; let \( \partial \Omega \) denote the boundary of \( \Omega \). The constitutive equations of thermoelasticity are taken then in the form

\[
\rho \ddot{u}_i = \left( C_{ijk\ell} u_{k,\ell} \right)_j - (m_{ij} T)_j, \quad (3.1)
\]
\[
\rho C_{ijl} + m_{ij} r_0 \delta_{ij,l} = (K_{ij} T)_i, \quad (3.2)
\]

where body forces and heat sources have been excluded. In these equations \( C_{ijk\ell} = C_{jik\ell} = C_{kij\ell}, m_{ij} = m_{ji}, K_{ij} = K_{ji} \) and \( C, \rho, C_{ijk\ell}, m_{ij} \) and \( K_{ij} \) are assumed to be smooth functions of \( x \).

Let now \( t_0 > 0 \). By a classical solution of the mixed initial-boundary value problem in \( \Omega \times (0,t_0) \) we mean a pair \((u,T)\) satisfying equations (3.1) and (3.2) together with the boundary conditions

\[
u = 0 \quad \text{on} \quad \partial \Omega \times (0,t_0) \quad \text{(closed boundary),} \quad (3.3)
\]
\[
T = 0 \quad \text{on} \quad \partial \Omega \times (0,t_0) \quad \text{(constant temperature);} \quad (3.4)
\]

and with initial conditions

\[
(u(x,0), \dot{u}(x,0), T(x,0)) = (u_0(x), \dot{u}_0(x), T_0(x)), \quad (3.5)
\]

where \( u_0(x), \dot{u}_0(x) \) and \( T_0(x) \) are given functions on \( \Omega \).

4. The Thermoelastic Problem as a Dynamical System

In this section we show, by recalling some results of DAFERMOS [3], that the generalized solutions of the mixed initial boundary value problem described above can be viewed on an appropriate Banach space as a dynamical
system (ZUBOV [9]). Once this is done, the application of Theorem 2.1 permits us to draw immediate conclusions on the asymptotic behavior of the solutions of our problem.

Consider the Sobolev spaces $W^{(k)}_2(\Omega)$ and $W^{(k)}_{20}(\Omega)$, $k = 1, 2, \ldots$ (see, for instance SOBOLEV [10, 11], AGMON [12]). Assume that

$$\text{ess inf } \rho(x) > 0, \text{ess inf } C_D(x) > 0, \quad (4.1)$$

$$K_{ij} \xi_i \xi_j \geq C_1 \xi_i \xi_j, \quad C_1 > 0 \quad \text{constant}, \quad (4.2)$$

(a reformulation of the Clausius-Duhem inequality; LANDAU and LIFSHITZ [13])

and for all $v_1 \in W^{(1)}_{20}(\Omega)$

$$\int_\Omega C_{ij} k_{ij} v_i, j v_k, k \, dx \leq C_2 \int_\Omega v_i, j v_i, j \, dx, \quad C_2 > 0 \quad \text{constant}, \quad (4.3)$$

a general property of the tensor of the elastic modulii for infinitesimal elasticity (TRUESDELL and NOLL [14]).

Define now the spaces $H_0(\Omega) \approx W^{(1)}_{20}(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ with norm

$$| (v_i, w_i, R) |^2 = \int_\Omega \left[ \rho v_i, w_i, j v_k, \xi^2 + \frac{\rho C_D R^2}{R} \right] dx$$

and $H(\Omega) = W^{(1)}_{20}(\Omega) \times W^{(1)}_{20}(\Omega)$. Define the map $P: H_0(\Omega) \rightarrow H(\Omega)$ sending $(v_i, w_i, R) \in H_0(\Omega)$ onto $(u_i, T) \in H_1(\Omega) \subset H(\Omega)$ where $(u_i, T) \in W^{(1)}_{20}(\Omega) \times W^{(1)}_{20}(\Omega)$ is defined by the solution of the system

$$\int_\Omega C_{ij} k_{ij} u_i, k \theta_i, j \, dx = -\int_\Omega \rho v_i, \theta_i + m_{ij} T \theta_i, j \, dx$$

$$\int_\Omega K_{ij} T, j \, dx = \int_\Omega \rho C_D R + m_{ij} y v_i, j \, dx$$

for every $D \in W^{(1)}_{20}(\Omega)$. The mapping $P$ is linear, well defined on $H_0(\Omega)$ and one to one. Hence, defining $P_m \circ \cdots \circ P \circ P$ let $H_m(\Omega)$ denote the range of the map $P_m$. It is clear that $P_m^{-1}$ exists and maps $H_m(\Omega)$ onto $H_0(\Omega)$.

Let $\psi \in H_m(\Omega)$ and define $|\psi|_m = |P_m^{-1} \psi|_0$. Then,
Lemma 4.1 (DAFERMOS [3]). $H_m$ is a Banach space with norm $\| \cdot \|_m$. $H_0(\Omega) \supset H(\Omega) \supset \ldots \supset H_m(\Omega)$ algebraically and topologically. Furthermore, $H_m(\Omega)$ is dense in $H_\ell(\Omega)$ for $m > \ell$ and the imbedding $I: H_m(\Omega) \to H_\ell(\Omega)$ is compact.

Let us now define appropriately a **generalized solution** of our problem:

**Definition 4.1.** $(u_i, \dot{u}_i, T)$ will be called a **generalized solution** of (3.1) - (3.5) on $\Omega \times (0, t_0)$ if for all smooth test functions $(v_i, R)$ with compact support on $\Omega$ and vanishing on $\Omega \times 0$,

\[
\int_0^t \int_\Omega \left[ (t-t_0)[\rho \ddot{u}_i \dot{v}_i - C_{ijk} \dddot{v}_i, j \dot{v}_i, j + m_{ij} \dot{v}_i, j + \frac{\rho c_D}{\gamma_0} Tr + m_{ij} u_i, j \dot{R} + \rho \ddot{u}_i, j + \frac{c_D}{\gamma_0} Tr + m_{ij} u_i, j \dot{R} - \frac{1}{\gamma_0} \int_0^t (K_{ij} \dot{R}, i, j, T) dt \right] dx dt = \tag{4.4}
\]

With this definition it follows that:

**Theorem 4.1 (DAFERMOS [3]).** Under assumptions (4.1) - (4.3) the triple $(u_i, \dot{u}_i, T)$ describes a dynamical system on $H_m(\Omega)$, $m = 0, 1, 2, \ldots$, where $(u_i, \dot{u}_i, T)$ is the generalized solution to the equations of linear thermoelasticity satisfying equation (4.4). Furthermore, for $t$ in $(0, t_0)$

\[
\left| (u_i, \dot{u}_i, T)(t) \right|^2_m + \frac{1}{\gamma_0} \int_0^t \int_\Omega K_{ij} \dot{u}_i, j, T, \dot{u}, j, T, \dot{T} dx dt = \left| (u_i, \dot{u}_i, T_0) \right|^2_m \tag{4.5}
\]

where $T^{(m)}(x, t)$ denotes the generalized $m$th derivative in time of $T(x, t)$.

5. Stability Analysis

The problem of thermoelastic stability has now been put in a setting
appropriate for the application of Theorem 2.1, which allows us to obtain
stability results in a simple and direct manner.

For this purpose, fix \( m \geq 1 \). Then, by Theorem 4.1 and (4.5) it
follows that for any initial data in \( H_m(\Omega) \) the trajectory \((u_1, \hat{u}_1, T)\) will lie in a bounded set of \( H_m(\Omega) \) for any \( t > 0 \). Hence, by Lemma 4.1 the
trajectory remains in a compact set \( G \) of \( H_\beta(\Omega) \), \( \beta < m \). But then all the hy-
pothesis of Theorem 2.1 are met with \( \mathcal{B} = H_\beta(\Omega) \). For simplicity let \( \ell = 0 \)
and \( V = |(u_1, \hat{u}_1, T)|^2 \). From (4.2) and (4.5) it immediately follows that
\[
\dot{V} = -\frac{1}{2} \int_{\Omega} K_{ij}^{1}, T, T, j^2 \, dx \leq -c_2 \|T\|_{L_2}^2, \quad c_2 > 0;
\]
therefore the set \( S \) is the set
\[
S = \{(u_1, \hat{u}_1, T) \in H_0(\Omega) \mid \|T\|_{L_2} = 0\}. \]
Let now \( M^+ \) be the largest positively
invariant set in \( S \). By the definition of generalized solution (4.4) it
follows that \( M^+ = \{(u_1, \hat{u}_1, T) \in H_0(\Omega) \mid \|T(\cdot, t)\|_{L_2} = 0 \text{ for } t \geq 0\} \). Choosing
now \( v_1 \equiv 0 \) in (4.4) it follows that for \((u_1, \hat{u}_1, T) \in M^+ \) it is necessary
that
\[
\int_{t_0}^{t} \int_{\Omega} \frac{d}{dt}(t-t_0)R|m_{ij}u_{i,j}dxdt = -t_0 \int_{\Omega} m_{ij}u_{i,j}R \, dx
\]
for every test function \( R \). Choosing this function as \( R(x,t) = \frac{\omega(t)\eta(x)}{t-t_0} \)
where \( \eta(x) \) is an arbitrary test function and \( \omega(t) \) is the \( C^\infty \) "bump"
function of Serrin [15], it follows that
\[
\int_{\Omega} m_{ij}(x)u_{i,j}(x,t)dx = \int_{\Omega} m_{ij}(x)u_{i,j}(t_0)dx, \quad t \geq 0
\]
for \((u_1, \hat{u}_1, T) \) in \( M^+ \). Hence, Theorem 2.1 applied to this context yields the
desired result:

Theorem 5.1: For any initial data \((u_{o1}, \hat{u}_{o1}, T) \) in \( H_m \), \( m \geq 1 \), and
under assumptions (4.1) - (4.3) \((u_1, \hat{u}_1, T)(t) \) approaches asymptotically the set
\[
\{(v_1, \hat{v}_1, R) \in W^{(1)}(\Omega) \times L_2(\Omega) \times L_2(\Omega) \mid \int_{\Omega} m_{ij}(x)v_{i,j}(x,t)dx = \int_{\Omega} m_{ij}(x)u_{i,j}(t_0)dx, \quad t \geq 0, \ R = 0\}
\]
in the norm of the space \( W^{(1)}(\Omega) \times L_2(\Omega) \times L_2(\Omega) \).
References


