PROBABILITY OF
A PURE EQUILIBRIUM POINT IN
n-PERSON GAMES

Melvin Dresher

This research is supported by the United States Air Force under Project RAND—Contract No. F11620-67-C-0015—monitored by the Directorate of Operational Requirements and Development Plans, Deputy Chief of Staff, Research and Development, Hq USAF. Views or conclusions contained in this study should not be interpreted as representing the official opinion or policy of the United States Air Force.

DISTRIBUTION STATEMENT
This document has been approved for public release and sale; its distribution is unlimited.

BEST AVAILABLE COPY

The RAND Corporation

1700 Main St • Santa Monica • California • 90406
DISCLAIMER NOTICE

THIS DOCUMENT IS BEST QUALITY AVAILABLE. THE COPY FURNISHED TO DTIC CONTAINED A SIGNIFICANT NUMBER OF PAGES WHICH DO NOT REPRODUCE LEGIBLY.
This Rand Memorandum is presented as a competent treatment of the subject, worthy of publication. The Rand Corporation vouches for the quality of the research, without necessarily endorsing the opinions and conclusions of the author.

Published by The RAND Corporation
This Memorandum continues Project RAND's program of research into the theory of games and its applications. It extends recent work done on n-person games and reported in RM-5543-PR, RM-5567-PR, and RM-5438-PR.

The result obtained herein, namely, that a large nonzero-sum n-person game chosen at "random" is likely to have a pure strategy equilibrium point, may have important implications. Every game has a mixed strategy equilibrium point, but it is not clear how or why one would ever use such a solution concept in a real-world situation. Not only is an optimal mixed strategy very difficult to compute, but decisionmakers are reluctant to make operational use of the notion because it means leaving the decision to chance. This Memorandum indicates that if the players have many strategies, a mixed strategy is rarely an optimal one. Thus many game theory models may take on additional significance.
SUMMARY

A "random" n-person noncooperative game—the game that prohibits communication and therefore coalitions among the n-players—is shown to have a pure strategy solution with a high probability. A solution of a game is an equilibrium point or a set of strategies, one for each player, such that if n - 1 players use their equilibrium strategies then the n-th player has no reason to deviate from his equilibrium strategy. It is shown that the probability of a solution in pure strategies for large random games converges to $1 - \frac{1}{e}$ for all $n \geq 2$. 
# CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>GAMES AND TRUNCATIONS</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>EQUILIBRIUM POINT</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>RANDOM GAMES</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>EXISTENCE OF t EQUILIBRIUM POINTS</td>
<td>7</td>
</tr>
<tr>
<td>6</td>
<td>EQUILIBRIUM POINTS IN TWO-PERSON GAMES</td>
<td>8</td>
</tr>
<tr>
<td>7</td>
<td>EQUILIBRIUM POINTS IN THREE-PERSON GAMES</td>
<td>9</td>
</tr>
<tr>
<td>8</td>
<td>PURE EQUILIBRIUM POINTS IN n-PERSON GAMES</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>REFERENCES</td>
<td>21</td>
</tr>
</tbody>
</table>
1. INTRODUCTION

The concept of a solution, or optimal strategy, frequently used for an n-person noncooperative game is the equilibrium strategy or equilibrium point. In order to assure the existence of a solution it is necessary to introduce mixed strategies. Except for the 2-person zero-sum game, however, it is generally very difficult to compute an optimal mixed strategy. Further, the decision-maker is reluctant to accept the operational notion of a mixed strategy.

These limitations of mixed strategies lead naturally to the hope that mixed strategy solutions are rarely required, or that a game chosen at random will in fact possess a pure strategy solution. For a 2-person zero-sum game this hope is not fulfilled; for large matrices in such games it is almost certain that the solution will be a mixed strategy, or the chance of a pure strategy solution is almost negligible.

It was conjectured that the optimal strategy of an n-person game would have a similar property. The present paper shows that with respect to solution, the n-person game is different from the 2-person zero-sum game. It is shown that the probability of a solution in pure strategies is quite large, in fact converging to

\[ 1 - e^{-1} = .632^+ \]
for large games. Further, this result is the same regardless of the number of players, two or more.
2. GAMES AND TRUNCATIONS

In the normal form of an n-person noncooperative game the i-th player (i ≤ n) has $m_i$ strategies which we label $u_i (1 ≤ u_i ≤ m_i)$. A play of a game can be represented by an n-vector $U = (u_1, u_2, \ldots, u_n)$, giving us $\prod_{i=1}^{n} m_i$ = $m$ possible plays. For each play $U$ and each player $i$ there exists a payoff $M_i(U)$, representing the payoff to the i-th player for the play $U$. There are therefore $n^n$ payoffs.

We now define a truncation of a play with respect to the i-th player to be an n - 1 vector

$$U_i = (u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n).$$

A truncation of a play leaves out the i-th player's strategy or

$$U = (U_i, u_i).$$
3. EQUILIBRIUM POINT

Nash first introduced the notion of an equilibrium point, and he showed that every game possesses such a point in mixed strategies. An $n$-vector of pure strategies $\mathbf{u}^* = (u_1^*, u_2^*, \ldots, u_n^*)$ is an equilibrium point in pure strategies if for each $i \leq n$ and $u_i \in m_i$,

$$M_i(\mathbf{u}^*) \geq M_i(\mathbf{u}_i^*, u_i).$$

Equivalently, we have, for each $i \leq n$,

$$M_i(\mathbf{u}^*) = \max_{u_i \in m_i} M_i(\mathbf{u}_i^*, u_i).$$

If the above condition is satisfied, $\mathbf{u}^*$ will be referred to as a pure equilibrium point or PE solution or just PE. For a 2-person zero-sum game a PE solution is the same as a saddle-point. We also call a PE point a solution of the $n$-person game.
4. RANDOM GAMES

It is well-known that PE solutions are rare for 2-person zero-sum games. For example, the probability that a "random" 2-person zero-sum game has a PE solution is

\[
\frac{m_1! \cdot m_2!}{(m_1+m_2-1)!}.
\]

This result exhibits the need for mixed strategies, even if the number of strategies for each player isn't very large in the 2-person zero-sum game.

It is natural to inquire about the need for mixed strategies in arbitrary n-person games. Is it likely that we can get by with pure strategies? To answer this inquiry we analyze "random games."

We define a random n-person game by the following properties:

(i) The n payoffs \( M_i(U) \) are independent random variables.

(ii) For each i, the payoffs \( M_i(U) \) have the same continuous probability distribution.

From the above definition of a random game it follows that the n payoffs are distinct in such a game. Further, the probability that a random n-person game has a PE solution is now well-defined.

Let \( E(U) \) be the event that \( U \) is a PE solution of the game. Define the following probabilities
\[ S_1 = \sum_j \Pr\{E(U^j)\}, \quad S_2 = \sum_{j,k} \Pr\{E(U^j)E(U^k)\}, \]
\[ S_3 = \sum_{j,k,i} \Pr\{E(U^j)E(U^k)E(U^i)\}, \ldots \]

Let \( P_n(m_1, m_2, \ldots, m_n) \) be the probability that a random \( n \)-person game, where the \( n \) players have \( m_1, m_2, \ldots, m_n \) strategies, respectively, has at least one PE solution. Then

\[ P_n(m_1, m_2, \ldots, m_n) = \Pr\{ \sum_U E(U) \} \]

Then by the so-called method of inclusion and exclusion

\[ P_n(m_1, m_2, \ldots, m_n) = \sum_{t=1} (-1)^{t+1} S_t. \]

Since the events are equally-likely, we have that

\[ S_t = \frac{N_t}{t}, \]

where \( N_t \) is the cardinality of the family of all sets which have \( t \) equilibrium points, or

\[ (3) \quad P_n(m_1, m_2, \ldots, m_n) = \sum_{t=1} (-1)^{t+1} N_t \cdots t. \]
5. EXISTENCE OF t EQUILIBRIUM POINTS

In order to determine $N_t$ we shall derive a condition that a game have $t$ equilibrium points. Our definition of equilibrium point and random game yields the following.

Theorem 1. A necessary and sufficient condition that $U^1, U^2, \ldots, U^t$ are $t$ equilibrium points of an $n$-person game is that

$$U^1_i, U^2_i, \ldots, U^t_i$$

are distinct for each $i \leq n$.

Proof: Suppose

$$U^1_i = U^2_i.$$

Then since $U^1$ and $U^2$ are equilibrium points

$$M_i(U^1) = \max_{u_i \in M_i} M_i(U^1_i, u_i)$$

$$= \max_{u_i \in M_i} M_i(U^2_i, u_i) = M_i(U^2),$$

contradicting the implication that all $n$ payoffs are distinct.

Since the $U$'s are n-vectors and the $U_i$'s are $(n-1)$-vectors, the theorem states that each pair of $U$'s must differ in at least two of their $n$-components in order to be PE solutions.
6. EQUILIBRIUM POINTS IN TWO-PERSON GAMES

If \( n = 2 \), a play of the game can be represented by a 2-vector \( U = (a, f) \). In order for \((a_1^1, f_1^1), (a_2^2, f_2^2), \ldots, (a^t, f^t)\) to be \( t \) equilibrium points, then from Theorem 1 it follows that

\[
\begin{align*}
a_1^1, a_2^2, \ldots, a^t & \text{ are distinct} \\
f_1^1, f_2^2, \ldots, f^t & \text{ are distinct.}
\end{align*}
\]

To compute \( N_t \), we observe that \( t \) distinct \( a \)'s can be chosen in \( \binom{m_1}{t} \) ways and \( t \) distinct \( f \)'s can be chosen in \( \binom{m_2}{t} \) ways, and then the two sets can be paired off in \( t! \) ways. Thus

\[
N_t = \binom{m_1}{t} \binom{m_2}{t} t!
\]

and

\[
P_2(m_1, m_2) = \sum_{t=1} \frac{(-1)^{t+1}}{t!(t^t)} (m_1^t) (m_2^t) t! (m_1 m_2)^{-t}.
\]

This result was first obtained by K. Goldberg, A. J. Goldman and M. Newman. They also obtained the asymptotic value of \( P_2(m_1, m_2) \).
7. EQUILIBRIUM POINTS IN THREE PERSON-GAMES

If \( n = 3 \), it is convenient to decompose the set of \( m_1m_2m_3 \) points into \( m_1m_2 \) sets of the form \( S_{ij} \). Each member \( U = (u_1, u_2, u_3) \) of \( S_{ij} \) is such that \( u_1 = i \), \( u_2 = j \), \( u_3 = m_3 \). Thus each set \( S_{ij} \) contains \( m_3 \) points. Now each \( S_{ij} \) can contain at most one equilibrium point. Therefore \( N_t \) can be determined by the following process:

(i) Choose \( t \) sets \( S_{i_1j_1}, S_{i_2j_2}, \ldots, S_{i_tj_t} \) from the \( m_1m_2 \) sets \( S_{ij} \).

(ii) Choose one member from each of these \( t \) sets so that the \( t \) choices are PE points.

Let \( \mu(t) \) be the number of ways of choosing \( t \) PE points satisfying (ii) above—i.e., \( \mu(t) \) is the number of ways of choosing \( t \) equilibrium points from \( t \) given sets \( S_{i_1j_1}, S_{i_2j_2}, \ldots, S_{i_tj_t} \). We have then

\[
N_t = \binom{m_1m_2}{t}\mu(t).
\]

For example, if \( t = 1 \), \( \mu(1) = m_3 \), and

\[
N_1 = (m_1m_2)m_3 = \ldots.
\]

If \( t = 2 \), we have

\[
\mu(2) = m_3^2 \quad \text{if} \quad i_1 \neq i_2, j_1 \neq j_2
\]

\[
\mu(2) = (m_3 - 1)m_3 \quad \text{if} \quad i_1 = i_2 \text{ or } j_1 = j_2
\]
Let \( \mu_t \) represent the number of ways of choosing \( t \) equilibrium points from the \( t \) sets \( S_{i_1 j_1}, S_{i_2 j_2}, \ldots, S_{i_t j_t} \) given that \( t - 1 \) equilibrium points have been chosen from the \( t - 1 \) sets \( S_{i_1 j_1}, S_{i_2 j_2}, \ldots, S_{i_{t-1} j_{t-1}} \). Then we have \( \mu_1 = m_3 \) and

\[
(6) \quad \mu(t) = \mu_t \mu(t-1) = \prod_{k=1}^{t} \mu_k.
\]

It is evident that \( \mu_t \) also represents the number of ways of choosing the \( t \)-th PE from the set \( S_{i_t j_t} \) given that \( (t-1) \) PE points have been chosen from the \( (t-1) \) sets \( S_{i_1 j_1}, S_{i_2 j_2}, \ldots, S_{i_{t-1} j_{t-1}} \). Hence \( \mu_t \) must be bounded as follows:

\[
(7) \quad m_3 - t + 1 \leq \mu_t \leq m_3.
\]

For example, to compute \( \mu_2 \), we have

\[
\mu_2 = m_3 \quad \text{if} \quad i_1 \neq i_2, \ j_1 \neq j_2
\]

\[
= m_3 - 1 \quad \text{if} \quad i_1 = i_2 \quad \text{or} \quad j_1 = j_2.
\]

Now the weights attached to the first value, \( m_3 \), is \((m_1 - 1)(m_2 - 1)\) while the weights attached to second value is \( m_1 + m_2 - 2 \). Therefore we have
\[ \mu_2 = m_3 - \frac{m_1 + m_2 - 2}{m_1 m_2 - 1} \]

We can now compute

\[ \mu(2) = m_3 \left( \frac{m_1 m_2 m_3 - m_1 - m_2 - m_3 + 2}{m_1 m_2 - 1} \right) \]

\[ = m_3 \left( \frac{r - S + 2}{m_1 m_2 - 1} \right) \]

where \( S = m_1 + m_2 + m_3 \).

Substituting in (5) we get

\[ N_2 = \left( \frac{m_1 m_2}{2} \right) \mu(2) = \frac{r(r - S + 2)}{2} \]

To compute \( \mu_3 \) we need to examine four cases

\[ \mu_3 = m_3 - 2 \quad \text{if } i_1 = i_2 = i_3 \text{ and } j_1, j_2, j_3 \text{ distinct,} \]

\[ \text{or if } j_1 = j_2 = j_3 \text{ and } i_1, i_2, i_3 \text{ distinct} \]

\[ \mu_3 = m_3 - 1 \quad \text{if } i_1 = i_2 \neq i_3, j_1, j_2, j_3 \text{ distinct} \]

\[ \text{or if } j_1 = j_2 \neq j_3, i_1, i_2, i_3 \text{ distinct} \]

\[ \mu_3 = m_3 \quad \text{if } i_1, i_2, i_3 \text{ distinct, } j_1, j_2, j_3 \text{ distinct} \]

\[ \mu_3 = \frac{(m_3 - 1)^2}{m_3} \quad \text{if } i_1 = i_2 \neq i_3 \text{ and } j_1 = j_3 \neq j_2 \]

\[ \text{or if } j_1 = j_2 \neq j_3 \text{ and } i_1 = i_3 \neq i_2 \]
The weights associated with each of the four above values of $\mu_3$ are, respectively

$$\begin{align*}
& (m_1 - 1)(m_1 - 2) + (m_2 - 1)(m_2 - 2) \\
& 2(m_1 - 1)(m_2 - 1)^2 + 2(m_1 - 1)^2(m_2 - 1) \\
& (m_1 - 1)(m_2 - 1)(m_1m_2 - m_1 - m_2) \\
& 2(m_1 - 1)(m_2 - 1)
\end{align*}$$

The sum of the above weights is $(m_1m_2 - 1)(m_1m_2 - 2)$.

Using the above weights and values of $\mu_3$ we obtain an average value of $\mu_3$ as a function of $m_1$, $m_2$, $m_3$. In particular, if $m_1 = m_2 = m_3 = m$, this average value is

$$\mu_3 = m - \frac{2(2m^3 - 5m + 1)}{m(m + 1)(m^2 - 2)}.$$

In terms of the above value of $\mu_3$, we can now evaluate

$$N_3 = (\begin{pmatrix} m_1m_2 \\ 3 \end{pmatrix})\mu(3)$$

$$= \frac{m_1m_2 - 2}{3} N_2 \mu_3$$

$$= \frac{(m_1 - 2m_3)(m_1 - S + 2)}{6m_3} \mu_3.$$
In a similar manner we can compute recursively the values of \( N_t \) and then compute the required probability

\[
P_3(m_1, m_2, m_3) = \sum_{t=1}^{\infty} (-1)^{t+1} N_t,^{-t}
\]

(8)

\[
= \sum_{t=1}^{\infty} (-1)^{t+1} \frac{m_1 m_2}{m_3} \frac{\mu_k}{t^{k+1}}
\]

It is of interest to determine the asymptotic value of \( P_3(m_1, m_2, m_3) \) as the number of strategies increase for each player. We note that the absolute value of the \( t \)-th term of the series for \( P_3 \) is

\[
N_{t{-}}^{-t} = \left( \frac{m_1 m_2}{m_3} \right)^t \frac{\mu_k}{t^{k+1} m_1 m_2 m_3}
\]

where

\[
m_3 - k+1 = \mu_k = m_3 \quad \text{if} \quad k = m_3
\]

Hence we have for \( k = t - m_1 m_2 \)

\[
(1 - \frac{k-1}{m_1 m_2})(1 - \frac{k-1}{m_3}) = \frac{(m_1 m_2 - k+1)\mu_k}{m_1 m_2 m_3} = (1 - \frac{k-1}{m_1 m_2})
\]

Thus
From the above inequality, it follows that

\[
\lim_{k=1} \frac{t}{\prod_{m_1, m_2, m_3} (1 - \frac{k}{m_1}) (1 - \frac{k}{m_2})} \leq \lim_{k=1} \frac{t}{\prod_{m_1, m_2, m_3} \frac{(m_1 m_2 - k + 1) \mu_k}{m_1 m_2 m_3}} \leq \lim_{k=1} \frac{t}{\prod_{m_1, m_2} (1 - \frac{k}{m_1})}.
\]

or

\[
\lim_{m_1, m_2, m_3} N_{t,-t} = \frac{1}{t^r}.
\]

Hence we get the asymptotic value of the probability

\[
\lim_{m_1, m_2, m_3} P_3(m_1, m_2, m_3) = \sum_{t=1}^{\infty} \left( \frac{-1}{t^r} \right)^{t+1} = 1 - e^{-1}.
\]
8. PURE EQUILIBRIUM POINTS IN n-PERSON GAMES

We now evaluate the probability of a PE solution in a random n-person game, where the i-th player has \( m_i \) strategies. In such a game the set of \( = m_1m_2 \ldots m_n \) points can be decomposed into \( m_1m_2 \ldots m_{n-1} = M \) sets of the form \( S_{i_1i_2 \ldots i_{n-1}} \) where each set contains \( m_n \) points.

Each member \( U = (u_1, u_2, \ldots, u_n) \) of \( S_{i_1i_2 \ldots i_{n-1}} \) is such that \( u_1 = i_1, u_2 = i_2, \ldots, u_{n-1} = i_{n-1} \), and \( u_n = m_n = m \). Thus each set contains \( m \) points.

From Theorem 1 it follows that each set \( S_{i_1i_2 \ldots i_{n-1}} \) can contain at most one PE point. Therefore choosing \( t \) equilibrium points from the points is equivalent to choosing \( t \) of the \( M \) sets and then choosing one point from each of these \( t \) chosen sets. Again, let \( \mu(t) \) be the number of ways of choosing \( t \) equilibrium points from the \( t \) chosen sets (we emphasize that only one point may be chosen from each set). Then, we have

\[
N_t = \binom{M}{t} \mu(t)
\]

As in the previous section let \( \mu_t \) be the number of ways of choosing \( t \) equilibrium points from the \( t \) sets

\[
S_{i_1i_2 \ldots i_{n-1}} = T_1, S_{j_1j_2 \ldots j_{n-1}} = T_2, \ldots, S_{t_1t_2 \ldots t_{n-1}} = T_t.
\]
given that \( t - 1 \) equilibrium points have been chosen from the \( t - 1 \) sets \( T_1, T_2, \ldots, T_{t-1} \). It follows that

\[
(9) \quad \mu(t) = \mu_t \mu(t-1) = \prod_{k=1}^{t} \mu_k.
\]

In making the above choices we need to choose the \( t \)-th PE point from the set \( T_t \) which contains \( m \) points. We thus have the following inequality

\[
(10) \quad m - t + 1 = \mu_t = m.
\]

The required probability of a PE point in the random game is given by

\[
P_n(m_1, m_2, \ldots, m_n) = \sum_{t=1}^{N} (-1)^{t+1} N_t \cdot \mu_t = \sum_{t=1}^{N} (-1)^{t+1} \binom{M}{t} (M-n)^{-t} \prod_{k=1}^{t} \mu_k.
\]

We may write this probability as

\[
(11) \quad P_n(M, m) = \sum_{t=1}^{M} (-1)^{t+1} \binom{M}{t} M^{-t} \prod_{k=1}^{t} \frac{\mu_k}{m}.
\]

where \( \mu_k \) is a function of \( m_1, m_2, \ldots, m_{m-1}, m \).

For each \( M \) and \( m \) we can compute \( P_n(M, m) \) by first computing \( \frac{\mu_k}{m} \) where \( k = t \). From the definition of \( \mu_t \)
we have \( \frac{\mu_1}{m} = 1 \). In order to get \( \frac{\mu_2}{m} \) we note that

\[
\mu_2 = m \quad \text{if } i_1 \neq j_1, i_2 \neq j_2, \ldots, i_{n-1} \neq j_{n-1}
\]

\[
\mu_2 = m - 1 \quad \text{if } i_1 = j_1, \text{ or } i_2 = j_2, \ldots, \text{ or } i_{n-1} = j_{n-1}.
\]

The weight associated with \( \mu_2 = m \) is

\[
(m_1 - 1)(m_2 - 1) \ldots (m_{n-1} - 1) = D.
\]

The weight associated with \( \mu_2 = m - 1 \), is \( m - D - 1 \). We thus get

\[
\mu_2 = m - \frac{n - D - 1}{M - 1},
\]

and

\[
\frac{\mu_2}{m} = 1 - \frac{n - D - 1}{m(M - 1)} = 1 - \frac{mM - D - 1}{mM - m}.
\]

In a similar manner we can compute \( \frac{\mu_3}{m}, \frac{\mu_4}{m}, \ldots, \frac{\mu_k}{m} \) and then obtain \( P_n \). Of course, the computation of \( \frac{\mu_k}{m} \) becomes more cumbersome with each value of \( k \). However, \( P_n \) has an asymptotic value given by

**Theorem 2.** For all \( n \)-person games \((n \geq 2)\)

\[
\lim_{M \to \infty} P_n(m_1, m_2, \ldots, m) = 1 - e^{-1}.
\]

**Proof:** Equation (11) can be written as
\[
P_n(M, m) = \sum_{t=1}^{t-l} (-1)^{t+1} \frac{t-1}{t!} \left(1 - \frac{i}{M}\right)^{t-k} \frac{\mu_k}{m}.
\]

Hence we have

\[
(12) \quad P_n(M, m) - (1 - e^{-1}) = \sum_{t=0}^{t-l} (-1)^{t} \left[ 1 - \sum_{i=1}^{t-1} \frac{i}{M} \frac{\mu_k}{m} \right].
\]

Now let

\[
t(M, m) = \prod_{i=1}^{t-1} \left(1 - \frac{i}{M}\right)^{t-k} \frac{\mu_k}{m}.
\]

It is clear from (10) that for all \(t\),

\[
0 \leq t(M, m) \leq 1.
\]

Now for all \(i \leq T \leq M\) we have

\[
1 \geq 1 - \frac{i}{M} > 1 - \frac{T}{M}.
\]

Hence

\[
\prod_{i=1}^{t-1} \left(1 - \frac{i}{M}\right) > \left(1 - \frac{T}{M}\right)^{t-1} > \left(1 - \frac{T}{M}\right)^T \quad \text{for} \quad t \leq T \leq M.
\]

We also have from (10) that
\[ \frac{\mu_k}{m} \geq 1 - \frac{k-1}{m} > 1 - \frac{t}{m} > 1 - \frac{T}{m} \text{ for } k < t < T. \]

Therefore

\[ \sum_{k=1}^{\infty} \frac{\mu_k}{m} > (1 - \frac{T}{m})^T \text{ for } t < T < m. \]

Substituting the above inequalities in the definition of \( \hat{\tau}(M,m) \) we have

\[ \hat{\tau}(M,m) \geq (1 - \frac{T}{M}) (1 - \frac{T}{m}) \text{ for } t < T < M \]

and \( t < T < m \)

\[ \geq (1 - \frac{T^2}{M}) (1 - \frac{T^2}{m}) \]

\[ \geq (1 - \frac{T^2}{M} - \frac{T^2}{m}) \]

Now \( T \) is arbitrary but \( T < M \) and \( T < m \). Suppose we restrict \( T \) so that \( T^3 < M \), and \( T^3 < m \), then \( \frac{T^2}{M} < \frac{1}{T} \) and \( \frac{T^2}{m} < \frac{1}{T} \), and we obtain the inequality

\[ 1 \geq \hat{\tau}(M,m) \geq 1 - \frac{2}{T} \text{ for } t < T < T^3 < M \]

and \( t < T < T^3 < m \).

Or
\[ 0 \leq 1 - \lambda_t(M,m) < \frac{2}{T} \text{ for } t < T < T^3 < M \]
and \( t < T < T^3 < m \).

Returning to (12) we have

\[
\left| P_n(M,m) - (1 - e^{-1}) \right| \leq \sum_{t=0}^{T} \frac{1}{t!} \left( \frac{2}{T} \right) + \sum_{t>T} \frac{(-1)^t}{t!} \left| 1 - \lambda_t(M,m) \right| \\
\leq \frac{2}{T} e + \sum_{t>T} \frac{(-1)^t}{t!} \left| 1 - \lambda_t(M,m) \right|.
\]

The second term represents the "tail" of a converging alternating series, hence there exists \( \varepsilon > 0 \) such that

\[
\sum_{t>T} \frac{(-1)^t}{t!} \left| 1 - \lambda_t(M,m) \right| < \varepsilon
\]
giving us

\[
\left| P_n(M,m) - (1 - e^{-1}) \right| < \frac{2}{T} e + \varepsilon.
\]

Let \( T > \frac{2\varepsilon}{e} \), then

\[
\left| P_n(M,m) - (1 - e^{-1}) \right| < 2\varepsilon,
\]

which proves the theorem. It is of interest to note that Theorem 2 requires only that two of the \( n \) players sets of strategies be infinite.
REFERENCES


A random n-person noncooperative game—that game that prohibits communication and therefore coalitions among n players—is shown to have a pure strategy solution with a high probability. A solution of a game is an equilibrium point or set of strategies, one for each player, such that if n - 1 players use their equilibrium strategies, then the n-th player has no reason to deviate from his equilibrium strategy. It is shown that the probability of a solution in pure strategies for large random games converges to $1 - \frac{1}{e}$ for all n greater than or equal to 2.