Relationships Among Some Notions of Bivariate Dependence

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RELATIONSHIPS AMONG SOME NOTIONS OF BIVARIATE DEPENDENCE

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ABSTRACT

A random variable $T$ is left tail decreasing in a random variable $S$ if $P[T \leq t \mid S \leq s]$ is non-increasing in $s$ for all $t$, and right tail increasing in $S$ if $P[T > t \mid S > s]$ is non-decreasing in $s$ for all $t$. We show that either of these conditions implies that $S,T$ are associated, i.e. $\text{Cov}(f(S,T), g(S,T)) \geq 0$ for all pairs of functions $f,g$ which are non-decreasing in each argument. No two of these conditions for bivariate dependence are equivalent. Applications of these and other conditions for dependence in probability, statistics, and reliability theory are considered in Lehmann (1966) *Ann. Math. Statist.* and Esary, Proschan, and Walkup (1966) Boeing documents D1-82-0567, D1-82-0578.
1. **Introduction**

Lehmann ([2], 1966) defines two random variables $S, T$ to be *positively quadrant dependent* if

$$P[S \leq s, T \leq t] \geq P[S \leq s] \cdot P[T \leq t]$$

for all $s, t$; and $T$ to be *positively regression dependent on $S$* if $P[T \leq t \mid S = s]$ is non-increasing in $s$ for all $t$ (with reference for the latter definition to Tukey, 1958, [3]). Esary, Proschan, and Walkup ([1], 1966) define $S, T$ to be *associated* if

$$\text{Cov}[f(S,T), g(S,T)] \geq 0$$

for all pairs of functions $f, g$ which are non-decreasing in each argument, and such that $E[f(S,T), g(S,T)]$ exist. Lehmann also mentions the type of dependence characterized by

$$(1.1) \quad P[T \leq t \mid S \leq s] \text{ is non-increasing in } s \text{ for all } t.$$ 

If condition (1.1) holds, we say that $T$ is *left tail decreasing in $S$*. A condition similar to (1.1) is

$$(1.2) \quad P[T > t \mid S > s] \text{ is non-decreasing in } s \text{ for all } t.$$ 

If condition (1.2) holds, we say that $T$ is *right tail increasing in $S$*.

Among $T$ *positively regression dependent on $S$* (we write $\text{PRD}(T \mid S)$), $T$ *left tail decreasing in $S$* ($\text{LTD}(T \mid S)$), and $S, T$ *positively*
quadrant dependent \((\text{PQD}(S,T))\) the implications

\[
\text{PRD}(T|S) \Rightarrow \text{LTD}(T|S) \Rightarrow \text{PQD}(S,T)
\]

hold. The implications are strict, i.e., no two of the conditions are equivalent \([2]\). Among \(\text{PRD}(T|S), S, T\) associated \((\text{A}(S,T))\), and \(\text{PQD}(S,T)\) the strict implications

\[
\text{PRD}(T|S) \Rightarrow \text{A}(S,T) \Rightarrow \text{PQD}(S,T)
\]

hold \([1]\).

In this note we study the unresolved relationships in this set of conditions for bivariate dependence, particularly the relationship of \(\text{LTD}(T|S)\) and \(\text{RTI}(T|S)\) to \(\text{A}(S,T)\), and extend the structure of strict implications to

\[
\text{PRD}(T|S) \Rightarrow \begin{cases} 
\text{LTD}(T|S) \\
\text{RTI}(T|S)
\end{cases} \Rightarrow \text{A}(S,T) \Rightarrow \text{PQD}(S,T).
\]

2. LTD, RTI, and PRD.

Condition \((1.1, \text{LTD}(T|S))\) can be restated as \(P[T > t \mid S < s]\) is non-decreasing in \(s\) for all \(t\). Then by elementary manipulation condition \((1.1)\) is equivalent to

\[(2.1) \quad P[T > t \mid S < s_1] \leq P[T > t \mid s_1 < S < s_2] \quad \text{for all} \quad t \quad \text{and} \quad s_1 < s_2.\]

Condition \((1.2, \text{RTI}(T|S))\) is equivalent to

\[(2.2) \quad P[T > t \mid s_1 < S < s_2] \leq P[T > t \mid s_2 < S] \quad \text{for all} \quad t \quad \text{and} \quad s_1 < s_2.\]
These expressions give a convenient way of viewing the joint condition
\([\text{LTD}(T|S) \text{ and } \text{RTI}(T|S))]\).

Using conditions (2.1) and (2.2) it is immediate that
\(\text{PRD}(T|S) \Rightarrow [\text{LTD}(T|S) \text{ and } \text{RTI}(T|S)]\), since for any interval \(I\)
\[
\Pr[T > t | S \in I] = \int_{s \in I} \Pr[T > t | S = s] dP[S = s] / P[S \in I].
\]
(cf. [2]).

It is known (e.g. see [1]) that all of the conditions for bivariate
dependence considered in this note are equivalent for \(2 \times 2\) distributions
(we say that \(S, T\) have an \(n \times m\) distribution if \(S\) has \(n\) values,
\(T\) has \(m\) values). To show that the implication \(\text{PRD}(T|S) \Rightarrow \text{LTD}(T|S)\) is
strict Lehmann uses a \(3 \times 3\) example. To show that the implication
\(\text{PRD}(T|S) \Rightarrow [\text{LTD}(T|S) \text{ and } \text{RTI}(T|S)]\) is strict we use a \(4 \times 2\) example,
since \(\text{PRD}(T|S) \Rightarrow [\text{LTD}(T|S) \text{ and } \text{RTI}(T|S)]\) for any \(3 \times m\) distribution
by conditions (2.1) and (2.2). We let \(S\) take values \(s_1 < s_2 < s_3 < s_4\),
each with probability \(1/4\). We let \(T\) take values \(t_1 < t_2\), with
\[
\Pr[T = t_2 | S = s_4] = p_4. \quad \text{If } p_1 = .4, p_2 = .6, p_3 = .5, p_4 = .7, \text{ we}
\text{have } [\text{LTD}(T|S) \text{ and } \text{RTI}(T|S)] \text{ but not } \text{PRD}(T|S).
\]

If in the example above \(p_1 = .4, p_2 = .6, p_3 = .5, p_4 = .5\), we
have \(\text{LTD}(T|S)\) but not \(\text{RTI}(T|S)\). If \(p_1 = .5, p_2 = .5, p_3 = .4,\)
\(p_4 = .6\), we have \(\text{RTI}(T|S)\) but not \(\text{LTD}(T|S)\).

3. LTD, RTI, and A.

By elementary manipulation condition (2.1, \(\text{LTD}(T|S)\)) is equivalent
to

(3.1) \( P[t < T, S \leq s_1] \cdot P[T \leq t, S \leq s_2] \leq P[T \leq t, S \leq s_1] \cdot P[t < T, s_1 < S \leq s_2] \)

for all \( t \) and \( s_1 < s_2 \).

Condition (2.2, \( RTI{T|S} \)) is equivalent to

(3.2) \( P[t < T, s_1 < S \leq s_2] \cdot P[T \leq t, s_2 < S] \leq P[T \leq t, s_1 < S \leq s_2] \cdot P[t < T, s_2 < S] \)

for all \( t \) and \( s_1 < s_2 \).

In [1] it is shown that association \( (A(S,T)) \) is equivalent to

(3.3) \( P[\gamma(S,T) = 1, \delta(S,T) = 0] \cdot P[\gamma(S,T) = 0, \delta(S,T) = 1] \)

\( \leq P[\gamma(S,T) = 0, \delta(S,T) = 0] \cdot P[\gamma(S,T) = 1, \delta(S,T) = 1] \)

for all pairs \( \gamma, \delta \) of binary functions which are non-decreasing in each argument.

A function is binary if it takes only the values 0 and 1.

We consider the 3 \times 3 distribution

\[
\begin{array}{ccc}
T & = & t_3 \\
& & p_{13} & 0 & 1/4 \\
T & = & t_2 \\
& & 0 & 1/4 & 0 \\
T & = & t_1 \\
& & 1/4 & 0 & p_{31} \\
S & = & s_1 & S & = & s_2 & S & = & s_3
\end{array}
\]

where \( s_1 < s_2 < s_3 \) and \( t_1 < t_2 < t_3 \). If \( p_{13} = p_{31} = 1/8 \), we have \( A(S,T) \) but neither \( LTD(T|S) \) nor \( RTI(T|S) \) (cf. [1]). If \( p_{13} = 0 \), \( p_{31} = 1/4 \), we have \( LTD(T|S) \) but not \( RTI(T|S) \). If \( p_{13} = 1/4, p_{31} = 0 \), we have \( RTI(T|S) \) but not \( LTD(T|S) \) (cf. [2]).
We now proceed toward a proof of the implication $RTI(T|S) \Rightarrow A(S,T)$. Once this is accomplished, the implication $LTD(T|S) \Rightarrow A(S,T)$ follows, since $LTD(T|S) \Leftrightarrow RTI(-T|-S) \Rightarrow A(-S,-T) \Leftrightarrow A(S,T)$.

Given random variables $S, T$ we choose fixed $s_1 < s_2 < \cdots < s_n$ and $t_1 < t_2 < \cdots < t_m$. We define discrete random variables $S^*, T^*$ by

$$S^* = 0 \text{ if } S \leq s_1 \quad T^* = 0 \text{ if } T \leq t_1$$
$$1 \text{ if } s_1 < S < s_2 \quad 1 \text{ if } t_1 < T < t_2$$
$$\vdots \quad \vdots$$
$$n \text{ if } s_n < S \quad m \text{ if } t_m < T.$$

It is shown in [1] that $A(S,T)$ is equivalent to $A(S^*,T^*)$ for all choices of $n, m$ and $s_1, \ldots, s_n, t_1, \ldots, t_m$. It is clear that $RTI(T|S) \Rightarrow RTI(T^*|S^*)$. Thus we only need to show that $RTI(T^*|S^*) \Rightarrow A(S^*,T^*)$.

Justified by the preceding remarks, we assume from now on that $S$ is discrete with the values $0, 1, \ldots, n$ and that $T$ is discrete with the values $0, 1, \ldots, m$. Also from now on we make the convention that $\gamma, \delta$ are binary, non-decreasing functions of $s = 0, 1, \ldots, n$ and $t = 0, 1, \ldots, m$.

We say that $(s_0, t_0)$ is a boundary point of $\{\gamma = 0\} = \{(s,t) \mid \gamma(s,t) = 0\}$ if $\gamma(s_0, t_0) = 0$ and $\gamma(s_0+1, t_0+1) = 1$.

Lemma 1.

Let $(s_2, t_2)$ be a boundary point of both $\{\gamma = 0\}$ and $\{\delta = 0\}$. Then $RTI(T|S)$ implies
(3.4) \[ P[\gamma(S,T) \neq \delta(S,T), s_1 < S \leq s_2] \cdot P[\gamma(S,T) \neq \delta(S,T), s_2 < S] \]
\[ \leq P[\gamma(S,T) = 0, \delta(S,T) = 0, s_1 < S \leq s_2] \cdot P[\gamma(S,T) = 1, \delta(S,T) = 1, s_2 < S] \]
for all \( s_1 < s_2 \).

Proof.

Since \( \gamma(s_2, t_2) = \delta(s_2, t_2) = 0 \), \( \gamma(s, t) = \delta(s, t) = 0 \) for all \( s \leq s_2 \), \( t \leq t_2 \). Since \( \gamma(s_2 + 1, t_2 + 1) = \delta(s_2 + 1, t_2 + 1) = 1 \), \( \gamma(s, t) = \delta(s, t) = 1 \) for all \( s_2 < s \), \( t_2 < t \). Thus \( \{\gamma(s, t) \neq \delta(s, t), s_1 < s \leq s_2\} \subset \{s_1 < s \leq s_2, t \leq t_2\} \).

Also \( \{\gamma(s, t) = 0, \delta(s, t) = 0, s_1 < s \leq s_2\} \supset \{s_1 < s \leq s_2, t \leq t_2\} \) and \( \{\gamma(s, t) = 1, \delta(s, t) = 1, s_2 < s\} \supset \{s_2 < s, t_2 < t\} \). Inequality (3.4) follows from condition (3.2, RTP(T|S)).

For fixed \( s \) either (a) \( \gamma(s, t) \geq \delta(s, t) \) for all \( t \), or (b) \( \gamma(s, t) \leq \delta(s, t) \) for all \( t \). It is clear that we can find an alternating partition of \([0,n]\), i.e. a partition of \([0,n]\) into intervals \( I_1, I_2, \ldots, I_k \) such that either (a) holds for all \( s \in I_j \), or (b) holds for all \( s \in I_j \), \( j = 1, \ldots, k \), and such that if (a) holds on \( I_j \) (or (b) holds on \( I_j \)), then (b) holds on \( I_{j+1} \) ((a) holds on \( I_{j+1} \), \( j = 1, \ldots, k - 1 \).

Lemma 2.

Let \( I_1, I_2, \ldots, I_k \) be an alternating partition of \([0,n]\). Let \( s_j = \max\{s | s \in I_j\} \), \( t_j = \max\{t | \gamma(s_j, t) = \delta(s_j, t) = 0\} \). Then the points \( (s_j, t_j), j = 1, \ldots, k - 1 \), are boundary points of both \( \{\gamma = 0\} \) and \( \{\delta = 0\} \).
Proof.

Suppose, to fix a case, that (a) holds on $I_j$. Then
\[ \gamma(s_j, t_j) = 0, \delta(s_j, t_j) = 0 \text{ and } \gamma(s_j, t_j+1) = 1 \]
from the definition of $s_j$ and $t_j$. Since \( \gamma(s_j+1, t_j+1) \geq \gamma(s_j, t_j+1), \gamma(s_j+1, t_j+1) = 1 \).

Then since (b) holds on $I_{j+1}$, \( \delta(s_{j+1}, t_{j+1}) \geq \gamma(s_{j+1}, t_{j+1}) \), so that finally \( \gamma(s_{j+1}, t_{j+1}) = 1, \delta(s_{j+1}, t_{j+1}) = 1 \).\]

Theorem.

RTI($T|S$) implies A($S,T$).

Proof.

With reference to condition (3.3, A($S,T$)) let $p_{ij} = P[\gamma(S,T) = i, \delta(S,T) = j], i,j = 0,1$. Let $I_1, I_2, ..., I_k$ be an alternating partition of $[0,n]$. Let $a_j = P[\gamma(S,T) \neq \delta(S,T), s \in I_j]$, $b_j = P[\gamma(S,T) = 0, \delta(S,T) = 0, s \in I_j]$, and $c_j = P[\gamma(S,T) = 1, \delta(S,T) = 1, s \in I_j], j = 1, ..., k$. In view of Lemma 2 we can apply Lemma 1 (with the interval $(s_1, s_2)$ of Lemma 1 taken to be $I_j$) to obtain
\[ a_j(a_{j+1} + ... + a_k) \leq b_j(c_{j+1} + ... + c_k) \quad j = 1, ..., k - 1. \]

Now $p_{10} = \sum_{j=1}^{k} e_j a_j$, $p_{01} = \sum_{j=1}^{k} (1 - e_j) a_j$, where $e_j = 1$ if \( \gamma \geq \delta \) on $I_j$, $e_j = 0$ if \( \gamma \leq \delta \) on $I_j$. Also $p_{00} = \sum_{j=1}^{k} b_j$ and $p_{11} = \sum_{j=1}^{k} c_j$.

Then
\[ p_{10}p_{01} \leq \sum_{i<j} a_i a_j \leq \sum_{i<j} b_i c_j \leq p_{00}p_{11}. \]

Thus condition (3.3, A($T|S$)) is verified.||
REFERENCES

