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ESTIMATING THE MEAN AND STANDARD DEVIATION FROM A BRUCETON STATISTICAL ANALYSIS

LEONARD SHAINHEIT

AUGUST 1966

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BY

LEONARD SHAINHEIT

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SUMMARY

The maximum likelihood equations used to determine estimates of the mean and standard deviation for the Bruceton method of statistical analysis were solved numerically. A digital computer algorithm was developed to reduce the time associated with the numerical calculations. The method of successive approximations (Newton-Raphson method) was suitable. Test data was evaluated to determine the relative difference between values of the means and standard deviation as obtained from the numerical solution of the maximum likelihood equations and as obtained from formulae approximating these equations. Results of the analysis showed that the differences between means were slight while those between standard deviations were more pronounced. Calculated high and low probability levels, because of their dependence on the mean and standard deviation, differed widely from one method of calculation to the other. The analysis also showed that the most adverse effects occurred for the smallest samples. Because the maximum likelihood equations yield better estimates of the mean and standard deviation than can be obtained from equations approximating them -- especially for small samples -- solution of the maximum likelihood equations by the Newton-Raphson method is advocated for determining the best estimates of the population mean and standard deviation.
CONCLUSIONS

The maximum likelihood equations used to estimate the mean and standard deviation in conjunction with a Bruceton analysis are readily solvable by the Newton-Raphson method of successive approximations with the aid of the digital computer codes of Appendix A and B.

The code of Appendix B appears to be superior based on higher reliability and safety criteria.

RECOMMENDATION

Because of its apparent superiority, the code of Appendix B should be used to calculate the maximum likelihood estimates of the mean and standard deviation for samples analyzed by the Bruceton method.
The Bruceton method of statistical analysis is used to determine the probability of explosion or non-explosion for a given explosive item as a function of input stimuli. The procedures used for calculating the statistical parameters (mean, standard deviation, percent points, etc.) are based on approximations to the maximum likelihood equations which will be discussed at greater length in the Study section. However, the validity of the approximations fall off rapidly with decrease in sample sizes less than 100. Consequently, it is desirable to use the maximum likelihood equations for small sample analysis.

Before the advent of the digital computer, exact solutions of the maximum likelihood equations required complex, wholly impractical computational schemes. As a result of the difficulties formerly encountered in applying the likelihood theory, approximate methods were always used to estimate the maximum likelihood quantities. In the case of small sample analysis, blatant deviations frequently occurred between sample and population statistical parameters. The Franklin Institute of Philadelphia, Pennsylvania investigated the affect of approximate methods on the data obtained from small samples and their findings indicate that the expected deterioration of the approximate formulae as applied to small samples, does in fact occur.

It must be emphasized that in any statistical analysis large samples are preferable. But if small samples must be used in a Bruceton test, the use of maximum likelihood techniques become in a realistic sense, mandatory. Furthermore, the maximum likelihood values associated with small samples should be treated with care, since small samples tend to yield poor statistical results. This is especially true for this particular problem.
STUDY

A method of statistical analysis frequently utilized in evaluating explosive devices is known as the Bruceton Staircase Sensitivity Test. The method was developed by the Explosives Research Laboratory of Bruceton, Pennsylvania. It is used to ascertain the safe and sure-fire energy levels associated with a given explosive item. In performing the experimental procedure the tester is first required to home in on a 50% firing level (mean) and on an approximate standard deviation (interval between consecutive energy levels). This can be accomplished easily if the past history of the explosive under investigation is available. Otherwise trial and error techniques are adequate. Once having chosen the 50% firing level and the step interval (based on an approximation of the standard deviation) the tester may then proceed as follows:

An attempt is made to actuate the explosive using the 50% energy estimate. If successful a second specimen is tested at an energy level one step interval below the 50% value. On the other hand, if the specimen failed to detonate, the second explosive item is tested at an energy level one step interval above the 50% level. Thus denoting any test level as \( H_i \) and the step interval as \( d \), then \( H_{i+1} = H_i - d \), for the case where the \( i \) th test produced an explosion and \( H_{i+1} = H_i + d \) if the \( i \) th test produced a non-explosion.

In this way, a sequence of explosions (xs) and non-explosions (0s) will be generated. Figure 1 shows a typical sequence. It may be noted that the bulk of the testing occurs in the neighborhood of the mean energy level. Thus a minimal number of tests will permit a good estimation of the mean. However, because of the approximate nature of the methods of calculation used to estimate the mean and standard deviation, samples of at least 100 specimens are required. Also, sensitivity data does not consist of exact measurements. Consequently the information contained in them is reduced by about one-half that which would be contained in measured data of a comparable nature.
TYPICAL SEQUENCE OF EXPLOSIONS (X) AND NONEXPLOSIONS (O) GENERATED IN A BRUCETON ANALYSIS

FIGURE 1
To reduce the bias associated with the Bruceton method, large sample sizes (at least 100 specimens) should be used. For very expensive explosive devices, large sample sizes are prohibitive. Consequently, tests are conducted using smaller sample sizes often 50 specimens or less. But it is known that for sample sizes below 50, the approximations of the mean and standard deviation become rather poor. Therefore, more elaborate methods using the maximum likelihood equations for estimating the mean and standard deviation must be used. However, these equations cannot be solved in closed form by any direct method (Equation 47, p. 38 of Reference 4). Of course, numerically the equations can be solved and the maximum likelihood estimates of the mean and standard deviation determined.

Owing to the time involved in the numerical solution of the maximum likelihood equations, no attempt was made to utilize them until the advent of the digital computer. However, once the computer algorithm developed in this report was established it became a rather simple task to demonstrate the effect of sample size on the calculated mean and standard deviation especially since it is this parameter which strongly affects the outcome of the extrapolation calculations. That is, the validity of the approximation formulae for estimating high and low percent points is doubtful -- especially if small sample sizes are used. Therefore, the purpose of this report is to present a numerical procedure for solving the maximum likelihood equations and suitable for application to a digital computer. The effect of sample size on the results of both the approximate and the numerical techniques will be emphasized.

To determine the maximum likelihood equations for estimating the mean and standard deviation of a normally distributed random variable \( x \) with mean \( \mu \) and variance \( \sigma^2 \), a random sample of size \( N \) is drawn from the population and tested in the manner described so that a sequence of ones and zeros is obtained. If:

- \( N_1 = \) total number of ones
- \( N_0 = \) total number of zeros
- \( H_i = \) energy at test level \( i \)
- \( N_i = \) number of ones at \( H_i \)
- \( M_i = \) number of zeros at \( H_i \)
then
\[ N_i = \sum N_i \]
\[ N_0 = \sum M_i \]
and
\[ N = N_0 + N_1 \]

Further, it may be assumed \( N_1 < N_0 \), then the probability that an \( X \) will be a one at the \( H_i \) level is:

\[ P_i = \frac{1}{\sqrt{2\pi s}} \int_{-\infty}^{H_i} e^{-\frac{(x-A)^2}{2s^2}} \, dx \]

and the probability that \( X \) will be a zero is: \( Q_i = 1 - P_i \).

Then, the probability of generating a sequence of \( N_1 \) ones and \( N_0 \) zeros is obtained from:

\[ P = K \int \prod_{i} P_i^{N_i} Q_i^{M_i} \]

where \( K \) is a function of the \( N_i \) and \( M_i \). This probability is based on generating a sequence of \( N \) independent outcomes (ones and zeros) from a multi-nomial distribution. Although this assumption is not strictly applicable to the current problem, only slight adjustments need be made to the general theory to apply it in the asymptotic case. This is denoted by the likelihood:

\[ L = \sum N_i \log P_i Q_i^{-1} \]

and differentiating with respect to \( A \) and \( S \) maximized likelihood equations are obtained. Before proceeding further the following derivatives are defined:

\[ \frac{\partial P_i}{\partial A} = \frac{1}{s} Z_i \]
\[ \frac{\partial P_i}{\partial S} = -\frac{H_i - A}{s^2} Z_i \]
\[ \frac{\partial Q_i}{\partial A} = \frac{1}{S} Z_i \]

and

\[ \frac{\partial Q_i}{\partial S} = \frac{H_i - A}{S^2} Z_i \]

where

\[ Z_i = \frac{1}{\sqrt{2\pi S}} e^{-\frac{(H_i - A)^2}{2S^2}} \]

Then, the maximum likelihood equations for estimating the mean and standard deviation are:

\[ \frac{\partial L}{\partial A} = \frac{1}{S} \sum_i N_i \left[ \frac{Z_i - 1}{Q_{i-1}} \cdot \frac{Z_i}{P_i} \right] = 0 \quad (1) \]

and

\[ \frac{\partial L}{\partial S} = \frac{1}{S^2} \sum_i N_i \left[ \frac{(H_{i-1} - A)Z_{i-1}}{Q_{i-1}} \cdot \frac{(H_i - A)Z_i}{P_i} \right] = 0 \quad (2) \]

These equations cannot be solved by direct methods. However, if good estimates of the mean and standard deviation are available a method of successive approximations (Newton-Raphson method) is applicable. The method follows: Let the maximum likelihood Equation 1, 2.

\[ F_1 (S, A) = 0 \quad (3) \]
\[ F_2 (S, A) = 0 \quad (4) \]

Also, let \( A_0, S_0 \) be approximate solutions of Equation (3) and (4) such that the true roots of Equation (3) and (4), \( A \) and \( S \) are related to \( A_0 \) and \( S_0 \) by:

\[ S = S_0 + H \]
\[ A = A_0 + K \]
where \( H \) and \( K \) are the corrections for \( S_0 \) and \( A_0 \), respectively.

Rewriting Equation (3) and (4):

\[
F_1 (S_0 + H, A_0 + K) = 0
\]

(5)

\[
F_2 (S_0 + H, A_0 + K) = 0
\]

(6)

Equation (5) and (6) may be expanded by Taylor's theorem for a function of two variables giving:

\[
F_1 (S_0 + H, A_0 + K) = 0 = F_1 (S_0, A_0) + H \left( \frac{\partial F_1}{\partial S} \right)_0 + K \left( \frac{\partial F_1}{\partial A} \right)_0 + \text{Terms in Higher Powers of } H, K
\]

(7)

\[
F_2 (S_0 + H, A_0 + K) = 0 = F_2 (S_0, A_0) + H \left( \frac{\partial F_2}{\partial S} \right)_0 + K \left( \frac{\partial F_2}{\partial A} \right)_0 + \text{Terms in Higher Powers of } H, K
\]

(8)

Since the values of \( H \) and \( K \) are small, their squares, products, and higher powers are neglected so that:

\[
F_1 (S_0, A_0) + H \left( \frac{\partial F_1}{\partial S} \right)_0 + K \left( \frac{\partial F_1}{\partial A} \right)_0 = 0
\]

(7)

\[
F_2 (S_0, A_0) + H \left( \frac{\partial F_2}{\partial S} \right)_0 + K \left( \frac{\partial F_2}{\partial A} \right)_0 = 0
\]

(8)

Using Cramer's rule (assuming \( D \neq 0 \)) solution for \( H \) and \( K \) are obtained from:

\[
H_0 = \frac{\text{DET} \begin{bmatrix} -F_1(S_0, A_0) & (\frac{\partial F_1}{\partial A})_0 \\ -F_2(S_0, A_0) & (\frac{\partial F_2}{\partial A})_0 \end{bmatrix}}{D}
\]

(9)
and

\[ K_0 = \frac{\begin{bmatrix} (\frac{\partial F_1}{\partial S})_0 & -F_1(S_0, A_0) \\ (\frac{\partial F_2}{\partial S})_0 & -F_2(S_0, A_0) \end{bmatrix}}{D} \] 

(10)

where

\[ D = \begin{bmatrix} (\frac{\partial F_1}{\partial S})_0 & (\frac{\partial F_1}{\partial A})_0 \\ (\frac{\partial F_2}{\partial S})_0 & (\frac{\partial F_2}{\partial A})_0 \end{bmatrix} \]

Using the values of H and K obtained above new approximations are gotten for S and A as:

\[ S_1 = S_0 + H_0 \]

\[ A_1 = A_0 + K_0 \]

If the process is repeated using \( S_1 \) and \( A_1 \), new corrections, \( H_1 \) and \( K_1 \), are obtained. Continuing in this manner, a predetermined degree of convergence may be achieved. The subscripts 0, 1, \ldots, n, refer to the values \( A, S, K, \) and \( H \) to be used. That is \( \left( \frac{\partial F_1}{\partial S} \right)_0 \)

\[ \frac{\partial F_1}{\partial S} \]

means that \( \frac{\partial F_1}{\partial S} \) is to be evaluated for \( S = S_0, A = A_0 \), and so on.

Also, examination of Equation (7), (8), (9) and (10) shows that for large \( D \) and correspondingly small \( H \) and \( K \), the assumptions made are applicable and convergence should be expected.

In accordance with this method, Equation (1) and (2) must be differentiated partially with respect to \( S \) and \( A \). However, before proceeding the following quantities are needed:
\[ \frac{\partial Z_i}{\partial S} = \left[ \frac{(H_i - A)^2 - S^2}{S^3} \right] z_i \]
\[ \frac{\partial Z_i}{\partial A} = \left( \frac{H_i - A}{S^2} \right) z_i \]
\[ \frac{\partial (\phi_i)}{\partial A} = -\frac{Z_i}{Q_i^2 S} \]
\[ \frac{\partial (\phi_i)}{\partial S} = -\left( \frac{H_i - A}{Q_i S} \right) z_i \]
\[ \frac{\partial (\beta_i)}{\partial A} = \frac{Z_i}{P_i S} \]
\[ \frac{\partial (\beta_i)}{\partial S} = \left( \frac{H_i - A}{P_i S} \right) z_i \]

Then for Equation (1):

\[ \frac{\partial F_i}{\partial S} = \frac{1}{S} \sum_i N_i \left[ \frac{\partial Z_i}{\partial S} \left( \frac{1}{Q_i} \right) + \frac{\partial (\phi_i)}{\partial S} z_i \right] - \frac{1}{S^2} \sum_i N_i \left[ \frac{Z_i}{Q_i} \frac{\partial (\beta_i)}{\partial S} \right] - \frac{1}{S^2} \sum_i N_i \left[ \frac{\partial Z_i}{\partial (\beta_i)} \frac{Z_i}{P_i} \right] \]

and

\[ \frac{\partial F_i}{\partial A} = \frac{1}{S} \sum_i N_i \left[ \frac{\partial Z_i}{\partial A} \left( \frac{1}{Q_i} \right) + \frac{\partial (\phi_i)}{\partial A} z_i \right] - \frac{1}{S^2} \sum_i N_i \left[ \frac{Z_i}{Q_i} \frac{\partial (\beta_i)}{\partial A} \right] - \frac{1}{S^2} \sum_i N_i \left[ \frac{\partial Z_i}{\partial (\beta_i)} \frac{Z_i}{P_i} \right] \]
Repeating the preceding for Equation (2) the following results are obtained:

\[
\frac{\partial^2 f^2}{\partial S^2} = \frac{1}{S^2} \sum_i N_i \left[ \frac{(H_i - A)}{Q_{i-1}} \frac{\partial Z_{i-1}}{\partial S} + (H_i - A) \frac{\partial \left( \frac{1}{P_i} \right)}{\partial S} Z_{i-1} - \left( \frac{H_i}{P_i} \right) \frac{Z_{i-1}}{Q_{i-1}} - \frac{(H_i - A)}{Q_i} \frac{\partial Z_i}{\partial S} \right] - \frac{Z_{i-1}}{Q_{i-1}} \frac{\partial A}{\partial S} - A \frac{\partial Z_{i-1}}{\partial S} + \frac{Z_{i-1}}{P_i} + AZ_i - \frac{H_i}{P_i} \frac{\partial Z_i}{\partial S} - \left( \frac{H_i - A}{P_i} \right) \frac{\partial Z_i}{\partial A} \right].
\]

and

\[
\frac{\partial^2 f^2}{\partial A^2} = \frac{1}{S^2} \sum_i N_i \left[ \frac{(H_i - A)}{Q_{i-1}} \frac{\partial Z_{i-1}}{\partial A} + (H_i - A) \frac{\partial \left( \frac{1}{P_i} \right)}{\partial A} Z_{i-1} - \left( \frac{H_i}{P_i} \right) \frac{Z_{i-1}}{Q_{i-1}} - \frac{(H_i - A)}{Q_i} \frac{\partial Z_i}{\partial A} - \frac{Z_{i-1}}{Q_{i-1}} \frac{\partial A}{\partial A} - A \frac{\partial Z_{i-1}}{\partial A} + \frac{Z_{i-1}}{P_i} + AZ_i - \frac{H_i}{P_i} \frac{\partial Z_i}{\partial A} - \left( \frac{H_i - A}{P_i} \right) \frac{\partial Z_i}{\partial A} \right].
\]

Using Equation (9) and (10) repeatedly, corrections (H and K) are calculated for S and A until the desired degree of convergence is achieved. Of course, owing to the lengthy calculations required by this method, a digital computer program for working out the details was essential. Otherwise, the method would prove impractical.
Two similar FORTRAN codes for calculating the results of a Bruceton analysis are in Appendix A and B. The two programs are basically the same. A comparison of them follows.

The main program entitled BRUCE in both codes is used to read in information pertaining to reliability and confidence levels. Then subroutine FOREST is called. (This subroutine is a modification of a FORTRAN code written by Forrest L. McMains. (Reference 3).

Subroutine FOREST is used to read in the basic data generated in the Bruceton analysis and then calculate estimates of the mean and standard deviation from approximations of the maximum likelihood equations. However, the estimates returned to BRUCE differ for the two codes. Using the code of Appendix A, the estimates of \( A \) and \( S \) are calculated for both the number of fails and fires -- the one obtained for the smaller of the preceding quantities being returned. Actually, the choice is made in BRUCE after returning from FOREST. A subroutine called SWAP is called when the number of fails is less than the number of fires. In this event the number of fails will be retained for further calculation. If the number of fires is less than or equal to the number of fails no transfer to subroutine SWAP will take place. Thus all calculations using the code in Appendix A depend on the smaller of the two quantities, number of fails and number of fires. This approach is advocated in Reference 4.

Using the code of Appendix B, the number of fires and fails are equally weighted regardless of which is smaller. The basis for this approach is described in Reference 3. Both the number of fires and fails are used throughout the program. Subroutine FOREST computes approximations to the mean and standard deviation by first calculating estimates of these quantities based on the number of fails and fires alone and then averaging them. These averages are weighted so that the quantities calculated using the smaller value between the number of fires or fails is more heavily favored.

Thus using either code approximate estimates of \( A \) and \( S \) are obtained using FOREST. After writing out the information read in plus the starting values of \( S \) and \( A \) just chosen, a subroutine entitled PROB is called. This subroutine calculates the probabilities \( P_i \) and \( Q_i \) associated with each energy level \( H_i \). The procedure followed in this subroutine is a modification of an existing program
used for calculating the probabilities of normally distributed random variables (Reference 2). The program which was modified is used to calculate probabilities from the normal probability integral defined as:

\[ P(x) = \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} e^{-\frac{1}{2}t^2} dt. \]

The integral gives the probability that the normalized random variable \( t \) will be between \(-t\) and \( t \) on the normal probability curve shown in Figure 2. That is, the probability \( P(x) \) is equal to the area under the curve from \(-t\) to \( t \). For small \( t \) the integral above can be evaluated from a power series:

\[ P(x) = t \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \left[ 1 - \frac{t^2}{2 \cdot 1} \cdot \frac{t^4}{2 \cdot 2 \cdot 1 \cdot 3} - \frac{t^6}{2 \cdot 3 \cdot 3 \cdot 5} + \cdots \right]. \]

For \( t > 2.90 \) a more efficient method of evaluating \( P(x) \) is achieved by using the asymptotic series defined as:

\[ P(x) \approx 1 - \left( \frac{2}{\pi} \right) e^{-\frac{1}{2}t^2} \left( 1 - \frac{t^2}{6} + \frac{1.3}{t^4} - \frac{1.3.5}{t^6} + \frac{1.3.5.7}{t^8} - \cdots \right). \]

For small \( t \), \( t < 1.20 \), the economy of calculating the normal probabilities is enhanced if use is made of telescoping the power series. For all terms beyond and including \( t^8 \), use is made of the telescoping effect produced by:

\[ t^8 = \frac{1}{128} (256t^2 - 160t^4 + 32t^2 - 1 + T8) \]

which is deduced from Chebyshev polynomials (Reference 2). For the range \( 1.2 < t < 2.9 \) application of the original power series is satisfactory.
THE PROBABILITY THAT \( X \) LIES BETWEEN \(-t\) AND \( t\) IS DEFINED AS:

\[
P(X) = \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} e^{-\frac{1}{2} t^2} dt.
\]

NORMAL FREQUENCY FUNCTION FOR THE NORMALIZED RANDOM VARIABLE \( t \).

FIGURE 2
For each range, \( t < 1.20 \), \( t > 2.90 \), and \( 1.20 < t < 2.90 \), probabilities were calculated and compared with those recorded in statistical tables. Because of the good agreement between the calculated and tabulated values and the relative ease with which a desired degree of accuracy was achieved, the approach using approximating series was employed.

However, as indicated, the method had to be modified. The reason for this is evident from Figure 3. It may be seen that the desired probability is equal to the area under the probability curve from \(-\infty\) to \( t \). Because of the symmetrical nature of the normal probability curve, the following technique was applicable. The area under the curve was obtained for the interval \(-t\) to \( t\). This value was halved and either subtracted from \(0.5\) (for negative \( t \)) or added to \(0.5\) if \( t \) was positive. Thus, the probability for \(-\infty\) to \( t \) was determined.

In addition, the subroutine PROB calculates the normal random variable associated with the random variable \( x \) from:

\[
t = \frac{x - A}{S}
\]

where \( A \) is the mean and \( S \) is the standard deviation. In terms of the symbols used in the code:

\[
Y_i = \frac{H_i - A}{S}
\]

The value of \( Z \) is adjusted too, using the expression:

\[
z_i = \frac{1}{\sqrt{2\pi}} e^{-\frac{(H_i - A)^2}{2S^2}}
\]

each time PROB is called. This is necessary since \( Z_i = f(A, S) \).

The corrections for \( S \) and \( A \) are calculated using the current values of \( P \), \( Q \), and \( Z \) as obtained from the subroutine PROB. The new values of \( S \) and \( A \) are used to evaluate \( F_1(S, A) \) and \( F_2(S, A) \) and then compare these with zero. If both are sufficiently close to zero, the order of accuracy may be determined by the User. For instance, \( 1 \times 10^{-3} \) and \( 1 \times 10^{-4} \) were chosen for \( F_1(S, A) \) and \( F_2(S, A) \), respectively, for the code shown in Appendix A while
THE PROBABILITY THAT X LIES BETWEEN AND IS DEFINED IS:

\[ P(X) = \frac{f(t)}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{1}{2} t^2} \, dt \]

NORMAL FREQUENCY FUNCTION FOR THE NORMALIZED RANDOM VARIABLE t

FIGURE 3
1 x 10^{-4} and 1 x 10^{-5} were chosen for the code of Appendix B. The mean, standard deviation, \( F_1(S, A) \) and \( F_2(S, A) \) are then written out and transfer is made to subroutine SOLVE. This subroutine is used to calculate such statistical quantities as the standard error of the mean, standard error of the standard deviation, the sure-fire probability point and a corresponding safe or NO-FIRE probability level. If the values of \( F_1(S, A) \) and \( F_2(S, A) \) are not sufficiently close to zero, however, the iterations are continued so that new corrections for \( A \) and \( S \) are obtained. This process of iteration is allowed to continue until \( F_1(S, A) \) and \( F_2(S, A) \) become as small as desired or until the number of iterations reach some predetermined value chosen by the User (1,000 iterations were chosen in the code shown in Appendix A, 100 for the code shown in Appendix B).

The values of \( F_1(S, A) \), \( F_2(S, A) \), the next to the last values of \( S \) and \( A \), the final values of \( S \) and \( A \) and the number of iterations are written out for each run. This is done so that if all 1,000 (or 100) iterations are carried out because the error in either \( F_1(S, A) \) or \( F_2(S, A) \) is larger than the bound specified for each, one can observe two successive values of \( S \) and \( A \) and decide whether convergence has actually been achieved. The results as computed by each code for three tests are shown at the end of the respective code. Note that for these three tests only two, three or four iterations were required. This rapid convergence is due to the good approximation of the mean and standard deviation obtained using subroutine FOREST.

Once the maximum likelihood estimates of \( A \) and \( S \) are approximated, subroutine SOLVE is called for purposes of estimating the standard errors associated with \( A \) and \( S \) and calculating the sure-fire and safe or NO-FIRE energy levels. To estimate the standard errors two parameters \( G \) and \( H \) must be determined. These are defined by:

\[
G^2 = \frac{\sum W_i / \sum W_i \left[ \frac{Z_i^2}{\sigma_i^2} + \frac{Z_i^2}{\rho_i^2} \right]}{
}
\]

and

\[
H^2 = \frac{\sum W_i / \sum W_i \left[ \frac{Z_i^2}{\sigma_i^2} + \frac{Z_i^2}{\rho_i^2} + \frac{Z_i^2}{\psi_i^2} \right]}
\]
where:

\[ w_i = \frac{\phi_i}{N \cdot \rho_i}, \quad i > 0 \]
\[ w_0 = 1 \]
\[ w_i = \frac{\phi_i}{\beta_i}, \quad i < 0. \]

Subroutine SOLVE makes use of the definitions of \( G \) and \( H \) and calculates these two quantities directly. Furthermore it calculates the high and low probability points and the other secondary statistics (standard errors).

The standard error of the mean is calculated from 
\[ \sigma_m = \frac{SG}{\sqrt{N}} \],
the standard error of the standard deviation from 
\[ \sigma_s = \frac{SN}{\sqrt{N}} \],
the standard error at the \( P \) percent energy level from 
\[ \sigma_P = S \cdot \frac{G^2 + K^2 + H^2}{N - 1} \].
In the above, \( N \) is the total number of fires and \( K \) is a constant associated with a particular percent point. The sure-fire and safe or NO-FIRE energy levels are then calculated from:

\[
\text{Sure-Fire} = A + KS \tag{11}
\]
\[
\text{NO-FIRE} = A - KS \tag{12}
\]

for the 50% confidence band. To add confidence to these reliability points, adjustment to Equation (11) and (2) are made using Sure-Fire = Sure-Fire + \( C \frac{N + C}{N} \rho \) and NO-FIRE = NO-FIRE - \( C \frac{N + C^1}{N} \rho \),

where \( C \) and \( C^1 \) are constants associated with a particular confidence interval. That is, the estimates of the sure-fire and NO-FIRE energy levels should be correct \( P \) percent of the time or the risk of being wrong is anticipated for only \( 1-P \) percent of the time (\( P = 90, 95, 99, \ldots \)).

As stated earlier, the code in Appendix A computes all quantities based on the smaller value between the number of fires or fails obtained in the Bruceton test. On the other hand, the code in Appendix B computes all quantities without regard to which value is smaller except in weighing averages in which case the smaller is more heavily favored.

Finally, the main program writes out the desired statistical values.
Regardless of which code is used the data cards are arranged identically. The following describes this arrangement:

**Card 1** -- Six fields of 10 columns each are reserved. Columns 1-10 and 11-20 are reserved for constants associated with the reliability and confidence level, respectively. Columns 21-30 and 31-40 are reserved for specifying the reliability and confidence level, respectively. Columns 41-50 are reserved for specifying the NO-FIRE energy level. Columns 51-60 are reserved for a second constant associated with the confidence level.

**Card 2** -- Four fields, one of 20 columns, one of 5 columns and 2 of 10 columns each are reserved. The first field is used for identification. The second contains the number of energy levels in the test. The third and fourth contain the lowest energy level in log units and the difference between successive energy levels (step interval) in log units, respectively.

**Card 3** -- Twelve fields, alternating between eight and four columns each are reserved. The fields of eight columns each contain the energy levels in ascending order. The corresponding number of fires at each energy level is specified in the right adjacent field of four columns.

**Card 4** -- Six fields of 10 columns each are reserved. The number of fails at each energy level -- starting at the lowest energy level -- is placed in these fields from left to right.

Table II on Page 52 and Page 20 and 21 of Reference 4 contain the constants associated with several reliability and confidence levels.

This example will demonstrate the input data cards for an actual set of test data. The table below shows the typical results of a Bruceton test. It is desired to calculate the 99.9% sure-fire and safe-fire reliability levels at the 95% confidence level:
From this table and Reference 4 the data cards for input may be prepared.
Card 1
Card Columns

1 1011 2021 3031 4041 5051 6061-80
"3.090" "1.96" "99.9" "95.0" "0.1" "1.2"

Card 2
Card Columns

1 2021 2526 3536 4546 80
"M2 Squib Test D" "5" "2.57287" "0.02"

Card 3
Card Columns

1 89 1213 2021 2425 3233 3637 4445 4849 5657 6061 80

Card 4
Card Columns

1 1011 2021 3031 4041 5051 80
"4." "10." "10." "2." "0."
The output using these data cards was obtained for both codes and is shown at the end of each (Appendix A and B).

The code of Appendix B appears to be superior. First, both the number of fails and numbers of fires are utilized in computing the various statistical quantities. Thus, a larger effective sample size is employed which for non-exact measurements is advantageous. Also, examination of the results at the end of each code shows that a higher ALL-FIRE and lower NO-FIRE are obtained using Code B if the number of fires and fails differ. This is desirable for increased reliability and safety. If the number of fires and fails are equal, the results using either code are identical. Consequently, for the two codes use of that of Appendix B is advocated.

In certain instances, the estimates of A and S from the first two moments (approximation method) are poor approximations to the maximum likelihood values. In particular when the step size between test levels becomes large (d > 2S) and when small sample sizes are used (<50) the approximations to A and S become poor -- that is, the maximum likelihood estimates of A and S are not well approximated by the first two moments.

If a difference in the two estimates does exist, the maximum likelihood quantities should be used because they are expected to be better than those generated by the first two moments. Furthermore, the standard errors of the mean and standard deviation are based on their respective maximum likelihood estimates and will not be applicable to the approximation values when a large difference exists. This means that the calculated extremes (sure-fire and NO-FIRE energy levels) will be poor estimates of the true quantities as well.

To determine the effect of sample size on the estimates of the standard deviation, a series of tests were conducted using sample sizes of 20, 30, 50 and 100 items. The mean and standard deviation for each sample were calculated by estimating the maximum likelihood quantities and by utilizing the first two moments. Because the Bruceton method is constructed to provide test data near the mean, that quantity is nearly always a good estimate of the population value. However, the standard deviation is not so well determined. Therefore, in what follows, special attention is paid to the effect of sample size on the standard deviation only.
Appendix C shows calculated values of the mean and standard deviation as determined by the maximum likelihood equations and from the first two moments. It is immediately obvious that little difference exists between the means as calculated by either method. But for small samples the standard deviation as calculated from the first two moments differs quite appreciably from that obtained utilizing the maximum likelihood equations. As stated, standard errors are based on the maximum likelihood quantities. In turn, the extremes of the probability distribution are likewise influenced because of their dependence on the mean, standard deviation and associated standard errors.

It is apparent that samples containing over 100 items should be used. The use of maximum likelihood theory for the cases in which the sample size is less than 100 is equally advocated. Specifically, Table 9 shows that for samples of 20 and 30 items large differences between the standard deviations occur frequently. For samples of size 50, better precision is achieved. Nevertheless, some poor results are still evident. For samples of size 100 even closer agreement between the two methods of calculation is achieved and it is anticipated that larger sample sizes would produce results differing only slightly from one method to the other.

It should be pointed out that the good results in the 50 item samples were due primarily to a knowledge of the past history of the test vehicle. However, the usual situation is one in which little or no data is available for the item since it is the purpose of these tests to generate such data. In these cases it is imperative to use large randomly chosen samples or if this is not practical, the maximum likelihood equations should be applied in calculating statistical quantities.

An example will best illustrate the effect of using small sample data in calculating the extremes of a population. For a sample of size 30, these were the results:
### Maximum Approximate Likelihood Equations

<table>
<thead>
<tr>
<th></th>
<th>Maximum Likelihood Equations</th>
<th>Approximate Formulae</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>2.20095909</td>
<td>2.18977</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.18969359</td>
<td>0.14878</td>
</tr>
<tr>
<td>Standard Error of Mean</td>
<td>0.041085</td>
<td>0.03283</td>
</tr>
<tr>
<td>Standard Error of Standard Deviation</td>
<td>0.073707</td>
<td>0.05342</td>
</tr>
<tr>
<td>Sure-Fire Level</td>
<td>3.27356932</td>
<td>2.78843</td>
</tr>
<tr>
<td>NO-FIRE Level</td>
<td>1.12834880</td>
<td>1.59111</td>
</tr>
</tbody>
</table>

These values are recorded in log volts. If the sure-fire and NO-FIRE levels are converted to volts, the potential danger becomes obvious—that is, using maximum likelihood equations the sure-fire level is about 1,877 volts as against 615 volts using the approximation, and the corresponding NO-FIRE levels are 13.5 volts and 39 volts, the former value was calculated from the maximum likelihood equations. Consequently, within the probability limits indicated, the sure-fire and NO-FIRE levels differ roughly by a factor of 3. Practically, this means that for this case in which a firing voltage of about 1,877 volts is required to provide sure-fire conditions, only 615 volts might be made available by a User relying on approximate formulas only. For similar reasons, safe voltage levels may be inaccurately specified. Thus, anticipated reliabilities are not closely achieved.

Although the discrepancies between the two methods become less pronounced with increasing sample size, maximum likelihood methods should be employed in view of their assumed superiority and the availability of a digital computer algorithm.
REFERENCES


APPENDIX A

FORTRAN CODE FOR DETERMINING THE MEAN AND STANDARD DEVIATION FROM THE SOLUTION OF THE MAXIMUM LIKELIHOOD EQUATIONS BY MEANS OF THE METHOD OF SUCCESSIVE APPROXIMATIONS (METHOD A)
C BRUCE STATIS MAX LIKE ESTIMATORS
DIMENSION H(99), P(99), Q(99), XN(99), X(99), Y(99), W(99)
DIMENSION, DZS(99), DZA(99), DQS(99), DQA(99), DPS(99), DPA(99)
DIMENSION, XNF(99), X(199), W(99)
COMMON GSQRD, RSQRD, SIGSQA, SIGSQS, SIGMAP, F1R50
COMMON FN0TS, FIREK, FUNQK
C 1S NO. OF ENERGY LEVELS, S = STEP SIZE (DIST. BETWEEN 2 ENERGY LEVELS
C A=MIDDLE ENERGY LEVEL (IF I IS ODD) OR AVG. OF MIDDLE 2 ENERGY LEVELS
C
READ (5,2) RELK,C0NK,RELV,C0NLP, RELV, CST
CALL FOREST (XN, XNF, I, ANTH, "FIRE", SAFIRE, SFILE, IDENT, SFAIL)
DO 7855 J = 1, I
7855 H(J) = ALOG10 (ANTHC(J))
C H(J) = ENERGY LEVELS, XN(J) = NO. OF FIRES AT A GIVEN H(J), Y(J) = NO.
C 1 RANDOM VARIABLE, X(J) = ABS. VALUE OF Y(J), P(J) = PROB. OF FIRE,
C Q(J) = PROB. OF NO FIRE, Z(J) = DERIVATIVE OF P(J)
SUM = 0.0
SUMNF = 0.0
DO 1216 J = 1, I
SUM = SUM + XN(J)
1216 SUMNF = SUMNF + XN(J)
IF (SUMM - SUMNF) .LT. 8571, 1758
1758 A = AFFAIR
S = SFIRE
GO TO 8578
8578 IF (SUMM .GT. SUMNF) CALL SWAP (XN, XNF, X)
WRITE (6,10) (IDENT(J), J = 1, 120)
10 FORMAT(11H1/1X,12HENERGY LEVEL,25X,18HNO. OF FIRES/FAILS//)
WRITE (6,11) (H(J), XN(J), J = 1, I)
11 FORMAT(43X,F10.5,30X,F4.0//)
WRITE (6,12) (1/J, 1, J = 1, I)
12 FORMAT(5JX,7HAVG. = ,F1J, 5/J50X, 17HAPPRX STDN DEV = ,F10.5//50X, 23
1HN0. OF ENERGY LEVELS = ,14//)
C COUNT OF THE NO. OF ITERATIONS
ITCNT = J
C CALL PR0B (H, X, Y, Z, A, S, P, Q, 1)
C BEGIN SUCCESSIVE APPROXIMATIONS (NEWTON RAPHSON METHOD)
C DERIVATIVE OF Z WITH RESPECT TO S AND A
C DERIVATIVE OF Q WITH RESPECT TO S AND A
C DERIVATIVE OF P WITH RESPECT TO S AND A
D0 436 J = 1, I
DZS(J) = (H(J) - A)**2 / S**3 * Z(J)
DZA(J) = (H(J) - A) / (S**3) * Z(J)
DQS(J) = -(H(J) - A) / (Q(J) * S) * Z(J)
DQA(J) = -Z(J) / (U(J) * S**2)
DPS(J) = (H(J) - A) / (P(J) * S**2) * Z(J)
DPQ(J) = Z(J) / (P(J) * P(J) * S)
CONTINUE
SUM = 0.0
BRUCE  -  EFN  SOURCE STATEMENT  -  IFNIS -

SUM2=0.0
SUM3=0.0
SUM4=0.0
SUM5=0.0
SUM6=0.0
SSUM=0.0
DO 555 J=2,1,
   SUM1=SUM1+XN(J)*((DZS(J-1)/Q(J-1)-DZS(J)/P(J))-
   Z(J))
   SUM2=SUM2+XN(J)*((Z(J-1)/Q(J-1)-Z(J))/P(J))
   SUM3=SUM3+XN(J)*((DZA(J-1)/Q(J-1)-DZA(J)/P(J))-
   Z(J))
   SUM4=SUM4+XN(J)*((H(J-1)-A)*DZS(J-1)/P(J)+
   (H(J-1)-A)*DPS(J)/Z(J))
   SUM5=SUM5+XN(J)*((H(J-1)-A)*Z(J-1)/Q(J-1)-
   (H(J-1)-A)*Z(J))/P(J))
   SUM6=SUM6+XN(J)*((H(J-1)-A)*DZA(J-1)/Q(J-1)+
   (H(J-1)-A)*Z(J-1)/Q(J-1)-Z(J))/P(J))
   A(J)=H(J)/Q(J)+Z(J)/P(J)+Z(J)/P(J)
   SUM=SUM+XN(J)*((Z(J-1)/Q(J-1)-(H(J-1)-A)*Z(J))/P(J))
$SUM=SUM+XN(J)*((Z(J-1)/Q(J-1)-(H(J-1)-A)*Z(J))/P(J))$
555 CONTINUE

C      DERIVATIVE OF F1(S,A) WITH RESPECT TO S AND A
C      DERIVATIVE OF F2(S,A) WITH RESPECT TO S AND A
C
FAS=F1(S,A)/S, SAS=F2(S,A)/S
DFO=S+1.0/S=SUM1-1.0/S**2  *SUM2
DFDO=1.0/S=SUM3
DSD=1.0/S**2 =SUM4-2.0/S**3 *SUM5
DSDA=1.0/S**2 =SUM6
FAS=1.0/S=SUM
SAS=1.0/S**2=SSUM

303 D=DFO*DSDA-DFDA*DSDA
C
C   CORRECTION FOR S
XH=(DFO*FAS-DSDA*FAS)/D
C   CORRECTION FOR A
XK=(DSDA*FAS-DFDA*FAS)/S
C
C   CORRECTED A
AA=A+XK
C
C   CORRECTED S
SS=S+XH

17 S=SS
A=AA
C
C   VALUE OF F1(A,S) AND F2(A,S) OBTAINED FROM FINAL A AND S
ZER01=0.0
ZER02=0.0
D0 69 J=2,1
   ZERO1=ZER01+XN(J)*((Z(J-1))/Q(J-1)-Z(J))/P(J))
69 ZER02=ZER02+XN(J)*((H(J-1)-A)*Z(J-1)/Q(J-1)-(H(J-1)-A)*Z(J))/P(J))
IF(ABS(ZER01)-1.E-3) 68,68,15
68 IF(ABS(ZER02)-1.E-4) 16,16,15
15 ITCNT = ITCNT + 1
   IF(ITCNT == 1000) 151,16,16
151 GO TO 6
16 END
BRUCE - EFN SOURCE STATEMENT - [FNIS] -

221HSTANDARD DEVIATION = '*.*F1.8*/
WRITE (6,111) ITCNT
111 FORMAT(52X,2H9.0, 3F ITERATIONS = ,I3//)
CALL SOLVE(P,Q,XN,Y,Z,I,A,S,H,RElK,C3NK,CST)
WRITE (6,77)
77 FORMAT(40X,49HCALCULATION OF ALL-ADN NO-FIRE ENERGY LEVELS//)
WRITE (6,971) SQRD,HSQRD,SIGSQA,SIGSQS,SIGMAP,FIR50, FNAT50
1,RELV,CONLV,FIREV,XRELV,C3NLV,FIRE
971 FORMAT(52X,11HHSQUARED = ,F14.6//5:5X,12HHSQUARED = ,F10.6//50X,29HS
1TAND. ERROR OF MEAN SQRD. = ,F13.6//35X,35HSTAND. ERROR OF STAND.
2DEV. SQRD. = ,F10.8//55X,35HSTAND. ERROR AT P PERCENT HEIGHT = ,F1
33.8//55X,33H99.9FIRE AT 5%CONFIDENCE LEVEL = ,F10.8//50X,32H0.1FIR
4E AT 5%CONFIDENCE LEVEL = ,F10.8//50X,F6.2,THFIRE AT,F6.2,19HC0NF
5ENCE LEVEL = ,F10.8//50X,F6.2,7THFIRE AT,F6.2,19HC0NFIDENCE LEVEL
6= ,F10.8//
GO TO 1
END
SUBROUTINE PGB(H,X,Y,Z,A,S,P,Q,I)
DIMENSION H(99),X(99),Y(99),Z(99),P(99),Q(99)
DO 7 J = 1, I
HMASQ = (H(J)-A)**2
EXPNT = HMASQ/(2.*S*S)
Z(J) = (1./2.5663)/EXP(EXPNT)
Y(J) = (H(J)-A)/S
X(J) = ABS(Y(J))
IF (X(J)-1.20) 21,21,22
21 XSQ = X(J)*X(J)
P(J) = 0.79788455*X(J)*(0.99999774-XSQ*(0.16659433-XSQ*(.2463831E-0
11-XSQ*.23974867E-02)))
IF (Y(J)) 400, 401, 402
400 P(J) = 0.5 - P(J)/2.0
G0 TO 403
401 P(J) = 0.5
G0 TO 403
402 P(J) = P(J)/2.0+.5
403 Q(J) = 1.0-P(J)
G0 TO 7
22 IF (X(J) = 2.96) 25, 26, 26
25 XSQ = X(J)*X(J)
P(J) = 1.0
PT = 1.0
FACT = 1.0
0DD = 3.0
29 PT = -PT*XSQ/(2.0*FACT)
T = PT/0DD
P(J) = P(J) + T
IF (ABS(T) = 0.03007) 63, 70, 70
70 FACT = FACT + 1.0
0DD = 200 + 2.0
G0 TO 29
63 P(J) = 0.79788455*X(J)*P(J)
IF (Y(J)) 500, 501, 502
500 P(J) = 0.5 - P(J)/2.0
G0 TO 503
501 P(J) = 0.5
G0 TO 503
502 P(J) = P(J)/2.0+.5
503 Q(J) = 1.0 - P(J)
G0 TO 7
26 RXSQ = 1.0/(X(J)*X(J))
P(J) = 1.0-3.79788455*EXP(-X(J)*X(J)/2.0)/X(J)*(1.0-RXSQ*(1.0-RXSQ
*: (3.0-RXSQ*(15.0-RXSQ*1.05.0))))
IF (Y(J)) 600, 601, 602
600 P(J) = 0.5 - P(J)/2.0
G0 TO 603
601 P(J) = 0.5
G0 TO 603
602 P(J) = P(J)/2.0+.5
603 Q(J) = 1.0 - P(J)
7 CONTINUE
RETURN
END
SUBROUTINE FORESTIX, FIX, N, X, UFIX, UFAIX, VFIX, IDENT, VFAX

ALCG(W) = .4343 * ALG(W)
ANT(W) = EXP(W / .4343)

DIMENSION (X(31), IDENT(20), FIX(99), FAX(99), S(30, 15)

READ (5, 639) IDENT(J), J = 1, 20; N, C, D

639. FORMAT (20A1, 15, ZFI0.0, 47)
READ (5, 637) (XI(I), FIX/I, I = 1, N)

637. FORMAT (6 (FB0.0, F4.0)),
READ (5, 638) (FAX(I), I = 1, N)

638. FORMAT (6FI0.0)

AFIX = 0.0
AFAX = 0.0
BFIX = 0.0
BFAX = 0.0
SFIX = 0.0
SFAX = 0.0
I = Q

3010 W = I
AFIX = AFIx + W * FIX(I + 1)
AFAX = AFAx + W * FAX(I + 1)
BFIX = BFIX + (W**2) * FIX(I + 1)
BFAX = BFAX + (I**2) * FAX(I + 1)
SFIX = SFIX + FIX(I + 1)
SFAX = SFAX + FAX(I + 1)
I = I + 1
IF (I = N + 1) 3010, 3010, 3011

1115 R = D
TFIX = ((SFX + BFIX - AFIx**2) / (SFX**2))
TFAx = ((SFX + BFAX - AFAx**2) / (SFAx**2))

13 UFIx = C*R + (AFIX / SFIX - 0.5)
UFAX = C*R + (AFAx / SFAX - 0.5)
IF (TFIX < 0.3) 118, 23, 23

118 CIX = TFIX * 100.
LIX = CIX

4111 WEEI = 500.
I = 0

112 WEEI = AMI N1(WEEI, ABS(UFIX - ALCG(X(I + 1))))
I = I + 1.
IF (I = N + 1) 112, 112, 113

113 RIX = ABS(WEEI / R)
FORTES - EFN SOURCE STATEMENT - IFN(S) -

KIX = IFIX (RIX * 22.) + 1
VFIX = (S(LIX,KIX)/100.)*R

2424 IF(TFA-X<0.3)117,2525,2525
117 CAX=TFA-X100.
LAX=CAX

114 WEEAX=500.
I=0

115 WEEAX = MIN(WEEAX,ABS(UFAX-ALCG(X(I+1))))
I=I+1
IF(I-N+1)115,115,116

116 RAX=ABS(WEEAX/R)
KAX = IFIX (RAX*22.)+1
VFAX = (S(LAX,KAX)/100.)*R
G0 TO 26

23 VFIX=1.62*R*(TFIX+0.29)*SQRT(SFIX/(SFIX-1.))
G0 TO 2424.

2525 VFAX=1.62*R*(TFAX+0.29)*SQRT(SFAX/(SFAX-1.))

26 RETURN
FND
Subroutine SWAP (XN, XNF, I)

Dimension XN(99), XNF(99)

DO 4143  J = 2, I

4143  XN(J) = XNF(J-1)

RETURN

END
SUBROUTINE SOLVE (P,Q,N,Y,Z,I,A,S,H,RELK,C0NK,CST )
COMMON GSQRD,HSQRD,SIGSQA,SIGSQS,SIGMAP,FIR50, FN0T50, FIREK,FN0TK
DIMENSION H(99),P(99),Y(99),Z(99),W(99)
SUMNJ = 0.0
W(1) = 1.0
WJSUM = 0.0
HSUM = 0.0
DO 51 J=2,I
  51 W(J)=W(J-1)*P(J-1)*W(J-1)
DO 61 J=1,I
  SUMNJ = SUMNJ + XN(J)
  WJSUM = WJSUM + W(J)
  GSUM = GSUM + W(J-1)*Z(J-1)**2/Q(J-1)**2+Z(J)**2/P(J)**2
  HSUM = HSUM+W(J-1)*(Y(J-1)**2+Z(J-1)**2/Q(J-1)**2+Y(J)**2*Z(J)**2/ P(J)**2)
GSQRD = WJSUM/GSUM
HSQRD = WJSUM/HSUM
SIGSQA = S*S/SUMNJ*GSQRD
SIGSQS = S*S/SUMNJ*HSQRD
SIGMAP = SQRT(SIGSQA*RELK**2*SIGSQS )
FIR50 = A+RELK*S
FN0T50 = A-RELK*S
FIREK = FIR50+C0NK*SUMNJ+CST)/SUMNJ)*SIGMAP
FN0TK = FN0T50-C0NK*(SUMNJ+CST)/SUMNJ)*SIGMAP
RETURN
END
### M2 Squib Test D

**Calc. of Mean and Std. Deviation by Soln. of Max. Likelihood Eqns.**

<table>
<thead>
<tr>
<th>Energy Level</th>
<th>No. of Fires/Fails</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.57287</td>
<td>0.</td>
</tr>
<tr>
<td>2.59329</td>
<td>4.</td>
</tr>
<tr>
<td>2.61278</td>
<td>10.</td>
</tr>
<tr>
<td>2.63246</td>
<td>9.</td>
</tr>
<tr>
<td>2.65321</td>
<td>1.</td>
</tr>
</tbody>
</table>

**Avgs.**

- \( \text{Avg.} = 2.60870 \)
- \( \text{Apprx \ Stnd. Dev.} = 0.02159 \)

**No. of Energy Levels**

- 5

**Zeros**

- \( \text{Zer01} = -0.52526593E-04 \)
- \( \text{Zer02} = 0.28066803E-04 \)

**NlMean**

- 2.60874474

**Mean**

- 2.60874474

**NlStnd. Dev.**

- 0.02079978

**Standard Deviation**

- 0.02079978

**No. of Iterations**

- 2

**Calculation of All-Fire and No-Fire Energy Levels**

- \( \text{GSquared} = 0.696013 \)
- \( \text{HSquared} = 0.791929 \)

- **Stnd. Error of Mean Sqr.** = 0.00001234
- **Stnd. Error of Stnd. Dev. Sqr.** = 0.00001428
- **Stnd. Error at P Percent Height** = 0.01220044

- **99.9% Fire at 50% Confidence Level** = 2.67301604
- **0.1% Fire at 50% Confidence Level** = 2.54447341

- **99.9% Fire at 95.00% Confidence Level** = 2.69812453
- **0.1% Fire at 95.00% Confidence Level** = 2.51936489
**M6 Cap Test A**

**CALC. OF MEAN AND STD. DEVIATION BY SOLN. OF MAX. LIKELIHOOD EQNS.**

<table>
<thead>
<tr>
<th>ENERGY LEVEL</th>
<th>NO. OF FIRES/FAILS</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.14301</td>
<td>0.</td>
</tr>
<tr>
<td>0.16435</td>
<td>2.</td>
</tr>
<tr>
<td>0.18469</td>
<td>9.</td>
</tr>
<tr>
<td>0.20683</td>
<td>10.</td>
</tr>
<tr>
<td>0.22531</td>
<td>3.</td>
</tr>
</tbody>
</table>

**AVG. = 0.18602**

**APPRX STND DEV = 0.02279**

**NO. OF ENERGY LEVELS = 5**

**ZER01 = 0.19371510E-05**

**ZER02 = 0.22351742E-07**

**NLMEAN = 0.18685731**

**MEAN = 0.18685731**

**NLSTND. DEV. = 0.02369443**

**STANDARD DEVIATION = 0.02369443**

**NO. OF ITERATIONS = 3**

**CALCULATION OF ALL-FIRE AND NO-FIRE ENERGY LEVELS**

**G SQUARED = 0.846374**

**H SQUARED = 1.498057**

**STAND. ERROR OF MEAN SQRD. = 0.00001980**

**STAND. ERROR OF STAND. DEV. SQRD. = 0.00003504**

**STAND. ERROR AT P PERCENT HEIGHT = 0.01882551**

**99.99%FIRE AT 95.00%CONFIDENCE LEVEL = 0.26007312**

**99.99%FIRE AT 95.00%CONFIDENCE LEVEL = 0.11364149**

**99.99%FIRE AT 95.00%CONFIDENCE LEVEL = 0.29881602**

**0.10%FIRE AT 95.00%CONFIDENCE LEVEL = 0.07489859**
M6 Cap Test E

CALC. OF MEAN AND STD. DEVIATION BY SOLN. OF MAX. LIKELIHOOD EQNS.

<table>
<thead>
<tr>
<th>ENERGY LEVEL</th>
<th>NO. OF FIRES/FAILS</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.43993</td>
<td>0.</td>
</tr>
<tr>
<td>2.42488</td>
<td>2.</td>
</tr>
<tr>
<td>2.44091</td>
<td>9.</td>
</tr>
<tr>
<td>2.45484</td>
<td>12.</td>
</tr>
<tr>
<td>2.46982</td>
<td>2.</td>
</tr>
</tbody>
</table>

AVG. = 2.44130
APPRX STND DEV = 0.01477
NO. OF ENERGY LEVELS = 5
ZERO1 = 0.26762486E-04
ZERO2 = 0.75111166E-06
NLMEAN = 2.44135469
MEAN = 2.44135469
NLSTND. DEV. = 0.01453041
STANDARD DEVIATION = 0.01453041
NO. OF ITERATIONNS = 2

CALCULATION OF ALL-FIRE AND NO-FIRE ENERGY LEVELS

GSQUARED = 0.785916
HSQUARED = 1.171202
STAND. ERRØR OF MEAN SQRD. = 0.00000663
STAND. ERRØR OF STAND. DEV. SQRD. = 0.00000949
STAND. ERRØR AT 99 PERCENT HEIGHT = 0.01005381
99.9% FIRE AT 90% CONFIDENCE LEVEL = 2.48625368
99.9% FIRE AT 95% CONFIDENCE LEVEL = 2.50690502
0.1% FIRE AT 95% CONFIDENCE LEVEL = 2.37580431
APPENDIX B

FORTRAN CODE FOR DETERMINING THE MEAN AND STANDARD DEVIATION FROM THE SOLUTION OF THE MAXIMUM LIKELIHOOD EQUATIONS BY MEANS OF THE METHOD OF SUCCESSIVE APPROXIMATIONS (METHOD B)
BRUCE - EFNS

C BRUCE STATIS MAX LIKE ESTIMATORS
DIMENSION H(99), Z(99), P(99), Q(99), XN(99), X(99), Y(99), W(99).
DIMENSION DZS(99), DZA(99), DQS(99), DQA(99), DPS(99), DPA(99)
DIMENSION XNF(99), ANTH(50), IDENT(2)
COMMON GSQRD, HSQRD, SIGSQA, SIGSQJ, SIGMAP, FIR50
COMMON FNST5, JFIREK, FN5TK
1=NO. OF ENERGY LEVELS, S=STEP SIZE(DIST. BETWEEN 2 ENERGY LEVELS.
A=MIDDLE ENERGY LEVEL(IF I IS ODD) OR AVG. OF MIDDLE 2 ENERGY LEVEL.
1 IS EVEN
READ (5, 2) RELK, CONK, RELV, CONLV, XRELV, CST
CALL FOREST(XN, XNF, I, ANTH, A, S, IDENT)
7855 J = 1, I
7855 H(J) = ALOG(ANTH(J))
C H(J) = ENERGY LEVELS, XN(J) = NO. OF FIRES AT A GIVEN H(J), Y(J) = NO.
1 RANDOM VARIABLE, X(J) = ABS. VALUE OF Y(J), P(J) = PROB. OF FIRE.
C Q(J) = PROB. OF NO FIRE, Z(J) = DERIVATIVE OF P(I)
WRITE (6, 10) IDENT(J), J = 1, 2)
10 FORMAT (1H1/1X, 2JA2/2X661; CALC. OF MEAN AND STD. DEVIATION BY SOLN.
1. OF MAX. LIKELIHOOD EQNS.//)
WRITE (6, 10)
10 FORMAT (4OXF10.5, 33X, F4.0, 3X, F4.0//)
WRITE (6, 11) A, S, I
11 FORMAT (4OXF10.5, 33X, F4.0//)
WRITE (6, 12) A, S, I
12 FORMAT (50X97HAVG., F10.5//-), TX, 17HAPPRX STD DEV =, F10.5///50X, 23
1N. OF ENERGY LEVELS = 14//)
COUNT OF THE NO. OF ITERATIONS
ITCNT = 0
CALL PROB (H, X, Y, Z, A, S, P, Q)
BEGIN SUCCESSIVE APPROXIMATIONS(NEWTON-RAPHSON METHOD)
DERIVATIVE OF Z WITH RESPECT TO T0 S AND A
DERIVATIVE OF Q WITH RESPECT TO T0 S AND A
DERIVATIVE OF P WITH RESPECT TO T0 S AND A
436 CONTINUE
SUM1 = 0.0
SUM2 = 0.0
SUM3 = 0.0
SUM4 = 0.0
SUM5 = 0.0
SUM6 = 0.0
SUM = 0.0
SUM = 0.0
SUM = 0.0
SUM = 0.0
555 J = 2, I
SUM1 = SUM1 * XNF(J-1) * (DZS(J-1) / Q(J-1) + DQS(J-1) * Z(J-1)) - XN(J) * (DZS(J)
1 / P(J) + DPS(J) * Z(J))
SUM2 = SUM2 * XNF(J-1) * (Z(J-1) / Q(J-1) - XN(J) * (Z(J) / P(J))
SUM3 * SUM3 * XNF(J-1) * (DZA(J-1) / Q(J-1) + DQA(J-1) * Z(J-1)) - XN(J) * (DZA(J)
43
BRUCE - EEF SOURCE STATEMENT - IFN(S) -

1/PIJ+DPJ(JZIJ)
SUM4=SUM4X2(J-1)+(H(J-1)-A)*DZ(S(J-1))Q(J-1)+H(J-1)-A*DQS(J-1)
1*Z(J-1)+XN(J-1)+(H(J-1)-A)*DZ(J/PJ)+(H(J-1)-A)*DQS(J-1)Z(J-1)
SUM5=SUM5+XN(F-1)+(H(J-1)-A)*Z(J-1)/Q(J-1)+XN(J-1)+(H(J-1)-A)*Z(J-1)
SUM6=SUM6+XN(F-1)+(H(J-1)-A)*Z(J-1)/Q(J-1)+XN(J-1)+(H(J-1)-A)*Z(J-1)

555 CONTINUE
DERIVATIVE OF F1(S,A) WITH RESPECT TO S AND A
DERIVATIVE OF F2(S,A) WITH RESPECT TO S AND A
FAS=F1(S,A), SAS=F2(S,A)
DFDS=1.0/S*SUM1-1.0/S**2 *SUM2
DFDA=1.0/S*SUM3
DSAS=1.0/S**2 *SUM4-1.0/S**2 *SUM5
DSDA=1.0/S**2 *SUM6
FAS=1.0/S*SUM
SAS=1.0/S**2 *SUM
303 D=DFPS*DSASA=DFDS*DSDA
CORRECTION FOR S
XH=(DFDA*SAS-DSDA*FAS)/D
CORRECTION FOR A
XA=(DSDS+FAS-DFDS*AAS)/D
CORRECTED A
AA=A+XK
CORRECTED S
SS=S+XH
17 S=SS
A=AA
VALUE OF F1(A,S) AND F2(A,S) OBTAINED FROM FINAL A AND S
ZER01=0.0
ZER02=0.0
D0 69 J=2,1
ZER1=ZER01+XN(F-1)+(Z(J-1)/Q(J-1))-XN(J-1)+(Z(J-1/PIJ))
69 ZER02=ZER02+XN(F-1)+(H(J-1)-A)*Z(J-1)/Q(J-1)-XN(J-1)+(H(J-1)-A)*Z(J-1/PIJ))
I=FABS(ZER01)-1,E-4) 60,68,15
68 I=FABS(ZER02)-1,E-5) 6,16,15
15 ITCN T=ITCN T+1
IF (ITCNT=100) 151,16,16
151 G0 TO 6
16 WRITE (6,6666) ZER01,ZER02,A,AA,S,SS
6666 FORMAT(5X,8H7ER01=,E20.8//5X,8H7ER02=,E20.8//5X,9HNL,MEAN=,F10.8//5X,15HLS,IND,DEV.=,F10.8//5X,22HSTANDARD,DEVIA;=,E10.8//5X)
WRITE (6,1111) ITCN T
1111 FORMAT(5X,2,8HITCN T=)
111 CALL SOLVE(P,X,Y,Z,1,A,S,H,RELK,CNK,CST,XNF)
WRITE (6,77)
77 FORMAT(40X,49HCALCULATION OF ALL-FIRE AND NO-FIRE ENERGY LEVELS//)
WRITE (6,971)SGRD,HSGRD,SSQ1,SSQ2,SSQ5,SSQ7,SSMAP,FIR50,FN050,
RELV,CNLV,FIREK,RELV,CNLV,FIRK
971 FORMAT(5X,11HGSQUARED=,F10.6//50X,10HISQUARED=,F10.6//50X,29HS}

44

GO TO 1
END
SUBROUTINE PR0B( H,X,Y,Z,A,S,P,G,I)
DIMENSION H(99),X(99),Y(99),Z(99),P(99),Q(99)
DO 7 J = 1, I
MMASQ = (H(J)-A)**2
EXPNT = MNASQ/(Z*S*S)
Z(J) = (1./2.5363)/EXP(EXPNT)
Y(J) = (H(J)-A)/S
X(J) = ABS(Y(J))
IF (X(J)-1.20) 21,21,22
21 XSQ = X(J)*X(J)
P(J) = 0.79788455*X(J)*(0.99999774-XSQ*(0.16659633-XSQ*(2.24638310E-Q-11-XSQ*2.3974976E-02))
IF (Y(J) ) 409, 401, 402
400 P ( J ) = 0.5 - P ( J ) / 2.5
G0 TO 403
401 P ( J ) = J .5
G0 TO 403
402 P(J)=P(J)/2.0+0.5
403 Q (J ) = 1.0-P ( J )
G0 TO 7
22 IF (X(J) - 2.95) 25,26,26
25 XSQ = X(J)*X(J)
P(J) = 1.0
PT = 1.0
FACT = 1.0
DD = 3.0
29 PT = -PT*XSQ/(2.0*FACT )
T = PT/DD
.P(J) = P(J) + T
IF (ARS(T) - 0.00007 ) 63 ,7) ,70
70 FACT = FACT + 1.0
DD = DD + 2.0
G0 TO 29
63 P(J) = 0.79788455*X(J)*P(J)
IF (Y(J) ) 500, 501, 502
500 P ( J ) = 0.5 - P ( J ) / 2.0
G0 TO 503
501 P ( J ) = 0.5
G0 TO 503
502 P(J)=P(J)/2.0+0.5
503 Q (J ) = 1.0-P ( J )
G0 TO 7
26 RXSQ = 1.0/(X(J)*X(J))
P(J) = 1.0-0.79788455*EXP(-X(J)*X(J)/2.0)/X(J)*(1.0-RXSQ*(1.0-RXSQ
1* (3.0-RXSQ*(15.0-RXSQ*105.0))))
IF (Y(J) ) 600, 601, 602
600 P ( J ) = 0.5 - P ( J ) / 2.0
G0 TO 603
601 P ( J ) = 0.5
G0 TO 603
602 P(J)=P(J)/2.0+0.5
603 Q (J ) = 1.0-P ( J )
7 CONTINUE
RETURN
END
SUBROUTINE FOREST(FIX,FAX,NX,UX,V,IDENT)
ALCG(W)=.4343*ALOG(W)
ANT(W)=EXP(W/4343)
DIMENSION NX(50),IDENT(20),FIX(99),FAX(99),S(30,15)
DATA S/64.0,25.0,40.0,43.0,46.0,49.0,51.0,16.0,2.0
15.0,27.0,31.0,35.0,37.0,42.0,44.0,47.0,49.0,51.0,12.0,25.0,26.0,28.0,29.0,31.0,33.0,37.0
24.0,36.0,39.0,40.0,43.0,46.0,48.0,49.0,51.0,9.0,25.0,27.0,28.0,31.0,32.0,34.0,36.0
3.0,37.0,49.0,41.0,42.0,44.0,46.0,49.0,51.0,4.0,25.0,26.0,27.0,28.0,4.0,30.0,31.0,32.0,33.0,35.0,36.0,37.0,38.0
41.0,25.0,26.0,27.0,28.0,29.0,30.0,31.0,32.0,33.0,34.0,35.0,36.0,37.0,38.0
6.0,40.0,41.0,42.0,43.0,45.0,46.0,47.0,49.0,51.0,52.0,25.0,26.0,27.0,29.0,29.0,29.0,30.0,31.0,32.0,33.0,34.0,35.0,36.0,37.0
97.0,38.0,39.0,40.0,41.0,43.0,44.0,45.0,47.0,49.0,50.0,51.0,53.0,26.0,26.0,27.0,28.0
A,29.0,30.0,31.0,32.0,34.0,35.0,36.0,37.0,39.0,39.0,40.0,41.0,42.0,44.0,44.0,44.0
B5.0,47.0,49.0,50.0,52.0,53.0,26.0,27.0,28.0,29.0,29.0,30.0,31.0,33.0,34.0,36.0
C,37.0,38.0,39.0,40.0,41.0,42.0,43.0,44.0,45.0,46.0,47.0,49.0,51.0,52.0,53.0,26.0,28.0
D,28.0,29.0,30.0,31.0,32.0,33.0,34.0,35.0,36.0,37.0,38.0,39.0,40.0,41.0,42.0,43.0
E4.4,45.0,46.0,47.0,49.0,50.0,52.0,54.0
READ(5,639)(IDENT(J),J=1,20),N,C,D
639 FORMAT(20A1,15,2F10.0)
READ(5,637) (XI(I),FIX(I),I=1,N)
637 FORMAT(6(F8.0,F4.0))
READ(5,638) (FAX(I),I=1,N)
638 FORMAT( 6F10.0 )
AFIX=0.0
AFAX = 0.0
BFIX=0.0
BFAX=0.0
SFIX=0.0
SFAX=0.0
I=0
3010 W=I
AFIX=AFIX+W*FIX(I+1)
AFAX=AFAX+W*FAX(I+1)
BFIX = BFIX + (W**2) * FIX(I+1)
BFAX = BFAX + (W**2) * FAX(I+1)
SFIX=SFIX+FIX(I+1)
SFAX=SFAX+FAX(I+1)
I=I+1
IF(I-N+1)3010,3010,1115
1115 R=0
TFIX=((-SFIX*BFIX)-AFIX**2)/(SFIX**2)
TFAX=(-(SFAX*BFAX)-AFAX**2)/(SFAX**2)
13 UFIX=CR*(AFIX/SFIX-0.5)
UFAX=CR*(AFAX/SFAX+.5)
1F(TFIX-.3)118,23,23
118 CIX=TFIX*100.
LIX=CIX
4111 WEEIX = 500.
I=0
112 WEEIX = AMIN(WEEIX,ABS(UFIX-ALCG(X(I+1))))
I=I+1
1F(I-N+1)112,112,113
113 RIX=ABS(WEEIX/R)

COMMON GSQRD, HSQRD, SIGSQA, SIGSQS, SIGMAP, FIR50, FNT50, FIREK, FN50K
.DIMENSION H(99), P(99), J(99), XN(99), Y(99), Z(99), W(99), XNF(99)
.SUMNFJ=0.0
.SUMNJ = 0.0
.WJSUM = 0.0
.GSUM = 0.0
.HSUM = 0.0
.IERO = 1/2 + 1
.I1 = IZER0 - 1
.DO 623 J = 1, I
.MJ = J - 1
.WJ = 1.0
.**IF(H(J) - H(IZER0 ) ) 358,347,223

358 DO 6262 K = J, IM1
6262 W(J) = W(J) * P(K) / Q(K)
.G0 TO 623
347 W(J) = 1.0
.G0 TO 623
223 DO 727 L = IZER0 , JH1
727 W(J) = W(J) * Q(L) / P(L)
623 CONTINUE
.D0 741 J = 1, I
.WJSUM = WJSUM + W(J)
.SUMNFJ=SUMNFJ+XNF(J)
741 SUMNJ = SUMNJ + XN(J)
.D0 61 J = 2, I
.GSUM = GSUM + W(J-1) *(Z(J-1)**2/Q(J-1)**2+Z(J)**2/P(J)**2)
61 HSUM = HSUM+W(J-1) *(Y(J-1)**2+Z(J-1)**2/Q(J-1)**2+Y(J)**2+Z(J)**2/P(J)**2)
1P(J)**2)
.GSQRD = WJSUM/GSUM
.HSQRD = WJSUM/HSUM
.SIGSQA=S*SUMNJ*GSQRD
.SIGSQS = S*SUMNJ*HSQRD
.SIGSF=S*SUMNFJ*GSQRD
.SIGSF=S*SUMNFJ*HSQRD
.SIGMAP = SQRT(SIGSQA+RELK**2*SIGSQS)
.SIGMAPF=SQR(T(SIGSF+RELK**2*SIGSQSF)
.FIR50 = A+RELK*S
.FNT50 = A-RELK*S
.FIREK = FIR50+C0NK*((SUMNJ+CST)/SUMNJ)*SIGMAP
.FN50K = FN50+SIGMAPF*(SUMNJ+CST)/SUMNJ
.FIREKF=FIR50+C0NK*((SUMNJ+CST)/SUMNJ)*SIGMAPF
.FN50KF=FN50K+C0NK*((SUMNFJ+CST)/SUMNFJ)*SIGMAPF
.SIGSQA=SORT((SIGSQA**2+SIGSF**2)**2/2.)
.SIGSQS=SORT((SIGSQS**2+SIGSQSF**2)**2/2.)
.FIREK=SQRT(FIREK**2+FIREKF**2)**2/2.
.FN50K=SQRT(FN50K**2+FN50KF**2)**2/2.
RETURN
.END
FORTES - EFN SOURCE STATEMENT - IFN(S) -

KIX = IFIX (RIX * 22. ) + 1
VFIX=( S(LIX,KIX)/100. )*R

2424 IF(TFAX-0.3)117,2525,2525
117 CAX=TFAX*100.
LAX=CAX
114 WEEAX=50.
I=0
115 WEEAX = AMIN1(WEEAX,ABS(UFAX-ALCX*I+1))
I=I+1
IF(I<=n+1)115,115,116
116 RAX=ABS(WEEAX/K)
KAX = IFIX (RAX*22.)+1
VFXA = (. S(LAX,KAX)/100. )*R
G0 T0 26
23 VFIX=1.62*R*(TFIX+0.029)*SQRT(SFIX/(SFIX-1.)).
G0 T0 2424
2525 VFXA=1.62*R*(TFAX+0.029)*SQRT(SFA/(SFA-1.))
26 U=((SFA*UFAX)+(SFIX*VFIX))/(SFA+SFIX)
V=SQRT((SFA*(VFA**2)+SFIX*(VFIX*2))/(SFA+SFIX))
RETURN
END
### M2 Squib Test D

**CALC. OF MEAN AND STD. DEVIATION BY SOLN. OF MAX. LIKELIHOOD EQNS.**

<table>
<thead>
<tr>
<th>ENERGY LEVEL</th>
<th>NO. OF FIRES/FAILS</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.57287</td>
<td>0.  4.</td>
</tr>
<tr>
<td>2.59329</td>
<td>4.  10.</td>
</tr>
<tr>
<td>2.61278</td>
<td>10. 10.</td>
</tr>
<tr>
<td>2.63246</td>
<td>9.   2.</td>
</tr>
<tr>
<td>2.65321</td>
<td>1.  0.</td>
</tr>
</tbody>
</table>

**AVG. = 2.60967**

**APPRX STND DEV = 0.02289**

**NO. OF ENERGY LEVELS = 5.**

\[ \text{LZERO1} = 0.16199425 \times 10^{-6} \]

\[ \text{LZERO2} = 0.44016633 \times 10^{-6} \]

\[ \text{NLMEAN} = 2.61094216 \]

**MEAN = 2.61094216**

\[ \text{NLSTND. DEV.} = 0.02415103 \]

**STANDARD DEVIATION = 0.02415103**

**NO. OF ITERATIONS = 3**

**CALCULATION OF ALL-FIRE AND NO-FIRE ENERGY LEVELS**

\[ \text{G} \times \text{SQUARED} = 0.826383 \]

\[ \text{H} \times \text{SQUARED} = 1.496004 \]

\[ \text{STAND. ERROR OF MEAN SQRD.} = 0.00001932 \]

\[ \text{STAND. ERROR OF STAND. DEV. SQRD.} = 0.00003498 \]

\[ \text{STAND. ERROR AT P PERCENT HEIGHT} = 0.01916319 \]

99.9% FIRE AT 50% CONFIDENCE LEVEL = 2.68556884

99.9% FIRE AT 95% CONFIDENCE LEVEL = 2.53631544

99.9% NO FIRE AT 50% CONFIDENCE LEVEL = 2.68556884

99.9% NO FIRE AT 95% CONFIDENCE LEVEL = 2.53631544

0.1% FIRE AT 95% CONFIDENCE LEVEL = 2.49772042

0.1% NO FIRE AT 95% CONFIDENCE LEVEL = 2.49772042
### ENERGY LEVELS

<table>
<thead>
<tr>
<th>ENERGY LEVEL</th>
<th>NO. OF FIRES/FAILS</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.14301</td>
<td>0.2.</td>
</tr>
<tr>
<td>0.16435</td>
<td>3.9.</td>
</tr>
<tr>
<td>0.18469</td>
<td>10.10.</td>
</tr>
<tr>
<td>0.20683</td>
<td>11.3.</td>
</tr>
<tr>
<td>0.22531</td>
<td>3.0.</td>
</tr>
</tbody>
</table>

### CALCULATION OF ALL-FIRE AND NO-FIRE ENERGY LEVELS

- **AVG.** = 0.18533
- **APPRX STND DEV** = 0.02337
- **NO. OF ENERGY LEVELS** = 5
- **ZER01** = 0.15720725E-05
- **ZER02** = 0.10360964E-07
- **NLMEAN** = 0.18430348
- **MEAN** = 0.18430348
- **NLSTND. DEV.** = 0.02588342
- **STANDARD DEVIATION** = 0.02588342
- **NO. OF ITERATIONS** = 4
- **GSQUARED** = 0.835367
- **HSQUARED** = 1.510502
- **STAND. ERROR OF MEAN SQRD.** = 0.00002206
- **STAND. ERROR OF STAND. DEV. SQRD.** = 0.00003989
- **STAND. ERROR AT P PERCENT HEIGHT** = 0.01945744
- **99.9FIRE AT 50CONFIDENCE LEVEL** = 0.26429325
- **0.1FIRE AT 50CONFIDENCE LEVEL** = 0.10432371
- **99.90FIRE AT 95.00CONFIDENCE LEVEL** = 0.305438
- **0.10FIRE AT 95.00CONFIDENCE LEVEL** = 0.063157
## Calculation of Mean and Standard Deviation by Solution of Maximum Likelihood Equations

### Energy Levels vs. No. of Fires/Fails

<table>
<thead>
<tr>
<th>Energy Level</th>
<th>No. of Fires/Fails</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.40993</td>
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<td>12.0</td>
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<tr>
<td>2.46982</td>
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- **Average (AVG.):** 2.44130
- **Approximate Standard Deviation (APPRX STND DEV):** 0.01477
- **No. of Energy Levels:** 5
- **Zero1:** 0.26669353E-04
- **Zero2:** 0.75314892E-06
- **NLMean:** 2.44135469
- **Mean:** 2.44135469
- **NLSSTND_DEV:** 0.01453041
- **Standard Deviation:** 0.01453041
- **No. of Iterations:** 2

### Calculation of All-Fire and No-Fire Energy Levels

- **G-Squared:** 0.785916
- **H-Squared:** 1.171202
- **Standard Error of Mean Squared:** 0.00000663
- **Standard Error of Standard Deviation Squared:** 0.00000989
- **Standard Error at 99.9% Confidence Level:** 0.01005038

- **99.9% Fire at 50% Confidence Level:** 2.48625368
- **0.1% Fire at 50% Confidence Level:** 2.39645568
- **99.9% Fire at 95.0% Confidence Level:** 2.50690502
- **0.1% Fire at 95.0% Confidence Level:** 2.37580431
### TABLE 1

**EFFECT OF SAMPLE SIZE ON THE MAXIMUM LIKELIHOOD ESTIMATES OF THE MEAN AND STANDARD DEVIATION AND ON THE APPROXIMATIONS OF THESE ESTIMATES**

<table>
<thead>
<tr>
<th>Lot Size</th>
<th>Maximum Likelihood</th>
<th>Approximation</th>
<th>Absolute Difference</th>
<th>Relative Difference</th>
<th>Mean</th>
<th>Absolute Difference</th>
<th>Relative Difference</th>
<th>Standard Deviation</th>
<th>Approximation</th>
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<tr>
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<td>2.19394</td>
<td>0.00358</td>
<td>0.17%</td>
<td>0.12787482</td>
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<td>0.010335</td>
<td>8.08</td>
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<td>6.22</td>
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Note: Values are in Log units.

Relative Difference = \( \frac{\text{Absolute Difference}}{\text{Maximum Likelihood Value}} \) \times 100
The maximum likelihood equations used to determine estimates of the mean and standard deviation for the Bruceton method of statistical analysis were solved numerically. A digital computer algorithm was developed to reduce the time associated with the numerical calculations. Test data was evaluated to determine the relative difference between values of the mean and standard deviation as obtained from the numerical solution of the maximum likelihood equations and as obtained from formulae approximating the maximum likelihood equations. The results showed that the differences between standard deviations were more pronounced.
### UNCLASSIFIED

**Security Classification**

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