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# MEASURABLE, TRANSFERABLE, COMPARABLE UTILITY AND MONEY

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# Measurable, Transferable, Comparable Utility and Money*

Martin Shubik

## 1. Introduction

Even now there still exists a certain amount of confusion concerning the meaning of "measurable utility" (or utility measurable up to a linear transformation) and its relation to a money of constant marginal utility. There are furthermore two other problems related, but basically different from the problem of the measurability of utility. They are the possibility of comparison between individuals; and the existence of a transferable commodity or "money" between individuals.

Although in general most economists doubt the validity or the use of making interpersonal comparisons, politicians are forced to do this every day. Furthermore in public policy and welfare decisions A is taxed in money to pay B in money. Whether it is possible to give to B at the same rate as we tax A is a problem we wish to examine in our investigation of the meaning of transferable utility.

The purpose of this discussion is to make clear the distinctions behind various assumptions; not to establish their "correctness" or

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falsehood. It should be stressed however, that without assumptions such as measurability, comparability and transferability, little meaning can be attached to concepts such as equality and fair division, or other such words which appear daily in the economic platforms of political parties.[1]

## 2. Utility Measures and the Individual

Dividing difficulties, this section concentrates solely upon problems involving the single individual, the measurability of utility and the role of money. In order to make our argument as simple as possible, we restrict ourselves for the most part to considering a world with only two goods and occasionally three.

### 2.1 Ordinal and Cardinal Utility

Suppose that there exists only two goods and that an individual has preferences for bundles of these two goods which can be described by the indifference curve analysis. For ease, suppose the indifference curves of the individual can be described by $xy$ where $x$ is the amount of the first commodity and $y$ the amount of the second. If we have only an ordinal measure for the individual then $\sqrt{xy}$, $xy$, $\log xy$ or $(xy)^2$ are all equally good descriptions. They all preserve the same

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1. On fair division, see Luce, R. D. and Raiffa H., Games and Decisions, New York, Wiley, 1957, Ch. VI and Shubik, M., Strategy and Market Structure, New York, Wiley, 1959, Appendix B.

ordering of preference between any two bundles; and as can be easily seen the marginal rate of substitution between $x$ and $y$ is the same under

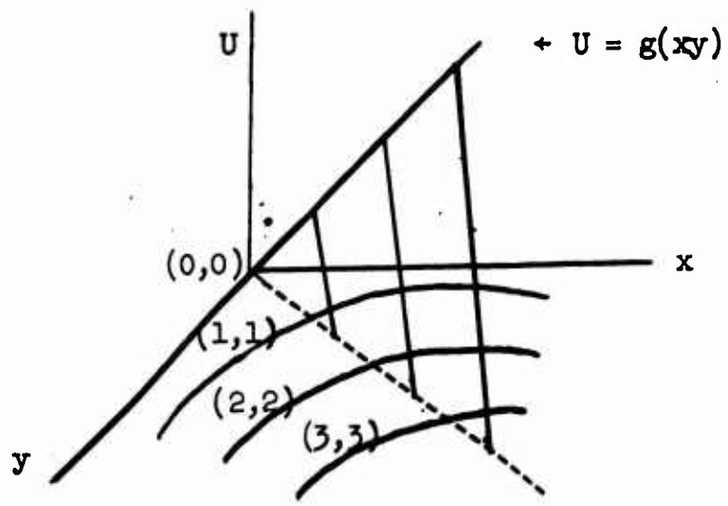


Figure 1

each of these transformations; for example

$$\frac{\partial \sqrt{xy}}{\partial x} \Big/ \frac{\partial \sqrt{xy}}{\partial y} = \frac{\partial \log xy}{\partial x} \Big/ \frac{\partial \log xy}{\partial y} = y/x \ .$$

The meaning of an ordinal scale is that in the u-axis in Figure 1 any order preserving transformation $g$, can be made without changing the contours in the $x$ and $y$ plane. Only the curvature of

the function labelled $u = g(xy)$ will change. The case of $u=(xy)$ is illustrated. If we had used $u = \sqrt{xy}$ then for any fixed ratio between $x$ and $y$ this curve would be a straight line. This can be seen immediately by considering the bundles (1,1), (2,2) and (3,3). In the first scale, the values of $u$ are 1, 4 and 9; and in the second scale the values are 1, 2 and 3.

The meaning of the existence of a cardinal utility has nothing whatsoever to do with the shape of the indifference curves, in other words, the contour may in the $x$ and $y$ plane remain the same. The restriction is in the $u$ axis. Instead of being able to change a utility function $U(x,y)$ for another one $g(U(x,y))$, if $U(x,y)$ satisfies then only other utility functions of the form $aU(x,y) + b$ will do.

Going back to the original example, suppose we have a set of indifference curves which can be represented by $xy$, $\sqrt{xy}$ or $g(xy)$ to be more general; if we consider the possibility of choice involving risk, we will be able to show that only one of these forms and linear transformations of it will fit the facts. As all bundles on the same indifference curve will have the same utility whether there exists an ordinal or a cardinal utility we need only examine risk choices involving as "prizes" a bundle of goods on different indifference curves. Let us consider four prizes (0,0),[1] (1,1), (2,2) and (3,3) call them $A_1$, $A_2$, $A_3$, and $A_4$. Suppose we offered the individual a series of gambles

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1. If we consider (0,0) there is some difficulty in defining marginal rate of substitution as this is a degenerate case; however, for our purposes this difficulty is not relevant at this time.

and discovered in our experiment the following information: He is indifferent between

$$A_2 \quad \text{and} \quad ({}^8/_9 A_1 \,, \quad {}^1/_9 A_4) \quad \text{and also}$$

$$\text{between} \quad A_3 \quad \text{and} \quad ({}^5/_9 A_1 \,, \quad {}^4/_9 A_4) \,.$$

If we regard $A_1$, the prize of (0,0) as having a zero utility and $A_4$ as having a utility of 1, then from the above we can call the utility of $A_2$, ${}^1/_9$ and $A_3$, ${}^4/_9$. But only the utility functions $U = xy$ or $U' = axy + b$ will satisfy this. $\sqrt{xy}$, $(xy)^2$ and so forth will <u>not</u> be consistent with our observations. This means that except for a number "b" which fixes the zero point on the scale and for a number "a" which fixes the size of the unit we can attach specific "altitude" numbers to the indifference curves. Here the value of $b = 0$ for (0,0) is fairly natural; had we called $A_4 = 9$ then $A_2$ would have been 1 and $A_3$ would have been 4. This is the same type of measure that exists for temperature. The difference between the fahrenheit and centigrade scales is that for the first $a = {}^9/_5$ and $b = 32$ and for the second $a = 1$ and $b = 0$; thus $5°c = {}^9/_5 (5) + 32 = 41°F$.

## 2.2 The Marginal Utility of Money

We now examine a completely different problem. This is the marginal utility of money. There are three references of interest, they are: Bernouilli, Marshall's implicit assumption of a constant marginal utility of money for the consumer and the work of Friedman and Savage in

the measurement of the utility for money. It appears that in each case, the money involved is the institutional "stuff" called money by the man on the street and not a poorly conceived aggregation called "buying power" or income or budget constraint which conceals the difference between money as an ordinary commodity, a special commodity or a numeraire and an institutionalized fiction.

It is possible to make a very naive assumption that money is some sort of special "utility pill" and that the individual has a linear utility for it. This would mean that $U(x) = ax + b$ where $x$ is the amount of money. If $U(0) = 0$ then $U(x) = ax$ where $a$ is the parameter describing the unit of measurement. We could choose $a = 1$ giving $U(x) = x$.

Bernouilli[2] observed that if this were true a paradox would arise. Suppose an individual with a linear utility for money of $U(x)$ were offered the following game. He is to toss a coin until the first head appears; if it appears on the nth trial he is paid $2^n$ and the game ends. What is the value of this game?

$$V = \frac{1}{2}(2) + \frac{1}{(2)^2}(2)^2 + \ldots + \frac{1}{(2)^n} + \ldots$$

$$= 1 + 1 + \ldots + 1 + \ldots \to \infty .$$

2. Daniel Bernouilli, Saint Petersburg Paradox. See Econometrica, Vol. 22, No. 1, 23-36, 1954.

The amount that an individual with utility $U(x) = x$ should be willing to pay for this gamble appears to be unbounded. Clearly however, most individuals would not even pay very much for a chance to play this game hence, the assumption of $U(x) = x$ does not appear to be tenable. Bernouilli suggested $U(x) = \log_{10} x$ . This gives:

$$V = \frac{1}{2} \log_{10} 2 + \ldots \frac{n}{2^n} \log_{10} 2 = 2 \log_{10} 2$$

which is a fairly small amount. His choice of the function $U(x) = \log_{10} x$ appears to be more or less arbitrary; (although the Weber-Fechner law is in its favor) however, his point that it is unreasonable to assume a linear utility for money over a large range appears to have been well established.

By considering the possibility of gambles for money, Friedman and Savage[3] were able to suggest a utility function for money of the sort indicated in Figure 2 below An argument involving the change in social

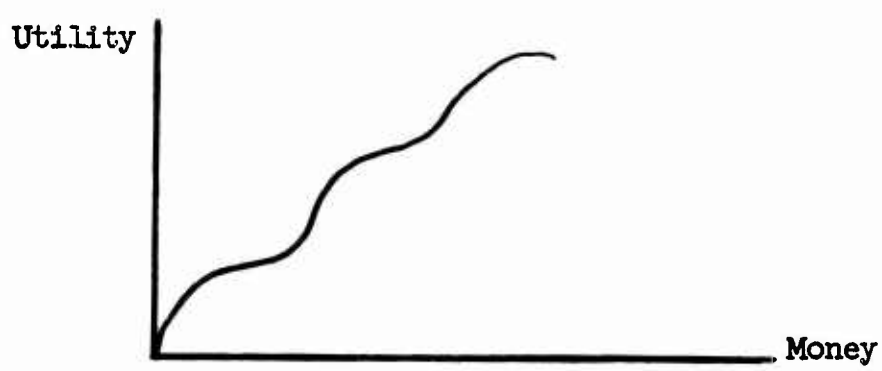


Figure 2

3. Friedman, M. and Savage, L. J., "The Utility Analysis of Choices Involving Risk", Journal of Political Economy, Vol, LVI, 1948, pp 279-304.

status was advanced to explain the inflexions. This does not concern us here. It is merely to be observed that the function is not linear.

While Bernouilli and Friedman and Savage have been dealing with the utility of money over a large range, it may be argued that Marshall[4] in his partial equilibrium analysis was only concerned with the change in the utility of money over a small range; the amount spent on one consumer item. If this were correct then suppose that the consumer's world consisted of only two commodities one good and the other a "money". A constant marginal utility of money implies that there is no income substitution effect. At all "incomes" the individual will buy the same amount of the good. By the word income in this context we mean the amount of money initially available to the consumer. Figure 3 illustrates this. Suppose that y is "money" then the assumption of a constant marginal utility for money

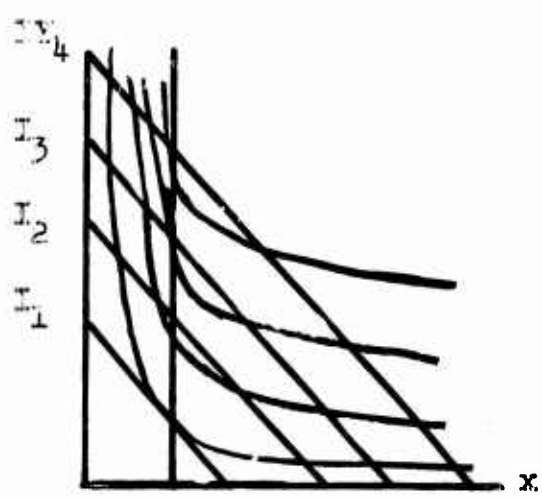


Figure 3

4. Marshall, A., Principles of Economics (8th Ed., MacMillan: London 1920), Book III, Ch. III and VI.

calls for the utility function $U(x,y)$ to be of a very special type it can be decomposed so that $U(x,y) = \tilde{U}(x) + \lambda y$ from which we see immediately that $\frac{\partial U}{\partial y} = \lambda$, a constant marginal utility for money. For the income levels $I_1, I_2, I_3, I_4$ and so forth the points of tangency of the budget constraints with the indifference map lie on a vertical line. The utility contours are parallel.

We must distinguish between two assumptions. They are (1) the marginal utility of money is constant; or (2) the marginal rate of substitution between money and other goods depends only on the other goods. Both of these assumptions call for $U(x,y) = \tilde{U}(x) + \lambda y$ however the first is much stronger than the second. The first limits the possibilities of transformations in the utility third dimension, while the second does not. This can be seen in Figure 4 and is also illustrated in the example calculated where we assume $U(x,y) = x^2 + y$.

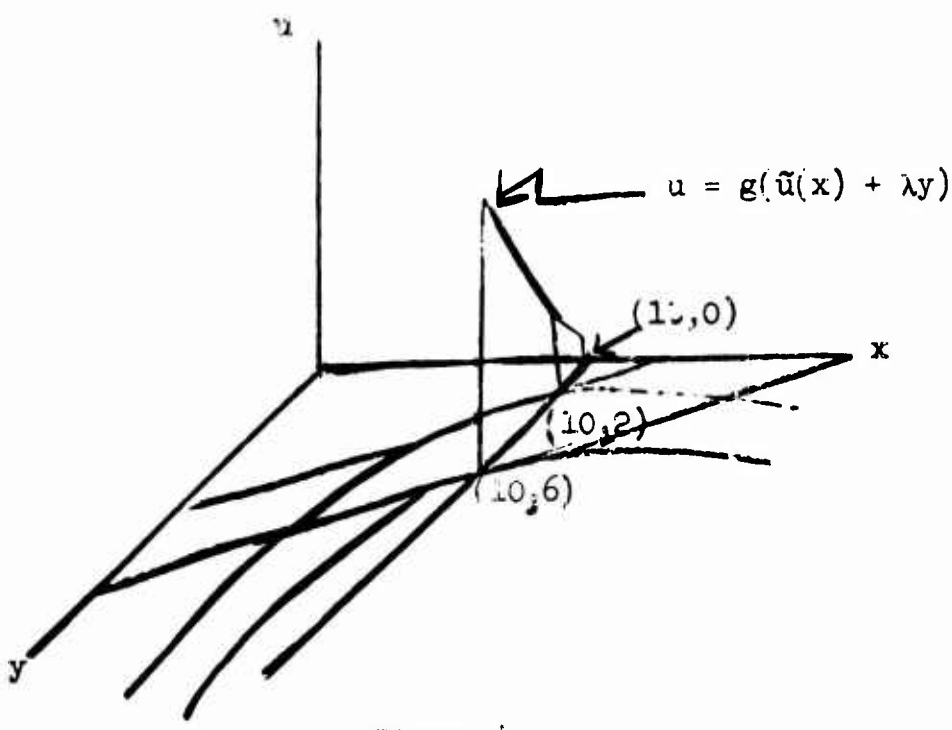


Figure 4

If we want the marginal utility of money to remain constant then we cannot subject $U(x,y)$ to any monotonic transformation. The only ones which will satisfy are linear ones, in other words, $g(U(x,y)) = aU(x,y) + b$ . Here we see $\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}(a(\tilde{U}(x) + \lambda y) + b) = a\lambda$ a constant. A transformation such as $g(\tilde{U}(x) + \lambda y) = (\tilde{U}(x) + \lambda y)^2$ does not preserve the constancy of the marginal utility of money. The example of $x^2 + y$ shows this.

$$\frac{\partial(x^2 + y)}{\partial y} = 1 \quad \text{but} \quad \frac{\partial(x^2 + y)^2}{\partial y} = 2(x^2 + y) .$$

Hence the assumption of a constant marginal utility for a special commodity money is equivalent to the existence of a linear utility measure and the curve in the $u$ dimension of Figure 4 will be restricted to the form $U = a(\tilde{U}(x) + \lambda y) + b$ and if the amount of $x$ is held constant then this is a straight line, showing the linear relationship between money and utility.

If we only wish to assume that the marginal rate of substitution between money and other goods is independent of the amount of money then any transformation $g(\tilde{U}(x) + \lambda y)$ where $g$ is monotonic will be feasible. For example suppose $g(x^2 + y) = (x^2 + y)^2$

$$\frac{\partial g}{\partial x} \Big/ \frac{\partial g}{\partial y} = \frac{4(x^2 + y)x}{2(x^2 + y)} = 2x$$

and
$$\frac{\partial(x^2 + y)}{\partial x} \Big/ \frac{\partial(x^2 + y)}{\partial y} = 2x \ .$$

Here although we can assign to each indifference curve in the $x$ and $y$ plane a number in terms of this $y$ commodity we cannot say that there exists a linear utility measure. In order to do that we would have to perform some experiments with gambles between the outcomes, say (10,0), (10,2), (10,6) and others as shown in Figure 4. Call (10,0) $A_1$, (10,2) $A_2$, (10,6) $A_3$ and (10,10) $A_4$; if we found $A_2$ to be indifferent to $(\frac{4}{5} A_1, \frac{1}{5} A_4)$ and $A_3$ indifferent to $(\frac{2}{5} A_1, \frac{3}{5} A_4)$ then the only utility scales satisfactory would be of the form $a(x^2 + y) + b$ . If on the other hand we found $A_2$ indifferent to $(\frac{1696}{2100} A_1, \frac{404}{2100} A_4)$ and $A_3$ to $(\frac{864}{2100} A_1, \frac{1236}{2100} A_4)$ then the scales would be of the form $a(x^2 + y)^2 + b$ .

It should be noted that in the first case; that of constant marginal utility, a linear measurement of utility was obtained without using probabilities and investigating gambles, in the second case in order to obtain the measure we utilized gambles.

## 2.3 Summary

If there exist $n$ commodities and an individual is assumed to have only an ordinal measure on them we can represent his preferences by

$U(x_1, x_2, \ldots, x_n)$ or by any transformed utility function of the form $g + U(x_1, x_2, \ldots, x_n))$ where $g$ is monotonic.

If we take into consideration the possibility of choice among risky outcomes we find that it becomes possible to limit the description of an individual's preference system to a specific $U(x_1, x_2, \ldots, x_n)$ or a related utility function of the form $aU(x_1, x_2, \ldots, x_n) + b$ .

If we make the very special assumption that there exists a commodity, say commodity $n$ for which the individual has a constant marginal utility this implies that $U(x_1, x_2, \ldots, x_{n-1}, x_n) = \tilde{U}(x_1, x_2, \ldots, x_{n-1}) + \lambda x_n$ and it <u>also</u> implies that only transformations of the type $aU(x_1, x_2, \ldots, x_{n-1}, x_n) + b$ will preserve the property of constant marginal utility for commodity $n$ . Here we obtain a linear measure of utility without considering alternatives involving risk.

We can make a special but less restrictive assumption that there exists a commodity, say commodity $n$ for which the individual's marginal rate of substitution is dependent only on the quantities held of the other commodities. This implies that $U(x_1, x_2, \ldots, x_{n-1}, x_n) = \tilde{U}(x_1, x_n, \ldots, x_{n-1}) + \lambda n_n$; however, the transformation $g(\tilde{U}(x_1, x_2, \ldots, x_{n-1}) + \lambda x_n)$ will be feasible. In order to obtain a measurable utility here we would have to consider alternatives involving risk.

## 3. Consumer Maximization and Budget Constraints

It has been a common practice to use a two commodity diagram to illustrate the manner in which a consumer choses his preferred consumption limited by his budget constraints. Often the demonstration is phrased in terms of an individual with a utility function $\varphi(x,y)$ and an income $I$ which he spends on the two commodities so that $I = p_1x+p_2y$ where $p_1$ and $p_2$ are the prices of the two commodities[5].

It is well known that for the consumer optimization problem where choice does not involve risky alternatives the assumption of a cardinal utility measure does not lead to results different from the assumption of an ordinal scale. Figure 5 is a

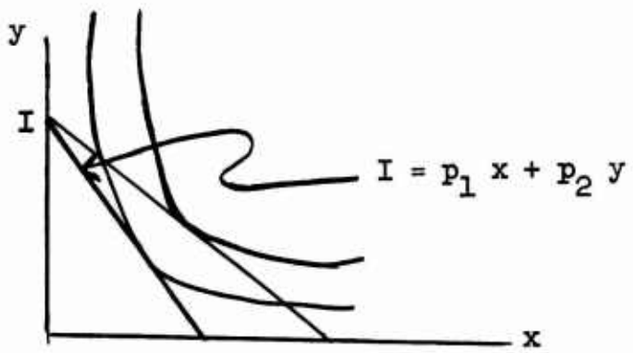


Figure 5

5. See for instance, Henderson, J. M. and Quandt, R. E., Microeconomic Theory (McGraw-Hill: New York 1958) Ch. 2.

standard diagram illustrating how a consumer will allocate his "income" $I$. But what is this income? How is it measured? There are several different views which can be taken. They depend upon the problem we are looking at and the degree of approximation we are interested in making. In a barter economy the income of the individual trader may be given in terms of the market value of his assets. Thus if we assume that he comes to market with $a$ units of the first and $0$ of the second commodity then his optimizations program is described as:

$$\text{Max} \quad \varphi(x,y)$$

$$\text{subject to} \quad p_1x + p_2y = p_1a.$$

If we wished to assume the existence of a special institutionally accepted commodity called money which everyone is willing to accept "in payment of all debts, public and private" then the optimizations program is:

$$\text{Max} \quad \varphi(x,y)$$

$$\text{subject to} \quad p_1x + p_2y = I$$

The difference between these two is seen when we wish to describe the effect of a "change in income" to the consumer. In the first case where we assume his initial resources are $(a,0)$ we could change his income by giving him a larger endowment of goods or by increasing the price ratio $p_1/p_2$. In the second case his endowment is $(0,0, I)$, we can increase his income by giving him more money or goods or by decreasing prices $p_1$ and/or $p_2$.

In the first case the price system is defined for 2 "real goods" and is homogeneous of order 0 ; in the second case it is defined for 2 "real goods" and money and is homogeneous of order 1 in the goods as the introduction of money as an entity with a real existence to the consumer adds a constraint to the price system.

Hicks has suggested [6] that the "income" illustrated on the y axes in Figure 4 may be regarded as a composite of all commodities other than x . We may interpret Marshall's view as that, with the added assumption that $\varphi(x,y) = f(x) + \lambda y$ as is shown in Figures 3 and 4 .

Each one of the four approaches noted above may be regarded as a different approximation. If we are interested in the one period behavior of a consumer in isolation then the first three are all equivalent. Marshall's approximation will hold only for goods where there is no income effect (up to the degree of approximation of interest) . If we wish to study general equilibrium analysis in a static world with no frictions, uncertainly or monetary controls where all individuals are assumed to act as pure competitors, then once more the first three models may be looked upon as equivalent, and the economic theorist may argue that as they are all equivalent, then by Occam's razor there is no need to introduce a model with fiat money and hence risk money illusions.

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6. Hicks, J. R., Value and Capital (2nd Ed. Oxford: London 1946) p. 33.

The understanding of the relationship between monetary economics and general equilibrium theories is still slight. It is beyond the scope of this article to add much analysis at this point. However, we should stress the view that all of the models dealt with are approximations and no one is more "institution-free" than any other. It appears that the models without an explicit fiat money are poorer approximations of the world we live in than ones with money. Peoples' incomes tend to be in money. At least for the study of individual behavior, the individual is so small compared to the economy that we could more easily give him some more green paper newly printed by the government, to increase his income as we could give him an extra horse to trade in Böhm Bowerk's horse market. Furthermore for the vast majority of the population the major or only commodity sold is labor and as can be seen from the periods of employment of individuals, formal or informal contractual arrangements determining money income from the sale of labor tend to be such that for the most part the individual knows his money income before he buys.

Without exploring the problem further at this time, we note that although the formal introduction of money as an $n + 1^{st}$ commodity appears to have no effect on general equilibrium analysis, it does make a difference if the economy is to be viewed as a non-cooperative game.[7] In the latter, Walras' law is not assumed. Supply does not necessarily have to equal demand, nor budget constraints always be met. In general they will, but when they fail monetary penalities specified by law are used on the individuals who cannot deliver what they have sold. Furthermore bankruptcy conditions are applied to those who are short of money.

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7. See Shapley, L. S. and Shubik, M.

## 4. Comparability, Transferability and Money

Our previous analysis has examined only the problems of utility measurement of the individual's preferences. As far as the economics of the isolated consumer are concerned, the existence of cardinal or ordinal utilities make no difference to the type of behavior we would predict. This is not the case for problems involving two or more players, such as bilateral monopoly for example.

### 4.1 Comparability of Utilities

If we limit ourselves to ordinal measures for every individual, clearly we are unable to make interpersonal comparisons. If $U_1(x_1, x_2 \ldots x_n)$ and $U_2(x_1, x_2, \ldots, x_n)$ are utility functions for two individuals we can make two monotonic transformations $g(U_1)$ and $f(U_2)$ in which the measures in each scale will have been changed.

What is the situation with a measurable utility? Given $U_1$ and $U_2$ we can still make transformations $a_1U_1 + b_1$ and $a_2U_2 + B_2$ of the utility scales and hence the statement that a certain outcome or bundle of goods is worth twice as much to A as it is to B cannot be made. It is interesting to observe that we can still make certain comparative statements if we are at least able to fix a common zero point.

Suppose that we can agree that in the measures of both players the bundle of $(0, 0, \ldots, 0)$ has zero utility $U_1(0, 0, \ldots, 0) = 0$ and $U_2(0, 0, \ldots, 0) = 0$. If we want this to hold then we have fixed the

parameters $b_1$ and $b_2$ in the set of linear transformations of utility as 0 . This leaves only the $a_1$ and the $a_2$ to be determined.

Consider two outcomes $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ . Even without comparing utilities we can make a statement such as x is worth twice as much as y in A's utility scale but x is worth four times as much as y in B's utility scale. For example, let the utility functions which can represent A's preferences be $a_1(x_1 x_2)^{1/2}$ and those for B, $a_2(x_1 x_2)$ where $a_1$ and $a_2$ are parameters which can have any positive value. Let the two outcomes or bundles of goods be (1,1) and (2,2) . Where in the scale of A's utility:

$$\frac{U_1(2,2)}{U_1(1,1)} = \frac{a_1 (2.2)^{1/2}}{a_1 (1.1)^{1/2}} = 2 ,$$

and in the scale of B's utility:

$$\frac{U_2(2,2)}{U_2(1,1)} = \frac{a_2 (2.2)}{a_2 (1.1)} = 4 .$$

Thus without comparing individual utilities it is possible to make a comparative statement about the size of changes in welfare referring to the utility scales of each individual.

For complete comparison of utility it becomes necessary to fix the values of the parameters $a_1$, $a_2$, $b_1$ and $b_2$. This leaves no degrees of freedom in the choice of utility functions between individuals. We could still have one overall linear transformation for all society but between all of m individuals the $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_m$ would be fixed. Consider two individuals, given specific utility functions $a_1U_1 + b$, and $a_2U_2 + b_2$, $k(a_1U_1 + b_1) + c$ and $k(a_2U_2 + b_2) + c$ would also serve.

Without comparability it is not possible to give meaning to joint maximization, as the sum $a_1U_1 + a_2U_2$ is not defined if $a_1$ and $a_2$ can be arbitrary. We have to make do with the much weaker and more general condition of Pareto optimality.

The assumption of the existence of a special money commodity with constant marginal utility does not imply comparability of utility, joint maximum is not defined, as can be seen in the example following: Let $U_1(x_1, x_2) = x_1^2 + 2x_2$ and $U_2(x_1, x_2) = x_1^4 + 3x_2$. The commodity 2 is a "money" to both in the sense that it has a constant marginal utility to both. As we have <u>not</u> assumed comparability then the functions $a_1(x_1^2 + 2x_2) + b_1$ and $a_2(x_1^4 + 3x_2) + b_2$ are also possible utility functions. Suppose the two wished to jointly maximize welfare from 2 units of each of the first and second goods. We consider two cases, the first where $a_1 = a_2 = 1$ and $b_1 = b_2 = 0$ and the second where $a_1 = 100$, $a_2 = 1$, $b_1 = b_2 = 0$ :

$$\max_{x_1} \max_{x_2} \left[ (x_1^2 + 2x_2) + ((2 - x_1)^4 + 3(2 - x_2)) \right] = (2)^4 + 3(2)$$

$$\max_{x_1} \max_{x_2} \left[ 100\ (x_1^2 + 2x_2) + ((2 - x_1)^4 + 3(2 - x_2) \right] = 100((2)^2 + 2(2)) \cdot$$

In the first case everything goes to the second person while in the second case everything goes to the first.

## 4.2 Transferability of Utilities

The phrase "transferability of utility" is possibly confusing. Even with an ordinal measure of utility, most things have positive value and can be transferred between individuals. In this sense almost everything has a utility and can be transferred. What is usually meant by the existence of a transferable utility is that there exists a commodity or symb h that the transfer of the commodity, piece of stone, box of gold r number in a bank account is done at a constant rate of marginal substitution to all parties involved.

We may consider two cases, they are:

(1) Transferability without comparability

and

(2) Transferability with comparability.

4.2.1. Transferability Without Comparability. Using the same function as in an example in 4.1 we consider the utility functions $a_1(x_1^2 + 2x_2) + b_1$ and $a_2(x_1^4 + 3x_2) + b_2$ . As there is no comparison of utilities between

the two individuals the parameters can take many values. Suppose that we can select a zero point on the utility scales of both. Say, to each individual (0,0) is worth zero in his own utility scale. This gives us $b_1 = b_2 = 0$ .

Suppose the first individual has an initial bundle of goods which we will call (A,B) and the second (C,D). Consider first a transfer of 1 and then k units of the second good from the first to the second person. We compare the situations of each before and after the transfers.

$$\frac{a_1(A^2 + 2(B-k)) - a_1(A^2 + 2B)}{a_1(A^2 + 2(B-1) - a_1(A^2 + 2B)} = \frac{-2a_1k}{-2a_1} = k$$

$$\frac{a_2(C^4 + 3(D+k)) - a_2(C^4 + 3D)}{a_2(C^4 + 3(D+1)) - a_2(C^4 + 3D)} = \frac{3a_2k}{3a_2} = k$$

We can say that the second commodity has the property such that to everyone at any level of asset holdings a transfer of k units will be worth k times more than a transfer of 1 unit in his utility scale.

4.2.2. <u>Transferability With Comparability</u>. With comparison possible, the $a_1$, $a_2$, $b_1$ and $b_2$ take on specific values. Suppose for example $a_1 = 2$, $a_2 = 1$, $b_1 = b_2 = 0$ . We have:

$$U_1(x_1, x_2) = 2(x_1^2 + 2x_2) = 2x_1^2 + 4x_2$$

$$U_2(x_1, x_2) = x_1^4 + 3x_2 .$$

We can now state that the transfer of $k$ units of the second good will result in a change of $4k$ "utiles" for the first and $3k$ utiles for the second.

A completely ideal money would not only have a constant marginal utility to all, but the same marginal utility to all thus, for example, if:

$$U_1(x_1, x_2) = \frac{1}{2}x_1^2 + x_2$$
$$U_2(x_1, x_2) = \frac{1}{3}x_1^4 + x_2$$

we could say that a transfer of a unit of the second commodity always results in the transfer of "1 utile."

4.2.3. Comparability Without Transferability. Suppose that two individuals are able to compare their value systems. Furthermore, suppose that they jointly inherit a very valuable painting, they are not permitted to sell it and neither of them has enough assets to buy the other one out. How can they share the painting? They physical properties of the situation cause difficulties in the sharing of this wealth.

If they cut the painting in half they would destroy its value to both of them, hence this is no answer. They could alternate their possession of the painting; however, if they lived far apart and transportation costs were very high this would not be practical. Another way would be to use a random device to decide who gets the picture. The

introduction of chance is another method for obtaining divisibility of otherwise indivisible items. Figure 6 below shows the utility diagram for the two individuals. In the example we assume that they can decide that the whole picture is worth 100 to the first and 80 to the second, and that the cut-up painting is almost worthless to both of them. The

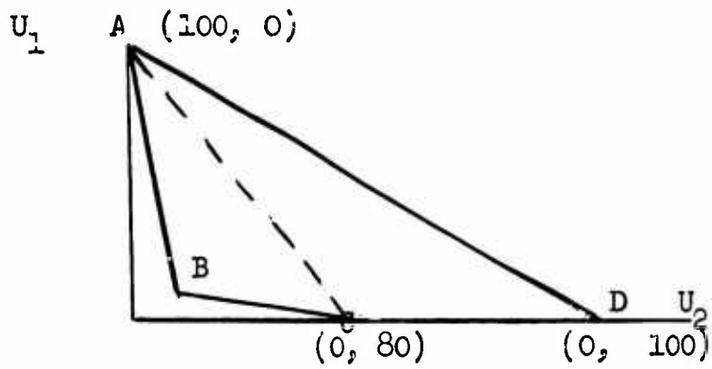


Figure 6

curve ABC represents the Pareto optimal surface or the values to both players of any division (a, 1-a) $0 \leq a \leq 1$, of the painting. The dotted line AC represents what they could obtain in expected utility by randomizing. If they used the probability mix $\gamma = 1/2$, $(1 - \gamma) = 1/2$ where $\gamma$ is the probability that the first keeps the painting, then the first has an expected gain of 50 utiles and the second 40 . If they used probabilities of $\gamma = 4/9$ and $(1 - \gamma) = 5/9$ they would both stand to gain the same amount of 400/9.

If there existed a linear utility money which both had in sufficient quantity, then the first would receive the painting and he would make a payment to the second along the line AD.

Without the use of gambles or of a side-payment commodity, even though the individuals can compare utilities the indivisibility makes it impossible to both jointly optimize and share.

A more general example is now considered. Suppose $U_1(x_1, x_2) = x_1x_2$ and $U_2(x_1, x_2) = x_2 - (x_1 - 4)^2$ defined for $0 \leq x \leq 4$. Suppose initial conditions are $(4,0)$ and $(0,16)$ respectively. Then including the constraints on the total amount of goods available the first will try to maximize $\tilde{U}_1 = (4 - x_1)x_2$ and the second $\tilde{U}_2 = (16 - x_2) - (x_1 - 4)^2 = 8x_1 - x_1^2 - x_2$.

The condition for the Pareto optimal surface is

$$\begin{vmatrix} \frac{\partial \tilde{u}_1}{\partial x_1} & \frac{\partial \tilde{u}_1}{\partial x_2} \\ \frac{\partial \tilde{u}_2}{\partial x_1} & \frac{\partial \tilde{u}_2}{\partial x_2} \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} -x_2 & 4 - x_1 \\ 8 - 2x_1 & -1 \end{vmatrix} = 0$$

or $x_2 = 2(4 - x_1)^2$. This gives us the equation for the physical outcomes on the Pareto optimal surface (in the case of two traders this includes the contract

curve of Edgeworth). It does not specify anything about the values of any outcome to the participants. As can be seen from the condition for the Pareto optimal surface the equation for the distribution of commodities is not affected by any monotonic transformations on the utility functions. Only ordinal properties are being used. However, the fact that this curve is not flat tells us that even if utility were measurable and comparable there exists no means by trade to necessarily encourage the participants to jointly maximize and share the proceeds. This result was first pointed out by Edgeworth in his discussion of the bargaining problem in bilateral monopoly.

In this example it is interesting to note that to the second trader, the second commodity could have served as a "money", but not to the first. The second commodity does not have a constant marginal utility to all.

Suppose that utilities could be measured and compared and that functions given for the traders are their unique utility functions. If they wished to jointly maximize, they would maximize:

$$(4 - x_1)x_2 + (16 - x_2) - (x_1 - 4)^2$$

which has a maximum on the contract curve at one end with

$x_1 = \frac{4}{\sqrt{3}}(\sqrt{3} - 1) \simeq 1.7$ and $x_2 = 10\ 2/3$ with a joint maximum of 24.5 .

Figure 5 shows root of the Pareto optimal surface mapped in the utility dimensions of the players. A small table with the

values of four points on the surface is given below.

| $x_1$ | $x_2$ | $u_1$ | $u_2$ | $u_1 + u_2$ |
|---|---|---|---|---|
| 1.7 | 10 2/3 | 24.5 | 0 | 24.5 |
| 2 | 8 | 16 | 4 | 20 |
| 3 | 2 | 2 | 13 | 15 |
| 4 | 0 | 0 | 16 | 16 |

We can see that for these functions the Pareto surface is concave. Merely by trading the two goods available the traders cannot take advantage of the comparability of their utility.

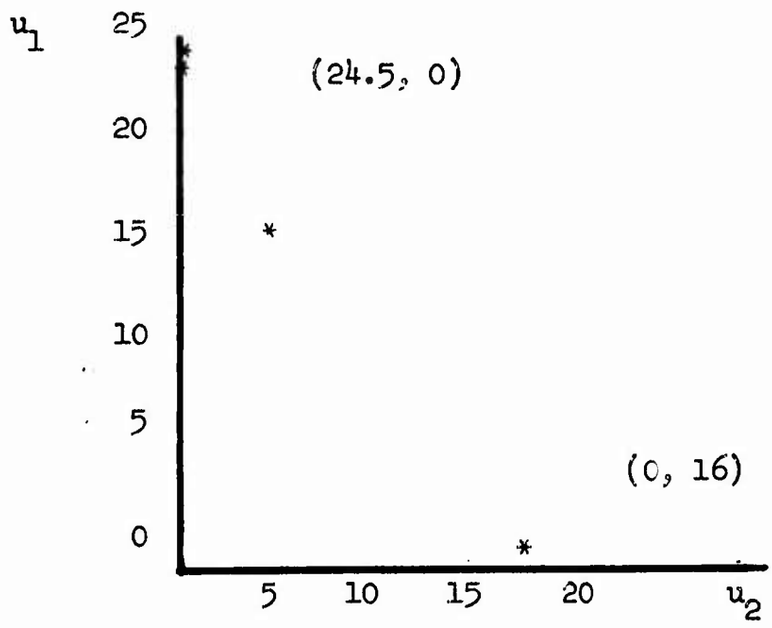


Figure 7

In a situation such as this if we only are interested in optimality

conditions or in studying a shadow price allocation mechanism, then even if the assumption of comparable utilities were reasonable it would not be useful as it is not used in those analyses. If, however, we were interested in some type of arbitration scheme even though a side-payment commodity with constant marginal utility to all does not exist the property of comparability might be used in selecting an outcome.

## 5. Transferability, Fiat and Commodity Money

### 5.1. Linearity of the Pareto Optimal Surface

In our discussion here unless otherwise stated, we confine ourselves to representations of the Pareto optimal surface in the dimensions of the distribution of the goods, not the utilities of the individuals. Both can be drawn but our interest is mainly directed to the commodity space. In terms of an Edgeworth box diagram shown in Figure 7 we are interested in the four dimensions of the $x_1^1$, $x_2^1$, $x_1^2$, $x_2^2$ plane and not in the two extra dimensions of $U_1$ and $U_2$

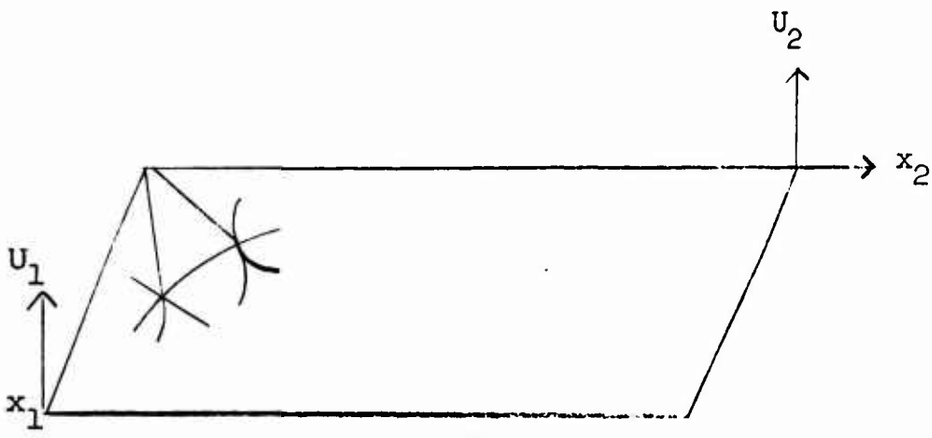


Figure 8

More generally suppose that a society has $n$ commodities which can be produced and $m$ individuals. There are three diagrams which can be drawn to represent the different aspects of production, distribution and utility. The first is the efficient production possibility set or the "commodity space". For example, if there were only two commodities with a production function linking them we would have an efficient set as shown in Figures 8a and 8b depending upon the existence of diminishing or constant marginal productivity.

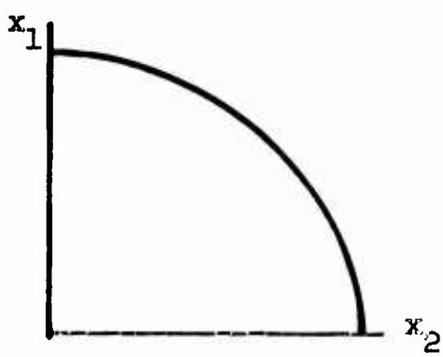


Figure 8a

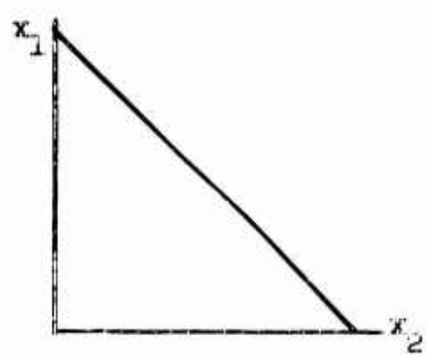


Figure 8b

The second is the distribution set which describes what each person possesses. This exists in a space of $m \times n$ dimensions. In the case of pure exchange the efficient production possibility set is clearly a single point in $n$ dimensions; and the efficient distribution set (a generalization of the Edgeworth contract curve) is a surface in a space with $m\,n - n = n(m-1)$ dimensions. For example consider the usual Edgeworth box $m = 2$ and $n = 2$. We are able to draw it on a piece of

paper because $n(m-1) = 2$ .

The third diagram is for the utility surface and exists in $m$ dimensions.

The linearity, or approximate linearity of the Pareto optimal surface in the distribution space in some or all of its dimensions is closely related to the question of the existence of a transferable utility. There is one special sufficient condition for flatness in $m$ dimensions (one for each player) and a sufficient condition for flatness in $m \times n$ dimensions is also known.

A question that is raised here is how flat or curved do we expect the Pareto optimal surface to be in an economy such as the United States or England? Our first difficulty is to decide upon the concept of curvature we wish to use in the $m \times n$ dimensions. Then there is the problem of trying to obtain even a crude estimate in terms of the gross features of an economy.

The reason for asking the question is that if the surface "is not too heavily curved," for many of the purposes of running an economy and especially welfare economics it may be useful to assume that as a first approximation it can be treated as flat in some region.

The two conditions we discuss are (1) the existence of a commodity which has a marginal rate of substitution independent of the quantity held and (2) the existence of a population with similar tastes. It is also conjectured that the presence of production processes with constant returns to scale tends to flatten out the Pareto optimal surface. The fourth point, a case for an approximation is discussed in 5.2 below.

### 5.1.1. A Linearly Transferable Commodity

If there exists a commodity $n$, such that the utility function for each individual $i$ can be expressed in the form $U_i(x_1, x_2, \ldots, x_{n-1}) + \lambda_i x_n$, then this will cause the Pareto optimal surface to be flat in dimensions involving trade in this commodity. For an example we consider:

$$U_1(x_1, x_2) = x_2 - (4 - x_1)^2 \text{ with initial holdings of } (4,0)$$

$$U_2(x_1, x_2) = \frac{16}{3} x_2 - (5 - x_1)^4 \quad " \quad " \quad " \quad " \quad " \quad (0,117.2)$$

Giving $U_1(A - x_1, x_2) = x_2 - x_1^2$

$$U_2(x_1, B - x_2) = \frac{16}{3}(117.2 - x_2) - (5 - x_1)^4$$

The initial holdings were selected so that $U_1(4,0) = 0$ and $U_2(0,117.2) = 0$. We obtain a degenerate form for the Pareto optimal surface involving only $x_1$ which amounts to saying that there is exactly one trade in $x_1$ which establishes the same marginal rate of substitution between goods 1 and 2 for both traders. Beyond that any payments made in terms of $x_2$ will be on the Pareto optimal surface, which will consist of all points of the form $(1, x_2)$ for the first and $(3, 117.2 - x_2)$ for the second as can be seen from the following:

$$\begin{vmatrix} -2x_1 & 1 \\ -4(5-x_1)^3 & -\frac{16}{3} \end{vmatrix} = 0 \quad \text{or} \quad \frac{32x}{3} = 4(5 - x_1)^3$$

which gives $x_1 = 3$.

Once more we must stress that in this case the existence of a linearly transferable commodity does not imply comparability or even cardinality of utilities ( had we assumed a constant marginal utility for $x_2$ this would have implied cardinality as was shown in 2.2 but not comparability).[8]

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8. The flatness of the Pareto optimal surface seems also to be related to the uniqueness of the competitive equilibrium. It is also known that if all utility functions are of the form $U_i(x_1 \ldots x_{n-1}) + \lambda_i x_n$ this is sufficient for a unique competitive equilibrium point.

It had been conjectured by this writer that if utility functions were of the form $U_i(x_1 \ldots x_{n-1}) + p_i(x_n)$ where $p_i(x_n)$ is a polynomial; that the number of competitive equilibria would be related to the degree of this polynomial. This conjecture was shown to be false by L. S. Shapley as follows:

It is only necessary to look at two commodities and see how the price ratio varies as we move along the contract curve. If this ratio can fluctuate arbitrarily many times, then arbitrarily many CE points are possible. (For if we look far enough away from the contract curve, there will be regions of potential starting points that will be swept over arbitrarily many times by the "price line" extended backwards from the contract curve.)

Let

$$u^i(x, y) = f_i(x) + y - a_i y^2 , \qquad i = 1, 2 .$$

The contract curve is given by

$$(1) \qquad \frac{p_x}{p_y} = \frac{f_1'(x)}{1 - 2a_1 y} = \frac{f_2'(A-x)}{1 - 2a_2(B-y)} ,$$

where (A, B) is the total bundle of goods in the market. Solving, we get

$$(2) \qquad y = \frac{f_2' - f_1' + 2a_2 B f_1'}{2a_2 f_1' + 2a_1 f_2'} .$$

Substituting (2) in (1) shows that the contract price ratio, as a function of $x$ has the form

$$p_x / p_y = c_1 f_1'(x) + c_2 f_2'(A-x) ,$$

where the constants $c_1$ and $c_2$ can be shown to be positive. But $f_1'(x)$ is an essentially arbitrary <u>decreasing</u> function of $x$ , and $f_2'(A-x)$ is an essentially arbitrary <u>increasing</u> function of $x$ . Hence, the ratio can fluctuate as much as we please, and arbitrarily many CE points can result.

5.1.2. Similarity of Tastes

The assumption of similarity of tastes does not imply that utilities can be compared. It does not even imply that they can be individually measured. It is the assumption that all individuals have the same preference ordering for all bundles of goods.

If tastes are the same and the utility functions are homogeneous of order 1 (this is an assumption that there is no income effect) then it can be shown that the Pareto optimal surface in the distribution space will be flat.[9] It was pointed out by L. S. Shapley that the word flat may be misleading in the sense that it is actually linearity of a set in dimensions much lower than that of the whole space. It was further suggested that utility functions of the form $W_i = U_i(x_1, x_2, \ldots x_n) + h(x_{n+1}, \ldots x_{n+m})$ where $h(x_{n+1}, \ldots x_{n+m})$ is the same for all individuals would also give a transferable utility. An example for two individuals is given. Suppose the initial conditions are that the first has $(A,0)$; in words, $A$ units of the first commodity and $0$ of the second. The second has $(0,B)$. Their preferences can each be represented by the utility function $ax_1^{\alpha} x_2^{1-\alpha}$. The equation in general for the Pareto optimal surface for two players with similar tastes is:

$$\begin{vmatrix} \dfrac{\partial U(A-x_1, x_2)}{\partial x_1} & \dfrac{\partial U(A-x_1, x_2)}{\partial x_2} \\[2ex] \dfrac{\partial U(x_1, B-x_2)}{\partial x_1} & \dfrac{\partial U(x_1, B-x_2)}{\partial x_2} \end{vmatrix} = 0 .$$

9. See Chipman, J. S., "A Survey of the Theory of International Trade: Part 2" Econometrica, Vol. 33, No. 4, October 1965, pp. 689-698.

Here

$$\begin{vmatrix} \dfrac{-\alpha a\, x_2^{1-\alpha}}{(A-x_1)^{1-\alpha}} & \dfrac{(1-\alpha)a(A-x)^{\alpha}}{x_2^{\alpha}} \\[2ex] \dfrac{\alpha a(B-x_2)^{1-\alpha}}{x_1^{1-\alpha}} & \dfrac{-(1-\alpha)ax_1^{\alpha}}{(B-x_2)^{\alpha}} \end{vmatrix} = 0$$

or

$$\frac{x_2^{1-\alpha}\, x_1^{\alpha}}{(A-x_1)^{1-\alpha}(B-x_2)^{\alpha}} = \frac{(A-x_1)^{\alpha}(B-x_2)^{1-\alpha}}{x_1^{1-\alpha}\, x_2^{\alpha}}$$

giving $(A-x_1)(B-x_2) = x_1x_2$ or $Bx_1 + Ax_2 = AB$ .

The slope of this straight line is affected only by the total amount of the commodities available.

## 5.2. The Role of Money

There are many properties of money and taxation as weapons in government policy; we are not concerned with these here. At most we wish to consider government only implicitly as a mechanism for redistributing money.

Many societies for many years have used either a fiat money or a commodity money such as gold or silver as a means of payment for individual commerce and for taxation. It is convenient; the possession of money enables individuals to achieve divisibility without resorting to temporal sharing or to probability devices. Its properties of easy storage and transportation are well known. What is its role as a measure of value?

Money is an invention of man. If it were possible to compare utilities and if there existed a substance which had a linear measure in the utility units, work in welfare policy in particular, and calculations in economics in general would be easier. As doubtful as our calculations are we tend to make them in terms of money.

If the world were smoothly organized and completely competitive we would not need to make welfare calculations because implicitly society would have made its welfare decision (although it would be using only the weak criterion of accepting whatever point a price ray selects on the Pareto optimal surface). This solution, as is well known needs no assumptions concerning measurability, comparability, transferability or the existence of money other than as a double-entry bookkeeping accounting device.[10]

The world however tends not to be completely competitive, even as a good first approximation. Taxes are paid and welfare distributions are made. The mechanism of voting, for example controls much of the distribution of wealth. For solutions to the distribution of wealth by means other than pure competition further consideration needs to be given to properties such as measurability, comparability and transferability. As we have noted these are three separate problems.

(1) Measurability usually hinges upon the acceptance of some plausible additional assumptions concerning the ordering of preferences with risk.

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10. We also know that the solution in general is not unique and that the different competitive equilibria can easily call for radically different distributions of income.

(2) Comparability cannot be considered without measurability (but not vice-versa). The existence of comparability, means that for $U_1, U_2, \ldots, U_m$ we are able to give a clear meaning to $U_1 + U_2 + U_3 + \ldots + U_m$ where $m$ is the number of individuals. In particular for some solution concepts in the theory of games use is made of:

$$\max_1 \max_2 \ldots \max_m \sum_{i=1}^{m} U_i ,$$

the joint maximum. This assumption is loaded with difficulties; although for certain purposes it might be regarded as a good first approximation.

(3) Transferability can be considered without comparability or even measurability. It in many ways is the most closely related assumption to monetary matters; as it is concerned with the availability of an efficient means of payment.

Transferability is equivalent to asking if the Pareto optimal surface in the dimensions of the distribution space is flat in some or all dimensions. Transferability with comparability is equivalent to having not only flatness in the Pareto optimal surface in the distribution space, but that the Pareto optimal surface drawn in the dimensions of the individuals utilities is also flat.

Although in the discussions of welfare economics and general equilibrium theory the Pareto optimal surface is often referred to, no questions have been asked as to what its shape and curvature might be for a society such as the United States. If we could show that in a neighborhood of sufficient size it could be treated as flat, this could be of considerable use in the consideration of welfare. "Sufficient size" needs to be judged in terms of the size of economic decisions as contrasted with total wealth. There are several factors which suggest that the flatness of the Pareto optimal surface is a reasonable first approximation. They are (1) similarity of tastes, (2) linearity of production processes, (3) Fiduciary relationships, (4) the high value of avoiding barter, and (5) the high ratio between long term assets and consumption goods.

Although A's tastes may differ from B's with respect to many items, as a first approximation, a grouping of individuals into a few classes with roughly the same tastes seems reasonable.

The presence of linear production functions tends to increase the substitutability between products which tends to flatten the Pareto optimal surface. The assumption that many production processes are representable by linear functions appears to be reasonable.

Much of the business in a modern society involves the handling of money not by the owners, but by trustees or fiduciaries. Friedman and Savage suggested a nonlinear curve for the individual's utility as is shown in Figure 2 (given that it has inflexions, a linear approximation to it over a large range might be adequate, but this is another question). It is an open empirical question as to what is the utility function of a fiduciary?

The assumption of linearity here seems more reasonable than for the individual.

A most important institutional reason for treating money as though it were a special "imaginary commodity" with a constant generalized purchasing power is the basic unfeasibility of running a complex economy by barter.[11] The productive value of calculating with a dollar and using as a first approximation the transferability implicit in the definition that "a dollar is a dollar," is considerable. This is equivalent to adapting the convention of "let us act as though the Pareto optimal surface were flat as a first approximation," in the money dimension.

In a modern capitalist economy the ratio between capital goods and consumption goods tends to be high (at a guess between 5 or 8 to 1) this means that much of economic activity is the exchange of certificates of ownership for nonconsumer goods; and much of this exchange is performed by fiduciaries such as governments, financial and nonfinancial corporations.

### 5.2.1. Fiat Money and Flatness on the Pareto Optimal Surface

Suppose that there is a region of the Pareto optimal surface in the distribution space which is flat but no "moneylike" commodity actually exists. If the competitive equilibrium were unique then for any taxation

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11. For instance, see Bagehot, W., Lombard Street, R. D. Irwin (New York, 1962) p. 63 (1st edition, London 1873).

and distribution policy if fiat money were issued against all assets (and velocity remained constant) then a constant amount of fiat money would be needed and regardless of distribution the total value of all assets in the economy would remain constant in the region and this number would become a relatively significant measure of overall welfare.

If the economy of a society such as the one noted above were viewed as a game rather than as a competitive market with redistribution, the government might print extra money rather than tax to try to attain its objectives and individuals would have monetary strategies to try to influence velocity. In such a model it would be necessary to specify bankruptcy laws and other financial institutional details.

Although flatness of the Pareto optimal surface implies the existence of transferability, flatness is not a sufficient condition for the uniqueness of the competitive equilibrium as is shown in Figure 9 . In this

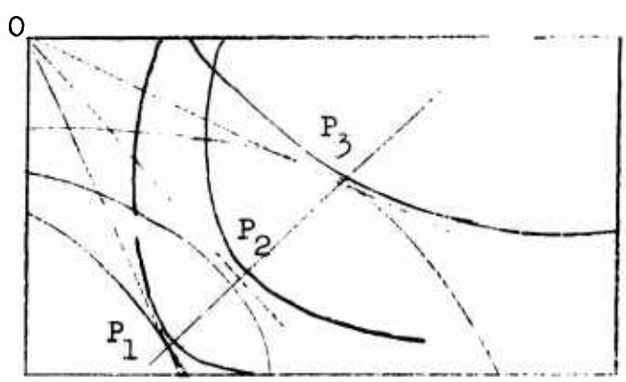


Figure 9

instance the total value of all assets in the economy would not remain constant under transfers. Diagram 9 was drawn with three equilibrium points. There is a conjecture since Marshall that the number of equilibrium points in an economy must be odd.

5.2.2. Commodity Money and a Flat Pareto Optimal Surface

If the transferable commodity is not symbolic, but is an actual good which enters the utility functions in the form that $u_i(x_1, x_2, \ldots, x_{n-1}, x_n) = \tilde{u}_i(x_1, x_2, \ldots, x_{n-1}) + \lambda_i x_n$ ; then if the Pareto optimal surface were not otherwise flat, the region of flatness would depend on the amount and distribution of the monetary commodity. If everyone has plenty of it, then a government redistribution of sufficiently small size would not effect the distribution of the first n-1 commodities or the price system. In this case the total value of all assets in terms of "gold" will remain constant. If there is a shortage of commodity money the price structure of the n-1 first commodities will be effected by taxation, hence the total value of all assets will change.

The "hot potato" velocity change effects, when the handling of fiat money is viewed as a noncooperative game, will not be seen here inasmuch as the "gold" has a utility of its own. As there is a fixed amount of the "gold" and it has a utility, then all individuals will in total be willing to hold the supply. If all act competitively then some may hold gold bars in their possession without ever using them in the market. This should not be confused with hoarding, which must be viewed strategically as an act in a "money game." Here the actual money commodity has value by itself and hence is worth having.[12]

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12. If we view the economy as a game of indefinite length, then even though fiat money has no intrinsic value an induced or derived utility may be found for it. Similarly steel mills may not enter directly into an individual's preference system but they too can be assigned a derived utility. The shape of the derived utility functions has only been partially explored.

## 6. Conclusions

The problems of ordinality, measurability, transferability and the use of money are different but highly related. Ordinality and cardinality concern individual preferences. Cardinality can be obtained from the ordinal properties of preferences either by considering behavior under risk, or by assuming a very special form for the utility functions (this form is related to but different from the assumption of something called money with a constant marginal utility to all).

Transferability concerns the existence of commodities or a surrogate commodity whose marginal rate of substitution is independent of the quantity possessed by any individual.

The same special assumption restricting the shape of the preference functions which is enough to establish cardinality, i.e., $u(x_1, x_2, \ldots, x_{n-1}, x_n) = u(x_1, x_2, \ldots, x_{n-1}) + \lambda x_n$ is enough to establish transferability. But this assumption is not necessary either for measurability or transferability.

The existence of transferability is equivalent to the Pareto optimal surface being flat in at least some part of the distribution space.

The existence of both transferability and comparability calls for the Pareto optimal surface to be flat in both the distribution space and the utility space.

The importance of different assumptions concerning measurability and comparability arises when welfare decisions employing criteria of

equality or fairness are applied; or when other solution concepts are utilized instead of the competitive market mechanism.

The flatness of the Pareto optimal surface with the $n^{th}$ good having the appropriate special properties gives rise to a natural commodity money. Flatness or approximate flatness without this means in some cases that a fiat money can be introduced to serve both as a medium for carrying on trade and taxation.

The introduction of fiat money when the Pareto optimal surface is flat is equivalent to introducing an $n+1^{st}$ "imaginary commodity" with a constant "generalized purchasing power."

Although it is conjectured that flatness in the distribution space is a good approximation in a large neighborhood, even if it were false, it is suggested that the investigation of the shape of induced utility functions and the shape of the Pareto optimal surface is an important preliminary to a positive welfare theory.