TECHNICAL REPORT

MATHEMATICAL MODELS FOR NAVIGATION SYSTEMS

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in accessible sources, but many are not readily available. Some are new, such as the expansion of \( t \): geodesic to second order in the flattening in both geodetic and parametric latitudes, and conversion formulas between the two forms.

Since the entire treatment is mathematical, an overall summary of the investigation is first presented to allow a quick assay of the topics and accomplishments. This summary is followed by a condensation of the formulas developed or included. The details of the actual development follow in three sections with computational examples in the Appendices.

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MATHEMATICAL MODELS
FOR
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OVERALL SUMMARY OF INVESTIGATIONS

Latitude

A loran station positioned on the auxiliary sphere of the ellipsoid of reference has as its
geoedetic latitude the angle at the equator made by that normal to the meridian which passes
through the station of the sphere. Its longitude will remain the same. See Figure 1, page 13.
Now this is analogous to the geodetic latitude of a subsatellite point, if the trajectory were
confined wholly to the surface of the auxiliary sphere. It is clearly not one of the three
commonly associated latitudes as shown in equation (1), page 12. Actually the relationship
between geocentric latitude on the sphere and geodetic latitude on the ellipsoid is given by
equation (2), page 12. From these one may find the maximum value of the difference, \( \Delta \phi \), and
the value of \( \phi \), the geodetic latitude, at which this maximum difference occurs, equations (3) –
(6), page 14. The expansions of \( \Delta \phi \) in series in terms of \( \phi \) were obtained and are given in
equations (7) – (20), pages 15, 16.

For computation of \( \phi \) as a function of \( \theta \), the geocentric latitude, it was necessary to employ
the Lagrange expansion formula and the resulting expansion and formulas are given in equations
(21) – (33), pages 16 to 18. Development of the series expansions for the height, \( h \), of the
auxiliary sphere above the ellipsoid is given in equations (43)– (48). See Figure 1, page 13
and pages 19,20. A summary of latitude formulas and a bibliography of pertinent references
are included, pages 21 –22.

The great circle track as determined by the geographical coordinates of two given points on the
auxiliary sphere. Parallels at a given distance from a great circle track.

As shown in figure 2, page 24, the treatment is somewhat different than the usual one in
that one works from the vertex of the great circle or the point where the great circle is or-
thogonal to a meridian. This simplifies computations and provides well balanced triangles
from which to compute. It also facilitates the computations for parallels at a given distance
from a fixed great circle track as shown in Figures 3 and 4, pages 26 and 27. See also
equations (1) – (13), pages 23–27.
A spherical rectangular coordinate system with a great circle base line as an axis.

Figure 5, page 29, shows, pictorially, this coordinate system developed on the great circle base line referenced to the vertex of the great circle base line. The conversion equations are developed in equations (14) to (26), pages 28 to 30.

Derivation of the equations of spherical hyperbolas and their plane equivalents.

Having established a spherical rectangular coordinate system we are in a position to derive the equations of spherical hyperbolas referenced to the system. This is done in both spherical rectangular coordinates and spherical polar form, equations (27) to (30), pages 31 to 34. See also figures 5, 6, and 7, pages 29, 32, 34.

The plane hyperbola equivalents are developed in equations (51) to (59), pages 35 and 36 and a comparison table of the spherical and plane equivalents is given as equation (60), page 37. See also Figures (8) and (9), pages 35 and 36.

An example of computations using the plane hyperbola approximation is given as Appendix 1, pages 99 to 110.

Distance computations and conversions; Azimuths; Associated geometrical quantities.

The classical "inverse" problem of geodesy was considered here since it is inherent in the electronic navigational systems problem. In the "inverse" problem, the latitudes and longitudes of each of two points are given from which the distance between the points and the azimuths at the two given points are to be determined.

The geodesic on the reference ellipsoid, other than meridians and circular equator, is a space curve, and its vertex (the latitude where it is orthogonal to a meridian) is not easily expressible in terms of the geographical coordinates (latitude and longitude) of two points on it. The actual length involves the evaluation of an elliptic integral, whose modulus depends on the latitude of the vertex of the geodesic. Iterative solutions have been devised as Helmert's, based on the earlier work of Bessel.

Approximations based on plane curves which are near the geodesic in length as the normal sections and the great elliptic arc have been devised. An investigation of these was made, including some extensions for instance in the series development for the great elliptic arc approximation. See pages 48 to 51 and Figure 15, page 50. Also their use and expression in terms of common computational parameters with some associated geometrical quantities useful in operational applications as the angle of depression of the chord below the horizon, the maximum separation between the chord and the surface, and the geographic coordinates of the point on the surface where maximum separation occurs.

An investigation of the expansion of the geodesic length in powers of the flattening was made which to first order in the flattening are the well-known, so-called Andoyer-Lambert
approximation formulas, one in terms of parametric latitude, the other in terms of geodetic
latitude. Since this Office uses the Andoyer-Lambert form in terms of parametric latitude, in
which geographic latitudes must first be converted to parametric, an investigation was made to
see if use of the parametric form to first order in the flattening was justified or necessary in
terms of operational requirements. This was done in connection with a review of an extensive
study by USAF (ACIC) of geodetic lines up to 6000 miles in length where the Andoyer-Lambert
approximation was recommended for such tasks as LORAN computing, since the errors in the
very near geodetic distances obtained are fairly constant on lines 50 to 6000 miles in length
and in all azimuths. The comparisons are given in tables 1—3, pages 65 to 67.

Since some of the excursions in the first order form were of the order of 10 meters, the
problem of obtaining the expansion of the geodesic to second order terms in the flattening was
examined. By introducing two parameters X and Y, in terms of the latitude of the vertex of the
great elliptic arc, it was found that the great elliptic arc approximation produced the so-called
Andoyer-Lambert first order approximations. (See pages 68–69.) Similarly they could be
produced by modification of the differential equation to the geodesic (See pages 69 to 74).

In review of an 1895 paper by the British Mathematician, A. R. Forsyth, by identifying his
fundamental approximation parameter as the vertex of the great elliptic arc, it was found that
he actually had both so-called Andoyer-Lambert first order expansions in the flattening, but
it had apparently not been recognized. Furthermore, he had an expansion to second order terms
in the flattening and in terms of geodetic latitude but it had two errors in the second order term.
After these had been detected and corrected, computations based on the resulting equations
give distances within a meter on all lines computed from 50 to 6000 miles. See pages 75 to 81.

Forsyth did not have the expansion to the geodesic in terms of parametric latitude to second
order terms in the flattening, so his results were extended to second order terms. See pages
79 to 90. Then transformation equations were developed to convert one form to the other as far
as second order terms in the flattening, pages 90 to 92, and finally the difference formulas for
the principal parameters, pages 92 to 93. As a result of this study, distance and azimuth
formulas are available in terms of easily computed parameters, in terms of either parametric
or geodetic latitude which will give distances over all lines within a meter and azimuths within
a second which is adequate for any operational requirement. A more detailed summary of the
investigations of this section with a bibliography of references is given on pages 93 to 97.
NEW LATITUDE FORMULAS

If $\theta$ is the geocentric latitude of a point $P(\cos \theta, \sin \theta)$ on the auxiliary sphere, then the corresponding geodetic latitude $\phi$ of $P$ at an altitude $h$ above the ellipsoid of reference as shown in Figure 1, is given by

$$\sin \Delta \phi = \sin(\phi - \theta) = \left(1 - e^2/2\right) \sin 2\phi = \left(1 - e^2 \sin^2 \phi\right)^{1/2}$$

$$\sin \Delta \phi = \sin 2\phi = \left(1 - e^2 \sin^2 \phi\right)^{1/2}$$

$$\Delta \phi = \left(1 - e^2 \sin^2 \phi\right)^{1/2}$$

$$\phi - \theta = \Delta \phi \text{ (in radians)} = \left(c_1 + c_1^2/8\right) \sin 2\phi - \left(c_4 + c_1^2 c_2/4\right) \sin 4\phi + \left(c_3 - c_1^2/24\right) \sin 6\phi$$

$$\Delta \phi \text{ (in seconds)} = (206,264.8062) \cdot \Delta \phi \text{ (in radians)}.$$
STANDARD LATITUDE FORMULAS

The three latitudes usually associated with the auxiliary sphere ellipsoid configuration as shown in Figure 1, are the geocentric, parametric, and geodetic represented here by \( \psi, \theta, \) and \( \phi_0 \) respectively and related through the equations

\[
\tan \psi / \tan \theta = \tan \theta / \tan \phi_0 = (1 - e^2) \theta,
\]

where \( e \) is the eccentricity of the meridian ellipse. The parametric latitude, \( \theta \), is also called here the geocentric latitude of points on the auxiliary sphere.

LATITUDES FOR CLARKE 1886 SPHEROID

Series representations, accurate to 0.001 second, for the differences in \( \phi, \phi_0, \theta, \psi \) are:

\[
\Delta \phi \text{ (seconds)} = \phi - \theta = 699.2540 \sin 2\phi - 0.5936 \sin 4\phi + 0.0004 \sin 6\phi
\]

\[
\Delta \phi \text{ (seconds)} = \phi - \theta = 699.2520 \sin 2\theta + 1.7769 \sin 4\theta + 0.0064 \sin 6\theta
\]

\[
\Delta \phi_0 \text{ (seconds)} = \phi_0 - \phi = 349.0318 \sin 2\theta + 1.4796 \sin 4\theta + 0.0061 \sin 6\theta
\]

\[
h \text{ (meters)} = 10,788.3852 - 10,811.2646 \cos 2\phi + 22.9147 \cos 4\phi - 0.0350 \cos 6\phi
\]

\[
\phi_0 - \psi = 700.4385 \sin 2\phi_0 - 1.1893 \sin 4\phi_0 + 0.0027 \sin 6\phi_0
\]

\[
\phi_0 - \psi = 700.4385 \sin 2\psi + 1.1893 \sin 4\psi + 0.0027 \sin 6\psi
\]

\[
\phi_0 - \theta = 350.2202 \sin 2\phi_0 - 0.2973 \sin 4\phi_0 + 0.0003 \sin 6\phi_0
\]

\[
\phi_0 - \theta = 350.2202 \sin 2\theta + 0.2973 \sin 4\theta + 0.0003 \sin 6\theta
\]

\[
\theta - \psi = 350.2202 \sin 2\theta - 0.2973 \sin 4\theta + 0.0003 \sin 6\theta
\]

\[
\theta - \psi = 350.2202 \sin 2\psi + 0.2973 \sin 4\psi + 0.0003 \sin 6\psi
\]

GREAT CIRCLE TRACK FORMULAS

First compute \( \lambda_0 \) and \( \theta_0 \) from

\[
\tan \lambda_0 = \frac{\tan \theta_2 \cos \lambda_1 - \tan \theta_1 \cos \lambda_2}{\tan \theta_1 \sin \lambda_2 - \tan \theta_2 \sin \lambda_1}
\]

\[
\cot \theta_0 = \cot \theta_1 \cos (\lambda_0 - \lambda_1) = \cot \theta_2 \cos (\lambda_0 - \lambda_2). \quad \text{(See Figure 2).}
\]

Then compute \( a_1 \) and \( a_2 \) from

\[
\sin a_1 = \frac{\cos \theta_0}{\cos \theta_1}, \quad \sin a_2 = \frac{\cos \theta_0}{\cos \theta_2}
\]

Next compute \( S_1 \) and \( S_2 \) from

\[
\tan S_1 = \cos a_1 \cot \theta_1, \quad \tan S_2 = \cos a_1 \cot \theta_2
\]

The computations for \( a_1, a_2, S_1 \) and \( S_2 \) are checked by

\[
\cos (\lambda_2 - \lambda_1) = \cos a_1 \cos a_2 + \sin a_1 \sin a_2 \cos (S_1 - S_2)
\]
For equally spaced intervals along the great circle track, for instance in 100 nautical mile intervals, let $S = S_i + 100K$, $K = 1, 2, 3, \ldots, n$. With these values of $S$ one computes successively corresponding values of $\theta', \lambda'$, and $\alpha'$ from

\[
\sin \theta' = \sin \theta_o \cos S, \tan (\lambda_o - \lambda') = \tan S / \cos \theta_o, \tan \alpha' = \cot \theta_o / \sin S
\]

and checks by means of $\sin \theta' \tan (\lambda_o - \lambda') \cdot \tan \alpha' = 1$.

**PARALLELS AT A GIVEN DISTANCE FROM THE GREAT CIRCLE TRACK**

To compute the coordinates $(\theta_p, \lambda_p)$ and $(\theta_p', \lambda_p')$ of points at a given distance $s$ from a given great circle track and symmetric with respect to it one computes (see Figure 3):

\[
\begin{align*}
\sin \theta_k &= A \cos S \pm B \quad \text{when } k = p, \text{ use } + \text{ sign} \\
\sin (\lambda_o - \lambda_k) &= C \sin S / \cos \theta_k \\
s & \text{and } \theta_o \text{ are the same as given under the great circle track formulas above and } A = C \sin \theta_o, B = \cos \theta_o \sin s, C = \cos s. \text{ The computations may be checked by } \\
\cos 2s &= \sin \theta_p \sin \theta_p' + \cos \theta_p \cos \theta_p' \cos (\lambda_p' - \lambda_p).
\end{align*}
\]

**SPHERICAL RECTANGULAR COORDINATE SYSTEM WITH A GREAT CIRCLE BASE LINE AS AN AXIS**

It is assumed that the base line has been established, that is the coordinates $(\theta_o, \lambda_o)$ of the vertex of the great circle base line have been computed from the coordinates of two given points $Q_1(\theta_1, \lambda_1)$, $Q_2(\theta_2, \lambda_2)$, see Figures 2 and 5.

Formulas for computing $y$, $S$, $x$ from $\theta$ and $\lambda$

\[
\begin{align*}
\sin y &= \cos \theta_o \sin \theta - \sin \theta_o \cos \theta \cos (\lambda_o - \lambda) \\
\tan S &= \frac{\sin \theta_o \cos \theta \sin (\lambda_o - \lambda)}{\sin \theta - \cos \theta \sin y} = \frac{\cos \theta \sin (\lambda_o - \lambda)}{\sin \theta_o \sin \theta + \cos \theta \cos \theta \cos (\lambda_o - \lambda)} \\
\sin x &= \sin (S - S_o) \cos y
\end{align*}
\]

Formulas for computing $S$, $\theta$, $\lambda$ from $x$ and $y$

Let $C = \cos y$, $D = \sin y$, $E = \sin x$, $A = C \sin \theta_o$, $B = D \cos \theta_o$, then

\[
\begin{align*}
S &= \arcsin (E/C) + S_o \\
\theta &= \arcsin (A \cos S + B) \\
\lambda &= \lambda_o - \arcsin (C \sin S / \cos \theta)
\end{align*}
\]
SPHERICAL HYPERBOLA FORMULAS AND PLANE EQUIVALENTS

**Spherical Plane**

(1) \( \tan^2 r = \frac{\tan^2 a (\sin^2 c - \sin^2 a)}{\sin^2 c \cos^2 a - \sin^2 a} \)

(2) \( \sin^2 x = \frac{\sin^2 a \cos^2 c}{\sin^2 c - \sin^2 a} \sin^2 y + \sin^2 a \)

(3) \( \tan R = \frac{\cos 2c \pm \cos 2a}{\sin 2c \cos \beta \pm \sin 2a} \)

(4) \( \tan^2(\beta/2) = \frac{\sin (c - a) \sin (R + c + a)}{\sin (c + a) \sin (R - c + a)} \)

In (1) and (2) the origin of coordinates is the midpoint of \( Q, Q \_2 \), see Figure 5. Equations (3) and (4) are two polar forms with origin at a focus \( Q, \) see Figures (5) and (6). Appendix 1 has computations based on the plane equivalent of (3).

DISTANCE AND AZIMUTH FORMULAS

Normal section azimuths (Geodetic latitude, \( \phi \))

\[
\cot \alpha_{AB} = \frac{\left[ \sin \phi_1 - (N_1 / N_2) \sin \phi_2 \right] e^2 \cos \phi_1 \sec \phi_2 + (\sin \phi_1 \cos \Delta \lambda - \tan \phi_2 \cos \phi_1)}{\sin \Delta \lambda}
\]

\[
\cot \alpha_{BA} = -\frac{\left[ \sin \phi_1 - (N_2 / N_1) \sin \phi_2 \right] e^2 \cos \phi_2 \sec \phi_1 + (\sin \phi_2 \cos \Delta \lambda - \tan \phi_1 \cos \phi_2)}{\sin \Delta \lambda}
\]

Normal Section Azimuths (parametric latitude \( \theta \))

\[
\cot \alpha_{AB} = \frac{\sin \theta_1 \cos \Delta \lambda - \cos \theta_1 \tan \theta_2 + e^2 (\sin \theta_2 - \sin \theta_1) \cos \theta_2 \sec \theta_2}{(1-e^2 \cos^2 \theta_2) \sin \Delta \lambda}
\]

\[
\cot \alpha_{BA} = -\frac{\sin \theta_2 \cos \Delta \lambda - \cos \theta_2 \tan \theta_1 + e^2 (\sin \theta_1 - \sin \theta_2) \cos \theta_1 \sec \theta_1}{(1-e^2 \cos^2 \theta_1) \sin \Delta \lambda}
\]

Great Elliptic Section Azimuths (Geodetic latitude \( \phi \))

\[
\cot \alpha_{AB} = (1 - e^2) \frac{N_1^2}{a^2} \frac{(\tan \phi_1 \cos \Delta \lambda - \tan \phi_2) \cos \phi_1}{\sin \Delta \lambda}
\]

\[
\cot \alpha_{BA} = (1 - e^2) \frac{N_2^2}{a^2} \frac{(\tan \phi_1 - \tan \phi_2 \cos \Delta \lambda) \cos \phi_2}{\sin \Delta \lambda}
\]

Great Elliptic Section Azimuths (parametric latitude \( \theta \))

\[
\cot \alpha_{AB} = \frac{(\tan \theta_1 \cos \Delta \lambda - \tan \theta_2) (\cos \theta_1) (1-e^2 \cos^2 \theta_1)^{1/2}}{\sin \Delta \lambda}
\]

\[
\cot \alpha_{BA} = \frac{(\tan \theta_1 - \tan \theta_2 \cos \Delta \lambda) (\cos \theta_2) (1-e^2 \cos^2 \theta_2)^{1/2}}{\sin \Delta \lambda}
\]
Great Elliptic Arc Distance

\[ s/a = (d_1 + d_2) - \frac{1}{2} k^2 [(d_1 + d_2) - \sin (d_1 + d_2) \cos (d_1 - d_2)] \]

\[- (1/128) k^4 [6(d_1 + d_2) - 8 \sin (d_1 + d_2) \cos (d_1 - d_2) + \sin 2(d_1 + d_2) \cos 2(d_1 - d_2)] \]

\[- (1/1536) k^4 [30(d_1 + d_2) - 45 \sin (d_1 + d_2) \cos (d_1 - d_2) + 9 \sin 2(d_1 + d_2) \cos 2(d_1 - d_2) \]

\[- \sin 3(d_1 + d_2) \cos 3(d_1 - d_2) \]  

Where in terms of geodetic latitude \( \phi \),

\[ k = (\sqrt{1 - e^2} a) \sin \phi, \quad d_1 = \arccos (N_1 \sin \phi_1 / N_0 \sin \phi_0), \]

\[ d_2 = \arccos (N_2 \sin \phi_2 / N_0 \sin \phi_0) \]

\[ \sin \phi_0 = \left[ \frac{J}{(J + \sin^2 \Delta \lambda)} \right]^{1/2}, \quad J = \tan^2 \theta_1 + \tan^2 \theta_2 - 2 \tan \theta_1 \tan \theta_2 \cos \Delta \lambda, \]

and in terms of parametric latitude \( \theta \)

\[ k = e \sin \theta_0, \quad d_1 = \arccos \left( \sin \theta_1 / \sin \theta_0 \right), \quad d_2 = \arccos \left( \sin \theta_2 / \sin \theta_0 \right) \]

\[ \sin \theta_0 = \left[ \frac{F}{(F + \sin^2 \Delta \lambda)} \right]^{1/2}, \quad F = \tan^2 \theta_1 + \tan^2 \theta_2 - 2 \tan \theta_1 \tan \theta_2 \cos \Delta \lambda. \]

Also in terms of parametric latitude \( \theta \), great elliptic arc distance

\[ s = a \left[ d - \frac{(e^2/8)(X_0 - Y \sin d)}{1 + \cos d} \right] \]

\[- \left( e^2/512 \right) [(6d - \sin 2d) X^2 - 8 \sin (d) XY + 2 \sin (2d) Y^2] \]

\[- \left( e^2/12288 \right) [3(10d - 3 \sin 2d) X^3 - 3(15 \sin d - \sin 3d) \sin 2d Y + 18(\sin 2d) X^2 Y - 4(\sin 3d) Y^3] \]

where \( X = \frac{(\sin \theta_1 + \sin \theta_2)^2}{1 + \cos d} + \frac{(\sin \theta_1 - \sin \theta_2)^2}{1 - \cos d} \)

\[ Y = \frac{(\sin \theta_1 + \sin \theta_2)^2}{1 + \cos d} - \frac{(\sin \theta_1 - \sin \theta_2)^2}{1 - \cos d} \]

where \( d_1, d_2 \) are spherical distances from \( P_1(\theta_1, \lambda_1) \) to \( P_2(\theta_2, \lambda_2) \) to the vertex \( P_0(\theta_0, \lambda_0) \).

NOTE: If \( e^2 = 2f \), the higher order terms in \( f \) then ignored, this becomes the so-called Andoyer-Lambert approximation in terms of parametric latitude.

GEODESIC IN TERMS OF GREAT ELLIPTIC ARC, IN GEOEDETIC LATITUDE WITH SECOND ORDER TERMS IN THE FLATTENING

Given the points \( P_1(\phi_1, \lambda_1), P_2(\phi_2, \lambda_2) \) on the reference ellipsoid, \( P_2 \) west of \( P_1 \), west longitudes considered positive.

With \( \phi_m = \frac{1}{2}(\phi_1 + \phi_2), \Delta \phi_m = \frac{1}{2}(\phi_2 - \phi_1), \Delta \lambda = \lambda_2 - \lambda_1, \Delta \lambda_m = \frac{1}{2}\Delta \lambda, \)

Let \( k = \sin \phi_m \cos \Delta \phi_m, K = \sin \Delta \phi_m \cos \phi_m, \)

\[ H = \cos^2 \Delta \phi_m - \sin^2 \phi_m = \cos^2 \phi_m - \sin^2 \Delta \phi_m \]

\[ L = \sin^2 \Delta \phi_m + H \sin^2 \Delta \lambda_m = \sin^2(d/2), 1 - L = \cos^2(d/2), \cos d = 1 - 2L, \]

\[ t = \sin^2 d = 4L(1 - L), U = 2k^2/(1 - L), V = 2K^2/L, X = U + V, Y = U - V, \]
\[ T = \frac{d}{\sin d} = 1 + \left( \frac{t}{6} \right) + 3\left( \frac{t^2}{40} \right) + 5\left( \frac{t^3}{112} \right) + 35\left( \frac{t^4}{1152} \right) + \left( \frac{1}{1 \text{ radian}} = 206,264.8062 \text{ seconds} \right) \]

\[ E = 30 \cos d, \quad A = 4T (8 + TE/15), \quad D = 4(6 + T^2), \quad B = -2D \]
\[ C = T - \frac{1}{2}(A + E), \quad f/4 = 0.000847518825, \quad f^2/64 = 0.179572039 \times 10^{-6} \text{ (Clarke 1866)} \]

\[ S = a \sin d \left[ T - \frac{f}{4} (TX - 3Y) + \left( \frac{f^2}{64} \right) (X(A + CX) + Y(B + EY) + DXY) \right], \]

\[ \sin (\alpha_2 + \alpha_1) = \left( K \sin \Delta \lambda \right)/L, \quad \sin (\alpha_2 - \alpha_1) = \left( k \sin \Delta \lambda \right)/(1 - L), \]
\[ \frac{1}{2}(\delta \alpha_2 + \delta \alpha_1) = -(f/2) H (T + 1) \sin (\alpha_2 + \alpha_1), \quad \frac{1}{2}(\delta \alpha_2 - \delta \alpha_1) = -(f/2) H (T - 1) \sin (\alpha_2 - \alpha_1), \]
\[ \alpha_{1-2} = \alpha_1 + \delta \alpha_1, \quad \alpha_{2-4} = \alpha_2 + \delta \alpha_2. \]

Additional check formulae

\[ X = \frac{(\sin \phi_1 + \sin \phi_2)^2}{1 + \cos d} + \frac{(\sin \phi_1 - \sin \phi_2)^2}{1 - \cos d} = 2 \sin^2 \phi_0 = 2F/(F + \sin^2 \Delta \lambda) \]
\[ Y = \frac{(\sin \phi_1 + \sin \phi_2)^2}{1 + \cos d} - \frac{(\sin \phi_1 - \sin \phi_2)^2}{1 - \cos d} = 2 \sin^2 \phi_0 \cos (d_1 + d_2) \]
\[ F = \tan^2 \phi_1 + \tan^2 \phi_2 - 2 \tan \phi_1 \tan \phi_2 \cos \Delta \lambda \]
\[ \cos (d_1 + d_2) = Y/X, \quad 1 + \cos d = 8k^2/(X + Y), \quad 1 - \cos d = 8K^2/(X - Y), \]
\[ \cos d = 4 \left( \frac{k^2}{X+Y} - \frac{X^2}{X-Y} \right), \quad 4 \left( \frac{k^2}{X+Y} + \frac{X^2}{X-Y} \right) = 1. \]

NOTE: If the second order term is ignored, the resulting equations are the equivalent of the so-called Andoyer-Lambert approximation in terms of geodetic latitude.

The quantities \( H, T, L, k, K \) enter into both distance and azimuth formulas. Distances are given within a meter and azimuths within a second over all lines in all latitudes and azimuths. Other advantages are (1) no conversion to parametric latitudes, (2) no square root calculations, (3) for desk computers the only tabular data required is a table of the natural trigonometric functions as Peter's eight place tables, (4) the formulas are adaptable to high speed computers. See Table 4 page 81 and Appendix 3, lines 12 through 16, for desk computer sample computations based on these formulas as checked against 5 Coast and Geodetic Survey specially computed lines. The mean difference for the 5 lines between true geodetic lengths and computed values was 0.15 meter with a maximum difference of 0.24 meter. The mean difference between true and computed azimuths was 0.59 second with a maximum difference of 0.93 second.

GEODESIC IN TERMS OF GREAT ELLIPTIC ARC, IN PARAMETRIC LATITUDE WITH SECOND ORDER TERMS IN THE FLATTENING

Given on the reference ellipsoid the points \( P_1(\theta_1, \lambda_1), P_2(\theta_2, \lambda_2); P_3 \) west of \( P_1 \), west longitudes considered positive. (Geodetic latitudes are converted to parametric by the relation \( \tan \theta = (1 - f) \tan \phi \) or an equivalent formula). With \( \theta_m = \frac{1}{2}(\theta_2 + \theta_1), \Delta \theta_m = \frac{1}{2}(\theta_2 - \theta_1), \Delta \lambda = \lambda_2 - \lambda_1, \Delta \lambda_m = \Delta \lambda/2; \)
let \( k = \sin \theta_m \cos \Delta \theta_m \), \( K = \sin \Delta \theta_m \cos \theta_m \),
\[
H = \cos^2 \Delta \theta_m - \sin^2 \theta_m = \cos^2 \theta_m - \sin^2 \Delta \theta_m ,
\]
\[
L = \sin^2 \Delta \theta_m + H \sin \Delta \lambda_m = \sin^2 \theta_m / 2, 1-L = \cos^2 \theta_m / 2 ,
\]
\[
\cos d = 1 - 2L, h = \sin^2 d = 4L(1-L), U = 2k^2/(1-L),
\]
\[
V = 2K^2/L, X = U+V, Y = U-V ,
\]
\[
T = d/\sin d = 1 + \left(1/(6)\right)h + \left(3/(40)\right)h^2 + \left(5/(112)\right)h^3 + \left(35/(2816)\right)h^5 + - - - - - ,
\]
\[
E_0 = -2 \cos d, D_0 = 4T^2, A_0 = -D_0E_0, B_0 = -2D_0, C_0 = T - \frac{1}{2}(A_0 + E_0),
\]
\[
S = a \sin d \left[ T - \left(f/4\right) (TX - Y) + \left(f^2/64\right) (A_0X + B_0Y + C_0X^2 + D_0XY + E_0Y^2) \right]
\]
\[
\sin (\alpha_2 + \alpha_1) = (K \sin A)/L, \sin (\alpha_2 - \alpha_1) = (k \sin A)/(1-L)
\]
\[
\frac{1}{2} (\delta_2 + \delta_1) = -(f/2) TH \sin (\alpha_1 + \alpha_2)
\]
\[
\frac{1}{2} (\delta_2 - \delta_1) = -(f/2) TH \sin (\alpha_2 - \alpha_1)
\]
\[
\alpha_2 \rightarrow \alpha_1 + \delta_1, \alpha_1 \rightarrow \alpha_2 + \delta_2
\]

Additional check formulae
\[
X = \frac{(\sin \theta_1 + \sin \theta_2)^2}{1 + \cos d} + \frac{(\sin \theta_1 - \sin \theta_2)^2}{1 - \cos d} = 2 \sin^2 \theta_0 = 2F/(F + \sin^2 \Delta \lambda)
\]
\[
Y = \frac{(\sin \theta_1 + \sin \theta_2)^2}{1 + \cos d} - \frac{(\sin \theta_1 - \sin \theta_2)^2}{1 - \cos d} = 2 \sin^2 \theta_0 \cos (d_1 + d_2)
\]
\[
F = \tan^2 \theta_1 + \tan^2 \theta_2 - 2 \tan \theta_1 \tan \theta_2 \cos \Delta \lambda
\]
\[
\cos \left(d_1 + d_2\right) = Y/X, 1 + \cos d = 8k^2/(X + Y), 1 - \cos d = 8K^2/(X - Y),
\]
\[
\cos d = 4 \left(\frac{k^2}{X+Y} - \frac{K^2}{X-Y}\right),
\]
\[
\frac{4}{X+Y} + \frac{K^2}{X-Y} = 1.
\]

NOTE: If the second order term is ignored, the resulting equations are the equivalent of the so-called Andoyer-Lambert approximation in terms of parametric latitude.

TRANSFORMATIONS: GEODETIC TO PARAMETRIC — PARAMETRIC TO GEODETIC

If primed quantities denote those in geodetic latitude, then the transformation equations are:

\[
d' = d - (f/2) Y \sin d + (f^2/16) [4Y(X-3) \sin d + (2Y^2 - X^2) \sin 2d],
\]
\[
\sin d' = \sin d - (f/4) Y \sin 2d
\]
\[
X' = X[1 + f (2 - X)]
\]
\[
Y' = Y[1 + f (2 - X)] + (f/2) (X^2 - Y^2) \cos d
\]
\[
d = d' + (f/2) Y' \sin d' + (f^2/16) [4Y' (X'-1) \sin d' + (2Y'^2 - X'^2) \sin 2d']
\]
\[
\sin d = \sin d' + (f/4) Y' \sin 2d'
\]
\[
X = X'[1 - f(2 - X')]
\]
\[
Y = Y'[1 - f(2 - X')] - (f/2) (X'^2 - Y'^2) \cos d'
\]

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DIFFERENCE FORMULAS TO SECOND ORDER IN THE FLATTENING

\[ d' - d = -(f/2) Y \sin d + (f^2/16) \left[ 4Y \ (X - 3) \sin d + (2Y^2 - X^2) \sin 2d \right], \]

\[ = -(f/2) Y' \sin d' - (f^2/16) \left[ 4Y' \ (X' - 1) \sin d' + (2Y'^2 - X'^2) \sin 2d' \right]; \]

\[ X' - X = fX \ (2 - X) \{ 1 + (f/2) \ (3 - 2X) \}, \]

\[ = fX' \ (2 - X') \{ 1 - (f/2) \ (1 - 2X) \}; \]

\[ Y' - Y = fY \ (2 - X) + (f/2) \ (X^2 - Y^2) \cos d \]

\[ + (f^2/8) \left[ 4Y \ (2 - X) \ (3 - 2X) \right. \]

\[ + (X^2 - Y^2) \left\{ (11 - 5X) \cos d + Y \ (1 - 3 \cos^2 d) \right\} \]

\[ = fY' \ (2 - X') + (f/2) \ (X'^2 - Y'^2) \cos d' \]

\[ - (f^2/8) \left[ 4Y' \ (2 - X') \ (1 - 2X') \right. \]

\[ + (X'^2 - Y'^2) \left\{ (25 - 3X') \cos d' + Y' \ (1 - 3 \cos^2 d') \right\} \]

CHORD DISTANCE, c

\[ c = a \left[ 1 - \cos \ (d_1 + d_2) \right] \left( 2 - k^2 \left[ 1 - \cos \ (d_1 - d_2) \right] \right)^{1/2} \]

Where in terms of geodetic latitude \( \phi \),

\[ d_1 = \text{arc cos} \ (N_1 \ \text{sin} \ \phi_1 / N_0 \ \text{sin} \ \phi_0), \]

\[ d_2 = \text{arc cos} \ (N_2 \ \text{sin} \ \phi_2 / N_0 \ \text{sin} \ \phi_0) \]

\[ k^2 = \left[ e^2 (1 - e^2) / a^2 \right] N_0^2 \ \text{sin}^2 \phi_0 \]

in terms of parametric latitude \( \theta \)

\[ d_1 = \text{arc cos} \ (\sin \ \theta_1 / \sin \ \theta_0), \]

\[ d_2 = \text{arc cos} \ (\sin \ \theta_2 / \sin \ \theta_0), \]

\[ k^2 = e^2 \ \text{sin}^2 \theta_0. \]

ANGLE OF DIP OF THE CHORD, \( \beta \)

\[ \sin \beta = \sqrt[1/2]{ \left( 1 - e^2 \right) \left[ 1 - \cos \ (d_1 + d_2) \right] \left( 2 - k^2 \left[ 1 - \cos \ (d_1 - d_2) \right] \right) \left( 1 - e^2 + k^2 \ \text{cos}^2 d_1 \right) } \]

with \( k, d_1, d_2 \) expressible in terms of either geodetic or parametric latitude as given above.

MAXIMUM SEPARATION OF CHORD AND ELLIPTIC ARC, \( H_0 \)

\[ H_0 = \frac{2ab_0}{c} \ \text{sin} \ \frac{1}{2} (d_1 + d_2) \left[ 1 - \cos \ \frac{1}{2} (d_1 + d_2) \right], \]

where \( c \) is the chord length as given above, \( b_0 = a \sqrt{1 - k^2}; \)

\( k, d_1, d_2 \) expressible in either parametric or geodetic latitude as given above.

GEOGRAPHIC COORDINATES OF POINT OF MAXIMUM SEPARATION

\[ \tan \phi = R / D, \] or \( \cos 2\phi = (D^2 - R^2) / (D^2 + R^2), \]

\[ \tan \lambda = (\cos \ \theta_2 \ \text{sin} \ \Delta \lambda) / (\cos \ \theta_1 + \cos \ \theta_2 \ \cos \ \Delta \lambda), \]

\[ R = \text{sin} \ \theta_1 + \text{sin} \ \theta_2, \]

\[ D = (0.996609925) \ (4 \ \text{cos}^2 \frac{1}{2} \ (d - R^2)) \left( 1 - \frac{c^2}{D^2} \right) \]

\( d \) is spherical distance between the points \( P_1 (\theta_1, \lambda_1), P_2 (\theta_2, \lambda_2) \) on the ellipsoid, \( \theta \) is parametric latitude, \( \Delta \lambda = \lambda_2 - \lambda_1 \). See Figure 23 for sample computation.
SECTION 1. LATITUDE FORMULAE

The auxiliary sphere, associated with an ellipsoid of reference, is the sphere tangent to the spheroid along the equator. If it is desired to work on this sphere with formulae for conversion to the spheroidal surface, then a correspondence between geocentric latitude \( \theta \) on the sphere and geodetic latitude \( \phi \) on the ellipsoid is needed. Longitudes will be the same.

Now there are three latitudes in geodetic usage associated with the auxiliary-sphere ellipsoid configuration as shown in Figure 1. The \( \theta \) as shown, and which we shall call geocentric latitude, is called the reduced or parametric latitude since it is the eccentric angle of the meridian ellipse. The angle \( \psi \), as shown, is called in geodetic nomenclature, the geocentric latitude since it is the angle measured from the center of the ellipsoid to the point \( R \) on the meridian from the equator. The angle \( \phi_0 \), as shown, is a geodetic latitude corresponding to \( \theta \). The three latitudes \( \psi, \theta, \phi_0 \) are related through the equations

\[
\tan \psi = \sqrt{1-e^2} \tan \theta = (1-e^2) \tan \phi_0
\]

or \( \tan \psi / \tan \theta = \tan \theta / \tan \phi_0 = \sqrt{1-e^2} \).

where \( e \) is the eccentricity of the meridian ellipse [1].

However, for working directly on the auxiliary sphere and transferring directly to the ellipsoid, if \( \theta \) is the geocentric latitude of the point \( P(a \cos \theta, a \sin \theta) \) on the auxiliary sphere, then the latitude actually corresponding on the spheroid is that found by dropping a perpendicular upon the meridian ellipse from \( P \) meeting the meridian in \( Q \) as shown in Figure 1, the normal making the angle \( \phi \) as shown with the equator. The distance \( PQ = h \), and \( \phi \) are needed for the conversion where \( 0 \leq h \leq a - b \), \( a \) and \( b \) the semimajor and semiminor axes of the spheroid. We now develop the necessary conversion formulas between \( \phi \) and \( \theta \).

The law of sines applied to triangles POT, POK of figure 1, yields

\[
\frac{Ne^2 \sin \phi}{\sin \Delta \phi} = \frac{h + N}{\cos \theta} = \frac{a}{\cos \phi}, \quad \frac{Ne^2 \cos \phi}{\sin \Delta \phi} = \frac{h + N(1-e^2)}{\sin \theta} = \frac{a}{\sin \phi},
\]

where \( N = a/\sqrt{1-e^2 \sin^2 \phi} \); \( e, a \) are the eccentricity and equatorial radius of the reference ellipsoid. \( \Delta \phi = \phi - \theta \).

*[1] Bracketed numbers refer to the list of references at the end of the section.
Figure 1. Latitude relationships in the auxiliary sphere-spheroid configuration.
From the first and last of either sets of equations (2) find

\[ \sin \Delta \phi = \frac{e^2}{2a} \quad \text{N} \sin 2\phi = \frac{e^2 \sin \phi \cos \phi}{\sqrt{1 - e^2 \sin^2 \phi}} . \quad (3) \]

To find the maximum value of \( \Delta \phi \) and the value of \( \phi \) at which the maximum occurs, one differentiates \( \Delta \phi = \arcsin \frac{e^2 \sin \phi \cos \phi}{\sqrt{1 - e^2 \sin^2 \phi}} \) to obtain

\[ \frac{d\Delta \phi}{d\phi} = \frac{e^2 (2 - e^2 \cos 2\phi + 2(2 - e^2) \cos 2\phi + e^2)}{(2 - e^2 + e^2 \cos 2\phi) \sqrt{2(2-e^2) - e^4 + 2e^2 \cos 2\phi + e^4 \cos^2 2\phi}} ; \quad (4) \]

neither factor of the denominator of (4) is zero for \( 0 \leq \phi \leq 90^\circ \). Hence to find the maximum from (4), place the numerator equal to zero and solve for \( 2\phi \) to obtain

\[ \cos 2\phi = 1 + 2 \left( \frac{\sqrt{1 - e^2} - 1}{e^2} \right) . \quad (5) \]

The flattening, \( f \), of the reference ellipsoid is given by \( f = (a-b)/a = 1 - b/a = 1 - \sqrt{1-e^2} \), whence \( e^2 = 2f - f^2 \), we can write

\[ \cos 2\phi = 1 - (1-\sqrt{1-e^2})/e^2 = 1 - 2f/(2f - f^2) = -1/(2f) \]

\[ \sin^2 2\phi = 1 - \cos^2 2\phi = 1 - f^2/(2-f)^2 = 4(1-f)/(2-f)^2 \]

\[ \sin^2 \phi = \frac{1}{2} - \frac{1}{2} \cos 2\phi = \frac{1}{2} + \frac{f}{2(2-f)} = \frac{1}{2-f} . \]

\[ 1 - e^2 \sin^2 \phi = 1 - f(2-f)/(2-f) = 1 - f . \]

from (3) \[ \sin^2 \Delta \phi = \frac{e^4}{4} \cdot \frac{\sin^2 2\phi}{1-e^2 \sin^2 \phi} = \frac{f^2(2-f)^2}{4(1-f)} \cdot \frac{1}{(2-f)^2} \cdot \frac{1}{1-f} \]

\[ \sin^2 \Delta \phi = f^2 \]

hence \( \sin \Delta \phi_{\max} = f = 0.0033900753 \) (Clarke 1866 ellipsoid). \[ \cos 2\phi = -0.001697914 \]

\[ \phi = 45^\circ \ 02' \ 55" \ 106' , \]

and \[ \Delta \phi_{\max} = 0^\circ \ 11' \ 39".255 , \quad (6) \]

\[ \theta = \phi - \Delta \phi = 44^\circ \ 51' \ 15".351 . \]

Now from (3) and \( \theta = \phi - \Delta \phi \) a complete table for corresponding latitudes can be computed readily since complete tables for \( N \) to 0.001 meter have been computed for most reference ellipsoids. [2]

To develop \( \sin \Delta \phi \) is a series for computation without the necessity of tables of \( N \), write (3) in the form \( \sin \Delta \phi = e^2 \sin \phi \cos \phi (1 - e^2 \sin^2 \phi)^{-1/2} \), then expand the radical by the binominal formula to get

\[ \sin \Delta \phi = e^2 \sin \phi \cos \phi \left(1 + \frac{e^2}{2} \sin^2 \phi + \frac{3}{8} e^4 \sin^4 \phi + \frac{5}{16} e^6 \sin^6 \phi \right) \]
\[
\text{\begin{align*}
= \frac{e^2}{2} \sin 2\phi + \frac{e^4}{2} \sin^3 \phi \cos \phi + \frac{3}{8} e^6 \sin^5 \phi \cos \phi + \frac{5}{16} e^8 \sin^7 \phi \cos \phi.
\end{align*}}
\]

(7)

now \( \sin^3 \phi \cos \phi = \frac{1}{4} \sin 2\phi - \frac{3}{16} \sin 4\phi \)

\[
\sin^5 \phi \cos \phi = \frac{5}{12} \sin 2\phi - \frac{1}{6} \sin 4\phi + \frac{1}{12} \sin 6\phi
\]

(8)

\[
\sin^7 \phi \cos \phi = \frac{7}{48} \sin 2\phi - \frac{7}{48} \sin 4\phi + \frac{3}{64} \sin 6\phi - \frac{1}{128} \sin 8\phi,
\]

and the values from (8) placed in (7) give

\[
\sin \Delta \phi = c_4 \sin 2\phi - c_2 \sin 4\phi + c_3 \sin 6\phi - c_4 \sin 8\phi,
\]

(9)

where \( c_1 = \frac{e^2}{2} + \frac{e^4}{8} + \frac{15}{12} e^6 + \frac{35}{64} e^8 \), \( c_2 = e^4 \), \( c_3 = e^6 + \frac{15}{12} e^8 \), \( c_4 = \frac{7}{48} e^8 \)

If \( \Delta \phi \) in radians is desired rather than \( \sin \Delta \phi \), then in the expansion

\[
\text{arc sin } x = x(1 + x^2 + \ldots)
\]

(10)

let \( x = \sin \Delta \phi \), whence \( \text{arc sin } x = \Delta \phi \) and

\[
\Delta \phi = \sin \Delta \phi \left(1 + \frac{\sin^3 \Delta \phi}{6} + \ldots\right).
\]

(11)

from (9) with \( e^2 = 0.006768657997 \), find

\[
c_1 = 0.003390074081, \quad c_2 = 0.000002878029,
\]

\[
c_3 = 3.665 \times 10^{-7}, \quad c_4 = 5 \times 10^{-12} \text{ (negligible)}.
\]

(12)

For estimation purposes the values in (12) may be written

\[
c_1 = 3 \times 10^{-7}, \quad c_2 = 3 \times 10^{-6}, \quad c_3 = 4 \times 10^{-9}
\]

\[
c_4 = 5 \times 10^{-12} \quad \text{ (negligible)}
\]

(13)

With the value of \( \sin \Delta \phi \) from (9) in terms of the estimation coefficients (13) we examine the term \( \sin^3 \Delta \phi/6 \) in (11), and find that (11) may be written \( \Delta \phi = \sin \Delta \phi + \frac{c_1^3}{6} \sin^3 2\phi - \frac{c_1^2 c_2}{2} \sin^2 2\phi \sin 4\phi \).

(14)

since \( \sin^3 2\phi = \frac{3}{4} \sin 2\phi - \frac{1}{4} \sin 6\phi \)

\[
\sin^2 2\phi \sin 4\phi = \frac{1}{2} \sin 4\phi - \frac{1}{4} \sin 8\phi,
\]

(15)

equation (14) may be written, with the value of \( \sin \Delta \phi \) from (9), as

\[
\Delta \phi (\text{radians}) = \left(c_1 + \frac{c_1^3}{8}\right) \sin 2\phi - \left(c_2 + \frac{c_1^2 c_2}{4}\right) \sin 4\phi + \left(c_3 - \frac{c_1^3}{24}\right) \sin 6\phi,
\]

(16)

or

\[
\Delta \phi (\text{seconds}) = (206,264.8062) \Delta \phi (\text{radians}),
\]

where \( c_1, c_2, c_3 \), are given by the expressions in (9) in terms of the eccentricity of the meridian ellipse.
We now check equations (9) and (17), using again values for the Clarke 1866 spheroid and for the maximum value of $\Delta \phi$.

From (9) and (12) we have

$$\sin \alpha = 3.390074081 \times 10^{-3} \sin 2\phi - 2.878029 \times 10^{-6} \sin 4\phi + 3.665 \times 10^{-9} \sin 6\phi. \quad (18)$$

From (12) and (17) find

$$\Delta \phi (\text{seconds}) = 699'2540 \sin 2\phi - 0'5936 \sin 4\phi + 0'0004 \sin 6\phi. \quad (19)$$

Now with $\phi = 45^\circ 02'55^\prime 106$ from (6), find $\sin 2\phi = + 0.99999856$, $\sin 4\phi = -0.00339575$,

$$\sin 6\phi = -0.99998703. \quad (20)$$

The values from (20) placed in (18) give

$$\sin \alpha = 0.0033900753 \text{ which checks the value found before in the 10th place. (See (6)).}$$

The values from (20) placed in (19) give $\Delta \phi (\text{seconds}) = 699'2530 + 0'0020 - 0'0004 = 699'2546$, or $11'39''255$ which is the value of $\Delta \phi_{\text{max}}$. (See (6)).

For explicit computation of $\phi$ as a function of $\theta$, we obtain the following development. From the second and third of each set of equations (2), find

$$h = a \cos \theta / \cos \phi = N e' + a \sin \theta / \sin \phi,$$

whence

$$\tan \phi = \tan \phi + (e^2 / a \cos \theta) (N \sin \phi) \quad (21)$$

or

$$\tan \phi = \tan \phi + \sqrt{1 + \tan^2 \theta} \left( \tan \phi / \sqrt{1 + (1 - e^2) \tan^2 \phi} \right).$$

(NOTE: Equation (21) also follows directly from (3) by expanding the left hand side and dividing every term by the product $\cos \phi \cos \theta$. $\sin \Delta \phi = \sin \phi \cos \theta - \cos \phi \sin \theta$.)

Now (21) is of the form

$$y = x + h (x) g (y)$$

and the Lagrange expansion formula may be used, [3].

Equation (21) may be written

$$y = x + e^2 (1 + x^2)^{1/2} \cdot \left[ y (1 + (1 - e^2) y^2)^{-1/2} \right]. \quad (22)$$

Where $y = \tan \phi$, $x = \tan \theta$, $h (x) = e^2 (1 + x^2)^{1/2}$, $g (y) = y (1 + (1 - e^2) y^2)^{-1/2}$.

By use of the Lagrange expansion formula, a function $f(y)$ which has a power series representation may be written

$$f(y) = f(x) + \sum_{n=1}^{\infty} \frac{h (x)}{n!} \frac{d (n-1)}{dx} \left[ f(x) \cdot g(x) \right] \quad (23)$$

With $y = \tan \phi$, $f(y) = \arctan y = \phi$; $x = \tan \theta$, $f(x) = \arctan x = \theta$, $f (x) = \frac{1}{1 + x^2} = \cos^2 \theta$,

equation (23) may be written

$$\Delta \phi = \phi - \theta = \sum_{n=1}^{\infty} \frac{e^n \sec^2 \theta}{n!} \frac{d (n-1)}{dx} G (\theta) \quad (24)$$

Where $G (\theta) = (\cos^2 \theta) (\tan \theta / \sqrt{1 + (1 - e^2) \tan^2 \theta})^n$, $\theta = \arctan x$. 

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First write \( G(\theta) \) in the form
\[
G(\theta) = (\cos^2 \theta) [\sin \theta (1 - e^2 \sin^2 \theta)^{-1/2}]^n.
\] 

We wish to retain terms to \( e^n \), but no higher. Hence we expand the radical in (25) to powers of \( e^6 \) since for \( n = 1 \), equation (25) will be multiplied by \( e^2 \) as seen from (24). Using the binomial formula for the expansion we can write (25) as
\[
G(\theta) = (\cos^2 \theta) (\sin \theta + \frac{1}{2} e^2 \sin^3 \theta + (\frac{3}{4}) e^4 \sin^5 \theta + (\frac{5}{16}) e^6 \sin^7 \theta)^n.
\] 

To retain terms in \( e^6 \) we will need the first four terms of the expansion (24) and hence three derivatives of (26). Now \( \theta = \text{arc tan} \frac{d}{dx} = \frac{1}{1 + x^2} = \cos^2 \theta \), \( \frac{d^2 \theta}{dx^2} = -2 \sin \theta \cos^2 \theta \),
\[
\frac{d^2 \theta}{dx^2} = 2(3 \sin^2 \theta - \cos^2 \theta) \cos^4 \theta.
\] 
\[
\frac{dG}{dx} = \frac{dG}{d\theta} \frac{d\theta}{dx} = \left( \frac{dG}{d\theta} \right) \cos^2 \theta
\] 
\[
\frac{d^2 G}{dx^2} = \left( \frac{d^2 G}{d\theta^2} \right) \frac{d\theta}{dx} + \left( \frac{dG}{d\theta} \right) \frac{d^2 \theta}{dx^2}
\]
\[
= \cos^2 \theta \left[ \left( \frac{d^2 G}{d\theta^2} \right) \cos \theta - 2 \left( \frac{dG}{d\theta} \right) \sin \theta \right]
\] 
\[
\frac{d^3 G}{dx^3} = \left( \frac{d^3 G}{d\theta^3} \right) \frac{d\theta}{dx} + 3 \left( \frac{d^2 G}{d\theta^2} \right) \left( \frac{d\theta}{dx} \right) \left( \frac{d^2 \theta}{dx^2} \right) + \frac{dG}{d\theta} \frac{d^3 \theta}{dx^3}
\]
\[
= \cos^2 \theta \left[ \left( \frac{d^3 G}{d\theta^3} \right) \cos^2 \theta - 6 \left( \frac{d^2 G}{d\theta^2} \right) \cos \theta \sin \theta + 2 \left( \frac{dG}{d\theta} \right) \left( 3 \sin^2 \theta - \cos^2 \theta \right) \right]
\] 

Because of the factor \( e^2 \) as a multiplier in (24), we can assume the following terms for (26) for \( n = 1, 2, 3, 4 \):
\[
\begin{align*}
n & \quad G(\theta) \\
1 & \quad (\cos^2 \theta) (\sin \theta + \frac{1}{2} e^2 \sin^3 \theta + (\frac{3}{4}) e^4 \sin^5 \theta + (\frac{5}{16}) e^6 \sin^7 \theta) \\
2 & \quad (\cos^2 \theta) (\sin \theta + e^2 \sin^3 \theta + e^4 \sin^5 \theta) \\
3 & \quad (\cos^2 \theta) (\sin^3 \theta + (\frac{3}{2}) e^2 \sin^5 \theta) \\
4 & \quad (\cos^2 \theta) (\sin^4 \theta)
\end{align*}
\] 

The terms of (24) are now formed by finding the derivatives of \( G(\theta) \) with respect to \( \theta \) using the appropriate form of \( G(\theta) \) from (30) and finding \( \frac{dG}{dx} \), \( \frac{d^2 G}{dx^2} \), \( \frac{d^3 G}{dx^3} \) by means of (27), (28), and (29).
Thus it is found that the first four terms of (24) are
\[ e^2 \sin \theta \cos \theta + \frac{1}{2} e^4 \sin^3 \theta \cos \theta + \frac{3}{8} e^6 \sin^5 \theta \cos \theta + \frac{5}{16} e^8 \sin^7 \theta \cos \theta; \]
\[ e^4 \sin \theta \cos \theta + (2e^6 - 2e^8) \sin^3 \theta \cos \theta + (3e^8 - 3e^6) \sin^5 \theta \cos \theta - 4e^3 \sin^3 \theta \cos \theta; \]
\[ e^6 \sin \theta \cos \theta + (5e^8 - 15e^6) \sin^3 \theta \cos \theta + (\frac{15}{4} e^6 - \frac{77}{4} e^8) \sin^5 \theta \cos \theta + \frac{3}{144} e^8 \sin^7 \theta \cos \theta; \]
\[ e^8 \sin \theta \cos \theta - 12e^4 \sin^3 \theta \cos \theta + 30e^6 \sin^5 \theta \cos \theta - 20e^8 \sin^7 \theta \cos \theta. \]

Adding corresponding terms of these we have
\[
\Delta \phi = \phi - \theta = (e^2 + e^4 + e^8) \sin \theta \cos \theta - [(3/2)e^4 + (23/6)e^6 + 7e^8] \sin^3 \theta \cos \theta
\]
\[ + [(77/24)e^6 + (55/4)e^8] \sin^5 \theta \cos \theta - (31/8) \sin^3 \theta \cos \theta. \]

Now \( \sin \theta \cos \theta = \frac{1}{2} \sin 2\theta \)
\[ \sin^3 \theta \cos \theta = \frac{1}{2} \sin 2\theta - \frac{1}{8} \sin 4\theta \]
\[ \sin^5 \theta \cos \theta = \frac{5}{32} \sin 2\theta - \frac{1}{8} \sin 4\theta + \frac{1}{32} \sin 6\theta \]
\[ \sin^7 \theta \cos \theta = \frac{7}{64} \sin 2\theta - \frac{7}{64} \sin 4\theta + \frac{3}{64} \sin 6\theta - \frac{1}{128} \sin 8\theta. \]

The values from (32) placed in (31) give finally
\[ \phi = \phi - \theta = C_1 \sin 2\theta + C_2 \sin 4\theta + C_3 \sin 6\theta + C_4 \sin 8\theta \]
where
\[ C_1 = \frac{1}{2}e^2 + (1/8) e^4 + (11/256) e^6 + (31/1024) e^8 \]
\[ C_2 = (3/16)e^4 + (5/64)e^6 + (25/1024)e^8 \]
\[ C_3 = (77/768)e^6 + (59/1024)e^8; \]
\[ C_4 = (127/2048)e^8. \]

Again for the Clarke 1866 spheroid
\[ e^2 = 0.006766857997, e^4 = 0.00004581473108, \]
\[ e^8 = 0.0000003101042459, \]
\[ e^6 = 0.00000000000000002098989584, \]
\[ e^8 = 0.00000000000000002098989584, \]
\[ C_1 = 3.390069228 \times 10^{-3}, C_2 = 8.614540216 \times 10^{-6}, \]
\[ C_3 = 3.12121 \times 10^{-8}, C_4 = 1.302 \times 10^{-10}. \]

We now check (33) directly from the maximum value of \( \Delta \phi \), the assumption being that if it holds for the maximum it will hold for all \( \Delta \phi \).

From (6) \( \theta = 44^\circ 51' 15.851 \), whence
\[ \sin 2\theta = 0.99998708, \sin 4\theta = 0.01016441, \sin 6\theta = -0.99988377, \sin 8\theta = -0.02032777. \]

With the values from (35) and (36) find
\[
C_1 \sin 2\theta = 0.0033900254283 \quad C_3 \sin 6\theta = -0.0000000312085
\]
\[
C_2 \sin 4\theta = 0.0000000875617 \quad C_4 \sin 8\theta = -0.0000000000026
\]
\[ 0.0033901129900 \quad -0.0000000312111 \]
\[ \Delta \phi \text{ (radians)} = 0.0033900817789 \]
\[ \Delta \phi \text{ (seconds)} = (0.0033900817789) (206,264.8062) = 699.2545611, \]
or \[ \Delta \phi_{\text{max}} = 11' 39" 255 \] which checks (6).
Note that the term $C_4 \sin 8\theta$ does not contribute to the result. Also, only eight place tables of trigonometric natural functions were used, \[4\].

Hence for geodetic latitude $\phi$ corresponding to geocentric latitude $\theta$ on the auxiliary sphere, the following formulas are sufficient for any spheroid of reference to 0.001 second:

\[
\Delta \phi \text{ (seconds)} = \phi - \theta = (206,264.8062) (C_1 \sin 2\theta + C_2 \sin 4\theta + C_3 \sin 6\theta)
\]

\[
C_1 = \frac{1}{2}e^2 + \frac{1}{8}e^4 + (11/256)e^6 + (31/1024)e^8, \quad C_2 = (3/16)e^4 + (5/4)e^6 + (25/1024)e^8, \quad C_3 = (77/768)e^6 + (59/1024)e^8, \quad e \text{ is eccentricity of the meridian}.
\]  

Now we have noted that the geocentric latitude $\theta$ as defined here is called the parametric or reduced latitude in geodetic nomenclature and has a corresponding geodetic latitude $\phi_0$ as shown in Figure 1. From (1) we see that they are related by the equation $\tan \phi_0 = (\tan \theta) / \sqrt{1 - e^2}$.  

For instance from (6) for $\theta = 44^\circ 51' 15.851'$ find from (38) that $\phi_0 = 44^\circ 57' 06.069'$. Also from (6), $\phi = 45^\circ 02' 55.106'$, whence for $\theta = 44^\circ 51' 15.851'$ we have $\Delta \phi_0 = \phi - \phi_0 = 0^\circ 0' 05' 49.037'$.  

Using the values from (34), equation (37) may be written for the Clarke 1866 spheroid as

\[
\Delta \phi \text{ (seconds)} = \phi - \theta = 699.2520 \sin 2\theta + 17769 \sin 4\theta + 0.0064 \sin 6\theta, \quad \Delta \phi = \frac{5}{4} 49^\circ 036' \quad \text{which is within 0.001 second of (39).}
\]

Subtracting (41) from (40) one finds

\[
\Delta \phi_0 = \phi - \phi_0 = 349.0818 \sin 2\theta + 1.4796 \sin 4\theta + 0.0061 \sin 6\theta.
\]

With $\theta = 44^\circ 51' 15.851'$ and the values from (28), equation (42) gives

\[
\Delta \phi_0 = 5' 49' 036' \quad \text{which is within 0.001 second of (39).}
\]

To develop $h$ in a power series in $\phi$, free of $N$ and $\theta$, refer again to Figure 1. If the tangent at $Q$ meets $OP$ in $P'$, then $PP' = a - (a^2/N) \sec \Delta \phi$, $h = PP' \cos \Delta \phi$, whence

\[
h/a = \cos \Delta \phi - a/N = \cos \Delta \phi - \sqrt{1 - e^2 \sin^2 \phi}
\]

With $\cos \Delta \phi = \sqrt{1 - \sin^2 \Delta \phi}$, and the value of $\sin \Delta \phi$ from (3), (44) may be written

\[
h/a = (1 - e^2 \sin^2 \phi)^{1/2} \left[1 - e^2 \sin^2 \phi (1 + e^2 \cos^2 \phi)\right]^{1/2} - 1 + e^2 \sin^2 \phi\]

The relation (45) may also be obtained directly from equation (2) by eliminating $\theta$ between the equations $a \cos \theta = (h + N) \cos \phi$ and $a \sin \theta = [h + N(1 - e^2)] \sin \phi$.

Expanding the two radicals by the binomial formula, (45) may be written

\[
h/a = (e^2/2 - e^4/2) \sin^2 \phi + [(5/8)e^4 - (7/8)e^6 - (1/8)e^8] \sin \phi
\]

\[+ [(9/16)e^6 - (1/4)e^8] \sin \phi + (53/128)e^8 \sin \phi.
\]
Now \( \sin^2 \phi = \frac{1}{2} (1 - \cos 2\phi) \)
\[
\sin^4 \phi = 3/8 - \frac{1}{2} \cos 2\phi + (1/8) \cos 4\phi
\]
\[
\sin^6 \phi = 5/16 - (15/32) \cos 2\phi + (3/16) \cos 4\phi - (1/32) \cos 6\phi
\]
\[
\sin^8 \phi = 35/128 - (7/16) \cos 2\phi + (7/32) \cos 4\phi - (1/16) \cos 6\phi + (1/128) \cos 8\phi
\]

and these values placed in (46) give
\[
h = a (d_1 - d_2 \cos 2\phi + d_3 \cos 4\phi - d_4 \cos 6\phi + d_5 \cos 8\phi)
\]
\[
d_1 = e^2/4 - e^4/64 - (3/256)e^6 - (233/16,384)e^8,
\]
\[
d_2 = e^2/4 + e^4/16 + 7e^6/512 + 3e^8/2048,
\]
\[
d_3 = 5e^6/64 + 11e^8/256 + 115e^10/4096
\]
\[
d_4 = 9e^6/512 + 37e^8/2048, d_5 = 53e^8/16,384
\]
a, e are the semimajor axis, eccentricity of the reference ellipsoid.

We now check (47) using the values of a and e for the Clarke 1866 spheroid. From (34) and (47) with \( a = 6,378,206.4 \) meters one has \( h(\text{meters}) = 10,788.3852 - 10,811.2646 \cos 2\phi + 22.9147 \cos 4\phi - 0.0350 \cos 6\phi \).

As a check, equation (48) should give
\[
h = a - b = 6,378,206.4 - 6,356,583.8 = 21,622.6 \text{ meters}
\]
when \( \phi = 90^\circ \). Placing \( \phi = 90^\circ \) in (48) gives
\[
h = 10,788.3852 + 10,811.2646 + 22.9147 + 0.0350 = 21,622.5995 \text{ meters}.
\]
Since we have the values of \( \theta \) and \( \phi \) for \( \Delta \phi_{\text{max}} \) from (6) we now check the value given by (48) against the closed formula (43),
\[
h = a \frac{\cos \theta}{\cos \phi} - N(\phi).
\]
\[
\phi = 45^\circ 02' 55.106, \cos \phi = 0.70650624, \cos 2\phi = -0.00169788
\]
\[
\cos 4\phi = -0.99999328, \cos 6\phi = +0.00509360.
\]
\[
\theta = 44^\circ 51' 15.03851, \cos \theta = 0.70890136, N(\phi) = 6,389,045.266.
\]
\[
h = a \frac{\cos \theta}{\cos \phi} - N(\phi) = (6,378,206.4) (0.70890136) / (0.70650624) - 6,389,045.266
\]
\[
= 6,399,829.094 - 6,389,045.266 = 10,783.828 \text{ meters}
\]
Equation (48) gives
\[
h = 10,788.3852 + 18.3562 - 22.9146 - 0.0002 = 10,783.827 \text{ meters},
\]
when \( \phi = 0, h = 0 \) and (48) gives
\[
h = 10,788.3852 - 10,811.2646 + 22.9147 - 0.0350 = +0.0003 \text{ meter}.
\]
Unless \( h \) were required to very high precision it is clear from the above checks that the formula (48) is adequate.
SUMMARY OF LATITUDE FORMULAE

If $\theta$ is the geocentric latitude of a point $P$ (a cos $\theta$, a sin $\theta$) on the auxiliary sphere, then the corresponding geodetic latitude $\phi$ of $P$ at an altitude $h$ above the ellipsoid reference, as shown in figure 1, is given by

$$\sin \Delta \phi = \sin (\phi - \theta) = (e^2/2a) N \sin 2\phi = (e^2 \sin \phi \cos \phi)/\sqrt{1 - e^2 \sin^2 \phi}$$

$$= c_1 \sin 2\phi - c_2 \sin 4\phi + c_3 \sin 6\phi - c_4 \sin 8\phi,$$

$$c_1 = e^2/2 + e^4/8 + 15e^6/256 + 35e^8/1024,$$

$$c_2 = e^4/16 + 3e^6/64 + 35e^8/1024$$

$$c_3 = 3e^6/256 + 15e^8/1024, c_4 = 5e^8/2048$$

$e =$ eccentricity of the meridian ellipse.

With the same coefficients as (49), we have

$$\Delta \phi \text{ (radians)} = (c_1 + c_1^3/8) \sin 2\phi - (c_2 + c_2^2/4 - c_3) \sin 4\phi + (c_3 - c_1^2/24) \sin 6\phi$$

and in seconds

$$\Delta \phi \text{ (seconds)} = (206,264.8062) \left[ (c_1 + c_1^3/8) \sin 2\phi - (c_2 + c_2^2/4 - c_3) \sin 4\phi + (c_3 - c_1^2/24) \sin 6\phi \right].$$

To express $\Delta \phi$ in terms of $\theta$, instead of $\phi$, we have the relation

$$\tan \phi = \tan \theta + (e^2/a \cos \theta) N \sin \phi$$

Which may be expanded by use of the Lagrange expansion formula to give

$$\Delta \phi = \phi - \theta = C_1 \sin 2\theta + C_2 \sin 4\theta + C_3 \sin 6\theta + C_4 \sin 8\theta$$

$$C_1 = e^2/2 + e^4/8 + 11e^6/256 + 31e^8/1024,$$

$$C_2 = 3e^4/16 + 5e^6/64 + 25e^8/1024,$$

$$C_3 = 77e^6/768 + 59e^8/1024, C_4 = 127e^8/2048.$$  

For checks within 0.001 second, (52) may be written $\Delta \phi \text{ (seconds)} = (206,264.8062)$

$$(C_1 \sin 2\theta + C_2 \sin 4\theta + C_3 \sin 6\theta)$$

with $C_1$, $C_2$, $C_3$ the same as in (52).

$$h/a = \cos \Delta \phi - a/N = (1 - e^2 \sin^2 \phi)^{-1/2}[1 - e^2 \sin^2 \phi (1 + e^2 \cos^2 \phi)]^{1/2} - 1 + e^2 \sin^2 \phi$$

$$h = a(d_1 - d_4) \cos 2\phi + d_4 \cos 4\phi - d_4 \cos 6\phi + d_4 \cos 8\phi$$

$$d_1 = e^2/4 - e^4/64 - 3e^6/256 - 233e^8/16,384$$

$$d_2 = e^2/4 + e^4/16 + 7e^6/512 + 3e^8/2048$$

$$0 \leq h \leq a - b$$

$$d_4 = 5e^4/64 + 11e^6/256 + 115e^8/4096$$

$$d_4 = 9e^6/512 + 37e^8/2048, d_4 = 53e^8/16,384$$

$a =$ radius of the auxiliary sphere (semimajor axis of the reference ellipsoid).
For the Clarke 1866 spheroid of reference we have from the above formulas:

\[ \Delta \phi \text{ (seconds)} = \phi - \theta = 699.2540 \sin 2\phi - 0.5936 \sin 4\phi + 0.0004 \sin 6\phi, \]  
\[ \Delta \phi \text{ (seconds)} = \phi - \theta = 699.2520 \sin 2\theta + 1.7769 \sin 4\theta + 0.0064 \sin 6\theta, \]  
\[ \Delta \phi_0 \text{ (seconds)} = \phi - \phi_0 = 349.0318 \sin 2\theta + 1.4796 \sin 4\theta + 0.0061 \sin 6\theta, \]  
\[ h \text{ (meters)} = 10,788.3852 - 10,811.2646 \cos 2\phi + 22.9147 \cos 4\phi - 0.0350 \cos 6\phi. \]  

(55)

For the Clarke 1866 spheroid, the maximum value of \( \Delta \phi \) was found to be 11' 39".255 at \( \phi = 45^\circ 02' 55".106 \).

The value of \( \Delta \phi_0 \), at this maximum of \( \Delta \phi \), was found to be 5' 49".037. Finally (58) was checked at \( \phi = 0, 90^\circ \) and \( \phi = 45^\circ 02' 55".106 \). At \( \phi = 90^\circ \), the check was within 0.0005 meter; at \( \phi = 0 \), it was within 0.0003 meter; at \( \phi = 45^\circ 02' 55".106 \), it was within 0.001 meter.

The following latitude formulae are from C & G.S. Special Publication No. 67, [5],

Where \( \phi_0, \psi, \theta \) are shown in figure 1.

\[ \psi - \psi = 700.4385 \sin 2\phi - 1.1893 \sin 4\phi + 0.0027 \sin 6\phi \]  
\[ \phi_0 - \psi = 700.4385 \sin 2\psi + 1.1893 \sin 4\psi + 0.0027 \sin 6\psi \]  
\[ \phi_0 - \theta = 350.2202 \sin 2\theta - 0.2973 \sin 4\theta + 0.0003 \sin 6\theta \]  
\[ \phi_0 - \theta = 350.2202 \sin 2\phi - 0.2973 \sin 4\phi + 0.0003 \sin 6\phi \]  
\[ \theta - \psi = 350.2202 \sin 2\theta - 0.2973 \sin 4\theta + 0.0003 \sin 6\theta \]  
\[ \theta - \psi = 350.2202 \sin 2\psi + 0.2973 \sin 4\psi + 0.0003 \sin 6\psi \]  

(59)

(60)

(61)

(62)

(63)

(64)

The above are the series expansions for the expressions given as equation (1) page 12, that is

\[ \tan \psi = \sqrt{1 - e^2} \tan \theta = (1 - e^2) \tan \phi_0. \]

(65)

REFERENCES

THE GREAT CIRCLE TRACK AS DETERMINED BY THE GEOGRAPHICAL COORDINATES OF TWO GIVEN POINTS ON THE AUXILIARY SPHERE

In figure 2, the two given points are \( Q_1(\theta_1, \lambda_1) \), \( Q_2(\theta_2, \lambda_2) \). The great circle track is then determined from the spherical triangle \( PQ_1Q_2 \). In order to simplify the computations and to have well balanced triangles from which to compute, one finds the point \( O(\theta_0, \lambda_0) \) where the great circle \( Q_1Q_2 \) is orthogonal to a meridian \( \lambda_0 \). One then works from the right spherical triangle \( POQ' \) by adding or subtracting increments of distance from \( S_1 = OQ_1 \) to get the distance \( S \). One always has then a strong right triangle \( POQ' \) from which to compute the latitude, longitude and azimuth \( \alpha \) of the point \( Q'(\theta', \lambda') \) on the base line \( Q_1Q_2 \).

DERIVATION OF FORMULAE

From right spherical triangle \( POQ' \)

\[
\cos (\lambda_0 - \lambda') = \tan\left(\frac{\pi}{2} - \theta_0\right) \cot\left(\frac{\pi}{2} - \theta'\right) = \cot \theta_0 \tan \theta'
\]

(1)

If the points \( Q_1 \) and \( Q_2 \) satisfy (1), we have by substituting their coordinates in (1)

\[
\cos (\lambda_0 - \lambda_1) = \cot \theta_0 \tan \theta_1, \tag{2}
\]

\[
\cos (\lambda_0 - \lambda_2) = \cot \theta_0 \tan \theta_2
\]

By forming the ratios of (2), expanding \( \cos (\lambda_0 - \lambda_1) \) and \( \cos (\lambda_0 - \lambda_2) \), dividing the left member numerator and denominator by \( \cos \lambda_0 \), one derives the formula

\[
\tan \lambda_0 = \frac{\tan \theta_1 \cos \lambda_1 - \tan \theta_1 \cos \lambda_2}{\tan \theta_1 \sin \lambda_2 - \tan \theta_2 \sin \lambda_1}
\]

(3)

Equations (2) may be written as

\[
\cot \theta_0 = \cot \theta_1 \cos (\lambda_0 - \lambda_1) = \cot \theta_2 \cos (\lambda_0 - \lambda_2)
\]

(4)

From right spherical triangle \( POQ' \) one has also

\[
\sin \alpha' = \frac{\sin\left(\frac{\pi}{2} - \theta_0\right)}{\sin\left(\frac{\pi}{2} - \theta'\right)} = \frac{\cos \theta_0}{\cos \theta'}
\]

(5)

\[
\cos \alpha' = \frac{\tan S}{\tan\left(\frac{\pi}{2} - \theta'\right)} = \tan S \tan \theta',
\]

(6)
Figure 2. The great circle track configuration.
\[
\sin \theta' = \cos S \sin \theta_0,
\]
(7)

\[
\tan (\lambda_0 - \lambda') = \frac{\tan S}{\sin(\frac{\pi}{2} - \theta_0)} = \frac{\tan S}{\cos \theta_0},
\]
(8)

\[
\tan \alpha' = \frac{\tan \left( \frac{\pi}{2} - \theta_0 \right)}{\sin S} = \frac{\cot \theta_0}{\sin S}
\]
(9)

\[
\sin \theta' = \cot (\lambda_0 - \lambda') \cot \alpha' \text{ or }
\tan \alpha' \sin \theta' \tan (\lambda_0 - \lambda') = 1
\]
(10)

From the oblique spherical triangle $PQ_1Q_2$ find
\[
\cos (\lambda_2 - \lambda_1) = -\cos (\pi - \alpha_2) \cos \alpha_1 + \sin (\pi - \alpha_2) \sin \alpha_1 \cos (S_1 - S_2)
\]
or
\[
\cos (\lambda_2 - \lambda_1) = \cos \alpha_1 \cos \alpha_2 + \sin \alpha_1 \sin \alpha_2 \cos (S_1 - S_2).
\]
(10.1)

Computations from the formulae
First compute $\lambda_0$ and $\theta_0$ from (3) and (4).
\[
\tan \lambda_0 = \frac{\tan \theta_0 \cos \lambda_1 - \tan \theta_1 \cos \lambda_2}{\tan \theta_1 \sin \lambda_2 - \tan \theta_2 \sin \lambda_1}
\]
cot \theta_0 = \cot \theta_1 \cos (\lambda_0 - \lambda_1) = \cot \theta_2 \cos (\lambda_0 - \lambda_2)

Next compute $\alpha_1$ and $\alpha_2$ from (5),
\[
\sin \alpha_1 = \frac{\cos \theta_0}{\cos \theta_1}, \quad \sin \alpha_2 = \frac{\cos \theta_0}{\cos \theta_2}
\]

Then $S_1$ and $S_2$ from (6)
\[
\tan S_1 = \cos \alpha_1 \cot \theta_1, \quad \tan S_2 = \cos \alpha_2 \cot \theta_2
\]
The computations for $\alpha_1$, $\alpha_2$; $S_1$ and $S_2$ are checked by (10.1)
\[
\cos (\lambda_2 - \lambda_1) = \cos \alpha_1 \cos \alpha_2 + \sin \alpha_1 \sin \alpha_2 \cos (S_1 - S_2).
\]

Now for equally spaced intervals along the great circle track, for instance in 100 nautical mile intervals, let $S = S_1 \pm 100k$.
\[
k = 1, 2, 3, \ldots \ldots \ldots N.
\]

With these values of $S$ one computes successively corresponding values of $\theta'$, $\lambda'$ and $\alpha'$
from equations (7), (8), and (9)
\[
\sin \theta' = \sin \theta_0 \cos S, \quad \tan (\lambda_0 - \lambda') = \frac{\tan S}{\cos \theta_0}, \quad \tan \alpha' = \frac{\cot \theta_0}{\sin S}
\]

These last computations are checked by (10)
\[
\sin \theta' \cdot \tan (\lambda_0 - \lambda') \cdot \tan \alpha' = 1.
\]
Figure 3. Parallels at a given distance from a great circle track.
PARALLELS AT A GIVEN DISTANCE FROM A GREAT CIRCLE TRACK

In Figure 3, the basic great circle track determined by $Q_1 (\theta_1, \lambda_1)$, $Q_2 (\theta_2, \lambda_2)$ is the same and the point $O(\theta_o, \lambda_o)$ is the same – (vertex of the great circle track). The point $P'$ is the pole of the great circle determined by $Q_1, Q_2$. The angle at $P'$ of the spherical triangle $PP'Q'$ is the distance $S = OQ'$ along the great circle track. If $p$ and $p'$ are points on the parallels at a distance $s$ from the great circle track, then the coordinates of $p$ and $p'$ can be computed from the two spherical triangles $PP'p$, $PP'p'$, (Figure 4).

From these triangles one has

$$\sin \theta_p = \cos \theta_o \sin s + \sin \theta_o \cos s \cos S$$

$$\sin \theta_{p'} = -\cos \theta_o \sin s + \sin \theta_o \cos s \cos S \quad (11)$$

$$\frac{\cos s}{\sin (\lambda_o - \lambda_p)} = \frac{\cos \theta_p}{\sin S}, \quad \frac{\cos s}{\sin (\lambda_o - \lambda_{p'})} = \frac{\cos \theta_{p'}}{\sin S} \quad (12)$$

From (11) and (12) one may write

$$\sin \theta_k = A \cos S \pm B$$

$$\sin (\lambda_o - \lambda_k) = C \sin S / \cos \theta_k \quad (13)$$

where $A = \sin \theta_o \cos s$, $B = \cos \theta_o \sin s$, $C = \cos s$.

$A$, $B$, $C$ are constants for a given $s$. When $k = p$, the $+$ sign is used in the first of equations (13). When $k = p'$, the $-$ sign is used.

The computations may be checked as before by means of the equation

$$\cos 2s = \sin \theta_p \sin \theta_{p'} + \cos \theta_p \cos \theta_{p'} \cos (\lambda_{p'} - \lambda_p).$$
A SPHERICAL RECTANGULAR
COORDINATE SYSTEM WITH A GREAT
CIRCLE BASE LINE AS AN AXIS

Figure 5 is a further elaboration of Figures 2 and 3. M is the midpoint of the spherical
segment \( Q_1Q_2 \). The section \( MP'P'' \) is perpendicular to the base line at \( M \). The general point
\( Q(\theta, \lambda) \) has for the foot of the perpendicular from \( Q \) upon the base line, the point \( Q'(\theta', \lambda') \) as
shown in figure 2. The great circle arc \( QQ' \) passes through \( P' \) and \( QQ' \) is taken for spherical
rectangular coordinate \( y \). The great circle perpendicular to the section \( MP'P'' \) and passing
through \( Q \) meets \( MP'P'' \) in \( T \). The distance \( OQ \) is \( S \) as shown in Figure 5. Note that the \( s \) of
Figure 3 in the \( y \) of Figure 5. The great circle arc \( QT \) is taken for \( x \). That is the spherical
rectangular system chosen is \( x = QT, y = QQ' \). Spherical polar coordinates are then \( r \) and \( \alpha \) as
shown in Figure 5, where \( r = MQ \), and \( \alpha \) is the angle between \( r \) and \( MQ' \).

From the right spherical triangles \( MQT, MQQ' \) one finds
\[
\sin x = \sin r \cos \alpha \\
\sin y = \sin r \sin \alpha \\
\]
whence
\[
\sin r = (\sin^2 x + \sin^2 y)^{1/2} \tag{14}
\]
\[
\tan \alpha = \sin y / \sin x, \tag{15}
\]
that is (14) and (15) represent the conversion formulas between the spherical rectangular and
spherical polar systems as given.

We now develop the coordinates \( x \) and \( y \) as functions of \( S \) and of \( \theta \) and \( \lambda \). Also \( \theta \) and \( \lambda \)
as functions of \( x \) and \( y \).

COMPUTATION OF \( S, x, y \), FROM \( \theta \) AND \( \lambda \)

Assume that the base line has been established, that is the coordinates \( \theta_0, \lambda_0 \) of the
vertex, \( 0 \), of the great circle base line have been computed from the coordinates of the two given
points \( Q_1(\theta_1, \lambda_1), Q_2(\theta_2, \lambda_2) \) by means of the equations as given on page 23. Then referring to
Figure 5, find in spherical triangles:
\[
\begin{align*}
PPQ: \quad \cos y \sin S &= \cos \theta \sin (\lambda_0 - \lambda), \tag{16} \\
&:\quad \sin y = \cos \theta_0 \sin \theta - \sin \theta_0 \cos \theta \cos (\lambda_0 - \lambda), \tag{17} \\
OPQ: \quad \cos f &= \sin \theta_0 \sin \theta + \cos \theta_0 \cos \theta \cos (\lambda_0 - \lambda), \tag{18} \\
OQQ': \quad \cos y \cos S &= \cos f, \tag{19} \\
TPQ: \quad \sin x &= \sin d \cos y. \tag{20}
\end{align*}
\]
Figure 5. Spherical rectangular coordinate system.
Dividing respective members of (16) and (19) find
\[ \tan S = \cos \theta \sin (\lambda_o - \lambda) / \cos f \]  
where \( \cos f \) is given by (18).

From (17) and (18) we have \( \sin \theta_o \cos f = \sin \theta - \cos \theta_o \sin y \) whence (21) may be written

\[ \tan S = \frac{\sin \theta_o \cos \theta \sin (\lambda_o - \lambda)}{\sin \theta - \cos \theta_o \sin y} \]  

Referring now to Figures 1 and 5, it is seen that \( d = MQ = S - \frac{1}{2}(S_1 + S_2) \), where \( S_1 \) and \( S_2 \) are the distances from \( O(\theta_o, \lambda_o) \) to \( Q_1 \) and \( Q_2 \) respectively.

Hence given the spherical curvilinear coordinates \( \theta, \lambda \) of a point \( Q(\theta, \lambda) \), to find \( S, x \) and \( y \) with \( \theta_o, \lambda_o, S_1, S_2 \) known, compute \( y \) and \( S \) from (17) and (21) or (22) and then \( x \) from (20), i.e.

\[ \sin y = \cos \theta_o \sin \theta - \sin \theta_o \cos \theta \cos (\lambda_o - \lambda) \]  
\[ \tan S = \frac{\sin \theta_o \cos \theta \sin (\lambda_o - \lambda)}{\sin \theta - \cos \theta_o \sin y} = \frac{\cos \theta \sin (\lambda_o - \lambda)}{\cos f} \]  

\[ \cos \theta \sin (\lambda_o - \lambda) \]  
\[ \sin \theta_o \sin \theta + \cos \theta_o \cos \theta \cos (\lambda_o - \lambda) \]  
\[ \sin y = \sin d \cos y = \sin \left[ S - \frac{1}{2}(S_1 + S_2) \right] \left( 1 - \sin^2 y \right)^{1/2} \]

**COMPUTATION OF \( S, \theta, \lambda \) FROM \( x \) AND \( y \)**

From equation (20) one has \( \sin d = \sin x / \cos y \) or \( \sin \left[ S - \frac{1}{2}(S_1 + S_2) \right] = \sin x / \cos y \) whence

\[ S = \arcsin \left( \frac{\sin x}{\cos y} \right) + \frac{1}{2}(S_1 + S_2). \]  

From equations (13) page 27,
\[ \sin \theta = A \cos S + B \]  
\[ \sin (\lambda_o - \lambda) = C \sin S / \cos \theta \]  

where \( A = C \sin \theta_o, B = D \cos \theta_o, C = \cos y, D = \sin y \)

Hence to compute \( S, \theta, \lambda \) from \( x \) and \( y \), first compute \( S \) from (24) and then \( \theta \) and \( \lambda \) from (25) i.e.:

let \( C = \cos y, D = \sin y, E = \sin x, A = C \sin \theta_o, B = D \cos \theta_o. \)

Then
\[ S = \arcsin \left( \frac{E}{C} \right) + \frac{1}{2}(S_1 + S_2) \]  
\[ \theta = \arcsin \left( A \cos S + B \right) \]  
\[ \lambda = \lambda_o - \arcsin \left( \frac{C \sin S / \cos \theta}{C} \right) \]  

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DERIVATION OF THE EQUATIONS TO SPHERICAL HYPERBOLAS

Having established a rectangular spherical coordinate system on a great circle base line, we are now in a position to develop the equations of spherical hyperbolas referred to our rectangular system. Referring again to Figure 5, we restrict the point Q(θ, ρ) or Q(x,y) to the locus defined by demanding that the distances σ and σ from the points Q and Q respectively satisfy the condition

\[ \sigma_1 - \sigma_2 = 2c/e = 2a \]  
\[ 2c = S_1 - S_2, \]  

where as before S, S are the distances of Q, Q respectively from O(0,0, X0); e is a number such that e > 1.

From the spherical triangles MQQ, MQQ one has

\[ \cos \sigma_2 = \cos r \cos c + \sin r \sin c \cos \alpha \]
\[ \cos \sigma_1 = \cos r \cos c - \sin r \sin c \cos \alpha \]  

Adding and subtracting respective members of (28) obtain

\[ \cos \sigma_1 + \cos \sigma_2 = 2 \cos r \cos c \]
\[ \cos \sigma_1 - \cos \sigma_2 = -2 \sin r \sin c \cos \alpha \]  

By well known trigonometric identities and condition (27), equations (29) may be written

\[ \cos \sigma_1 + \cos \sigma_2 = 2 \cos \sigma_1 \cos \sigma_2 = 2 \cos \sigma_1 \cos \sigma_2 \cos a = 2(\cos r)(\cos c), \]
\[ \cos \sigma_1 - \cos \sigma_2 = 2 \sin \sigma_1 \cos \sigma_2 \sin \sigma_1 - \cos \sigma_2 = 2 \sin \sigma_1 \cos \sigma_2 \sin a = -2(\sin r)(\sin c) \cos \alpha, \]
or

\[ \cos \sigma_1 + \sigma_2 = \cos r \cos c / \cos \alpha, \]
\[ \sin \sigma_1 + \sigma_2 = \sin r \sin c \cos \alpha / \sin \alpha. \]  

Squaring and adding respective members of (30), get

\[ (\cos^2 r) (\cos^2 c / \cos^2 \alpha) + (\sin^2 r \cos^2 \alpha) (\sin^2 c / \sin^2 \alpha) = 1. \]  

Now in (31) place \( \cos^2 r = 1/(1 + \tan^2 r), \)
\[ \sin^2 r = \tan^2 r / (1 + \tan^2 r), \]
whence (31) may be written

\[ \tan^2 r = \tan^2 a \left( \cos^2 a - \cos^2 c \right) / \left( \sin^2 c \cos^2 a - \sin^2 \alpha a \right) = \tan^2 a \left( \sin^2 c - \sin^2 \alpha \right) / \left( \sin^2 c \cos^2 \alpha - \sin^2 \alpha \right) \]  

Now (32) is the polar form of the equation to the spherical hyperbola.

From conversion formulas (15) we have

\[ \tan^2 r = (\sin^2 x + \sin^2 y) / (1 - \sin^2 x - \sin^2 y), \]
\[ \cos^2 a = \sin^2 x / (\sin^2 x + \sin^2 y) \]  

31
and substitutions for $\tan^2r$, $\cos^2a$ from (33) in (32) give the rectangular equation to the spherical hyperbola

$$\sin^2x = \frac{\sin^2a \cos^2c}{\sin^2c - \sin^2a} \cdot \sin^2y + \sin^2a.$$  (34)

**THE POLAR EQUATION OF SPHERICAL HYPERBOLAS WITH ORIGIN AT A FOCUS**

If we choose the given point $Q_1(\theta_1, \lambda_1)$ of the great circle base line as origin of coordinates and a focus, then the following figure may be abstracted from Figure 5:

![Figure 6.](image)

The polar radius is now $R = \sigma_2$, $\beta$ is the angle between $R$ and $Q_1Q'$, $k = Q_1Q' = S' - S$. From spherical triangle $Q_2QQ_1$ we find $\cos \sigma_1 = \cos R \cos 2c - \sin R \sin 2c \cos \beta$, (35)

and from (27) $\sigma_1 - R = 2a$, whence

$$\cos (\sigma_1 - R) = \cos \sigma_1 \cos R + \sin \sigma_1 \sin R = \cos 2a,$$  (36)

$$\sin (\sigma_1 - R) = \cos \sigma_1 \sin R + \sin \sigma_1 \cos R = \sin 2a.$$

Multiply the first of (36) by $\sin R$, the second by $\cos R$ and add respective members to solve for

$$\sin \sigma_1 = \cos 2a \sin R + \sin 2a \cos R.$$  (37)

Square and add respective members of (35) and (37) to get

$$(\cos R \cos 2c - \sin R \sin 2c \cos \beta)^2 + (\cos 2a \sin R + \sin 2a \cos R)^2 = 1.$$  (38)

Multiply every term of (38) by $\sec^2R$, whence it may be written

$$(\cos 2c - \tan R \sin 2c \cos \beta)^2 + (\cos 2a \tan R + \sin 2a)^2 = \sec^2R = 1 + \tan^2R.$$  (39)

Expanding (39) and writing as a quadratic in $\tan R$ find

$$\tan^2R (\sin^22c \cos^2\beta - \sin^22a) + 2\tan R (\sin 2a \cos 2a - \sin 2c \cos 2c \cos \beta) + \cos^22c - \cos^22a = 0.$$  (40)
Now equation (40) factors into \[\tan R (\sin 2c \cos \beta + \sin 2a) - (\cos 2c + \cos 2a)\].

\[\tan R (\sin 2c \cos \beta - \sin 2a) - (\cos 2c - \cos 2a)\] = 0. \hspace{1cm} (41)

Whence

\[
\tan R = \frac{\cos 2c + \cos 2a}{\sin 2c \cos \beta + \sin 2a}, \quad \tan R = \frac{\cos 2c - \cos 2a}{\sin 2c \cos \beta - \sin 2a}
\]

or

\[
\tan R = \frac{\cos 2c \pm \cos 2a}{\sin 2c \cos \beta \pm \sin 2a}, \hspace{1cm} (42)
\]

where either the (two plus signs) or (two minus) signs are taken together.

Equation (42) is the polar equation to spherical hyperbolas referred to a focus as pole.

We now derive expressions for the spherical rectangular coordinates \(x, y\) as functions of the polar coordinates \(R, \beta\).

From right triangles \(WPQ, WQQ_{1}, Q_{1}QQ^{'\prime}\) (Figure 6) find

\[
\sin x = \sin R \cos \beta,
\]
\[
\sin y = \sin R \sin \beta.
\] (43)

\[
\sin x = \sin k \cos y;
\]
\[
\cos R = \cos k \cos y.
\] (44)

Equations (43) are similar to equations (14) and provide the conversions from polar to rectangular coordinates, i.e. from (43)

\[
\sin R = (\sin^2 x + \sin^2 y)^{1/2},
\]
\[
\tan \beta = \sin y / \sin x.
\] (45)

Since moving the origin from \(M\) to \(Q_{1}\) (see Figure 5) is only a translation along the \(x\)-axis, there is no change in \(y\), but \(x\) is changed. Hence from (44) and the relations (23) and (26) we can write when the origin is at \(Q_{1}\), \(k = S - S_{1}\):

FORMULAS FOR COMPUTATION OF \(S, x, y\) FROM \(\theta\) AND \(\lambda\)

\[
\sin y = \cos \theta \sin \theta - \sin \theta \cos \theta \cos (\lambda_{0} - \lambda)
\]
\[
\tan S = \frac{\sin \theta \cos \theta \sin (\lambda_{0} - \lambda)}{\sin \theta - \cos \theta \sin y} = \frac{\cos \theta \sin (\lambda_{0} - \lambda)}{\cos \theta}
\] (46)

\[
\sin y = \frac{\cos \theta \sin (\lambda_{0} - \lambda)}{\sin \theta \sin \theta \cos \cos \theta \cos (\lambda_{0} - \lambda)}
\]

\[
\sin x = \sin k \cos y = \sin (S - S_{1}) \cos y
\]

FORMULAS FOR COMPUTATION OF \(S, \theta, \lambda\) FROM \(x\) AND \(y\)

Let \(C = \cos y, D = \sin y, E = \sin x, A = C \sin \theta, B = D \cos \theta,\) then

\[
S = \arcsin (E/C) + S_{1}
\]
\[ \theta = \arcsin (A \cos S + B) \]  
\[ \lambda = \lambda_0 - \arcsin (C \sin S / \cos \theta) \]  

AN ALTERNATIVE EQUATION TO THE SPHERICAL HYPERBOLA WITH ORIGIN AT A FOCUS

If \( S = \frac{1}{2}(a_0 + b_0 + c_0) \) in the spherical triangle

\[
\begin{align*}
X &= k - \arcsin \left( \frac{C \sin S}{\cos \theta} \right) \\
\end{align*}
\]

Figure 7.

then \( \tan^2 \frac{1}{2} \theta = \frac{\sin(s - b_o) \sin(s - c_o)}{\sin \theta \sin \sin(s - a_o)} \), [6].

Referring to figure 6, \( a_o = \sigma_i \), \( b_o = 2c \), \( c_o = R \) and from (27) we have the conditions

\[
\begin{align*}
\sigma_i - R &= 2a, \quad \sigma_i + R = 2(R + a) \\
\end{align*}
\]

Hence

\[
\begin{align*}
s &= \frac{1}{2}(\sigma_i + R) + c = R + a + c, \\
s - a_o &= \frac{1}{2}(R - \sigma_i) + c = c - a, \\
s - b_o &= R + a - c, \quad S - c_o = c + a \\
A &= \pi - \beta, \tan \frac{1}{2} \theta = \tan \left( \frac{\pi}{2} - \beta / 2 \right) = \cot \beta / 2 \\
\end{align*}
\]

With the values from (49) placed in (48) find

\[ \tan^2 \beta / 2 = \frac{\sin(c - a) \sin(R + c + a)}{\sin(c + a) \sin(R - c + a)} \],

which is the desired alternative form, [7].

CORRESPONDING PLANE HYPERBOLA EQUIVALENTS

For the plane case and analogous reference system, Figure 5 becomes
Given the condition $\sigma_1 - \sigma_2 = 2a$

By the law of cosines applied to triangles MQQ_1 MQQ_2

\[ \sigma_1^2 = r^2 + c^2 - 2rc \cos \alpha, \quad \sigma_1^2 = r^2 + c^2 + 2rc \cos \alpha \]

whence $\sigma_1^2 + \sigma_2^2 = 2(r^2 + c^2), \quad \sigma_1^2 \sigma_2^2 = (r^2 + c^2)^2 - 4r^2 c^2 \cos^2 \alpha \quad \text{(51)}$

Now by squaring both sides of $\sigma_1 - \sigma_2 = 2a$ obtain

\[ \sigma_1^2 - 2\sigma_1 \sigma_2 + \sigma_2^2 = 4a^2 \text{ whence} \]

\[ (\sigma_1^2 + \sigma_2^2 - 4a^2)^2 = 4\sigma_1^2 \sigma_2^2 \quad \text{(52)} \]

With the values of $\sigma_1^2 + \sigma_2^2, \sigma_1^2 \sigma_2^2$ from (51) placed in (52) obtain

\[ [2(r^2 + c^2) - 4a^2]^2 = 4[(r^2 + c^2)^2 - 4r^2 c^2 \cos^2 \alpha]. \quad \text{(53)} \]

Expanding (53) find

\[ r^2 c^2 \cos^2 \alpha - a^2 r^2 - a^2 c^2 + a^4 = 0 \]

or

\[ r^2 = \frac{a^2(c^2 - a^2)}{c^2 \cos^2 \alpha - a^2} \quad \text{(54)} \]

To transform to rectangular equation we have $x = r \cos \alpha, y = r \sin \alpha$, or $r^2 = x^2 + y^2$,

tan $\alpha = \frac{y}{x}, \quad \cos^2 \alpha = x^2/(x^2 + y^2)$ and these values of $r^2$ and $\cos^2 \alpha$ placed in (54) give

\[ x^2 = \frac{a^2 y^2}{c^2 - a^2} + a^2 \quad \text{(55)} \]

as corresponding rectangular equation.
If the focus $Q_1$ is to be the origin and $\sigma_1 = R$, the radius for polar coordinates, and $\beta$ the angle which $R$ makes with the positive x-axis, i.e. $\beta$ is the angle $QQ_1Q'$, then our plane figure is as follows:

![Diagram](image)

Figure 9.

By the law of cosines in triangle $Q_2QQ_1$

$$\sigma_1^2 = 4c^2 + R^2 + 4cR \cos \beta$$

(56)

From the condition $\sigma_1 - R = 2a$, $\sigma_1 = R + 2a$, and this value of $\sigma_1$ placed in (56) gives

$$(R + 2a)^2 = 4c^2 + R^2 + 4cR \cos \beta$$

which when expanded gives

$$R = \frac{a^2 - c^2}{c \cos \beta - a}$$

(57)

For the alternative form of (57), we have the well known formula

$$\tan^2 \frac{\beta}{2} = \frac{(s - b_o)(s - c_o)}{s(s - a_o)}$$

(58)

where $2s = a_o + b_o + c_o$.

Here $a_o = \sigma_1$, $b_o = R$, $c_o = 2c$, $A = \pi - \beta$.

Hence: $s = a + c + R$, $s - a_o = c - a$, $s - b_o = a + c$, $s - c_o = a - c + R$,

whence

$$\tan^2 \frac{\beta}{2} = \frac{(c - a)(R + c + a)}{(c + a)(R - c + a)}$$

(59)

which is an alternative form of (57).

Now (54), (55), (57) and (59) could have been obtained directly from (32), (34), (42) and (50) by replacing correctly the trigonometric functions of lengths by corresponding lengths, i.e.

$\tan a = \sin a = a$, $\cos a = 1$, etc. We place them side by side for direct comparison in the following table which will also serve as a summary for both:

<table>
<thead>
<tr>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_1^2 = 4c^2 + R^2 + 4cR \cos \beta$</td>
<td>By the law of cosines in triangle $Q_2QQ_1$</td>
</tr>
<tr>
<td>$R = \frac{a^2 - c^2}{c \cos \beta - a}$</td>
<td>Alternative form of (57)</td>
</tr>
<tr>
<td>$\tan^2 \frac{\beta}{2} = \frac{(s - b_o)(s - c_o)}{s(s - a_o)}$</td>
<td>Well known formula</td>
</tr>
<tr>
<td>$\tan^2 \frac{\beta}{2} = \frac{(c - a)(R + c + a)}{(c + a)(R - c + a)}$</td>
<td>Alternative form of (57)</td>
</tr>
</tbody>
</table>
SPHERICAL HYPERBOLA FORMULAS AND PLANE EQUIVALENTS, [7]

SPHERICAL

(1) \( \tan^2 r = \frac{\tan^2 a (\sin^2 c - \sin^2 a)}{\sin^2 c \cos^2 a - \sin^2 a} \)

(2) \( \sin^2 x = \frac{\sin^2 a \cos^2 c}{\sin^2 c - \sin^2 a} \sin^2 y + \sin^2 a \)

(3) \( \tan R = \frac{\cos 2c \pm \cos 2a}{\sin^2 c \cos \beta \pm \sin 2a} \)

(4) \( \tan^2 (\beta/2) = \frac{\sin (c - a) \sin (R + c + a)}{\sin (c + a) \sin (R - c + a)} \)

PLANE

(60)

\( r^2 = \frac{a^2(c^2 - a^2)}{c^2 \cos^2 a - a^2} \)

\( x^2 = \frac{a^2 y^2 + a^2}{c^2 - a^2} \)

\( R = \frac{a^2 - c^2}{c \cos \beta - a} \)

\( \tan^2 (\beta/2) = \frac{(c-a)(R + c + a)}{(c+a)(R - c + a)} \)

In (1) and (2) of equations (60), the origin of coordinates is the midpoint \( M_1 \), of the segment \( Q_1 Q_2 \), see Figure 5. (3) and (4) are two polar forms with origin at a Focus \( Q_1 \), see Figures (5) and (6).

REFERENCES


[7] Equations (32), (34), (42), (50) to spherical hyperbolas are essentially those given without derivation in LORAN, Pierce, McKenzie, Woodward, McGraw Hill 1948, pages 173, 175.
If we are given two points \( P_1(\phi_1, \lambda_1) \), \( P_2(\phi_2, \lambda_2) \) on the ellipsoid of reference as shown in Figure 10, we may compute distances and azimuths according to known or given elements. That is, we may compute the geographic coordinates of the point \( P_2(\phi_2, \lambda_2) \) if we know the geographic coordinates of \( P_1(\phi_1, \lambda_1) \) the distance between \( P_1 \) and \( P_2 \), and the azimuth from \( P_1 \) to \( P_2 \). This is the direct problem and the one most important in Geodesy relative to establishing triangulation control nets. If the coordinates of both \( P_1 \) and \( P_2 \) are given, the distance between them and the azimuths can be computed. This is the inverse problem, and the one concerned primarily in electronic positioning systems as Loran.

Since there are several possible curves connecting the points \( P_1 \) and \( P_2 \) on the ellipsoid along which distances would differ very little, for instance – the geodesic, the normal sections, the great elliptic arc, the curve of alinement, etc. – criteria for selection would be simplicity in computations relative to required accuracy. Also to be considered are other useful geometric quantities associated with the configuration and expressible in terms of common computational parameters. (See Figure 11).

The shortest distance is always the geodesic or the geodetic line between \( P_1 \) and \( P_2 \). It is usually a space curve (that is it has a first and second curvature at each point). For instance on the reference ellipsoid, the equator and the meridians are the only plane geodesics, [8].

Now in Figure 10, the point \( P_0(\phi_0, \lambda_0) \) is the vertex of the great elliptic arc, that is \( P_0 \) is the point where the great elliptic arc is orthogonal to a meridian. The geodesic, or geodetic line, between \( P_1 \) and \( P_2 \) also has a vertex where it is orthogonal to a meridian. Since the geodesic is a space curve and climbs nearer to the ellipsoid pole, \( T_0 \), than any of the other representative curves (if \( P_1 \) and \( P_2 \) were ends of a diameter of the equator, the geodesic would be the elliptic meridian through \( P_1 \) and \( P_2 \) since it is shorter than the equator), the vertex of the geodesic is closer to \( T_0 \) than is \( P_0 \). Unfortunately the geographic coordinates of the geodesic vertex cannot be expressed simply in terms of the geographic coordinates of \( P_1 \) and \( P_2 \), hence an approximation scheme, usually iterative, is used. [9] The computations are usually quite lengthy for long lines. Many schemes and formulae have been devised to approximate the geodesic and studies have been made comparing them. [21] The geodetic line is of most interest to the geodesist proper, since he is primarily concerned with closure on a particular ellipsoid of reference of large arcs and areas of triangulation, hence the geodesic or geodetic line and geodetic azimuths on the ellipsoid are consonant with his mathematical model.
Figure 10. Corresponding distances on the reference ellipsoid and the auxiliary sphere.
OPERATIONAL APPLICATIONS

Requirements, accuracy wise, with respect to geodetic data obviously depend on the particular guidance system employing it. If some guidance, particularly external, is to be provided a missile, its initial launch requirements are not as critical as say for a purely ballistic missile. Since it has yet to be demonstrated that the flight of missiles are geodesic or that the traces of the trajectories upon the ellipsoid of reference are geodesics, distances can be computed by any method which will give results within the capability of the particular system. Since alinement is usually with respect to a local vertical and a "bearing", the normal section azimuth, the angle of depression of the chord below the horizon and the maximum separation between the chord and the surface are all useful associated quantities which can be "integrated" in the computations for distance as will subsequently be shown in the discussion of distance computations along the great elliptic arc. This configuration is shown in Figure 11 as abstracted from Figure 10.

HYPERBOLIC MEASURING SYSTEMS

For Loran systems, the earth must be considered an oblate ellipsoid or spheroid, but the nearest hundred feet is probably close enough particularly on long lines. [7], page 170. Hence a computational system is desirable which provides modifications to spherical elements, i.e. functions of spherical arc lengths so that the auxiliary sphere of the particular spheroid of reference can be used since the hyperbolic propagation of systems as Loran may be worldwide as base lines are added or extended. Also to be considered is the use of such computational systems in local areas as for oceanographic surveying and corresponding adaptation to a local sphere of reference. Azimuth computations should be independent, except for dependence on spherical arc length, so that one can have readily the Normal plane section azimuths as well as geodetic azimuths. Finally the system should be easily adapted to local area work in terms of plane coordinates. This can probably best be accomplished through the series of projections, all conformal; spheroid to aposphere, aposphere to sphere, sphere to plane. [8].

The present investigation will center about the configuration depicted in Figure 12 which shows the relationships, exaggerated; between the Normal sections, The Great Elliptic Section, The Geodesic, and the Chord between two points \(Q_1, Q_2\) on the ellipsoid. We begin by deriving the formulae for the Normal Section Azimuths and the Great Elliptic Arc Azimuths.

NORMAL SECTION AZIMUTHS

The normal section azimuths are shown in Figure 13, as extended from Figure 11. The spheroid has been referred to its center as origin of rectangular coordinates, with the reference plane – \(xz\) containing the point \(Q_1(\phi_1, \lambda_1)\) as shown. The \(z\)-axis is the polar axis of the spheroid.
\( \alpha \) = Normal Section Azimuth at \( P_1 \) (from North)

\( S \) = Arc length - Geodetic distance

\( C \) = Chord length, \( P_1 \) \( P_2 \)

\( \beta \) = Angle of depression of \( C \) below horizon at \( P_1 \)

\( H_o \) = Maximum separation of arc \( S \) and chord \( C \)

Figure 11. Relationship between arc length, normal section azimuth, chord length, angle of depression of the chord below the horizon, maximum separation of arc and chord.

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Figure 12. Relationships relative to the pole on the ellipsoid of reference, of the geodesic, normal sections, and great elliptic section.
Figure 13. The normal section azimuths.
and the $y$-axis is then in the plane of the equator — the $xy$-plane is the equatorial plane of the ellipsoid. In this coordinate system the points $Q_1(\phi_1, \lambda_1), Q_2(\phi_2, \lambda_2)$ have the rectangular coordinates:

\[
\begin{align*}
Q_1: & \quad x_1 = N_1 \cos \phi_1 \\
& \quad y_1 = 0 \\
& \quad z_1 = N_1 (1 - e^2) \sin \phi_1
\end{align*}
\]

\[
\begin{align*}
Q_2: & \quad x_2 = N_2 \cos \phi_2 \cos \Delta \lambda \\
& \quad y_2 = N_2 \cos \phi_2 \sin \Delta \lambda \\
& \quad z_2 = N_2 (1 - e^2) \sin \phi_2
\end{align*}
\]

The rectangular equation to the ellipsoid is

\[
(1 - e^2) (x^2 + y^2) + z^2 - a^2 (1 - e^2) = 0, \tag{2}
\]

where $a, e$ are respectively the semimajor axis and eccentricity of the meridian ellipse.

The tangent plane to (2) at any point $(x_1, y_1, z_1)$ is

\[
(1 - e^2) (xx_1 + yy_1) + zz_1 - a^2 (1 - e^2) = 0. \tag{3}
\]

Hence the tangent plane at $Q_1$ is, from (1) and (3)

\[
x N_1 \cos \phi_1 + z N_1 \sin \phi_1 - a^2 = 0. \tag{4}
\]

The equation of the plane containing the normal at $Q_1$ and the point $Q_2$ is determined by $Q_2$ and the points $(N_1 e^2 \cos \phi_1, 0, 0), (0, 0, -N_1 e^2 \sin \phi_1)$, see Figure 13. With the coordinates of $Q_2$ from (1) we can write the equation as

\[
\begin{vmatrix}
 x & y & z & 1 \\
 N_2 \cos \phi_2 \cos \Delta \lambda & N_2 \cos \phi_2 \sin \Delta \lambda & N_2 (1 - e^2) \sin \phi_2 & 1 \\
 N_1 e^2 \cos \phi_1 & 0 & 0 & 1 \\
 0 & 0 & -N_1 e^2 \sin \phi_1 & 1 \\
\end{vmatrix} = 0,
\]

which upon expansion may be written

\[
A x + B y - C z - D = 0
\]

where

\[
A = N_2 \sin \phi_1 \cos \phi_2 \sin \Delta \lambda
\]

\[
B = (N_1 \sin \phi_1 - N_2 \sin \phi_2) e^2 \cos \phi_1 + N_2 (\sin \phi_2 \cos \phi_1 - \sin \phi_1 \cos \phi_2 \cos \Delta \lambda)
\]

\[
C = N_1 \cos \phi_1 \cos \phi_2 \sin \Delta \lambda
\]

\[
D = N_1 N_2 e^2 \sin \phi_1 \cos \phi_2 \cos \phi_2 \sin \Delta \lambda.
\]

Now the direction cosines $p, q, r$ of the intersection of two planes $A_1 x + B_1 y + C_1 z = D_1, A_2 x + B_2 y + C_2 z = D_2$ are given by

\[
p = (B_2 C_1 - B_1 C_2)/d, \quad q = (C_1 A_2 - A_1 C_2)/d, \quad r = (A_1 B_2 - A_2 B_1)/d
\]

where

\[
d = [(B_2 C_2 - B_1 C_1)^2 + (C_1 A_2 - A_1 C_2)^2 + (A_1 B_2 - A_2 B_1)^2]^{1/2}.
\]

Note from figure 13 that the tangent, $t_1$, to the meridian at $Q_1$ lies in the plane $y = 0$ and that defined by equation (4). To apply (6) to these two planes we have respectively

\[
A_1 = C_1 = D_1 = 0, \quad B_1 = 1; \quad A_2 = N_1 \cos \phi_1, \quad B_2 = 0, \quad C_2 = N_1 \sin \phi_1, \quad D_2 = a^2
\]

and (6) gives the direction cosines of $t_1$ as

\[
p_1 = \sin \phi_1, \quad q_1 = 0, \quad r_2 = -\cos \phi_1.
\]

(7)
From Figure 13, the tangent $t_2$ to the elliptic section lying in the plane (5) is the line of intersection of the planes (4) and (5). From (4) and (5) we have respectively $A_1 = N_1 \cos \phi_1$, $B_1 = \phi, C_1 = N_1 \sin \phi_1$; $A_2 = A, B_2 = B, C_2 = -C$ and applying (6) find the direction cosines of $t_2$ to be

$$P_2 = (-B \sin \phi_1)/d, \quad q_2 = (A \sin \phi_1 + C \cos \phi_1)/d, \quad r_2 = (B \cos \phi_1)/d$$

where $d = [B^2 + (A \sin \phi_1 + C \cos \phi_1)^2]^{1/2}$. (8)

The forward azimuth $\alpha_{AB}$ from $Q_1$ to $Q_2$, as shown in Figure 13, is the angle reckoned clockwise from south between the tangents $t_1$ and $t_2$. Hence from (7) and (8)

$$\cos \alpha_{AB} = p_1 p_2 + q_1 q_2 + r_1 r_2 = -B \sin^2 \phi_1 - B \cos^2 \phi_1 = -B/d,$$  

$$d = [B^2 + (A \sin \phi_1 + C \cos \phi_1)^2]^{1/2}$$

Since $\cot \alpha_{AB} = \cos \alpha_{AB} / (1 - \cos^2 \alpha_{AB})^{1/2}$ we have from (9) that

$$\cot \alpha_{AB} = -B/(d^2 - B^2)^{1/2},$$  

(10)

Now $d^2 - B^2 = B^2 + (A \sin \phi_1 + C \cos \phi_1)^2 - B^2 = (A \sin \phi_1 + C \cos \phi_1)^2$,

so $\sqrt{d^2 - B^2} = A \sin \phi_1 + C \cos \phi_1$ and (10) may be written

$$\cot \alpha_{AB} = -B/(A \sin \phi_1 + C \cos \phi_1).$$

With the values of $A, B, C$ from (5), equation (11) may be written as

$$\cot \alpha_{AB} = -[\sin \phi_2 - (N_1/N_2) \sin \phi_1]e^{2 \cos \phi_1 \sec \phi_2 + (\sin \phi_1 \cos \Delta \lambda - \tan \phi_2 \cos \phi_1)]$$

$$\sin \Delta \lambda$$

(12)

Referring again to figure 13, it is seen that from considerations of symmetry, we have only to interchange the subscripts 1 and 2 and change $\Delta \lambda$ to $-\Delta \lambda$ in (12) to obtain cotBA (the back azimuth on the other normal section). We thus obtain from (12)

$$\cot \alpha_{BA} = -[\sin \phi_1 - (N_2/N_1) \sin \phi_2]e^{2 \cos \phi_2 \sec \phi_1 + (\sin \phi_2 \cos \Delta \lambda - \tan \phi_1 \cos \phi_2)]$$

$$\sin \Delta \lambda$$

(13)

GREAT ELLIPTIC SECTION AZIMUTHS

Figure 14 shows the great elliptic section and azimuths as abstracted from Figure 12. The same coordinate system is used as in Figure 13 so that most of the equations developed with the normal section azimuths can be used. The angle $\alpha_{AB}$ between the tangents $t_1$ and $t_2$ is the forward azimuth required. We already have the direction cosines of $t_1$ see equations (7). The tangent $t_2$ is the intersection of the great elliptic plane with the tangent plane at $Q_1$, equation (4). The equation of the great elliptic plane through $Q_1, Q_2$, using equations (1), is given by the determinant
GREAT ELLIPTIC SECTION AZIMUTHS AND ASSOCIATED GEOMETRY

P - point of maximum separation, chord and arc
H_o - maximum separation of chord and arc

Figure 14. The great elliptic section azimuths.
which when expanded reduces to
\[ Ax + By - Cz = 0, \]
\[ A = (1 - e^2) \tan \phi \sin \Delta \lambda \quad \text{(14)} \]
\[ B = (1 - e^2) (\tan \phi_2 - \tan \phi_1 \cos \Delta \lambda) \]
\[ C = \sin \Delta \lambda \]

Since equation (11) was developed for generalized coefficients \( A, B, C \) we have only to substitute the values of \( A, B, C \) from (14) in (11) to obtain after some algebraic manipulation,

\[ \cot a_{AB} = (1 - e^2) \frac{N_2}{a} \frac{\left( \tan \phi_1 \cos \Delta \lambda - \tan \phi_2 \right) \cos \phi_1}{\sin \Delta \lambda} \quad \text{(15)} \]

By symmetrical interchange of subscripts and replacing \( \Delta \lambda \) by \(-\Delta \lambda \), we obtain \( \cot a_{BA} \) from (15) as

\[ \cot a_{BA} = (1 - e^2) \frac{N_2}{a} \frac{\left( \tan \phi_1 - \tan \phi_2 \cos \Delta \lambda \right) \cos \phi_2}{\sin \Delta \lambda} \quad \text{(16)} \]

Equations (15) and (16) represent the azimuths of the great elliptic section as shown in Figure 14.

NORMAL SECTION AND GREAT ELLIPTIC SECTION AZIMUTHS IN TERMS OF PARAMETRIC LATITUDE \( \theta \)

From the transformation equations\[ \tan \theta = \frac{(1 - e^2)^{1/2}}{a} \tan \phi, \cos \theta = \frac{N}{a} \cos \phi, \]
\[ \sin \theta = \frac{(1 - e^2)^{1/2}}{a} N \sin \phi, (1 - e^2 \cos^2 \theta)^{1/2} = \frac{(1 - e^2)^{1/2}}{a} N \]

applied to equations (12), (13), (15), (16) we have the normal section and great elliptic section azimuths in terms of parametric latitude.

Normal Section Azimuths in terms of \( \theta \).

\[ \cot a_{AB} = \frac{\sin \theta \cos \Delta \lambda - \cos \theta_1 \tan \theta_2 + e^2 (\sin \theta_2 - \sin \theta_1) \cos \theta_1 \sec \theta_2}{(1 - e^2 \cos^2 \theta_1)^{1/2} \sin \Delta \lambda} \]
\[ \cot a_{BA} = - \frac{\sin \theta_2 \cos \Delta \lambda - \cos \theta_2 \tan \theta_1 + e^2 (\sin \theta_1 - \sin \theta_2) \cos \theta_2 \sec \theta_1}{(1 - e^2 \cos^2 \theta_2)^{1/2} \sin \Delta \lambda} \quad \text{(17)} \]
Great Elliptic Section Azimuths in terms of $\theta$

\[
\cot \alpha_{AB} = \pm \frac{(\tan \theta_1 \cos \Delta \lambda - \tan \theta_2) \cos \theta_1 (1 - e^2 \cos^2 \theta_1)^{1/2}}{\sin \Delta \lambda}
\]

\[
\cot \alpha_{BA} = \pm \frac{(\tan \theta_1 - \tan \theta_2 \cos \Delta \lambda \cos \theta_2) (1 - e^2 \cos^2 \theta_2)^{1/2}}{\sin \Delta \lambda}
\]

(18)

GREATER ELLIPTIC ARC DISTANCE

Referring to Figure 9, it is seen that the great elliptic arc is orthogonal to a meridian at a point $P_0(\phi_0, \lambda_0)$ which is the vertex of the great elliptic arc determined by the points $P_1(\phi_1, \lambda_1), P_2(\phi_2, \lambda_2)$ on the ellipsoid. The equation of the great elliptic plane through $P_1$ and $P_2$ is given by equations (14). Now a meridional plane orthogonal to (14) has an equation of the form $Bx - Ay = 0$ and the rectangular coordinates of $P_0(\phi_0, \lambda_0)$ must satisfy both planes.

From (1), the rectangular coordinates of $P_0(\phi_0, \lambda_0)$ are

\[
x_0 = N_0 \cos \phi_0 \cos \lambda_0,
\]

\[
y_0 = N_0 \cos \phi_0 \sin \lambda_0,
\]

\[
z_0 = N_0 (1 - e^2) \sin \phi_0 \sin \lambda_0
\]

and these placed in $Bx - Ay = 0$ and (14) give

\[
B \cos \lambda_0 - A \sin \lambda_0 = 0,
\]

(19)

\[
A \cos \lambda_0 + B \sin \lambda_0 = C (1 - e^2) \tan \phi_0.
\]

From the first of (19) find $\tan \Delta \lambda_0 = B/A$, whence $\sin \Delta \lambda_0 = B/(A^2 + B^2)^{1/2}$ and these values placed in the second of (19) give $\tan \phi_0 = (A^2 + B^2)^{1/2}/C (1 - e^2)$,

\[
\tan \Delta \phi_0 = (\cot \phi_1 \tan \phi_2 - \cos \Delta \lambda)/\sin \Delta \lambda,
\]

\[
\tan \phi_0 = (\tan^2 \phi_1 + \tan^2 \phi_2 - 2 \tan \phi_1 \tan \phi_2 \cos \Delta \lambda)/\sin \Delta \lambda.
\]

From the second of equations (19), dropping the subscript zero and differentiating we obtain

\[
(-A \sin \Delta \lambda + B \cos \Delta \lambda) (d \Delta \lambda) = C (1 - e^2) \sec^2 \phi \, d \phi.
\]

(22)

By solving $A \cos \Delta \lambda + B \sin \Delta \lambda = C (1 - e^2) \tan \phi$ with the identity $\sin^2 \Delta \lambda + \cos^2 \Delta \lambda = 1$, find

\[
\sin \Delta \lambda = \frac{-BC(1 - e^2) \tan \phi + A [(A^2 + B^2) - C^2(1 - e^2)^2 \tan^2 \phi]^{1/2}}{A^2 + B^2},
\]

(23)

\[
\cos \Delta \lambda = \frac{-AC(1 - e^2) \tan \phi + B [(A^2 - B^2) - C^2(1 - e^2)^2 \tan^2 \phi]^{1/2}}{A^2 + B^2}.
\]
From (23) one has then

\[- A \sin \Delta \lambda + B \cos \Delta \lambda = [(A^2 + B^2) - C^2(1 - e^2)^2 \tan^2 \phi]^{1/2}\]

and this value placed in (22) gives

\[
(d \Delta \lambda) = \frac{C(1 - e^2) \sec^2 \phi \, d \phi}{[(A^2 + B^2) - C^2(1 - e^2)^2 \tan^2 \phi]^{1/2}}
\]

whence, by means of relations (20) and trigonometric identities,

\[
(d \Delta \lambda)^2 = \frac{C^2(1 - e^2)^2 \sec^4 \phi \, d \phi^2}{A^2 + B^2 - C^2(1 - e^2)^2 \tan^2 \phi} = \frac{\sec^4 \phi \, d \phi^2}{A^2 + B^2 - \tan^2 \phi}
\]

\[
= \frac{\sec^4 \phi \, d \phi^2}{\tan^2 \phi_0 - \tan^2 \phi} = \frac{\sec^4 \phi \, d \phi^2}{\sec^2 \phi_0 - \sec^2 \phi}.
\]

Now the linear element of the spheroid is, [8] page 62,

\[
ds^2 = \left[ \sec^2 \phi \, d \phi^2 + \left( \frac{N}{R} \right)^2 (d \Delta \lambda)^2 \right] R^2 \cos^2 \phi,
\]

where \( R = a(1 - e^2)/(1 - e^2 \sin^2 \phi)^{1/2} = \frac{1 - e^2}{a^2} N^3; \ N = a/(1 - e^2 \sin^2 \phi)^{1/2} \)

Now from (25) and (26) it is seen that we will be able to express the quantity in brackets in terms of \( \sec \phi \) and \( \sec \phi_0 \) since

\[
\left( \frac{N}{R} \right)^2 \frac{(1 - e^2 \sin^2 \phi)^2}{(1 - e^2)^2} = \frac{[(1 - e^2) \sec^2 \phi + e^2]^2}{(1 - e^2)^2 \sec^4 \phi}.
\]

With the values of \((d \Delta \lambda)^2\) and \(\left( \frac{N}{R} \right)^2\) from (25) and (27), the linear element (26) may be be written

\[
ds^2 = \left[ \sec^2 \phi + \frac{[(1 - e^2) \sec^2 \phi + e^2]^2}{(1 - e^2)^2 (\sec^2 \phi_0 - \sec^2 \phi)} \right] \left( R^2 \cos^2 \phi \, d \phi^2 \right).
\]

If the quantity in brackets is given a common denominator, then (28) may be written as

\[
ds^2 = \frac{(1 - e^2) \sec^2 \phi \left[(1 - e^2) \sec^2 \phi_0 + 2e^2\right] + e^4}{(1 - e^2)^2 (\sec^2 \phi_0 - \sec^2 \phi)} \left( R^2 \cos^2 \phi \, d \phi^2 \right).
\]

To bring (29) into manageable form we place \( k = \frac{\sqrt{1 - e^2}}{a} N_0 \sin \phi_0 \), and

\[
\cos d = \frac{N \sin \phi}{N_0 \sin \phi_0}.
\]

(Nota de que \( k = e_0 \), is the eccentricity of the great elliptic arc. See Figure 15.)
Great Elliptic Section

Major semiaxis is $a$

Minor semiaxis is $b_0 = a\sqrt{1-e^2}\sin^2\theta_0$

$a, e$ are semimajor axis and eccentricity of the ellipsoidal meridian

$\theta_0$ is the geocentric latitude of the vertex $P_0$ of the Great Elliptic Section

$e_0$ is the eccentricity of the Great Elliptic

$e_0 = \left(\frac{a^2 - b_0^2}{a}\right)^{1/2} = e \sin \theta_0 = \left(\frac{\sqrt{1-e^2}}{a}\right) N_0 \sin \phi_0$

Coordinates of $P_0$ are $P_0 (a \cos \theta_0 \cos \lambda_0, a \cos \theta_0 \sin \lambda_0, b_0 \sin \theta_0)$ or in terms of geodetic latitude $\phi_0$

$P_0 (N_0 \cos \phi_0 \cos \lambda_0, N_0 \cos \phi_0 \sin \lambda_0, N_0 (1-e^2) \sin \phi_0)$

Figure 15. Elements of the great elliptic section.
From the first of (30), placing \( N_o = a/(1 - e^2 \sin^2 \phi_o) \) and solving for \( \sec^2 \phi_o \) find
\[
\sec^2 \phi_o = (1 - e^2 + k^2)/(1 - e^2) (1 - k^2/e^2).
\] (31)

With the value of \( N_o \sin \phi_o \) from the first of (30) placed in the second find
\[
N \sin \phi = (ak/e \sqrt{1 - e^2}) \cos \theta \quad \text{and with } N = a \sqrt{1 - e^2 \sin^2 \phi}, \quad \text{solving for } \sec^2 \phi \text{ find}
\]
\[
\sec^2 \phi = \frac{1 - e^2 + k^2 \cos^2 \theta}{(1 - e^2) [1 - (k^2/e^2) \cos^2 \theta]}
\] (32)

By differentiating \( N \sin \phi = (ak/e \sqrt{1 - e^2}) \cos \theta \) obtain
\[
(N \sin \phi) \frac{d\phi}{d\theta} = -(ak/e \sqrt{1 - e^2}) \sin \theta \cos \theta
\] (33)

Since \( (N \sin \theta) \frac{d\phi}{d\theta} \), equation (33) may be written
\[
\frac{R \cos \phi}{1 - e^2} \frac{d\phi}{d\theta} = -(ak/e \sqrt{1 - e^2}) \sin \theta \cos \theta
\] or finally
\[
(R^2 \cos^2 \phi d\phi) = (1 - e^2) a^2 (k^2/e^2) \sin^2 \theta \cos \theta d\theta
\] (34)

Now from (31) and (32) find
\[
\sec^2 \phi_o - \sec^2 \phi = \frac{(k^2/e^2) \sin \theta d\theta}{(1 - e^2) (1 - k^2/e^2) [1 - (k^2/e^2) \cos^2 \theta]}
\] (35)

and the numerator of (29) becomes
\[
(1 - e^2) \sec^2 \phi_o \cos \theta \sec^2 \phi_o + 2e^2] + e^4 = \frac{1 - k^2 + k^2 \cos^2 \theta}{(1 - k^2/e^2) [1 - (k^2/e^2) \cos^2 \theta]}
\] (36)

With the values from (34), (35), (36) the linear element (29) becomes
\[
ds^2 = \frac{1 - k^2 + k^2 \cos^2 \theta}{(1 - k^2/e^2) [1 - (k^2/e^2) \cos^2 \theta]} \cdot (1 - e^2)
\]
\[
a^2(k^2/e^2) \sin \theta \cos \theta \sin \theta \cos \theta \delta \theta = a^2(1 - k^2 + k^2 \cos^2 \theta) \delta \theta
\]
\[
ds^2 = a^2(1 - k^2 \sin^2 \theta) \delta \theta
\] (37)

Now equation (37) is the usual elliptic integral form with modulus \( k \), and we write
\[
s = a \left[ \int_0^{d_1} + \int_0^{d_2} \right] (1 - k^2 \sin^2 \theta)^{1/2} \delta \theta
\] (38)

where \( k = (e \sqrt{1 - e^2}/a) N_o \sin \phi_o \), the modulus of the elliptic integral, and
\[d_1 = \cos^{-1} (N_1 \sin \phi_1/N_o \sin \phi_o), \quad d_2 = \cos^{-1} (N_2 \sin \phi_2/N_o \sin \phi_o). \quad (k \text{ is equal to } e_0 \text{ the eccentricity of the great elliptic arc — see Figure 15}).

The integrand of (38) may be expanded by the binomial formula and integrated term by term to obtain an approximation formula for direct computation. To 6th order terms in \( k: (1 - k^2 \sin^2 \theta)^{1/2} = 1 - \frac{1}{2} k^2 \sin^2 \theta - (1/8) k^4 \sin^4 \theta - (1/16) k^6 \sin^6 \theta -
\] (39)
Making the identity substitutions
\[
\sin^2 d = \frac{1}{2} - \frac{1}{2} \cos 2d, \quad \sin^4 d = (\frac{3}{8}) - \frac{1}{2} \cos 2d + (\cos 4d)/8
\]
\[
\sin^6 d = (\frac{5}{16}) - (\frac{15}{32}) \cos 2d + (\frac{3}{16}) \cos 4d - (\frac{1}{32}) \cos 6d, \text{ in (39) and integrating}
\]
term by term according to (38) one obtains
\[
s/a = (d_1 + d_2) - \frac{1}{2}k^2 [(d_1 + d_2) - \frac{1}{2} \sin (2d_1 + \sin 2d_2)]
- (\frac{1}{8})k^4 [(\frac{3}{8})(d_1 + d_2) - \frac{1}{2}(\sin 2d_1 + \sin 2d_2) + (\frac{1}{32})(\sin 4d_1 + \sin 4d_2)]
- (\frac{1}{16})k^6 [(\frac{5}{16})(d_1 + d_2) - (\frac{15}{64})(\sin 2d_1 + \sin 2d_2) + (\frac{3}{64})(\sin 4d_1 + \sin 4d_2)]
- (\frac{1}{192})(\sin 6d_1 + \sin 6d_2)] \quad (40)
\]
By means of the identity \(\sin x + \sin y = 2 \sin \frac{1}{2}(x + y) \cos \frac{1}{2}(x - y)\), equation (40) may be written finally as
\[
s/a = (d_1 + d_2) - \frac{1}{2}k^2 [(d_1 + d_2) - \sin (d_1 + d_2) \cos (d_1 - d_2)]
- (1/128)k^4 [6(d_1 + d_2) - 8 \sin (d_1 + d_2) \cos (d_1 - d_2) + 2(\sin d_1 + \sin d_2) \cos 2(d_1 - d_2)]
- (1/1536)k^6 [30(d_1 + d_2) - 45 \sin (d_1 + d_2) \cos (d_1 - d_2) + 9 \sin 2(d_1 + d_2) \cos 2(d_1 - d_2)
- \sin 3(d_1 + d_2) \cos 3(d_1 - d_2)], \quad (41)
\]
a and \(e\) are semimajor axis and eccentricity of the meridian ellipse, \(k = (e^2/1 - e^2/a) N_o \sin \phi_o\) (\(k = e_o\), the eccentricity of the great elliptic arc), \(\phi_o\) is the vertex of the great elliptic arc as given by (21). \(d_1 = \arccos (N_1 \sin \phi_1 / N_o \sin \phi_o)\), \(d_2 = \arccos (N_2 \sin \phi_2 / N_o \sin \phi_o)\). When \(\phi_o = 90^\circ\); equation (41) gives a meridian arc of the spheroid. When \(\phi_o = 0\), an arc of the equator or circle of radius \(a\) is given. Formula (41) thus consists of a circular arc and successive corrective terms.

To examine the contribution of the terms in (41) take the case \(\phi_1 = \phi_2 = 0\), \(\phi_o = 45^\circ\), \(d_1 = d_2 = 90^\circ\) which will give the semilength of the great ellipse making an angle of \(45^\circ\) with the equator. For the Clarke 1866 spheroid, \(e^2 = 6.7668657997 \times 10^{-3}\), \(a = 6,378,206.4\) meters. From (41) we have then
\[
1\text{st term } a \times (d_1 + d_2) = 20,037,773 \text{ meters}
\]
\[
2\text{nd term } -a \times 2.65804 \times 10^{-3} = -16,954 \text{ meters}
\]
\[
3\text{rd term } -a \times 0.17 \times 10^{-5} = -11 \text{ meters}
\]
\[
4\text{th term } -a \times 0.24 \times 10^{-8} = -0.015 \text{ meters}
\]

When \(\phi_o = 90\), \(\phi_1 = \phi_2 = 0\), \(d_1 + d_2 = \pi\), and (41) reduces to the usual formula for length of the semimeridian from equator to equator through the pole \(s = a \pi [1 - \frac{1}{4}e^2 - (\frac{3}{64})e^2 - (\frac{5}{256})e^6 - \cdots]\).
GREAT ELLIPTIC ARC LENGTH IN TERMS OF PARAMETRIC LATITUDE $\theta$

Equation (41) gives the arc length, but the modulus $k$, $d_1$, and $d_2$, and vertex $\phi_0$ must be expressed in terms of parametric latitude, $\theta$, if the geographic latitudes $\phi_1$, $\phi_2$ of the given points $P_1$, $P_2$ have been first converted to parametric latitudes $\theta_1$, $\theta_2$.

The relationships $\tan \phi = \frac{\tan \theta}{\sqrt{1 - e^2}}$, $N \sin \phi = \frac{a}{\sqrt{1 - e^2}} \sin \theta$, applied to $k = \left(\frac{e}{\sqrt{1 - e^2}}a\right) N \sin \phi_0$, $d_1 = \arccos \left(\frac{N \sin \phi_1}{N_0 \sin \phi_0}\right)$, $d_2 = \arccos \left(\frac{N \sin \phi_2}{N_0 \sin \phi_0}\right)$, and the last of equations (21) give

$e_0 = k = e \sin \theta_0$, $d_1 = \arccos \left(\frac{N \sin \phi_1}{N_0 \sin \phi_0}\right)$, $d_2 = \arccos \left(\frac{N \sin \phi_2}{N_0 \sin \phi_0}\right)$,

$tan \theta_0 = \left(\frac{\tan^2 \theta_1 + \tan^2 \theta_2 - 2 \tan \theta_1 \tan \theta_2 \cos \Delta \lambda}{\tan^2 \theta_1 + \tan^2 \theta_2 - 2 \tan \theta_1 \tan \theta_2 \cos \Delta \lambda + \sin^2 \Delta \lambda}\right)^{1/2}$.

Equations (41) and (42) give then the arc length along the great elliptic arc when geographic latitudes have been converted to parametric latitudes.

THE CHORD DISTANCE

The chord distance between the points $Q_1 (x_1, y_1, z_1)$, $Q_2 (x_2, y_2, z_2)$ as shown in Figures (13) and (14) is given by the usual distance formula where the coordinates may be expressed in terms of either $\phi$ or $\theta$, that is from (1)

$x_1 = N_1 \cos \phi_1$, $y_1 = 0$, $z_1 = N_1 (1 - e^2) \sin \phi_1$ (in terms of $\phi$)

$x_2 = N_2 \cos \phi_2 \cos \Delta \lambda$, $y_2 = N_2 \cos \phi_2 \sin \Delta \lambda$, $z_2 = N_2 (1 - e^2) \sin \phi_2,$ (43)

or $x_1 = a \cos \theta_1$, $y = 0$, $z = a \sqrt{1 - e^2} \sin \theta_1$ (in terms of $\theta$)

$x_2 = a \cos \theta_2 \cos \Delta \lambda$, $y_2 = a \cos \theta_2 \sin \Delta \lambda$, $z_2 = a \sqrt{1 - e^2} \sin \theta_2$.

Applying the distance formula to each set of formulas in (43) for coordinates one obtains

$C = [(N_1 \cos \phi_1 - N_2 \cos \phi_2 \cos \Delta \lambda)^2 + N_2^2 \cos^2 \phi_2 \sin^2 \Delta \lambda + (1 - e^2)^2 (N_1 \sin \phi_1 - N_2 \sin \phi_2)^2]^{1/2}$

and in terms of $\theta$

$C = a [(\cos \theta_2 \cos \Delta \lambda - \cos \theta_1)^2 + \cos^2 \theta_2 \sin^2 \Delta \lambda + (1 - e^2) \sin (\theta_2 - \sin \theta_1)^2]^{1/2}$ (45)

In (45), expand the quantities in the brackets combining terms to obtain

$C = a \left[2 - 2 (\sin \theta_1 \sin \theta_2 + \cos \theta_1 \cos \theta_2 \cos \Delta \lambda) - e^2 (\sin \theta_2 - \sin \theta_1)^2\right]^{1/2}.$ (46)

Now $\cos (d_1 + d_2) = \sin \theta_1 \sin \theta_2 + \cos \theta_1 \cos \theta_2 \cos \Delta \lambda$ and with $\sin \theta_1 = \sin \theta_0 \cos d_1$, $\sin \theta_2 = \sin \theta_0 \cos d_2$, $k^2 = e^2 \sin^2 \theta_0$ from (42), equation (46) can be written

$C = a \left[2 \left(1 - \cos (d_1 + d_2)\right) - k^2 (\cos d_1 - \cos d_2)^2\right]^{1/2}.$ (47)
With the identity \((\cos d_1 - \cos d_2)^2 = [1 - \cos (d_1 + d_2)] [1 - \cos (d_1 - d_2)]\), we can write (47) finally as

\[
C = a \left[1 - \cos (d_1 + d_2) \right] \{2 - k^2 [1 - \cos (d_1 - d_2)] \}^{1/2}.
\] (48)

Now (48) gives the chord length no matter which latitude is used, \(\phi\) or \(\theta\), since for \(\phi\):
\[
d_1 = \arccos (\sin \phi_1 / \sin \phi_0), \quad d_2 = \arccos (\sin \phi_2 / \sin \phi_0),
\]
\[
k^2 = \left[ e^2 (1 - \varepsilon^2) / a^2 \right] N_1^2 \sin^2 \phi_0; \text{ while for } \theta:
\]
\[
d_1 = \arccos (\sin \theta_1 / \sin \theta_0), \quad d_2 = \arccos (\sin \theta_2 / \sin \theta_0), \quad k^2 = \varepsilon^2 \sin^2 \theta_0. \quad \text{Also (41) and (48) make it possible to prepare a computing form in terms of either } \phi \text{ or } \theta \text{ with corresponding azimuth forms from equations (12), (13), (15), (16), (17), (18).}
\]

THE ANGLE BETWEEN THE CHORD AND THE HORIZON AT A GIVEN POINT OF THE ELLIPSOID

Referring to Figure 13, it is seen that the angle \(\beta\) is determined by a perpendicular, \(u\), from \(Q_2\) upon the tangent at \(Q_1\) and the chord \(c\). That is \(\sin B = u/c\).

Now the length of \(u\) is obtained by normalizing the equation of the tangent plane at \(Q_1\), equation (4), and substituting the coordinates of the point \(Q_2\) from (1):

\[
u = \frac{1}{N_1} \left[ a^2 - N_1 N_2 \cos \phi_1 \cos \phi_2 \cos \Delta \lambda - (1 - \varepsilon^2) N_1 N_2 \sin \phi_1 \sin \phi_2 \right].
\] (49)

We can express \(u\) in parametric latitude, \(\theta\), since \((1 - \varepsilon^2) N_1 N_2 \sin \phi_1 \sin \phi_2 = a^2 \sin \theta_1 \sin \theta_2 N_1 N_2 \cos \phi_1 \cos \phi_2 = a^2 \cos \theta_1 \cos \theta_2, N_1 = (a/\sqrt{1 - \varepsilon^2}) \sqrt{1 - \varepsilon^2 \cos^2 \theta_1}, \) i.e.

\[
u = a \sqrt{1 - \varepsilon^2} \frac{1 - \sin \theta_1 \sin \theta_2 + \cos \theta_1 \cos \theta_2 \cos \Delta \lambda}{\sqrt{1 - \varepsilon^2 \cos^2 \theta_1}}.
\] (50)

Referring to equation (46) and the discussion there, \(\cos (d_1 + d_2) = \sin \theta_1 \sin \theta_2 + \cos \theta_1 \cos \theta_2 \cos \Delta \lambda, \)
\(\sin \theta_1 = \sin \theta_0 \cos d_1, \kappa = e \sin \theta_0\) and (50) can be written in the form

\[
u = b \frac{1 - \cos (d_1 + d_2)}{(1 - \varepsilon^2 + k^2 \cos^2 d_1)^{1/2}},
\] (51)

Where \(b = a \sqrt{1 - \varepsilon^2}\) is the minor semiaxis of the reference ellipsoid. From (48) and (51) we have then

\[
\sin \beta = \frac{u}{c} = \left\{ \frac{(1 - \varepsilon^2) [1 - \cos (d_1 + d_2)]}{(2 - k^2 [1 - \cos (d_1 - d_2)]) (1 - \varepsilon^2 + k^2 \cos^2 d_1)} \right\}^{1/2}.
\] (52)

and thus \(\sin \beta\) is expressed in the same quantities as the distance and chord lengths; see equations (41) and (48).
MAXIMUM SEPARATION OF CHORD AND ELLIPTIC ARC

In Figure 14, $H_0$ is the maximum separation between the great elliptic arc and the chord. As shown, this occurs when the tangent to the ellipse is parallel to the chord. Also when this occurs the center of the ellipse, the midpoint of the chord, and the point $P$ on the curve are collinear, [10]. Hence the geographic coordinates of the point $P$ can be found from the intersection of the meridian through $Q$ and the plane of the great elliptic section.

The coordinates of $Q$, the midpoint of the chord $Q_1Q_2$, are

\[
\begin{align*}
Q &= \left\{ \begin{array}{l}
\left(\frac{a}{2}\right) \cos \theta_2 \cos \Delta \lambda + \cos \theta_1 \\
\left(\frac{b}{2}\right) \sin \theta_1 + \sin \theta_2
\end{array} \right.
\end{align*}
\]

and the meridian through $Q$ has the equation \((\cos \theta_2 \sin \Delta \lambda) x - (\cos \theta_1 + \cos \theta_2 \cos \Delta \lambda) y = 0\). (53)

The equation to the plane of the great elliptic arc in terms of parametric latitude is

\[
Ax + By + Cz = 0,
\]

where

\[
A = b \tan \theta_1 \sin \Delta \lambda, \quad B = b (\tan \theta_2 - \tan \theta_1 \cos \Delta \lambda), \quad C = -a \sin \Delta \lambda
\]

(Compare equation (14), where it is in terms of geodetic latitude $\phi$). Now the point $P$ on the ellipsoid must satisfy both equations (53) and (54) if it is to be the required point $P$ on the great elliptic arc. This leads to the equations

\[
\cos \theta_2 \sin \Delta \lambda \cos \lambda - (\cos \theta_1 + \cos \theta_2 \cos \Delta \lambda) \sin \lambda = 0,
\]

\[
A \cos \lambda + B \sin \lambda + C \tan \theta = 0,
\]

where $A, B, C$ are those of equation (54).

Solving (55) for $\lambda$ and $\theta$ find,

\[
\begin{align*}
\lambda &= \arctan \left[ \frac{(\cos \theta_2 \sin \Delta \lambda)/(\cos \theta_2 \cos \Delta \lambda + \cos \theta_1)}{1 + \cos (d_1 + d_2)} \right], \\
\theta &= \arctan \left[ \frac{(\tan \theta_1 \sin \Delta \lambda) \cos \lambda + (\tan \theta_2 - \tan \theta_1 \cos \Delta \lambda) \sin \lambda}{\sin \Delta \lambda} \right], \\
\theta &= \arctan \left[ \frac{\tan \theta_2 \sin \lambda + \tan \theta_1 \sin (\Delta \lambda - \lambda)}{\sin \Delta \lambda} \right],
\end{align*}
\]

\[
\theta = \arctan \left[ \frac{(\sin \theta_1 + \sin \theta_2)/(\cos^2 \theta_1 + \cos^2 \theta_2 + 2 \cos \theta_1 \cos \theta_2 \cos \Delta \lambda)^{1/2}}{\sin \Delta \lambda} \right].
\]

We have seen that

\[
\begin{align*}
\cos (d_1 + d_2) &= \sin \theta_1 \sin \theta_2 + \cos \theta_1 \cos \theta_2 \cos \Delta \lambda \\
\sin \theta_1 &= \sin \theta_0 \cos d_1, \quad \sin \theta_2 = \sin \theta_0 \cos d_2
\end{align*}
\]

whence we can express

\[
\begin{align*}
\cos^2 \theta_1 + \cos^2 \theta_2 + 2 \cos \theta_1 \cos \theta_2 \cos \Delta \lambda &= [1 + \cos (d_1 + d_2)][2 - \sin^2 \theta_0 \{1 + \cos (d_1 - d_2)\}], \\
\sin \theta_1 + \sin \theta_2)^2 &= \sin^2 \theta_0 \left[ 1 + \cos (d_1 + d_2) \right] [1 + \cos (d_1 - d_2)]
\end{align*}
\]

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and the last equation of (56) may be written

\[ \theta = \arctan \frac{\sin \theta_0 \sqrt{1 + \cos (d_1 - d_2)}}{\sqrt{2 - \sin^2 \theta_0 [1 + \cos (d_1 - d_2)]}} . \] (58)

It is known that \( H_0^2 = PP'^2 \) will be given by

\[ H_0^2 = \left[ (y - y_1) r - (z - z_1) q \right]^2 + \left[ (x - x_1) r - (z - z_1) p \right]^2 + \left[ (x - x_1) q - (y - y_1) p \right]^2, \]

where \( x, y, z \) are coordinates of \( P \); \( x_1, y_1, z_1 \) are coordinates of \( Q \); and \( p, q, r \) are direction cosines of the chord \( c = Q_1Q_2 \) [11]. See Figure 14.

From (56) and (58) we can express the rectangular coordinates of \( P \) as

\[ P: \]

\[ x = a \cos \theta \cos \lambda = \frac{a}{\sqrt{2}} \cos \theta_1 + \cos \theta_2 \cos \Delta \lambda \]

\[ y = a \cos \theta \sin \lambda = \frac{a}{\sqrt{2}} \cos \theta_2 \sin \Delta \lambda \]

\[ z = b \sin \theta = \frac{b}{\sqrt{2}} \sin \theta_1 + \sin \theta_2 \] (60)

If the coordinates from (1) are converted to parametric latitude they will be \( Q_1 \left( a \cos \theta_1, 0, b \sin \theta_1 \right) \), \( Q_2 \left( a \cos \theta_2 \cos \Delta \lambda, a \cos \theta_2 \sin \Delta \lambda, b \sin \theta_2 \right) \) whence the direction cosines of the chord \( c = Q_1Q_2 \) are

\[ p = \frac{a}{c} \left( \cos \theta_2 \cos \Delta \lambda - \cos \theta_1 \right) \]

\[ q = \frac{a}{c} \cos \theta_2 \sin \Delta \lambda \]

\[ r = \frac{b}{c} \left( \sin \theta_2 - \sin \theta_1 \right) \] (61)

From (60) and the coordinates of \( Q_1 \left( a \cos \theta_1, 0, b \sin \theta_1 \right) \) we have

\[ x - x_1 = \frac{a}{\sqrt{2} R_0} \left( \cos \theta_1 + \cos \theta_2 \cos \Delta \lambda \right) - a \cos \theta_1 \]

\[ y - y_1 = \left( \cos \theta_2 \sin \Delta \lambda \right)/\sqrt{2} R_0 \]

\[ z - z_1 = \frac{b}{\sqrt{2} R_0} \left( \sin \theta_1 + \sin \theta_2 \right) - b \sin \theta_1 \] (62)

Where \( R_0 = \sqrt{1 + \cos (d_1 + d_2)} = \sqrt{2} \cos \frac{1}{2}(d_1 + d_2) \).

With the values from (61) and (62) the expression (59) is formed to give

\[ H_0^2 = \frac{a^2 (\sqrt{2} - R_0)^2}{c^2 R_0^2} \cos^2 \theta_1 \cos^2 \theta_2 \left[ b^2 (\tan^2 \theta_1 + \tan^2 \theta_2 - 2 \tan \theta_1 \tan \theta_2 \cos \Delta \lambda) + a^2 \sin^2 \Delta \lambda \right] \] (63)

56
Where $R_o = [1 + \cos (d_1 + d_2)]^{1/2} = \sqrt{2} \cos \frac{1}{2}(d_1 + d_2)$.

Using the relationships (42), (48), (57) equation (63) can be solved for $H_o$ in any of the following several forms:

$$H_o = \frac{b_o (\sqrt{2} - \sqrt{1 + \cos (d_1 + d_2)})}{\sqrt{2 - k^2 [1 - \cos(d_1 - d_2)]}}, \quad (64)$$

$$= \frac{ab_0}{c} \left( \frac{\sqrt{2}}{R_o} - 1 \right) \sin (d_1 + d_2),$$

$$= \frac{2ab_0}{c} \sin \frac{1}{2}(d_1 + d_2) [1 - \cos \frac{1}{2}(d_1 + d_2)].$$

Where $R_o = \sqrt{1 + \cos (d_1 + d_2)} = \sqrt{2} \cos \frac{1}{2}(d_1 + d_2)$

$b_o = \sqrt{1 - k^2} = a \sqrt{1 - e_o^2}$ = minor semiaxis of the great elliptic arc — see Figure 15. Thus $H_o$ is also expressed in quantities common with other elements of the great elliptic arc — see equations (41), (48), and (52).

A COMPUTING FORM FOR GREAT ELLIPTIC ARC LENGTH AND ASSOCIATED ELEMENTS

Since the computations to be discussed with the great elliptic arc approximation and the Andoyer-Lambert approximation both involve corrections to spherical elements, the basic spherical approximation is reviewed in Figure 16, and basic spherical formulae listed.

Now from (42) write

$$\sin^2 \theta_o = K/(K + 1),$$

$$K = (A \tan \theta_1 + B \tan \theta_2) / \sin^2 \Delta \lambda \quad (65)$$

$$A = \tan \theta_1 - \tan \theta_2 \cos \Delta \lambda, \quad B = \tan \theta_2 - \tan \theta_1 \cos \Delta \lambda. \quad (66)$$

Azimuth equations (17) become

$$\cot a_{AB} = D_1 (R_1 - B), \quad \cot a_{BA} = D_2 (A - R_2)$$

$$D_1 = \cos \theta_1/T_1 \sin \Delta \lambda, \quad D_2 = \cos \theta_2/T_2 \sin \Delta \lambda \quad (67)$$

$$R_1 = C/\cos \theta_2, \quad R_2 = -C/\cos \theta_1$$

$$C = e^2 (\sin \theta_2 - \sin \theta_1)$$

$$T_1 = (1 - e^2 \cos^2 \theta_1)^{1/2}, \quad T_2 = (1 - e^2 \cos^2 \theta_2)^{1/2}$$

Equation (41) becomes

$$s = a (H + U_1 + U_2 + U_3) \quad (68)$$

where

$U_1 = -N_1 (H - Q_1), \quad U_2 = -N_2 (6H - 8Q_1 + Q_2),$

$U_3 = -N_3 (30H - 45Q_1 + 9Q_2 - Q_3)$$

$k^2 = e^2 \sin^2 \theta_o = e_o^2$ (eccentricity of the great elliptic arc).
Figure 16. Elements of polar spherical triangles.

\[
\cot A = \frac{\cos \phi \tan \beta - \sin \phi \cos \Delta \lambda}{\sin \Delta \lambda}
\]

\[
\cot B = \frac{\cos \phi \tan \beta - \sin \phi \cos \Delta \lambda}{\sin \Delta \lambda}
\]

\[
\cos (d_1 + d_2) = \sin \phi \sin \Delta + \cos \phi \cos \Delta \cos \lambda
\]

\[
\sin (d_1 + d_2) = \frac{\cos \delta_1 \sin \Delta \lambda}{\sin \beta} = \frac{(\cos \delta_2 \sin \Delta \lambda)}{\sin A}
\]

\[
\sin \gamma = \sin \delta_1 \cos d_1, \quad \sin \gamma = \sin \delta_2 \cos d_2
\]

NOTE: \( Q_0 \) may be external to \( Q_1Q_2 \), i.e., if either \( A \) or \( B \) is greater than 90°.
\[ N_1 = k^{3/4}, \ N_2 = k^{4}/48 = 1/3 \ N_1^{-2}, \ N_3 = k^{4}/1536 = (1/3) \ N_1 \ N_2, \]

\[ Q_1 = \sin H \ \cos P, \quad Q_2 = \sin 2H \ \cos 2P, \quad Q_3 = \sin 3H \ \cos 3P, \quad H = d_1 + d_2, \quad P = d_1 - d_2. \]

\[ d_1 \text{ and } d_2 \text{ are computed from} \]

\[ \cos 2d_1 = 2(1 - \cos^2 \theta_1)/\sin^2 \theta_0 - 1 \]

\[ \cos 2d_2 = 2(1 - \cos^2 \theta_2)/\sin^2 \theta_0 - 1 \]

(69)

since \( \cos^2 \theta_1 \) and \( \cos^2 \theta_2 \) are already needed for \( T_1 \) and \( T_2 \), (67) above, and the use of \( \sin^2 \theta_0 \) eliminates the computation of the square root of \( K/(K + 1) \). A check is provided by

\[ \sin (d_1 + d_2) = \sin \theta_1 \ \sin \theta_2 + \cos \theta_1 \ \cos \theta_2 \ \cos \Delta \lambda. \]

From (48) the equation of the chord may be written

\[ c = a(VW)^{1/2}, \ V = (1 - \cos H), \ W = 2 - k^2 R, \ R = (1 - \cos P). \]

(70)

From (51) and (52) in terms of the symbols used above find

\[ u = bV/T_1 \quad \sin \beta = bV/cT_1 = \frac{b}{T_1} \sqrt{\frac{V}{W}}. \]

(71)

From (64) in terms of the above symbols find \( H_o = \frac{2ab_o}{c} (\sin \frac{1}{2}H) (1 - \cos \frac{1}{2}H), \)

(72)

\[ b_o = a\sqrt{1 - k^2}, \ k^2 = e^2 \ \sin^2 \theta. \]

Figure 17, shows equations (65) through (72) arranged for computing and a computation performed on the line Moscow to Cape of Good Hope. On the form find the geodetic distance, the normal section azimuths, the chord distance, the angle between the chord and the horizon at \( P_1 \), and the maximum separation of the chord and surface. The following table lists these values and gives a comparison with the distances computed by the rigorous Helmert method and the Andoyer-Lambert Approximation. Note that the geographic coordinates of the point \( P(\phi, \lambda) \) where the maximum chord separation from the surface occurs may be computed from (56), (58), and already computed quantities in Figure (17).

**MOSCOW TO CAPE OF GOOD HOPE**

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<tbody>
<tr>
<td>10,102,069.91</td>
<td>5454.6814</td>
<td>Great Elliptic</td>
<td>15° 46' 56&quot; 744</td>
<td>190° 39' 27&quot; 350</td>
<td>Great Elliptic Section</td>
<td></td>
</tr>
<tr>
<td>10,102,069.06</td>
<td>5454.6809</td>
<td>Helmert</td>
<td>15° 49' 57&quot; 607</td>
<td>190° 41' 29&quot; 799</td>
<td>Normal Section</td>
<td></td>
</tr>
<tr>
<td>10,102,065.28</td>
<td>5454.6789</td>
<td>Andoyer-Lambert</td>
<td>15° 48' 17&quot; 674</td>
<td>190° 39' 32&quot; 208</td>
<td>Geodetic</td>
<td></td>
</tr>
</tbody>
</table>

CHORD DISTANCE 9,068,419.05 meters 4896.5546 n.m.

(MAXIMUM CHORD SEPARATION) 1,906,854.55 meters 1029.6191 n.m.

CHORD DEPRESSION ANGLE 45° 32' 37" 462.
Computations for distance, Normal Section Azimuths, Chord length, Angle of Depression of the Chord, Maximum Separation distance of chord and arc. Based on Great Elliptic Section Approximation to geodesic. Clarke 1866 Spheroid.
\[ a = 6,378,206.4 \text{ meters}, \ b = 6,356,583.8 \text{ meters}, \ e^2 = 6.76686580 \times 10^{-3}, \ 1 \text{ radian} = 206,264,8062 \text{ sec.} \]

| \( \phi_1 \) | 95.46 | 19,500 | 1 (A) | Moscow |
| \( \phi_2 \) | 33.56 | 08,500 | 2 (B) | Cape of Good Hope |
| \( \tan \phi_1 \) | 1.462 | 15.17 | \( \tan \theta = 0.996149255 \) |
| \( \tan \phi_2 \) | 1.462 | 15.17 |
| \( \tan \theta_1 \) | 0.8375 | 99.25 |
| \( \tan \theta_2 \) | 0.8375 |
| \( \sin \theta_1 \) | 0.564 | 32.69 |
| \( \sin \theta_2 \) | 0.564 |
| \( \cos \theta_1 \) | 2.001 | 32.82 |
| \( \cos \theta_2 \) | 2.001 |
| \( e^2 = \sin \theta_1 + \sin \theta_2 \) | \( e^2 = (1-e^2 \cos^2 \theta) \) |
| \( A = \tan \theta_1 - \tan \theta_2 \) | \( A = (1-e^2 \cos^2 \theta)^{1/2} \) |
| \( B = \tan \theta_2 - \tan \theta_1 \) | \( B = (1-e^2 \cos^2 \theta)^{1/2} \) |
| \( K = (A \tan \theta_1 + B \tan \theta_2) / \sin^2 \lambda \) | \( K = (A \tan \theta_1 + B \tan \theta_2) / \sin^2 \lambda \) |
| \( C = e^2 (\sin \theta_1 - \sin \theta_2) / \sin^2 \lambda \) | \( C = e^2 (\sin \theta_1 - \sin \theta_2) / \sin^2 \lambda \) |
| \( \cot \alpha(AB) = D_1 \) | \( \cot \alpha(AB) = D_1 \) |
| \( \cos \alpha = 2(1 - \cos^2 \theta) / \sin^2 \theta - 1 \) | \( \cos \alpha = 2(1 - \cos^2 \theta) / \sin^2 \theta - 1 \) |
| \( \cos \beta = 2(1 - \cos^2 \theta) / \sin^2 \theta - 1 \) | \( \cos \beta = 2(1 - \cos^2 \theta) / \sin^2 \theta - 1 \) |
| \( \sin \) | \( \sin \) |
| \( \sin H = \cos \theta_2 / \cos \theta_1 \) | \( \sin H = \cos \theta_2 / \cos \theta_1 \) |
| \( \cos H = \cos \theta_1 / \cos \theta_2 \) | \( \cos H = \cos \theta_1 / \cos \theta_2 \) |
| \( \sin 2H = 0.024 \times 16.8 \) | \( \cos 2H = 0.102 \times 16.8 \) |
| \( \sin 3H = 0.006 \times 16.8 \) | \( \cos 3H = 0.006 \times 16.8 \) |
| \( Q_1 = \sin H \cos P \) | \( Q_1 = \sin H \cos P \) |
| \( U_1 = -N \) | \( U_1 = -N \) |
| \( U_2 = -N \) | \( U_2 = -N \) |
| \( U_3 = -N \) | \( U_3 = -N \) |
| \( \Sigma = H + U_1 + U_2 + U_3 \) | \( \Sigma = H + U_1 + U_2 + U_3 \) |
| \( V = 1 - \cos H \) | \( V = 1 - \cos H \) |
| \( W = 1 - \cos H \) | \( W = 1 - \cos H \) |
| \( \Sigma = 2H + Q_1 + Q_2 + Q_3 \) | \( \Sigma = 2H + Q_1 + Q_2 + Q_3 \) |
| \( \beta = \) | \( \beta = \) |

Figure 17.
Figures 18 and 19 show the great elliptic arc formulae for distance arranged with geodetic azimuth formulae and the computations for distance and azimuth over the two lines  
(1) MOSCOW TO CAPE OF GOOD HOPE and (2) RAMEY AFB to MOUNTAIN HOME AFB.

No square roots are involved and only eight place tables of trigonometric functions, as Peters, are needed in addition to the constants for a particular spheroid of reference. The comparison with the Helmert rigorous and Andoyer-Lambert approximation is:

<table>
<thead>
<tr>
<th>Line</th>
<th>Distance (meters)</th>
<th>Method</th>
<th>Forward Az.</th>
<th>Back Az.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>10,102,069.91</td>
<td>Great Elliptic Arc</td>
<td>15° 48' 17'' 519</td>
<td>190° 39' 32'' 109</td>
</tr>
<tr>
<td></td>
<td>10,102,069.06</td>
<td>Helmert</td>
<td>15° 48' 17'' 674</td>
<td>190° 39' 32'' 208</td>
</tr>
<tr>
<td></td>
<td>10,102,065.28</td>
<td>Andoyer-Lambert</td>
<td>15° 48' 17'' 518</td>
<td>190° 39' 32'' 110</td>
</tr>
<tr>
<td>(2)</td>
<td>5,304,035.439</td>
<td>Great Elliptic Arc</td>
<td>131° 52' 34'' 985</td>
<td>285° 10' 06'' 370</td>
</tr>
<tr>
<td></td>
<td>5,304,032.437</td>
<td>Helmert</td>
<td>131° 52' 35'' 29</td>
<td>285° 10' 06'' 65</td>
</tr>
<tr>
<td></td>
<td>5,304,030.844</td>
<td>Andoyer-Lambert</td>
<td>131° 52' 35'' 043</td>
<td>285° 10' 06'' 869</td>
</tr>
</tbody>
</table>

REVIEW OF FORMER STUDIES

The Air Force Aeronautical Charting and Information Center made an extensive study of the Inverse Problem of Geodesy (1956–1957), over lines 50 to 6000 miles, [12]. A review of this study indicates favorably the use of the so called Andoyer-Lambert Formulae relative to requirements for Hyperbolic Electronic Systems since (1) they give very nearly geodetic distance with about the same error over all lines from 50 to at least 6000 miles, (2) azimuths are within about a second of true geodetic azimuths over all lines, (3) no tabular data for a particular spheroid is needed, (4) the only table of mathematical functions required is a table of the natural trigonometric functions as Peters eight place tables, (5) no root extraction is involved in the computations. The formulae are thus quite adaptable to small electric desk calculators or larger high speed digital machines. However, in review it seemed unnecessary to convert geographic coordinates to parametric before making the computations, hence a series of computations were made over the ACIC chosen lines for direct comparison. A representative group from 50 to 6000 miles was selected and additional comparisons were made against two lines whose true geodetic lengths and azimuths were known. No lines of 0° azimuth (meridional sections) were used because this is the trivial or limiting case and extensive tables of meridional distances for all reference ellipsoids are available or quite simple computation formulae are available for computing meridional arcs. The spherical formulae used are:
COMPUTATIONS, DISTANCE, AZIMUTHS
Great Elliptic Arc, Geodetic Azimuths
Clarke 1866 Ellipsoid; a = 6,378,206.4 meters, e² = 6.6786580 × 10⁻³,
f/2 = 0.00169503765, 1 radian = 206,264.8062 seconds, 1852 meters = 1 n. m.

\[ \phi_1 = 55° 45' 19.50'' \]
\[ \phi_2 = 63° 56' 03.50'' \]
\[ \tan \phi_1 = 1.468.95 \]
\[ \tan \phi_2 = -0.672.8415 \]
\[ \tan \phi_3 = 0.8355.24 \]
\[ \sin \phi_1 = 0.5564.98 \]
\[ \cos \phi_1 = 0.3269 \]
\[ \cos^2 \phi_1 = 0.3181.12 \]

1. (A) MOSCOW
\[ \lambda_1 = 60° 34' 15.45'' \]
\[ \Delta \lambda = \lambda_2 - \lambda_1 = 1.19.05 \]
\[ \Delta \phi = 0.3270.99 \]
\[ \cos \Delta \lambda = 0.941 \]
\[ \sin \Delta \phi = 0.990 \]

2. (B) Cape of Good Hope
\[ \lambda_2 = 18° 28' 41.40'' \]
\[ \Delta \lambda = \lambda_2 - \lambda_1 = 1.19.05 \]
\[ \Delta \phi = 0.3270.99 \]
\[ \cos \Delta \lambda = 0.941 \]
\[ \sin \Delta \phi = 0.990 \]

\[ A = \tan \theta_1 - \tan \phi_2 \cos \Delta \phi + 2.079.68 \]
\[ B = \tan \phi_2 - \tan \phi_2 \cos \Delta \phi - 2.079.68 \]

\[ K = \sin^2 \phi_1 = K(1 + \cos \phi_1) \]
\[ \cos 2\phi_1 = 2(1 - \cos^2 \phi_1)/V_0 - 1 \]
\[ \cos 2\phi_2 = 2(1 - \cos^2 \phi_2)/V_0 - 1 \]

\[ \cos 2d_1 = 2(1 - \cos^2 \theta_1)/V_0 - 1 \]
\[ \cos 2d_2 = 2(1 - \cos^2 \theta_2)/V_0 - 1 \]

\[ \sin H = \tan \theta_1 + B \tan \theta_2 ) \sin^2 \Delta \phi \]
\[ \sin 2H = 2P \tan \theta_1 \tan \theta_2 \]

1. (A) MOSCOW
\[ \lambda_1 = 55° 45' 19.50'' \]
\[ \Delta \lambda = \lambda_2 - \lambda_1 = 1.19.05 \]
\[ \Delta \phi = 0.3270.99 \]
\[ \cos \Delta \lambda = 0.941 \]
\[ \sin \Delta \phi = 0.990 \]

2. (B) Cape of Good Hope
\[ \lambda_2 = 18° 28' 41.40'' \]
\[ \Delta \lambda = \lambda_2 - \lambda_1 = 1.19.05 \]
\[ \Delta \phi = 0.3270.99 \]
\[ \cos \Delta \lambda = 0.941 \]
\[ \sin \Delta \phi = 0.990 \]

\[ A = \tan \theta_1 - \tan \phi_2 \cos \Delta \phi + 2.079.68 \]
\[ B = \tan \phi_2 - \tan \phi_2 \cos \Delta \phi - 2.079.68 \]

\[ K = \sin^2 \phi_1 = K(1 + \cos \phi_1) \]
\[ \cos 2\phi_1 = 2(1 - \cos^2 \phi_1)/V_0 - 1 \]
\[ \cos 2\phi_2 = 2(1 - \cos^2 \phi_2)/V_0 - 1 \]

\[ \cos 2d_1 = 2(1 - \cos^2 \theta_1)/V_0 - 1 \]
\[ \cos 2d_2 = 2(1 - \cos^2 \theta_2)/V_0 - 1 \]

\[ \sin H = \tan \theta_1 + B \tan \theta_2 ) \sin^2 \Delta \phi \]
\[ \sin 2H = 2P \tan \theta_1 \tan \theta_2 \]

\[ \cot A = B \cos \phi_1 / \sin \Delta \phi = 3.541.89 \]
\[ P = \int \frac{H''}{
\cos \theta_1 \sin \phi_1 \}
\[ \cot B = A \cos \phi_2 / \sin \Delta \phi = 3.26.35 \]

\[ \cot A = 3.541.89 \]
\[ P'' = \int \frac{H''}{
\cos \theta_1 \sin \phi_1 \}
\[ \cot B = 3.26.35 \]

\[ \delta A = 1.4 \] 0.146 \]
\[ \delta B = 1.4 \] 0.146 \]
\[ \delta A'' = P'' \cos \phi_1 \sin 2A + 1.38 \] \] 935 \]
\[ \delta B'' = P'' \cos \phi_1 \sin 2A - 92 \] \] 888 \]

\[ \theta = 180° - (A - \delta A) \]
\[ \theta = 180° - (A - \delta A) \]

\[ a_{BA} = 180° + B - \delta B \]
\[ a_{BA} = 180° + B - \delta B \]
## COMPUTATIONS, DISTANCE, AZIMUTHS

Great Elliptic Arc, Geodetic Azimuths

Clark 1866 Ellipsoid: \( a = 6,378,206.4 \) meters, \( e^2 = 6.7686580 \times 10^{-3} \)

\( f/2 = 0.00169503765 \), 1 radian = 206,264.8062 seconds, 1852 meters = 1 n. m.

<table>
<thead>
<tr>
<th>( \phi )</th>
<th>( \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 A</td>
<td>( ) Ramey Air Force Base</td>
</tr>
<tr>
<td>2 B</td>
<td>Mountain Home AFB</td>
</tr>
<tr>
<td>1</td>
<td>69</td>
</tr>
<tr>
<td>2</td>
<td>115</td>
</tr>
</tbody>
</table>

\( \Delta \lambda = \lambda_2 - \lambda_1 = 48 \) 45 24.4

2. Always west of 1.

\( \sin \Delta \lambda = 0.75 \) 91180

\( \cos \Delta \lambda = 0.659 \) 25687

\( \sin^2 \Delta \lambda = 0.565 \) 38038

\( \lambda_2 = \tan \theta - \tan \lambda \) cos \( \lambda \) + 0.911 52944

\( \cos^2 \lambda = \tan \theta = \tan \lambda \) cos \( \lambda \) + 0.911 52944

\( \cos (d_1 + d_2) = \sin \theta \sin \lambda + \cos \theta \cos \lambda \cos \Delta \lambda = 0.623 62978 \) \( \cot A = M \cos \theta / \sin \Delta \lambda = 1/289192.28 \)

\( \sin (d_1 + d_2) = \cos \theta \sin \Delta \lambda / \sin B = \cos \theta_2 \sin \Delta \lambda / \sin A = 0.739 39825 \) \( \cot B = N \cos \theta_2 / \sin \Delta \lambda = -2.272 93825 \)

2d = 2 sin \( \theta_2 / \sin \lambda - 1 = -600.97037 \) cos 2d = 2 sin \( \theta_2 \) / \( \sin \lambda - 1 = 0.851 91908 \) A = 105 15 51.929

B = 105 15 51.929

\( d_1 \) and \( d_2 \) are always in the first or second quadrant. If \( \lambda > 90^\circ \), \( |d_1| > |d_2| \), \( d_1 > 0 \), \( d_2 < 0 \).

2d = 126 56 21938 2d = 31 24 44.339 H = \( d_1 + d_2 = 47 \) 10 48.600 P = d1 - d2 79 15 33.139

sin H = 233 39825 P = 138.62618 Q = sin H cos P + 139 9892 \( H_r \) (radians) + 8.82 19689

\( Q = 2H + 985 24670 \) cos 2P = 930 53528 \( k^2 = e^2 \) \( V_0 = 5.394 6921 \times 10^{-3} \)

sin 3H = 0.601 24803 \( k^2 = e^2 \) \( V_0 = 5.394 6921 \times 10^{-3} \)

sin 3H = 383 20682 Q = sin 3H cos 3P - 320.58955 N = k^2 / 4

U_1 = -N_H - Q_1 = -589 29984 x 10^{-3} \( U_2 = -N_2 (6H + 3Q_1 + 1) - 2669 x 10^{-6} \) N_2 = N_1 / 8 9.003 x 10^{-6}

\( U_2 = -N_2 (30H + 45Q_1 + 9Q_2 - Q_3) = 274 x 10^{-9} \)

\( \Sigma = H_r + U_1 + U_2 + U_3 = 1318 \) 589 300 s = a \( \Sigma = 5.304 \) 035 439 meters 2863.9500 n.m.

\( T = (f/2) H'' / \sin H = 933 429 \) sin 2A = 944 174 10 sin (B - 508 3061)

\( \delta A = T \cos^2 \theta_2 \sin^2 B = 107.7130 \)

\( T = 105 15 58.929 \)

\( \delta B = T \cos^2 \theta_1 \sin 2A = 352.059 \)

(\( B - \delta B) = 105 10 6.870 \)

\( 46 \) 07 25.015

\( a_{AB} = 180^\circ - (A - \delta A) = 181 52 34.975 \)

\( a_{BA} = 180^\circ + (B - \delta B) = 285 10 06.870 \)

Figure 19.
Spherical Formulae (see Figure 16)

\[
\begin{align*}
\cos d &= \sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2 \cos \Delta \lambda \\
\sin A &= (\cos \phi_2 \sin \Delta \lambda)/\sin d, \quad \sin B = (\cos \phi_1 \sin \Delta \lambda)/\sin d \\
\cot A &= (\cos \phi_1 \tan \phi_2 - \sin \phi_1 \cos \Delta \lambda)/\sin \Delta \lambda \\
\cot B &= (\cos \phi_2 \tan \phi_1 - \sin \phi_2 \cos \Delta \lambda)/\sin \Delta \lambda \\
\sin d &= (\cos \phi_1 \sin \Delta \lambda)/\sin B = (\cos \phi_2 \sin \Delta \lambda)/\sin A.
\end{align*}
\]

(73)

The Andoyer-Lambert correction [13] for distance is:

\[
\delta d = -\frac{f}{4} \left[ \frac{d + 3 \sin d}{1 - \cos d} (\sin \phi_1 - \sin \phi_2)^2 + \frac{d - 3 \sin d}{1 + \cos d} (\sin \phi_1 + \sin \phi_2)^2 \right],
\]

(74)

where \(d\) is spherical distance from (73) and \(s = a(d + \delta d)\), \(f\) is the flattening, \(f = (a - b)/a\),

where \(a, b\) are the semiaxes of the reference ellipsoid (\(a\) is the radius of the auxiliary sphere).

Now (73) and (74) are essentially the same as used for several years in Loran computations
except for the conversion to parametric latitudes which is not required with these formulas.
The only difference in the appearance of the formulas is in the term \(3 \sin d\) in (74) which is
simply \(\sin d\) in the formulae for parametric latitude, [14].

The corrections to the spherical angles \(A\) and \(B\) as given by (73) to get geodesic azimuths
are, [13]:

\[
\begin{align*}
\delta A &= \frac{f}{2} \left[ \frac{\cos^2 \phi_1 \sin 2B - \cos^2 \phi_1 \sin 2A}{\sin d} \right], \\
\delta B &= \frac{f}{2} \left[ \cos^2 \phi_2 \sin 2B - \frac{d}{\sin d} \cos^2 \phi_1 \sin 2A \right],
\end{align*}
\]

(75)

the geodetic azimuths being then

\[
\alpha_{\text{AB}} = 180^\circ - A + \delta A, \quad \alpha_{\text{BA}} = 180 + B + \delta B.
\]

The formulae as given by (73), (74), (75) were arranged in computing forms to make the
check computations of the ACIC chosen lines. Note that the azimuths as given in the ACIC
publications differ by \(180^\circ\) from the usual geodetic azimuths and the forward and back azimuths
are interchanged from the conventions used in the check computations. The lines chosen are
shown in TABLE 1, the comparisons are given in TABLES 2 and 3, while the actual computations
are in Appendix 2.
<table>
<thead>
<tr>
<th>Line No.</th>
<th>Az.</th>
<th>Terminus</th>
<th>Origin</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Lat.</td>
<td>Long.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Lat.</td>
<td>Long.</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>45</td>
<td>40</td>
<td>18</td>
<td>40 30</td>
</tr>
<tr>
<td>2</td>
<td>90</td>
<td>10</td>
<td>18</td>
<td>9 59</td>
</tr>
<tr>
<td>3</td>
<td>90</td>
<td>70</td>
<td>18</td>
<td>69 48</td>
</tr>
<tr>
<td>4</td>
<td>45</td>
<td>10</td>
<td>18</td>
<td>13 04</td>
</tr>
<tr>
<td>5</td>
<td>45</td>
<td>70</td>
<td>18</td>
<td>73 35</td>
</tr>
<tr>
<td>6</td>
<td>90</td>
<td>40</td>
<td>18</td>
<td>39 37</td>
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<td>7</td>
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<td>40</td>
<td>18</td>
<td>44 54</td>
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<tr>
<td>8</td>
<td>45</td>
<td>70N</td>
<td>18W</td>
<td>76 00</td>
</tr>
<tr>
<td>9</td>
<td>90</td>
<td>40N</td>
<td>18W</td>
<td>27 49</td>
</tr>
<tr>
<td>10</td>
<td>45</td>
<td>40N</td>
<td>18W</td>
<td>35 18</td>
</tr>
<tr>
<td>11</td>
<td>50</td>
<td>43 03 19.6</td>
<td>115 52 54.7</td>
<td>18 29</td>
</tr>
<tr>
<td>12</td>
<td>10</td>
<td>33 56 03.5S</td>
<td>18 28 41.4E</td>
<td>55 45</td>
</tr>
</tbody>
</table>

1–10 From ACIC Reports 59 (page 39), 80 (page 23).


12 Cape of Good Hope to Moscow
<table>
<thead>
<tr>
<th>Line No.</th>
<th>Computed Distance ( S_c ) meters</th>
<th>True Distance ( S_t ) meters</th>
<th>( S_c - S_t = \Delta S ) meters</th>
<th>Computed ( \alpha_{AB}^c )</th>
<th>True ( \alpha_{AB}^t )</th>
<th>( \alpha_{AB}^c - \alpha_{AB}^t = \Delta \alpha_{AB} )</th>
<th>Computed ( \alpha_{BA}^c )</th>
<th>True ( \alpha_{BA}^t )</th>
<th>( \alpha_{BA}^c - \alpha_{BA}^t = \Delta \alpha_{BA} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>80,467.388</td>
<td>80,466.490</td>
<td>+0.898</td>
<td>45 26 00.443</td>
<td>45 26 01.692</td>
<td>-1.249</td>
<td>244 59 58.759</td>
<td>244 59 59.997</td>
<td>-1.238</td>
</tr>
<tr>
<td>2</td>
<td>160,935.945</td>
<td>160,932.956</td>
<td>+2.989</td>
<td>90 15 17.506</td>
<td>90 15 17.480</td>
<td>+0.026</td>
<td>270 00 00.023</td>
<td>270 00 00.000</td>
<td>+0.023</td>
</tr>
<tr>
<td>3</td>
<td>321,862.977</td>
<td>321,866.796</td>
<td>-3.819</td>
<td>97 52 01.112</td>
<td>97 52 01.063</td>
<td>+0.049</td>
<td>270 00 00.026</td>
<td>269 59 59.950</td>
<td>+0.076</td>
</tr>
<tr>
<td>4</td>
<td>482,794.743</td>
<td>482,798.163</td>
<td>-3.420</td>
<td>45 37 44.972</td>
<td>45 37 46.111</td>
<td>-1.139</td>
<td>224 59 58.629</td>
<td>224 59 59.732</td>
<td>-1.103</td>
</tr>
<tr>
<td>5</td>
<td>643,728.709</td>
<td>643,732.429</td>
<td>-3.720</td>
<td>58 50 30.885</td>
<td>58 50 31.600</td>
<td>-0.715</td>
<td>224 59 59.601</td>
<td>225 00 00.154</td>
<td>-0.553</td>
</tr>
<tr>
<td>6</td>
<td>804,664.697</td>
<td>804,664.762</td>
<td>-0.065</td>
<td>96 01 06.689</td>
<td>96 01 06.640</td>
<td>+0.049</td>
<td>270 00 00.073</td>
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<td>190 39 32.208</td>
<td>-0.753</td>
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TABLE 3

Error Summary

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<tr>
<th>Line No.</th>
<th>Azimuth</th>
<th>Terminal Latitude</th>
<th>( S = \text{distance} )</th>
<th>( \Delta S )</th>
<th>Relative distance error ( \Delta S_m/S_m )</th>
<th>( \Delta \alpha_{AB} = \Delta \alpha_{1-2} )</th>
<th>( \Delta \alpha_{BA} = \Delta \alpha_{2-1} )</th>
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<td>1</td>
<td>45</td>
<td>40N</td>
<td>80,466 43.5</td>
<td>+ 0.9 + 3.0</td>
<td>89,407</td>
<td>- 1.25**</td>
<td>- 1.24**</td>
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<td>160,933 86.9</td>
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<td>-11.1 -36.6</td>
<td>910,096</td>
<td>- 0.74</td>
<td>- 0.75</td>
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</table>

* Maximum distance error

** Maximum azimuth errors
INVESTIGATION OF HIGHER ORDER TERMS IN ANDOYER-LAMBERT APPROXIMATION

While either form of Andoyer-Lambert approximation is probably satisfactory in the "state of the art" in hyperbolic navigational systems development, the question arises as to the higher order terms in the flattening of the Andoyer-Lambert approximation and the possibility of a single set of formulae which will give distance within one meter and azimuth within one second over all geodetic lines on the spheroid. This would be a practical operational system particularly if it maintained the several attributes of the Andoyer-Lambert first order approximation.

HISTORICAL

Now Lambert, [13], never published his derivation but had equivalent formulae for a first order approximation several years before the publication posthumously in 1932 of Andoyer's sketch, [15], of the derivation of the form as given in equation (74). Andoyer's derivation employs a differential oblique spherical triangle and it is not clear how one would proceed to higher order terms in the flattening. It is believed that Andoyer's derivation is the only recognized published one in existence.

DERIVATION FROM THE GREAT ELLIPTIC ARC

Independent derivations of the Andoyer-Lambert approximations were sought in the hopes of discovering a simple method of arriving at higher order terms in the flattening. It was noticed that the computations using the Andoyer-Lambert approximations; the ratios \((d - \sin d)/(1 + \cos d)\), \((d + \sin d)/(1 - \cos d)\) were being used in forming computational parameters, [16]. It was decided to try the ratios

\[
\frac{(\sin \theta_1 + \sin \theta_2)^2/(1 + \cos d), (\sin \theta_1 - \sin \theta_2)^2/(1 - \cos d)}{76}
\]

with the hope of relating these to other parameters and identification of the Andoyer-Lambert approximations in some other extant series expansion as the great elliptic arc approximation. See equations (19) through (42).

From equations (42) we have

\[
\sin \theta_1 = \sin \theta_0 \cos d_1, \sin \theta_2 = \sin \theta_0 \cos d_2.
\]

From (77), by simple algebraic operations and trigonometric identities, we may express (76) as

\[
\frac{(\sin \theta_1 + \sin \theta_2)^2/(1 + \cos d)}{76} = 2 \sin^2 \theta_0 \cos^2 \frac{1}{2}(d_1 + d_2)
\]

\[
\frac{(\sin \theta_1 - \sin \theta_2)^2/(1 - \cos d)}{76} = 2 \sin^2 \theta_0 \sin^2 \frac{1}{2}(d_1 + d_2),
\]

68
where \( d = d_2 - d_1 \).

From (78) by adding and subtracting respective members, we write

\[
X = \frac{(\sin \theta_1 + \sin \theta_2)^2}{1 + \cos d} + \frac{(\sin \theta_1 - \sin \theta_2)^2}{1 - \cos d} = 2\sin^2 \theta_0
\]

\[
Y = \frac{(\sin \theta_1 + \sin \theta_2)^2}{1 + \cos d} - \frac{(\sin \theta_1 - \sin \theta_2)^2}{1 - \cos d} = 2\sin^2 \theta_0 \cos (d_1 + d_2),
\]

where \( d = d_2 - d_1 \).

The Andoyer-Lambert forms can now be written in terms of \( X \) and \( Y \) of (79) as

\[
S = a[d - (f/4) (Xd - Y \sin d)],
\]

\[
S = a[d - (f/4) (Xd - 3Y \sin d)],
\]

where in the second equation, the geodetic latitude, \( \phi \), is used in forming the \( X \) and \( Y \) of (79).

If in the expansion of the great elliptic arc, equation (41), we place \( d_1 = -d_1 \), and then \( d = d_2 - d_1 \), \( k = e \sin \theta_0 \), we obtain as far as sixth order terms in \( e \):

\[
S = a \left[ d - \frac{1}{6} e^2 \sin^2 \theta_0 [d - \sin d \cos (d_1 + d_2)] - \frac{1}{128} e^4 \sin^4 \theta_0 [(6d - 3 \sin d \cos (d_1 + d_2) + \sin 2d \cos 2(d_1 + d_2)]ight] - \frac{1}{1536} e^6 \sin^6 \theta_0 \left[ 30d - 45 \sin d \cos (d_1 + d_2) + 9 \sin 2d \cos 2(d_1 + d_2) - \sin 3d \cos 3(d_1 + d_2) \right]
\]

Using relations (79), equation (81) can be written:

\[
S = a \left[ d - \frac{1}{6} e^2 \sin^2 \theta_0 (Xd - Y \sin d) - \frac{e^4}{512} \left[(6d - \sin 2d) X^2 - 8(\sin d) XY + 2(\sin 2d) Y^2 \right]ight] - \frac{e^6}{12,288} \left[ 3(10d - 3 \sin 2d) X^3 - 3(15 \sin d - \sin 3d) X^2 Y + 18(\sin 2d) XY^2 - 4(\sin 3d) Y^3 \right]
\]

Note in (82) that if all terms above the first power in \( f \) are ignored (\( e^2 = 2f \)) equation (82) reduces directly to the Andoyer-Lambert form as given by the first of (80). Now it is known that the difference in lengths of the great elliptic arc and the geodesic is of 4th order in \( e \), [17], but the 6th order term will be useful for comparison later in the investigation.

**DERIVATION FROM MODIFIED DIFFERENTIAL EQUATIONS**

The corresponding differential triangles, auxiliary sphere, spheroid, where geodetic latitude has been converted to parametric arc, as abstracted from Figure (20):
Figure 20. Differential triangles, sphere and spheroid.
and since \( \alpha_c = \alpha_g \) (property of geodesics on surfaces of revolution, i.e. \( r \sin \alpha_c = r \sin \alpha_g \)),

\[ r = a \cos \theta, \quad ds/\partial \delta d = a(1 - e^2 \cos^2 \theta)^{1/2} \, d\theta/\partial \delta d = (1 - e^2 \cos^2 \theta)^{1/2}, \]

which may be written

\[ S = a(d + \delta d) = a \left[ d + \int_{d_1}^{d_2} \left[ (1 - e^2 \cos^2 \theta)^{1/2} - 1 \right] \, D\delta d \right]. \tag{83} \]

If (83) also represents the equator, then \( \delta d = 0 \), when \( \theta = \theta_0 = 0 \). Hence we add to the integrand \( 1 - (1 - e^2 \cos^2 \theta)^{1/2} \) to get

\[ S = a(d + \delta d) = a \left[ d + \int_{d_1}^{d_2} \left[ (1 - e^2 \cos^2 \theta)^{1/2} - (1 - e^2 \cos^2 \theta)^{1/2} \right] \, D\delta d \right], \tag{84} \]

and we note that when \( \theta = \theta_0 = 0 \), \( \delta d = 0 \); when \( \theta = \theta_0 \), \( s = d = \delta d = 0 \); when \( \theta_0 = \pi/2 \), \( d_1 = \theta_1 \), \( d_2 = \theta_2 \), \( D\delta d = d\theta \), \( d = \theta_2 - \theta_1 \), then (84) represents the meridian.

Expanding (84) to 6th order terms in \( e \), find

\[ S = a \left[ d - (e^2/2) \left( 1 + e^2/2 + 3e^4/8 \right) \int_{d_1}^{d_2} (\sin^2 \theta_0 - \sin^2 \theta) \, D\delta d \right] \]

\[ + (e^4/8) \left( 1 + 3e^2/2 \right) \int_{d_1}^{d_2} (\sin^4 \theta_0 - \sin^4 \theta) \, D\delta d \]

\[ - (e^6/16) \int_{d_1}^{d_2} (\sin^6 \theta_0 - \sin^6 \theta) \, D\delta d \tag{85} \]

Now from (77), \( \sin \theta = \sin \theta_0 \cos d \),

\[ \sin^2 \theta = \sin^2 \theta_0 \cos^2 d = \frac{\sin^2 \theta_0}{2} (1 + \cos 2d). \tag{86} \]

The value of \( \sin^2 \theta \) from (86) placed in (85) and the resulting integrations performed with respect to \( d \), leads to expressions in powers of the right hand quantities in (79) so that (85) may be written finally as

---

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\[ S = a \left[ \frac{d - (e^2/8) (1 + e^2/2 + 3e^4/8) (Xd - Y \sin d)}{1 + (10d + \sin 2d) X^2 + 2(\sin 2d) Y^2} \right] \]

Again if all terms above first order in \( f \) \( (e^2 = 2f) \) in (87) are ignored then the first two terms of (87) represent the Andoyer-Lambert form as given by the first of equations (80).

For the case where geographic latitudes, \( \phi \), are not first converted to parametric, but are considered spherical, the corresponding differential right triangles are:

We have for the approximation

\[ \text{Rd} \phi = ds \cos \alpha \]

or

\[ \text{Rd} \phi = ds \frac{d \phi}{D \delta d} \text{, placing } \cos a_g = \cos a_c = \frac{d \phi}{D \delta d} \]

\[ ds = R D \delta d = a(1 - e^2)(1 - e^2 \sin^2 \phi)^{-1/2} D \delta d. \] (88)

If (88) represents the equator, then when \( \phi = 0 \), \( ds = a D \delta d \). Hence add \( e^2 \cos^2 \phi_0 \) to the integrand of (88), to obtain

\[ \left( \frac{ds}{a} \right) = \left[ 1 - e^2(1 - e^2 \sin^2 \phi)^{-1/2} \right] + e^2 \cos^2 \phi_0 \]

Note the following for (89): When \( \phi = \phi_0 = 0 \), \( ds = a D \delta d \); when \( \phi_0 = \pi/2 \), \( D \delta d = d \phi \), equation (89) will represent the meridian.

Expanding (89) to 6th order terms in \( e \) get

\[ \left( \frac{ds}{a} \right) = \left[ 1 + (3/2)e^2 \sin^2 \phi + (15/8)e^4 \sin^4 \phi + (35/16)e^6 \sin^6 \phi \right] D \delta d \]

\[ - e^2 (1 + (3/2)e^2 \sin^2 \phi + (15/8)e^4 \sin^4 \phi) \]

which may be written in the integral form
From (77), with \( \theta \) replaced by \( \phi \), we have
\[
\sin^2 \phi = \frac{\sin^2 \phi_0}{2} (1 + \cos 2d),
\]
and with the aid of trigonometric identities we can find expressions for \( \sin^4 \phi \) and \( \sin^6 \phi \), i.e.
\[
\sin^2 \phi = \frac{\sin^2 \phi_0}{2} (1 + \cos 2d),
\]
\[
\sin^4 \phi = \frac{\sin^4 \phi_0}{8} (3 + 4 \cos 2d + \cos 4d),
\]
\[
\sin^6 \phi = \frac{\sin^6 \phi_0}{32} (10 + 15 \cos 2d + 6 \cos 4d + \cos 6d).
\]
The values of \( \sin^2 \phi \), \( \sin^4 \phi \), \( \sin^6 \phi \) from (92) placed in (91) give
\[
S = a \left[ d - \frac{(e^2/4) \int_{d_1}^{d_2} (2 \sin^2 \phi_0 - 3 \sin^2 \phi) D \delta d}{(1 - 3 \cos 2d)} \right] - (3e^4/64) \int_{d_1}^{d_2} \left[ (16 - 5 \sin^2 \phi_0) + (16 - 20 \sin^2 \phi_0) \cos 2d \right] \sin^2 \phi_0 D \delta d \]
\[
- (3e^4/512) \int_{d_1}^{d_2} \left[ (72 - 70 \sin^2 \phi_0) + (96 - 105 \sin^2 \phi_0) \cos 2d \right] \sin^4 \phi_0 D \delta d \]
\[
- (5e^6/1536) \int_{d_1}^{d_2} \left[ 216d (\sin^2 \phi_0)^2 - 210d (\sin^2 \phi_0)^3 + 288 \sin d (\sin^2 \phi_0) [\sin^2 \phi_0 \cos (d_1 + d_2)] \right] \sin^6 \phi_0 D \delta d \]
\[
- 315 \sin d (\sin^2 \phi_0)^2 \sin^2 \phi_0 \cos (d_1 + d_2) + 72 \sin 2d (\sin^2 \phi_0 \cos (d_1 + d_2))^3 \]
\[
- 126 \sin 2d (\sin^2 \phi_0) (\sin^2 \phi_0 \cos (d_1 + d_2))^3 - 36 \sin 2d (\sin^2 \phi_0)^2 \]
\[
+ 63 \sin 2d (\sin^2 \phi_0)^3 - 28 \sin 3d (\sin^2 \phi_0 \cos (d_1 + d_2))^3 \]
\[
+ 21 \sin 3d (\sin^2 \phi_0)^3 (\sin^2 \phi_0 \cos (d_1 + d_2)) \right].
\]
Integration of (93) with respect to \( d \) leads to:
\[
S = a \left[ d - \frac{(e^2/4) \{ d [\sin^2 \phi_0] - 3 \sin d [\sin^2 \phi_0 \cos (d_1 + d_2)] \}}{1 - 3 \cos 2d} \right] - (3e^4/128) \left[ 32d [\sin^2 \phi_0] - 30d [\sin^2 \phi_0]^2 + 32 \sin d [\sin^2 \phi_0 \cos (d_1 + d_2)] \right] \]
\[
- 40 \sin d [\sin^2 \phi_0] [\sin^2 \phi_0 \cos (d_1 + d_2)] \]
\[
- 10 \sin 2d [\sin^2 \phi_0 \cos (d_1 + d_2)]^2 + 5 \sin 2d [\sin^2 \phi_0]^2 \]
\[
- (5e^6/1536) \left[ 216d [\sin^2 \phi_0]^2 - 210d [\sin^2 \phi_0]^3 + 288 \sin d [\sin^2 \phi_0] [\sin^2 \phi_0 \cos (d_1 + d_2)] \right] \]
\[
- 315 \sin d [\sin^2 \phi_0]^2 [\sin^2 \phi_0 \cos (d_1 + d_2)] + 72 \sin 2d [\sin^2 \phi_0 \cos (d_1 + d_2)]^3 \]
\[
- 126 \sin 2d [\sin^2 \phi_0] [\sin^2 \phi_0 \cos (d_1 + d_2)]^3 - 36 \sin 2d [\sin^2 \phi_0]^2 \]
\[
+ 63 \sin 2d [\sin^2 \phi_0]^3 - 28 \sin 3d [\sin^2 \phi_0 \cos (d_1 + d_2)]^3 \]
\[
+ 21 \sin 3d [\sin^2 \phi_0]^3 (\sin^2 \phi_0 \cos (d_1 + d_2)) \right].
\]
From (79), with θ replaced by φ, we have

\[
X = \frac{(\sin \phi_1 + \sin \phi_2)^2}{1 + \cos d} + \frac{(\sin \phi_1 - \sin \phi_2)^2}{1 - \cos d} = 2[\sin^2 \phi_0] \tag{95}
\]

\[
Y = \frac{(\sin \phi_0 + \sin \phi_2)^2}{1 + \cos d} - \frac{(\sin \phi_1 - \sin \phi_2)^2}{1 - \cos d} = 2[\sin^2 \phi_0 \cos (d_1 + d_2)].
\]

Substituting from (95) in (94) we obtain finally

\[
S = a \left[ d - (e^2/8) (Xd - 3Y \sin d) \right. \\
- (3e^4/512) \left[ 64(Xd + Y \sin d) + (5 \sin 2d - 30d) X^2 \right] \\
- 40 (\sin d) XY - 10 (\sin 2d) Y^2 \] \\
- (5e^6/12,288) \left[ (432d - 72 \sin 2d) X^2 + 576 (\sin d) XY - 144 (\sin 2d) Y^2 \\
+ (63 \sin 2d - 210 d) X^3 + (21 \sin 3d - 315 \sin d) X^2 Y \\
- 126 (\sin 2d) XY^2 - 28(\sin 3d) Y^3 \right] \right]
\]

(96)

If, in (96), we place \(e^2 = 2f\), ignoring all terms above first order in \(f\), one obtains the second of equations (80), or the Andoyer-Lambert approximation in terms of geodetic latitude, \(\phi\).

Now the Andoyer-Lambert forms can be obtained from other modifications of differential equations. For instance if the differential for arc length along the geodesic is taken in the form, [8] page 64,

\[
ds = (N^2 \cos^2 \phi/N_0 \cos \phi_0) d\lambda, \quad N = a/(1 - e^2 \sin^2 \phi)^{1/2}; \tag{97}
\]

if the differential of arc length from (84), after converting to geodetic latitude is written

\[
ds = [(1 - e^2 \sin^2 \phi)^{-1/2} - (1 - e^2 \sin^2 \phi_0)^{-1/2}] \delta \phi \\
+ (1 - e^2)^{1/2} \left\{ (1 - e^2 \sin^2 \phi)^{-1/2} - (1 - e^2 \sin^2 \phi_0)^{-1/2} \right\} \delta \phi \tag{98}
\]

and if (97) and (98) are combined with the relationship \(d\lambda = (\sin \phi_0 / \cos \phi) \delta \phi\), one can write

\[
(ds/a) = D\delta \phi + \left[ (1 - e^2 \sin^2 \phi)^{-1} (1 - e^2 \sin^2 \phi_0)^{1/2} - 1 \\
+ (1 - e^2)^{1/2} \left\{ (1 - e^2 \sin^2 \phi)^{-1/2} - (1 - e^2 \sin^2 \phi_0)^{-1/2} \right\} \right] \delta \phi 
\]

Expanding the expressions in (99) to first order terms in \(f\), \(e^2 = 2f\), equation (99) can be written in the integral form

\[
S = a \left[ d - f \int_{d_1}^{d_2} (2 \sin^2 \phi_0 - 3 \sin^2 \phi) \delta \phi \right]. \tag{100}
\]

Comparison of equations (100) and (91) (with \(e^2 = 2f\)) shows that (100) will again give the second of equations (80) or the Andoyer-Lambert Approximation in terms of geodetic latitude.
DERIVATIONS FROM EXPANSIONS OF FORSYTH

In reviewing the literature on geodetic computation one finds that A. R. Forsyth, [18], as early as 1895 had given some series expansions for geodetic arc length in terms of the flattening and certain spherical and elliptic parameters. On page 120 of his treatise one finds the expression
\[ S_{12}/a = \nu_2' - \nu_1' - \frac{1}{4} c (\nu_2' - \nu_1') + (1/8) c (\sin 2 \nu_2' - \sin 2 \nu_1') \tag{101} \]

Now the correspondences between the parameters as used by Forsyth in deriving (101) and those used above in this investigation are to first order in \( f \):
\[ \nu_2' = d_2, \quad \nu_1' = d_1, \quad \nu_2 - \nu_1 = d_2 - d_1 = d, \quad c = 2 f \sin^2 \theta_0, \]
\[ \cos \nu_2' = \sin \nu_1' = \cos \lambda_2 - \lambda_1 = \Delta \lambda, \quad \cos \phi_1' = \cot \phi_0 \tan \phi_1 = \cos \phi_0 \cos d_1 \sec \phi_1 \]
\[ \sin \phi_1' = \sin d_1 \sec \phi_1, \quad \cos \phi_2' = \cot \phi_0 \tan \phi_2 = \cos \phi_0 \cos d_2 \sec \phi_2 \]
\[ \sin \phi_3' = \sin d_2 \sec \phi_2, \quad \cos \nu_1 = \cos d_1 = \sin \phi_1' \sin \phi_0, \]
\[ \cos \nu_2 = \cos d_2 = \sin \phi_2' / \sin \phi_0, \quad a_0 = \frac{\pi}{2} - \phi_0, \]
the relationship \( \sin a_0 \sin (\nu_2' - \nu_1') = \cos \phi_0 \sin d = \cos \phi_1 \cos \phi_2 \sin \Delta \lambda \) in the notation of this investigation.
Assurance that Forsyth's \( \alpha \) is the complement of the geodetic latitude, \( \phi_0 \), of the great elliptic arc is found from his expression, [18] page 106, which is

\[
\tan \alpha = \sin 2 \phi_0 / \left( \tan \lambda_1 + \tan \lambda_2 \right)^2 - 4 \tan \lambda_1 \tan \lambda_2 \cos^2 \phi_0 \right)^{1/2}.
\]

With equivalent substitutions from (103) and some trigonometric identities it will transform into

\[
\tan \phi_0 = (\tan^2 \phi_1 + \tan^2 \phi_2 - 2 \tan \phi_1 \tan \phi_2 \cos \Delta \lambda)^{1/2} / \sin \Delta \lambda
\]

which defines the vertex of the great elliptic arc. See equations (21) of this investigation.

A cursory check of the equations just preceding (102) in Forsyth's treatise revealed that the numerical coefficient of the second order term \( *1 \) in (102) should be 15/64 instead of 23/64. Then by use of relations (103) and (95) it was found that (102) could be written as

\[
S = a \left[ d - \left( f/4 \right) \left( Xd - 3Y \sin d \right) + \left( f^2/128 \right) \left( AX - BY - CX^2 + DY^2 + EXY + FX^2Y + GX^3 \right) \right] (104)
\]

where

\[
A = 64d + 16d^2 \cot d, \quad B = 96 \sin d + 16d^2 \csc d - 48 \sin^2 \Delta \lambda \csc d, \quad C = 30d + 15 \sin 2d + 8d^2 \cot d + 12 \sin^2 \Delta \lambda \cot d, \quad D = 30 \sin 2d, \quad E = 48 \sin d + 8d^2 \csc d - 36 \sin^2 \Delta \lambda \csc d,
\]

\[
F = 6 \sin^2 \Delta \lambda \csc d, \quad G = 6 \sin^2 \Delta \lambda \cot d.
\]

Note that the first two terms of (104) are exactly the Andoyer-Lambert form given by the second of equations (80). But we apparently also have the second order term in the flattening. Thus, Forsyth had both so-called Andoyer-Lambert approximation forms as early as 1895 but they had not been recognized as such.

Equation (104) was used to compute several lines of known lengths. On those in which the term \( *2 \) of (102) was small, an improvement would be obtained by including the second order terms. On others, the error introduced would outweigh the first order correction, which could mean, since equation (104) is a power series in \( f \), that the coefficient of the second order term in \( f \) is erroneous. Now examination of the second order terms of equations (82) and (96) shows no cubic terms in \( X \) and \( Y \) as are found in the second order term of (104). Hence Forsyth's paper [18], was reworked from the beginning and it was found that indeed the term \( *2 \) in (102) actually vanishes and reaffirmation was also made that the numerical coefficient of the term \( *1 \) of (102) should be 15/64 rather than 23/64. These errors are the result of carrying throughout the derivation the numerical factor 9/32 in the last term of the expression for \( \delta \), [18], section 17, page 98, when it should be 3/32. This affects the approximation equation for \( \tan \Phi \), section 22, page 104. In the last term, the factor \( -7 \sin^2 \alpha \) should be \( +5 \sin^2 \alpha \). This continues to be reflected through section 27, pages 111 to 115, until the term is actually seen to vanish in collecting the terms together on page 115. Also on page 115, omission of a factor \( 1/2 \) in use of a trigonometric identity in the third line from the bottom gave the printed value \( 1/4 \) for the numerical coefficient of
cos \^4a_0 \sin 4 \nu \text{ when it should be } 1/8. This leads in turn to the printed value 23/64 as given on page 116 when it should be 15/64.

After the two errors in Forsyth's second order term in f had been detected, two papers were found which are concerned with the Forsyth derivation, Wassef 1948, [19], and Gougenheim 1950, [20]. Wassef purports to give the corrected version of Forsyth's second order term but he includes the term "2 in (102) and he gives 15/23 for the numerical coefficient of "1 in (102). Hence Wassef's results are erroneous and useless. Gougenheim, unaware of Forsyth's work, had developed his formulae independently and he has the term "2 in (102) missing in his derivation and the correct numerical coefficient 15/64 for "1 of (102). His formula for the second order term is (in the notation of Forsyth)

\[\xi^2 \left\{- \frac{(\nu_2 - \nu_1)^2}{\cot \nu_2 - \cot \nu_1} \cos^2 a_0 \sin^2 a_0 + (1/16) (\nu_2 - \nu_1) (\cos^2 a_0 + 15 \cos^2 a_0 \sin^2 a_0) \right.\]

\[- (3/4) \cos^2 a_0 \sin^2 a_0 (\sin 2\nu_2 - \sin 2\nu_1) + (15/64) \cos^4 a_0 (\sin 4\nu_2 - \sin 4\nu_1)\]

Since the last two terms of (105) are the same as the last two of (102), as corrected, we have only to show that

\[(1/16) \cos^4 a_0 + \cos^2 a_0 \sin^2 a_0 = (1/16) (\cos^2 a_0 + 15 \cos^2 a_0 \sin^2 a_0)\]

\[1/(\cot \nu_1 - \cot \nu_2) = (\sin \phi_1 \sin \phi_2)/\sin 2\phi_0.\]

Writing the right member of the first of (106) as

\[(1/16) \cos^4 a_0 + (15/16) \cos^2 a_0 \sin^2 a_0 + (1/16) \cos^2 a_0 - (1/16) \cos^2 a_0 (1 - \sin^2 a_0)\]

\[= (1/16) \cos^4 a_0 + (1/16) \cos^2 a_0 + (15/16) \cos^2 a_0 \sin^2 a_0 - (1/16) \cos^2 a_0 - (1/16) \cos^2 a_0 \sin^2 a_0\]

\[= (1/16) \cos^4 a_0 + \cos^2 a_0 \sin^2 a_0.\]

From relations (103) we have

\[\sin a_0 \sin (\nu_2 - \nu_1) = \cos l_1 \cos l_2 \sin 2\phi_0 \text{ or} \]

\[\frac{\sin a_0}{\sin 2\phi_0} = \frac{\cos l_1 \cos l_2}{\sin (\nu_2 - \nu_1)} \]

\[\frac{\sin a_0 \sin \phi_1 \sin \phi_2}{\sin 2\phi_0} = \frac{\cos l_1 \sin \phi_1 \cdot \cos l_2 \sin \phi_2}{\sin \nu_1 \cos \nu_1 - \cos \nu_2 \sin \nu_1} = \frac{\cos l_1 \sin \phi_1}{\sin \nu_1} \cdot \frac{\cos l_2 \sin \phi_2}{\sin \nu_2} \]

\[\cot \nu_1 - \cot \nu_2 \]
From pages 111, 117 of Forsyth find:
\[
\tan \phi_1 \sin \alpha = \tan \nu_1, \cos \phi_1 = \tan \alpha \tan \nu_1, \cos \nu_1 \cos \alpha = \sin \nu_1,
\]
\[
\tan \phi_2 \sin \alpha = \tan \nu_2, \cos \phi_2 = \tan \alpha \tan \nu_2, \cos \nu_2 \cos \alpha = \sin \nu_2,
\]
whence
\[
\frac{\cos l_1 \sin \phi_1'}{\sin \nu_1} = \frac{\sin l_1}{\cos \nu_1 \cos \alpha} = 1, \tag{108}
\]
\[
\frac{\cos l_2 \sin \phi_2'}{\sin \nu_2} = \frac{\sin l_2}{\cos \nu_2 \cos \alpha} = 1.
\]

The values from (108) placed in (107) prove the second of (106) and thus Gougenheim's paper provides an independent check of the corrections given here to Forsyth's second order term.

Gougenheim also gave formulae for azimuths, convergence of the meridians, and difference in longitude between the spheroidal and spherical (elliptical) vertices of geodesics in terms of the same variables. The importance of Gougenheim's work has not been recognized. He has had a correct expansion including the second order term in the flattening, in print since 1950.

THE FORSYTH-ANDOYER-LAMBERT TYPE APPROXIMATION IN GEODETIC LATITUDE WITH SECOND ORDER TERMS

With the corrections to (102), i.e. with the numerical coefficient of *1 as 15/64 and the term *2 omitted, equation (102) may be written, with relations (103) and (95), as
\[
S = a[d - (f/4)(Xd - 3Y \sin d)] + (f^2/128)(AX + BY + CX^2 + DXY + EY^2)], \tag{109}
\]
where a, f are the semimajor axis and flattening of the reference ellipsoid; d is the spherical distance between the points \(P_1(\phi_1, \lambda_1), P_2(\phi_2, \lambda_2)\) on the ellipsoid given by some spherical formula as \(\cos d = \sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2 \cos \Delta \lambda; \) \(\phi\) is geodetic latitude, \(\lambda\) is longitude, \(\Delta \lambda = \lambda_2 - \lambda_1; A = 64d + 16d^2 \cot d, D = 48 \sin d + 8d^2 \csc d, B = -2D, E = 30 \sin 2d,\)
\(C = -(30d + 8d^2 \cot d + E/2), X = \frac{(\sin \phi_1 + \sin \phi_2)^2}{1 + \cos d} + \frac{(\sin \phi_1 - \sin \phi_2)^2}{1 - \cos d},\)
\(Y = \frac{(\sin \phi_1 + \sin \phi_2)^2}{1 + \cos d} - \frac{(\sin \phi_1 - \sin \phi_2)^2}{1 - \cos d}; d = d_2 - d_1, \) where \(d_1\) and \(d_2\) are spherical distances from the vertex of the great elliptic arc to the points \(P_1(\phi_1, \lambda_1), P_2(\phi_2, \lambda_2)\).

Now by factoring \(\sin d\) out of every term of (109) and using the azimuth formulae as given by Lambert, [13], we can, by means of trigonometric identities, arrange equations (109) in a form more convenient for computing as follows:
Given on the reference ellipsoid the points \( P_1(\phi_1, \lambda_1), P_2(\phi_2, \lambda_2), \) \( \phi \) is geodetic latitude, \( \lambda \) is longitude, \( P_2 \) is west of \( P_1 \) with west longitudes considered positive.

With \( \phi_m = (1/2) (\phi_1 + \phi_2), \Delta \phi_m = (1/2) (\phi_2 - \phi_1), \Delta \lambda = \lambda_2 - \lambda_1, \Delta \lambda_m = (1/2) \Delta \lambda; \) Let:

\[
\begin{align*}
   k &= \sin \phi_m \cos \Delta \phi_m, \quad K = \sin \Delta \phi_m \cos \phi_m, \\
   H &= \cos^2 \Delta \phi_m - \sin^2 \phi_m = \cos^2 \phi_m - \sin^2 \Delta \phi_m, \\
   L &= \sin^2 \Delta \phi_m + H \sin^2 \Delta \lambda_m = \sin^2(d/2), \quad 1 - L = \cos^2(d/2), \cos d = 1 - 2L, \quad t = \sin^2 d = 4L(1-L), \\
   U &= 2k^2/(1 - L), \quad V = 2K^2/L, \quad X = U + V, \quad Y = U - V, \\
   T &= d/\sin d = 1 + (t/6) + 3(t^2/40) + 5(t^3/112) + 35(t^4/1152) + 63(t^5/2816) + \ldots, \\
   E &= 30 \cos d, \quad A = 4T (8 + TE/15), \quad D = 4(6 + T^2), \quad B = -2D, \quad C = T - 3/2(A + E),
\end{align*}
\]

(110)

\[
\begin{align*}
   S &= a \sin d \left[ T - (f/4) (T^2 - 3Y) + (f^2/64) \{ X(A + CX) + Y (B + EY) + DXY \} \right]; \\
   \sin (\alpha_2 + \alpha_2) &= (K \sin \Delta \lambda)/L, \quad \sin (\alpha_2 - \alpha_1) = (k \sin \Delta \lambda)/(1 - L) \\
   (\frac{1}{2}) (\delta \alpha_2 + \delta \alpha_2) &= - (f/2) H (T + 1) \sin (\alpha_2 + \alpha_1), \quad (\frac{1}{2}) (\delta \alpha_2 - \delta \alpha_1) = -(f/2) H (T - 1) \sin (\alpha_2 - \alpha_1), \\
   a_{1-2} &= a_1 + \delta \alpha_1, \quad a_{2-2} = a_2 + \delta \alpha_2.
\end{align*}
\]

Note that the quantities \( H, T, L, k, K \) enter into both distance and azimuth formulas.

Figure (21) shows an arrangement of equations (110) for desk computing using an ordinary ten bank electric desk calculator and Peters eight place tables of trigonometric functions. It is arranged to show the contribution of both the first and second order terms in the flattening.

Table 4 summarizes the results of computations over 17 lines of known lengths and azimuths. The computations are given in Appendix 3. Part of these lines were used in the computations of Appendix 2. The first 11 lines are from two ACIC publications [12], lines 12 through 17 are Coast and Geodetic Survey specially computed lines, [22].

Note that all distances are within one meter and azimuths are within one second which was the objective since this is adequate for any operational requirement. Other advantages are (1) no conversion to parametric latitudes, (2) no square root calculation, (3) for desk computers the only tabular data required is a table of the natural trigonometric functions as Peters eight place tables, (4) the formulas are adaptable to high speed computers, (5) about the same accuracy is obtained over all lines in all azimuths and latitudes.

EXPANSION TO SECOND ORDER TERMS IN \( f \) USING PARAMETRIC LATITUDE

Foryth [18], gave an expansion of the geodesic to first order in the elliptic modulus

\[
c = (e^2 \cos^2 \alpha)/(1 - e^2 \sin^2 \alpha)\]

where \( \alpha \) is the complement of the parametric latitude of the vertex of the geodesic. (See pages 118–120 of his treatise). We will follow the Forsyth method and
DISTANCE COMPUTING FORM, FORSYTH-ANDOYER-LAMBERT
TYPE APPROXIMATION WITH SECOND ORDER TERMS
(No conversion to parametric latitudes)
Clarke Spheroid 1866, a = 6,378,206.4 meters
f/2 = 0.00169503765, f/4 = 0.000847518825, f^2/64 = 0.1795720390 \times 10^{-6}
1 radian = 206,264.8062 seconds

\[ \phi_1 = 8^\circ 58' 25'' \quad \lambda_1 = 79^\circ 34' 24'' \]
\[ \phi_2 = 21^\circ 26' 06'' \quad \lambda_2 = 158^\circ 01' 33.0'' \]
\[ \Delta \phi = \frac{1}{2}(\phi_1 + \phi_2) = 15^\circ 18' 15.5'' \quad \Delta \lambda = \lambda_2 - \lambda_1 = 78^\circ 27' 09.0'' \]
\[ \Delta \phi_m = \frac{1}{2}(\phi_2 - \phi_1) = 6^\circ 13' 30.5'' \quad \Delta \lambda_m = \frac{1}{2} \Delta \lambda = 39^\circ 13' 34.5'' \]
\[ \sin \phi = +0.26226170 \quad \sin \Delta \phi_m = +0.10853193 \quad \sin \Delta \lambda = +0.979753909 \]
\[ \cos \phi = +0.96499679 \quad \cos \Delta \phi_m = +0.99409297 \quad \sin \Delta \lambda_m = +0.63238428 \]
\[ k = \sin \phi \cos \Delta \phi_m + 0.260712512 \quad K = \sin \phi \cos \Delta \phi_m + 0.104732963 \]
\[ H = \cos^2 \Delta \phi_m - \sin^2 \phi = \cos^2 \phi_m - \sin^2 \Delta \phi_m + 0.949439630 \quad 1 - L = +0.62052783 \]
\[ L = \sin^2 \Delta \phi_m + H \sin \Delta \lambda_m = +0.373917217 \quad \cos d = 1 - 2L + 0.24105566 \]
\[ d = \frac{1.327342885}{\sin d} + 0.97051299 \quad T = d/\sin d + 1.367673822 \]
\[ V = 2k^2/(1-L) = 219074828 \quad E = 30 \cos d \quad +7.2316698 \]
\[ X = U + V = 2.768866675 \quad Y = U - V = +1.61262981 \quad D = 4(6 + T^2) = +31.48212675 \]
\[ A = 4T(8 + ET/15) + 4.13727825 \quad C = T - \frac{1}{2}(A + E) = -25.93455125 \quad B = -2D = -62.9642535 \]
\[ X(A + CX) = 111.125575321 \quad Y(B + EY) = -9.96573823 \quad DXY = 1.405726406 \]
\[ (TX - 3Y) = -1050.98286 \quad \delta f = - (f/4) (TX - 3Y) = +8.90728 \times 10^{-5} \]
\[ T + \delta f = +1.36776290 \quad S_1 = \sin d (T + \delta f) = 8.466, 618.26 \text{ meters} \]
\[ \Sigma = X(A + CX) + Y(B + EY) + DXY = 2.56857555 \quad \delta f^2 = -(f^2/64) \Sigma = +4.6124 \times 10^{-7} \]
\[ T + \delta f + \delta f^2 = +1.36776336 \quad S_2 = \sin d (T + \delta f + \delta f^2) = 8.466, 621.11 \text{ meters} \]
\[ \sin (a_2 + a_1) = (K \sin \Delta \lambda)/L = +0.270 \times 10^{-1} \]
\[ \sin (a_2 - a_1) = (K \sin \Delta \lambda)/(1-L) = +0.1164222 \]
\[ \frac{1}{2}(\delta a + a_2) = -(f/2) H (T + 1) \sin (a_2 + a_1) = -9.71808513 \times 10^{-4} \]
\[ \frac{1}{2}(\delta a - a_2) = -(f/2) H (T - 1) \sin (a_2 - a_1) = -2.3587677 \times 10^{-4} \]
\[ a_1 = 109^\circ 29' 54.018' \]
\[ a_1 + \delta a = 109^\circ 57' 16.959' \]
\[ a_1 = a_1 + \delta a \]
\[ a_1 = a_1 + \delta a_1 \]
\[ a_2 = 265^\circ 41' 25.179' \]
\[ a_2 = 265^\circ 37' 10.713' \]
\[ a_2 = a_2 + \delta a \]

\[ a_2 = a_2 + \delta a_2 \]

Figure 21.
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<th>Computed Length $S_1(\delta f)$</th>
<th>$S_2(\delta f^2)$</th>
<th>$S_1 - S$</th>
<th>$S_2 - S$</th>
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<th>Computed Azimuths</th>
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extend the results to second order in $c$ and subsequently to second order in $f$ since $c$ can be expressed as a series in $f$.

The quantities needed to achieve the approximation are found in or derived from the results of Forsyth's work, pages 86, 97-105. We list them here for reference in the development.

\[
\Phi = \phi + \frac{c}{2} u' \sec \alpha \tan \alpha \left[ 1 + \frac{c}{8} (1 - 6 \tan^2 \alpha) \right] \]

\[
u' = \nu' + c U + c^2 V \]

\[
\phi = \phi' + c \Omega + c^2 \Psi \]

\[
a = a_0 + cA \cot a_0 + c^2 B \]

\[
\cn u = \cos u' \{ 1 - \frac{1}{4} c \sin^2 u' - \frac{c^2}{64} \sin^2 u' \{ 7 + 4 \cos^2 u' \} \}
\]

\[
c = \frac{c^2 \cos^2 \alpha}{(1 - c^2 \sin^2 \alpha)}, \quad c^2 = 2f - f^2, \quad c^4 = 4f^2
\]

\[
c = 2f \cos^2 \alpha + f^2 \cos^2 \alpha (3 - 4 \cos^2 \alpha) \]

\[
\cos \theta = \cn u \cos \alpha
\]

\[
\tan \Phi = \tan u' \csc \alpha \left[ 1 + \frac{1}{4} c + (1/64) c^2 (9 - 2 \sin^2 u' - 4 \tan^2 a_0) \right]
\]

\[
\frac{s}{a} = (1 - e^2 \sin^2 \alpha)^{1/2} \csc u
\]

\[
= u' + \frac{c^2}{4} [\sin 2u' - (1 + 2 \tan^2 \alpha) u']
\]

\[
+ \frac{c^2}{64} \{ \sin 4u' + 4 \sin 2u' (1 - 2 \tan^2 \alpha) + \{ 8 \tan^2 \alpha (1 + 3 \tan^2 \alpha) - 3 \} u' \}
\]

\[
\sin \alpha = \sin a_0 \left[ 1 + c A \cot^2 a_0 + c^2 \cot a_0 \left( B - \frac{1}{2} A^2 \cot a_0 \right) \right]
\]

\[
\cos \alpha = \cos a_0 \left[ 1 - c A - c^2 \tan a_0 \left( B + \frac{1}{2} A^2 \cot^2 a_0 \right) \right]
\]

\[
\tan \alpha = \tan a_0 \left[ 1 + c A \csc^2 a_0 + c^2 \csc^2 a_0 \left( A^2 + B \tan a_0 \right) \right]
\]

\[
\sec \alpha = \sec a_0 \left[ 1 + c A + c^2 \tan a_0 \left( B + A^2 \cot a_0 \{ 1 + \frac{1}{2} \cot^2 a_0 \} \right) \right]
\]

\[
\csc \alpha = \csc a_0 \left[ 1 - c A \cot^2 a_0 - c^2 \cot a_0 \{ B - \frac{1}{2} A^2 \cot a_0 (1 + 2 \cot^2 a_0) \} \right]
\]

\[
\sin u' = \sin \nu' \left[ 1 + c U \cot \nu' + c^2 \left( V \cot \nu' - U^2/2 \right) \right]
\]

\[
\cos u' = \cos \nu' \left[ 1 - c U \tan \nu' - c^2 \left( V \tan \nu' + U^2/2 \right) \right]
\]

\[
\tan u' = \tan \nu' + c U \sec^2 \nu' + c^2 \sec^2 \nu' \left( V + U^2 \tan \nu' \right)
\]

\[
\sin 2u' = \sin 2\nu' \left( 1 + 2c U \cot 2\nu' \right) \text{ (to first order in } c)\]

\[
\tan \phi' = \tan \nu' \csc a_\nu, 1 + \tan^2 \nu' \csc^2 a_\nu = \sec^2 \phi'
\]

\[
U = -(A \cot \nu' + (1/8) \sin 2\nu'), \quad A = - \left( \nu'/2 \right) \tan \frac{5}{6} \tan \nu'
\]

\[
\Omega + (\nu'/2) \sin a_\nu \sec^2 a_\nu = - A \csc^2 a_\nu \cot \phi'
\]

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In these formulas, \( \alpha_0 \) is the complement of the parametric latitude of the vertex of the great elliptic arc. To see this, find on page 119 of Forsyth, the expression

\[
\sin \alpha = \frac{(\tan \phi_0)/(p \sec^2 \phi_0 - 1)}{(p' \sec^2 \phi_0 + 1)^{1/2}},
\]

where

\[ p = \sin^2 \frac{1}{2} (\theta_1 + \theta_2)/\sin \theta_1 \sin \theta_2 \]

\[ p' = \cos^2 \frac{1}{2} (\theta_1 + \theta_2)/\sin \theta_1 \sin \theta_2 \]

Now replace Forsyth's \( \theta_1 \) and \( \theta_2 \) by \( 90 - \theta_1, 90 - \theta_2 \) respectively and his \( \phi_0 \) by \( \Delta \lambda/2 \).

Then find:

\[ \tan \phi_0 = \tan (\Delta \lambda/2) = (1 - \cos \Delta \lambda)/\sin \Delta \lambda \]

\[ p \sec^2 \phi_0 - 1 = [(1 - \cos \Delta \lambda)/\sin^2 \Delta \lambda] (1 + \sec \theta_1 \sec \theta_2 - \tan \theta_1 \tan \theta_2) - 1 \]

\[ p' \sec^2 \phi_0 + 1 = [(1 - \cos \Delta \lambda)/\sin^2 \Delta \lambda] (-1 + \sec \theta_1 \sec \theta_2 + \tan \theta_1 \tan \theta_2) + 1 \]

The values from (113) placed in (112) give

\[ \sin \alpha_0 = \sin \Delta \lambda/\tan^2 \theta_1 + \tan^2 \theta_2 - 2 \tan \theta_1 \tan \theta_2 \cos \Delta \lambda + \sin^2 \Delta \lambda)^{1/2} \]

Now the right member of (114) is \( \cos \theta_0 \) where \( \theta_0 \) is the parametric latitude of the vertex of the great elliptic arc [17]. (See also GEODESICS AND PLANE ARCS ON AN OBLATE SPHEROID, L. E. Ward, American Mathematical Monthly, Aug.–Sept., 1943 page 427).

From 111a, 111b, 111c, 111m, 111n we have, retaining terms to \( c^2 \) inclusive:

\[ \Phi = \phi + c (\Omega + \frac{\nu'}{2} \sec a_o \tan a_o) \]

\[ + \frac{c^2}{2} [\Psi + \frac{1}{2} \sec a_o \tan a_o \{ U + \nu' (1 + \csc^2 a_o) + \frac{1}{8} \nu'(1 - 6 \tan^2 a_o) \}] \]

If \( R, S \) are the coefficients respectively of \( c \) and \( c^2 \) in (115), then

\[ \tan \Phi = \tan \phi' + c \sec^2 \phi' R + c^2 \sec^2 \phi' (S + R^2 \tan \phi') \]

With the values of \( R \) and \( S \) from (115) and the values of \( \Omega + \nu'/2 \sec a_o \tan a_o \) and \( U \) from 111t, cot \( \phi' \) from 111s, we can write (116) as

\[ \tan \Phi = \tan \phi' - c A \cot \nu' \csc a_o \sec^2 \phi' \]

\[ + \frac{c^2 \sec^2 \phi'}{2} \left[ \Psi + A^2 \cot \nu' \csc^3 a_o \right. \]

\[ + \frac{1}{2} \sin a_o \sec^2 a_o \left[ A [\nu' (1 + \csc^2 a_o) - \cot \nu'] \right. \]

\[ - \frac{1}{8} \sin 2\nu' + \frac{1}{8} (1 - 6 \tan^2 a_o) \right] \]
From \(111h, 111o, 111r\) we write a second formula for \(\tan \Phi\):

\[
\tan \Phi = \tan \nu' \csc \alpha_0 - cA \left( \csc^2 \nu' + \cot^2 \alpha_0 \right) \tan \nu' \csc \alpha_0 \\
+ c^2 \tan \nu' \csc \alpha_0 \left[ V \sec \nu' \csc \nu' - B \cot \alpha_0 + (9/64) + (1/32) \sin^2 \nu' \right] \\
+ \frac{A}{4} \left( 2 - \csc^2 \nu' \right) - (1/16) \sec^2 \alpha_0 \\
+ A^2 \left( \csc^2 \nu' \csc^2 \alpha_0 + \cot^4 \alpha_0 + \frac{1}{2} \cot^2 \nu' \right) \tag{118}
\]

From \(111g, 111e, 111k, 111p, 111q, 111t\) we can write:

\[
\cos \theta = \cos \alpha_0 \cos \nu' + c \cdot 0 \\
+ c^2 \cos \alpha_0 \cos \nu' \left( \frac{A}{4} \cos 2 \nu' - V \tan \nu' - \left( 5/64 \right) \sin^2 \nu' - \left( 3/32 \right) \sin^4 \nu' \right) \\
- B \tan \alpha_0 - A^2 \left( 1 + \frac{1}{2} \cot^2 \alpha_0 + \frac{1}{2} \cot^2 \nu' \right) \tag{119}
\]

Now in (119), the coefficient of \(c\) was zero as it should be and the coefficient of \(c^2\) must be zero since \(\cos \theta = \cos \alpha_0 \cos \nu'\). Placing the coefficient of \(c^2\) in (119) equal to zero find:

\[
- B \cot \alpha_0 = A^2 \left( 1 + \frac{1}{2} \cot^2 \alpha_0 + \frac{1}{2} \cot^2 \nu' \right) \cot^2 \alpha_0 - \frac{A}{4} \cos 2 \nu' \cot^2 \alpha_0 \\
\]

\[
+ V \tan \nu' \cot^2 \alpha_0 + \left( 5/64 \right) \sin^2 \nu' \cot^2 \alpha_0 + \left( 3/32 \right) \sin^4 \nu' \cot^2 \alpha_0 \tag{120}
\]

With the value of \(- B \cot \alpha_0\) from (120) placed in the second order term of (118) and with some manipulation through the identities \(111s\), we can write (118) as:

\[
\tan \Phi = \tan \nu' \csc \alpha_0 - cA \cot \nu' \csc \alpha_0 \sec^2 \Phi' \\
+ c^2 \csc \alpha_0 \sec^2 \Phi' \left[ A^2 \cot \nu' \left( 1 + \left( 3/2 \right) \cot^2 \alpha_0 \right) + V \right] \\
+ \frac{A}{4} \left( \sin 2 \nu' - \cot \nu' \right) + \left( 1/16 \right) \sin 2 \nu' \\
- \left( 3/256 \right) \sin 4 \nu' - \left( 1/32 \right) \sin 2 \nu' \tan^2 \alpha_0 \tag{121}
\]

From (117) and (121), since \(\tan \Phi' = \tan \nu' \csc \alpha_0\) from \(111s\), the coefficients of the terms in \(c\) and \(c^2\) must be respectively equal. Equating the second order terms in (117) and (121) and solving for \(V\) we find:

\[
V = \Psi \sin \alpha_0 - \frac{1}{2} A^2 \cot \nu' \cot \alpha_0 \\
+ \frac{A}{4} \left[ \left( 2 \nu' \tan^2 \alpha_0 \left( 1 + \csc^2 \alpha_0 \right) - \sin 2 \nu' + \cot \nu' \left( 1 - 2 \tan^2 \alpha_0 \right) \right) \right] \\
+ \frac{\nu'}{16} \tan^2 \alpha_0 \left( 1 - \tan^2 \alpha_0 \right) - \frac{\sin 2 \nu'}{16} + \frac{3 \sin 4 \nu'}{256} - \tan^2 \alpha_0 \sin 2 \nu' \tag{122}
\]

From \(111i, 111b, 111m, 111p, 111q\), the value of \(U\) in terms of \(A\) from \(111t\), and \(V\) from (122) we may write:
\[
\frac{S}{a} = \nu' + c \left[ \left( \frac{1}{8} \sin 2\nu' - A \cot \nu' - \frac{\nu'}{4} \left( 1 + 2 \tan^2 \alpha_a \right) \right] \\
+ c^2 \left[ \Psi \sin \alpha_a - \frac{1}{2} A^2 \cot^2 \alpha_a \cot \nu' + \frac{A}{4} \left( \sin 2\nu' - 2\nu' \right) \\
+ \left( \frac{1}{256} \right) \left[ 8 \sin 2\nu' \left( 1 - 3 \tan^2 \alpha_a \right) - \sin 4\nu' \right] + \left( \frac{3}{64} \right) \nu' \left( 4 \tan^2 \alpha_a - 1 \right) \right]
\]

Referring (123) to the end points of the geodesic arc we have:

\[
\frac{S}{a} = (\nu' - \nu'_1) + c \left[ \left( \frac{1}{8} \sin 2\nu' - 2\nu' \right) - A \left( \cot \nu' - \cot \nu'_1 \right) - \frac{1}{2}(\nu' - \nu'_1) \left( 1 + 2 \tan^2 \alpha_a \right) \right] \\
+ c^2 \left[ -\frac{1}{2} A^2 \cot^2 \alpha_a \left( \cot \nu' - \cot \nu'_1 \right) + \frac{A}{4} \left[ \left( \sin 2\nu' - 2\nu' \right) - 2(\nu' - \nu'_1) \right] \\
+ \left( \frac{1}{256} \right) \left[ 8 \left( 1 - 3 \tan^2 \alpha_a \right) \left( \sin 2\nu' - \sin 2\nu'_1 \right) - \left( \sin 4\nu' - \sin 4\nu'_1 \right) \right] \\
+ \left( \frac{3}{64} \right) (\nu' - \nu'_1) \left( 4 \tan^2 \alpha_a - 1 \right) \right]
\]

Note that the term \( \Psi \sin \alpha_a \) vanishes in (124).

From (125) we have from the expression for \( A \) that:

\[
- A \left( \cot \nu' - \cot \nu'_1 \right) = \frac{\tan^2 \alpha_a}{2} (\nu' - \nu'_1),
\]

\[
A = \frac{1}{2} (\nu' - \nu'_1) \tan^2 \alpha_a \left[ \cot (\nu' - \nu'_1) - \csc (\nu' - \nu'_1) \cos (\nu'_1 + \nu'_2) \right]
\]

We list also for reference the identities:

\[
\sin 2\nu' - \sin 2\nu'_1 = 2 \sin (\nu' - \nu'_1) \cos (\nu'_1 + \nu'_2),
\]

\[
\sin 4\nu' - \sin 4\nu'_1 = 2 \sin (2\nu' - \nu'_1) \left[ 2 \cos^2 (\nu'_1 + \nu'_2) - 1 \right]
\]

Applying (125) and (126) to (124) we obtain:

\[
\frac{S}{a} = (\nu' - \nu'_1) - \frac{(c/4)}{2} \left[ (\nu' - \nu'_1) - \sin (\nu' - \nu'_1) \cos (\nu'_1 + \nu'_2) \right]
\]

\[
+ c^2 \left[ \frac{A}{2} \sin (\nu' - \nu'_1) \cos (\nu'_1 + \nu'_2) - \frac{A}{4} (\nu' - \nu'_1) + \left( \frac{3}{64} \right) (\nu' - \nu'_1) \left( 4 \tan^2 \alpha_a - 1 \right) \right]
\]

Note that the first two terms of (127) are equivalent to Forsyth's equation, page 120 of his treatise.

Now for the value of \( c \), we find on page 97 of Forsyth, that for approximations involving

\( f^2 \) (second order in the flattening) a value of \( \alpha_a \) that is accurate up to \( f \) inclusive must be

substituted in the first term of \( c \). Hence from 111d, 111f, 111k we have

\[
c = 2f \cos^2 \alpha_a + 3f^2 \cos^2 \alpha_a - 4f^2 \cos^4 \alpha_a (1 + 2A).
\]

This value of \( c \) placed in (127) with the value of \( A \) from (125) gives:
\[ S = a (\nu_1' - \nu_2') - (f/2) \cos^2 a_0 [(\nu_2' - \nu_1') \sin (\nu_1' + \nu_2')] \] (129)

\[ + f^2 \begin{bmatrix}
\frac{1}{4}(\nu_2' - \nu_1')^2 \cot (\nu_2' - \nu_1') \cos^2 a_0 - \frac{1}{4}(\nu_2' - \nu_1')^2 \cot (\nu_2' - \nu_1') \cos^2 a_0 \\
- \frac{1}{4}(\nu_2' - \nu_1')^2 \csc (\nu_2' - \nu_1') \cos (\nu_1' + \nu_2') \\
+ \frac{1}{4}(\nu_2' - \nu_1')^2 \csc (\nu_2' - \nu_1') \cos^2 a_0 \cos (\nu_1' + \nu_2') \\
- (1/16) \sin 2 (\nu_2' - \nu_1') \cos a_0 \cos^2 (\nu_1' + \nu_2') \\
+ (1/16) (\nu_2' - \nu_1') \cos a_0 + (1/32) \sin 2(\nu_2' - \nu_1') \cos^2 a_0
\end{bmatrix}
\]

Now in (129) let \( a_0 = 90^\circ - \theta_o \), \( d_1 = \nu_1' \), \( d_2 = \nu_2' \), \( d = d_2 - d_1 = \nu_2' - \nu_1' \) and the equation becomes:

\[ \frac{S}{a} = d - (f/2) \left[ d \sin^2 \theta_o - \sin d \sin^2 \theta_0 \cos (d_1 + d_2) \right] \] (130)

Since \( \theta_0 \) is the parametric latitude of the vertex of the Great elliptic arc, we have (or may place)

\[ X = \frac{(\sin \theta_1 + \sin \theta_2)^2}{1 + \cos d} + \frac{(\sin \theta_1 - \sin \theta_2)^2}{1 - \cos d} = 2 \sin^2 \theta_o, \] (131)

\[ Y = \frac{(\sin \theta_1 + \sin \theta_2)^2}{1 + \cos d} - \frac{(\sin \theta_1 - \sin \theta_2)^2}{1 - \cos d} = 2 \sin^2 \theta_o \cos (d_1 + d_2) \]

From (131) \( \sin^2 \theta_o = X/2, \sin^2 \theta_o \cos (d_1 + d_2) = Y/2 \), and we can write (130) in the form:

\[ \frac{S}{a} = d - (f/4) (Xd - Y \sin d) \]

\[ + \left( f^2 /128 \right) \begin{bmatrix}
(16d^2 \cot d) X - (16d^2 \csc d) Y \\
+ (2d + \sin 2d - 8d^2 \cot d) X^2 \\
+ (8d^2 \csc d) XY - (2 \sin 2d) Y^2
\end{bmatrix} \] (132)

If we factor \( \sin d \) out of every term of (132), we can write:

\[ S = a \sin d \left[ T - (f/4)(TX - Y) + (f^2/64)(A_o X + B_o Y + C_o X^2 + D_o XY + E_o Y^2) \right] \]

\[ T = d/\sin d, \quad E_o = -2 \cos d, \quad A_o = - D_o E_o, \quad C_o = T - \frac{1}{2} (A_o + E_o), \] (133)

\[ D_o = 4T^2, \quad B_o = -2 D_o, \quad d \]

d is the spherical distance between the points \( P_1(\theta_1, \lambda_1) \) and \( P_2(\theta_2, \lambda_2) \)
given by some spherical formula as

\[ \cos d = \sin \theta_1 \sin \theta_2 + \cos \theta_1 \cos \theta_2 \cos \Delta \lambda, \Delta \lambda = \lambda_2 - \lambda_1. \]
COMPARISON WITH AN EXISTING EXPANSION

On page 8, GIMRADA Research Note No. 11, E. M. Sodano, April 1963 [23] one finds the following formula:

\[
\frac{S}{b_o} = (1 + f + f^2) \phi + a [(f + f^2) \sin \phi - (f^2/2) \phi^2 \csc \phi] \\
+ m \left( -\frac{f + f^2}{2} \phi - \frac{f + f^2}{2} \sin \phi \cos \phi + \frac{f^2}{2} \phi^2 \cot \phi \right) \\
+ m^2 \left( \frac{f^2}{16} \phi + \frac{f^2}{16} \sin \phi \cos \phi - \frac{f^2}{2} \phi^2 \cot \phi - \frac{f^2}{8} \sin \phi \cos^2 \phi \right) \\
+ m \left( \frac{f^2}{2} \phi^2 \csc \phi + \frac{f^2}{2} \sin \phi \cos^2 \phi \right) - a^2 (f^2/2) \sin \phi \cos \phi
\] (134)

Now the correspondence between the parameters as used in (133) and those of Sodano are:

\[
m(\text{Sodano}) = X/2, \quad a(\text{Sodano}) = \frac{1}{2} (Y + X \cos d), \quad \phi(\text{Sodano}) = d, \quad b_o(\text{Sodano}) = a(1 - f) \] (135)

(a is equatorial radius, f the flattening).

If we substitute from (135) into (134), retaining terms to \( f^2 \) inclusive, we can write (134) as

\[
\frac{S}{a} = d - (f/4) (Xd - Y \sin d) \\
+ (f^2/128) \left[ (16d^2 \cot d) X - (16d^2 \csc d) Y \\
+ (2d + \sin 2d - 8d^2 \cot d) X^2 \\
+ (8d^2 \csc d) XY - (2 \sin 2d) Y^2 \right]
\] (136)

Now comparing (132) and (136) find that the equations are identical which gives an independent check of Sodano’s inverse formula.

COMPUTING FORM IN TERMS OF PARAMETRIC LATITUDE

Given on the reference ellipsoid the points \( P_1(\theta_1, \lambda_1), P_2(\theta_2, \lambda_2); P_2 \) nearest of \( P_1 \), west longitudes considered positive. (Geodetic latitudes are converted to parametric by \( \tan \theta = (1-f) \tan \phi \) or an equivalent formula). Formulas (133) may be used as follows:

With \( \theta_m = \frac{1}{2}(\theta_1 + \theta_2), \Delta \theta_m = \frac{1}{2}(\theta_2 - \theta_1), \Delta \lambda = \lambda_2 - \lambda_1, \Delta \lambda_m \Rightarrow \frac{\Delta \lambda}{2} \)

let \( k = \sin \theta_m \cos \Delta \theta_m, K = \sin \Delta \theta_m \cos \theta_m, \)

\( H = \cos^2 \Delta \theta_m - \sin^2 \theta_m = \cos^2 \theta_m - \sin^2 \Delta \theta_m, \)

\( L = \sin^2 \Delta \theta_m + H \sin^2 \Delta \lambda_m = \sin^2 d/2, 1 - L = \cos^2 d/2, \)
\[
\cos d = 1 - 2L, \quad h = \sin^2 d = 4L(1 - L), \quad U = 2k^2/(1 - L),
\]
\[
V = 2k^2/L, \quad X = U + V, \quad Y = U - V
\]
\[
T = (d/\sin d) = 1 + (1/6)h + (3/40)h^2 + (5/112)h^3 + (35/2816)h^4 + \ldots
\]
\[
E_o = -2 \cos d, \quad A_o = -D_o F_o = -4E_oT^2, \quad D_o = 4T^2, \quad B_o = -2D_o, \quad C_o = T - \frac{1}{2}(A_o + E_o) \quad (137)
\]
\[
S = a \sin d \left[ T - \left(\frac{f}{4}\right)(TX - Y) + \left(\frac{f^2}{64}\right)(A_oX + B_oY + C_oX^2 + D_oXY + E_oY^2)\right] \]
\[
\sin (a_2 + a_1) = (K \sin \Delta \lambda)/L, \quad \sin (a_2 - a_1) = (k \sin \Delta \lambda)/(1 - L)
\]
\[
\frac{1}{2}(\delta a_2 + \delta a_1) = -\left(\frac{f}{2}\right) \sin (a_1 + a_2)
\]
\[
\frac{1}{2}(\delta a_2 - \delta a_1) = -\left(\frac{f}{2}\right) \sin (a_2 - a_1)
\]
\[
a_{2+1} = a_1 + \delta a_1, \quad a_{2-1} = a_2 + \delta a_2.
\]

The azimuth formulas of (137) are obtained by manipulation of expressions given on pages 126-128 of THE DISTANCE BETWEEN TWO WIDELY SEPARATED POINTS ON THE SURFACE OF THE EARTH, W. D. Lambert, Journal of the Washington Academy of Sciences, Vol. 32, No. 5, May 15, 1942, [13]. Note that in the distance and azimuth formulas of (137), the same quantities H, T, L, k, K are used.

Figure 22 in an example of the arrangements of equations (137) and computations for comparison with those of Figure 21, page 80. The results are:

<table>
<thead>
<tr>
<th>True distance</th>
<th>Geodetic Latitude</th>
<th>Parametric Latitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>meters</td>
<td>(\delta f)</td>
<td>(\delta f^2)</td>
</tr>
<tr>
<td>8,466,621.01</td>
<td>618.26</td>
<td>621.11</td>
</tr>
<tr>
<td>True Azimuths</td>
<td></td>
<td></td>
</tr>
<tr>
<td>109° 57' 17&quot;41</td>
<td>16° 86</td>
<td>16° 68</td>
</tr>
<tr>
<td>265° 37' 10&quot;59</td>
<td>10° 71</td>
<td>11° 37</td>
</tr>
</tbody>
</table>

As was to be expected both approximations are adequate within any operational requirements. The coefficients \(A_o, B_o, C_o, D_o, E_o\) of the parametric latitude form, Figure 22, are slightly less complicated than those of the geodetic form, Figure 21. But no conversion to parametric latitudes needs to be made for Figure 21. For purely geodetic computations further investigation needs to be made and it is suggested that computations be made using both forms against the computed lines contained in the revised issues of ACIC Reports 59 and 80, Sept. 1960 and December 1959.[12]
DISTANCE COMPUTING FORM, FORSYTH-ANOYER-LAMBERT
TYPE APPROXIMATION WITH SECOND ORDER TERMS

$$\tan \theta = 0.996609925 \tan \phi$$

Clarke Spheroid 1866, \(a = 6,378,206.4\) meters

\(f/2 = 0.00169503765, f/4 = 0.000847518825, f^2/64 = 0.1795720390 \times 10^{-6}\)

1 radian = 206,264.8062 seconds

| \(\phi_1\)  | 8 58 25.0 |
| \(\phi_2\)  | 21 26 06.0 |
| \(\theta_m\) = \(\frac{1}{2}(\theta_1 + \theta_2)\) | 15° 09' 22.644 |
| \(\Delta \theta_m\) = \(\frac{1}{2}(\theta_2 - \theta_1)\) | 6 12 45.386 |
| \(\sin \theta_m + 0.26145290\) | \(\sin \Delta \theta_m + 0.10821810\) |
| \(\cos \theta_m + 0.96521623\) | \(\cos \Delta \theta_m + 0.99412718\) |
| \(k = \sin \theta_m \cos \Delta \theta_m + 0.25991743\) | \(K = \sin \Delta \theta_m \cos \theta_m + 0.10445387\) |

\(H = \cos^2 \Delta \theta_m - \sin^2 \theta_m = \cos^2 \theta_m - \sin^2 \Delta \theta_m + 0.91993122\)

\(L = \sin^2 \Delta \theta_m + H \sin^2 \Delta \lambda_m + 0.37960074\)

\(d + 1.3276078324\) \(sin d + 0.97057512\) \(T = d/sin d + 1.367856856\)

\(U = 2k^2/(1 - L) + 0.2177857865\) \(V = 2K^2/L + 0.0574846667\) \(E = -2 \cos d - 0.48159704\)

\(X = U + V + 0.2752704532\) \(Y = U - V + 0.1603011198\) \(D = 4T^2 + 7.484129512\)

\(A = -DE = -4ET^2 + 3.604334620\) \(C = T - \frac{1}{2}(A + E) - 0.19351193\) \(B = -2D - 14.968259024\)

\(X(A + CX) + 0.977503686\) \(Y(B + EY) - 2.411804017\) \(DXY + 0.330245911\)

\(T + \delta f + 1.367673597\) \(\delta f = - (f/4) (TX - Y) - 1.83259 \times 10^{-4}\)

\(\Sigma = X(A + CX) + Y(B + EY) + DXY - 1.10405442\) \(\delta f^2 = + (f^2/64) \Sigma - 1.9826 \times 10^{-7}\)

\(T + \delta f + 1.367673399\) \(S_1 = a \sin d (T + \delta f) 8,466,622.30\) meters

\(\sin (\alpha_2 + \alpha_1) = (K \sin \Delta \lambda)/L + 0.26959808\)

| \(\sin (\alpha_2 - \alpha_1) = (k \sin \Delta \lambda)/(1 - L) + 0.41047190\) |

\(\frac{1}{2}(\delta \alpha_1 + \delta \alpha_2) = - (f/2) H T \sin (\alpha_2 + \alpha_1) - 5.75032185 \times 10^{-4}\)

\(\frac{1}{2}(\delta \alpha_2 - \delta \alpha_1) = - (f/2) HT \sin (\alpha_2 - \alpha_1) - 8.75500321 \times 10^{-4}\)

\(\alpha_1 = 109 56 14.701\)

\(\delta \alpha_1 = + 1 01.977\)

\(\alpha_{1-2} = 109 57 16.678\)

\(\alpha_{1-2} = \alpha_1 + \delta \alpha_1\)

\(\alpha_2 = 265 42 10.565\)

\(\delta \alpha_2 = - 4 59.195\)

\(\alpha_{2-1} = 265 37 11.370\)

\(\alpha_{2-1} = \alpha_2 + \delta \alpha_2\)

Figure 22
TRANSFORMATION FROM SECOND ORDER FORM IN GEODETIC LATITUDE
TO SECOND ORDER IN PARAMETRIC

In terms of geodetic latitude, the equations corresponding to (132) are:

$$\frac{s}{a} = d' - \frac{(f/4)}{(X'd' - 3Y'sin\ d')}
+ \frac{(f^2/128)}{(AX' + BY' + CX'Z + DX'Y' + EY'Z)}$$

$$A = 64d' + 16d'^2 cot\ d',\ B = -96 sin\ d' - 16d'^2 csc\ d',
C = -30d' - 15 sin\ 2d' - 8d'^2 cot\ d',
D = 48 sin\ d' + 8d'^2 csc\ d',\ E = 30 sin\ 2d'$$

(See Equation (109), page 78.

Equation (132) is written here in the form:

$$\frac{s}{a} = d - \frac{(f/4)}{(Xd - Y sin\ d)}
+ \frac{(f^2/128)}{(A_0X + B_0Y + C_0X^2 + D_0XY + E_0Y^2)}$$

$$A_0 = 16d^2 cot\ d,\ B_0 = -16d^2 csc\ d,\ C_0 = 2d + sin\ 2d - 8d^2 cot\ d,
D_0 = 8d^2 csc\ d,\ E_0 = -2 sin\ 2d$$

Now we should be able to find transformation equations of the form:

$$d' = d'(d, X, Y),\ X' = X'(X, Y, d),\ Y' = Y'(Y, X, d)$$

which when substituted in (138) should produce equations (139).

The definitions of $X'$, $Y'$ and $X, Y$ are:

$$X' = 2 \sin^2 \phi_0,\ X = 2 \sin^2 \theta_0$$

$$Y' = 2 \sin^2 \phi_0 \cos(d'_1 + d'_2),\ Y = 2 \sin^2 \theta_0 \cos(d_1 + d_2)$$

where $\phi_0, \theta_0$ are respectively geodetic, parametric latitude of the vertex of the great elliptic arc. From the equation $\tan \theta = (1 - f) \tan \phi_0$ or equivalent, we find:

$$\phi_0 = \theta_0 + f \sin \theta_0 \cos \theta_0 (1 + f \cos^2 \theta_0).$$

(142)

From the values indicated by Forsyth on page 120, of his treatise, to first order in $f$, extending the results to second order in $f$ we find:

$$d' = d - \frac{(f/2)}{Y sin\ d + (f^2/16)} [4Y (X - 3) sin\ d + (2Y^2 - X^2) sin\ 2d]$$

(143)

and to first order in $f$,

$$\cos (d'_1 + d'_2) = \cos (d_1 + d_2) + f \cos d \sin^2 \theta_o - f \cos d \sin^2 \theta_0 \cos^2 (d_1 + d_2).$$

(144)

From (142), to first order in $f$, find

$$2 \sin^2 \phi_0 = 2 \sin^2 \theta_0 (1 + 2f \cos^2 \theta_0).$$

(145)
From (143), to first order in \( f \), find
\[
\sin d' = \sin d - \left( \frac{f}{4} \right) Y \sin 2d
\]  
(146)

From (141), (144), and (145) find
\[
\begin{align*}
X' &= X + 2fX - fX^2 \\
Y' &= Y + 2fY - fXY + \left( \frac{f}{2} \right) (X^2 - Y^2) \cos d
\end{align*}
\]  
(147)

Hence the transformations (140) are from (143), (146), and (147) the following:
\[
\begin{cases}
  d' = d - \left( \frac{f}{2} \right) Y \sin d + \left( \frac{2}{16} \right) \left[ 4Y(X - 3) \sin d + (2Y^2 - X^2) \sin 2d \right] \\
  \sin d' = \sin d - \left( \frac{f}{4} \right) Y \sin 2d \\
  X' = X + 2fX - fX^2 \\
  Y' = Y + 2fY - fXY + \left( \frac{f}{2} \right) (X^2 - Y^2) \cos d
\end{cases}
\]  
(148)

Substitution of the relations (148) into (138) produces equations (139), providing a second check of Sodano’s formula for the inverse solution.

The inverse of the transformations (148) which will carry (139) into (138) are:
\[
\begin{cases}
  d = d' + (f/2) Y \sin d' + (f/16) \left[ 4Y(X' - 1) \sin d' + (2Y^2 - X^2) \sin 2d' \right] \\
  \sin d = \sin d' + (f/4) Y' \sin 2d' \\
  X = X' - 2fX' + fX^2 \\
  Y = Y' - 2fY' + fX'Y' + \left( \frac{f}{2} \right) (X' - Y') \cos d'
\end{cases}
\]  
(149)

DIFFERENCE FORMULAE FOR THE TWO SECOND ORDER FORMS

From equation (142) to second order in \( f \), find
\[
2 \sin^2 \phi_0 = 2 \sin^2 \theta_0 \left[ 1 + 2f - 2f \sin^2 \theta_0 + 3f^2 - 7f \sin^2 \theta_0 + 4f^2 \sin^4 \theta_0 \right],
\]  
(150)

and extending (144) to second order in \( f \)
\[
\cos (d^2 + d') = \cos (d_1 + d_2) + f \sin^2 \theta_0 \cos d \sin^2 (d_1 + d_2)
\]  
(151)

\[\begin{align*}
- \left( \frac{f^2}{2} \right) \sin^2 \theta_0 \sin^2 (d_1 + d_2) &+ \frac{1}{2} \sin^2 \theta_0 \cos (d_1 + d_2) \\
&+ \sin^2 \theta_0 \cos d - (3/2) \cos d \\
&+ (3/2) \sin^2 \theta_0 \cos 2d \cos (d_1 + d_2)
\end{align*}\]

From equations (148), by factoring \( \sin d \) out of every term of the expression for \( d' \), we can write:
\[
d' = \sin d \left[ T - (f/2) Y + (f/8) \left[ 2Y(X - 3) + (2Y^2 - X^2) \cos d \right] \right]
\]  
(152)

Since we can write \( X' = 2 \sin^2 \phi_0 \), \( X = 2 \sin^2 \theta_0 \), \( Y' = 2 \sin^2 \phi_0 \cos (d'_1 + d'_2) \), \( Y = 2 \sin^2 \theta_0 \cos (d_1 + d_2) \) we have from (150) and then combining (150) and (151) (multiplying respective members together)
\[ X' = X \left[ 1 + f \frac{(2 - X)(3 - 2X)}{(2 + X^2)} \right] \quad (153) \]

\[ Y' = Y \left[ 1 + f \frac{(2 - X)(X^2 - Y^2)}{(2 + X^2)} \cos d \right] + \left( \frac{f}{2} /8 \right) \left[ 4Y (2 - X)(3 - 2X) \right. \]
\[ \left. + (X^2 - Y^2) \{(11 - 5X) \cos d + Y (1 - 3 \cos^2 d)\} \right] \quad (154) \]

From Figure 22 we have
\[ X = 0.2752704532, \quad Y = 0.1603011198, \]
\[ \sin d = 0.97057512, \quad \cos d = 0.24079852, \quad (155) \]
\[ T = 1.367856856, \quad f = 0.0033900753, \]
\[ f/2 = 0.00169503765, \quad f^2/8 = 1.436576317 \times 10^{-6} \]

Using the values from (155) to compute \( d', X', Y' \) from (152), (153), (154) find:
\[ d' = (0.97057512) (1.367856856 - 2.717164 \times 10^{-4} - 1.2634 \times 10^{-6}) \]
\[ = (0.97057512) (1.3678583876) = 1.327342885; \quad (156) \]
\[ X' = (0.2752704532) (1.005871239) = 0.27688663; \]
\[ Y' = 0.1603011120 + 9.37275 \times 10^{-4} + 2.0440 \times 10^{-5} + 4.068 \times 10^{-6} = 0.16126290. \]

From Figure 21, \( d' = 1.327342885, X' = 0.27688668, Y' = 0.16126298 \) and comparing with the values from (156), we have a "flat" check for \( d, X, Y \) in the terms of \( d' \). Now the first significant figure of \( f^2 \) is 1 in the 5th decimal place and of \( f \) is 4 in the 8th decimal place. If seven place tables are used, terms in \( f^2 \) are sufficient. If eight figure tables are used, as Peters trigonometric functions, there is some uncertainty in the 8th place of decimals.

Now the corresponding formulas for \( d, X, Y \) in the terms of \( d', X', Y' \) are found similarly to be, to second order terms in \( f \) inclusive;
\[ d = \sin d' \left[ T' + (f/2) Y' + \left( \frac{f^2}{8} \right) \left[ 2 Y' (X' - 1) + (2Y'^2 - X'^2) \cos d' \right] \right] \]
\[ X = X' \left[ 1 + f (X' - 2) \left( 1 + (f/2) (2X' - 1) \right) \right] \quad (157) \]
\[ Y = Y' \left[ 1 - f (2 - X') \right] \left( 1 - 2X' \right) + \left( \frac{f^2}{8} \right) \left[ 4Y'(2 - X') (1 - 2X') \right. \]
\[ \left. + (X'^2 - Y'^2) \{(5 - 3X') (1 - 2X') \cos d' + Y'(1 - 3 \cos^2 d')\} \right] \]

From Figure 21 we have
\[ X' = 0.276886675, \quad Y' = 0.161262981, \quad (158) \]
\[ \sin d' = 0.97051129, \quad \cos d' = 0.24105566 \]
\[ T' = 1.36783822. \]

With the values of \( X', Y', \sin d', \cos d', T' \) from (158) and of \( f, f/2, f^2/8 \) from (155)
we find from (157) that
\[ d = (0.97051129) (1.367673822 + 2.73347 \times 10^{-5} - 3.44 \times 10^{-7}) \]
\[ d = (0.97051129) (1.36794682) = 1.32750747 \]
\[ X = (0.276886675) (0.994162934) = 0.27527047 \]
\[ Y = 0.161262981 - 9.42015 \times 10^{-6} - 2.0700 \times 10^{-5} + 8.68 \times 10^{-7} = 0.16030112. \]

From (155), \( X = 0.27527045, \ Y = 0.16030112, \) and from Figure 22, \( d = 1.327607832. \)

Comparing with (159) we have a difference in \( d \) of 1 in the 9th decimal place; in \( X \) and \( Y \) of 2 and 1 in the 8th decimal place respectively, which is within the computational uncertainties.

From (152), (153), (154), and (157) we can express the differences as functions of either set of variables:
\[ \Delta d = d' - d = - \left( \frac{f}{2} \right) Y \sin d + \left( \frac{f^2}{16} \right) [4Y (X - 3) \sin d + (2Y^2 - X^2) \sin 2d], \]
\[ = - \left( \frac{f}{2} \right) Y \sin d' - \left( \frac{f^2}{16} \right) [4Y' (X' - 1) \sin d' + (2Y'^2 - X'^2) \sin 2d'] \]
\[ \Delta X = X' - X = fX(2 - X) \{1 + \left( \frac{f}{2} \right) (3 - 2X)\}, \]
\[ = fX' (2 - X') \{1 - \left( \frac{f}{2} \right) (1 - 2X') \} \]
\[ \Delta Y = Y' - Y = fY (2 - X) + \left( \frac{f}{2} \right) (X^2 - Y^2) \cos d \]
\[ + \left( \frac{f^2}{8} \right) \left[ 4Y (2 - X) (3 - 2X) \right. \]
\[ + (X^2 - Y^2) \{ (11 - 5X) \cos d + Y (1 - 3 \cos^2 d) \}, \]
\[ = fY' (2 - X') + \left( \frac{f}{2} \right) (X'^2 - Y'^2) \cos d' \]
\[ - \left( \frac{f^2}{8} \right) \left[ 4Y' (2 - X') (1 - 2X') \right. \]
\[ + (X'^2 - Y'^2) \{ 2 (5 - 3X') \cos d' + Y' (1 - 3 \cos^2 d') \}. \]

SUMMARY OF DISTANCE COMPUTATIONS INVESTIGATION

As long as accuracy requirements remain within the range of the capabilities of the Andoyer-Lambert formulae, as exhibited in TABLE 3, they are quite adequate and it makes no difference if geographic latitudes are transformed to parametric latitudes first as far as accuracy requirements are concerned relative to hyperbolic electronic measuring systems. However, the formulae for geodetic azimuths are slightly more complicated in terms of geodetic latitude and some of the auxiliary quantities as chord length, dip of the chord, etc. are slightly less difficult to compute when expressed in terms of parametric latitude.

In order to arrange the computing in conformance with the Andoyer-Lambert formulae, equations (17), (48), (52), (56)), and (64) have been rearranged as follows to be expressible in common computational parameters:
The spherical length, \( d \), is determined from formulae as given with Figure 16,
\[
(d = d_1 - d_2);
\]
\[
\cot A = (\cos \theta_1 \tan \theta_2 - \sin \theta_1 \cos \Delta \lambda)/\sin \Delta \lambda
\]
\[
\cot B = (\cos \theta_1 \tan \theta_2 - \sin \theta_1 \cos \Delta \lambda)/\sin \Delta \lambda
\]
\[
\sin d = \cos \theta_1 \sin \Delta \lambda/\sin B = \cos \theta_2 \sin \Delta \lambda/\sin A;
\]
these will compensate for any unfavorable triangle geometry.

The Andoyer-Lambert Formulae are taken in the form [13]
\[
\delta d_r = -(f/8) (VQ^2/\sin^2 \frac{d}{2} + UR^2/\cos^2 \frac{d}{2})
\]

\[
(1) \quad s = a(d_r + \delta d_r), \quad Q = \sin \theta_1 - \sin \theta_1, \quad R = \sin \theta_1 + \sin \theta_2.
\]
\[
V = d_r + \sin d, \quad U = d_r - \sin d,
\]

With corresponding geodetic azimuths computed from
\[
T = (f/2) \frac{\sin \frac{d}{2}}{\sin d}, \quad \delta \alpha'' = T \cos^2 \theta_2 \sin 2B,
\]
\[
(2) \quad \delta B'' = T \cos^2 \theta_1 \sin 2A; \quad \alpha_{AB} = 180^\circ - A + \delta A; \quad \alpha_{BA} = 180^\circ + B - \delta B
\]

The Normal Section Azimuths may be written
\[
(3) \quad \cot_n \alpha_{AB} = -(\cot A)/T_1 + (e^2 Q \cos \theta_1)/(\sin \Delta \lambda)T_1 \cos \theta_2
\]
\[
\cot_n \alpha_{BA} = (\cot B/T_2 + (e^2 Q \cos \theta_2))/(\sin \Delta \lambda)T_2 \cos \theta_1
\]
\[
T_1 = (1 - e^2 \cos^2 \theta_1)^{1/2} \quad T_2 = (1 - e^2 \cos^2 \theta_2)^{1/2}
\]

The chord length becomes
\[
(4) \quad c = a (4 \sin^2 \frac{d}{2} - e^2 Q^2)^{1/2}
\]

The angle of dip of the chord may be written
\[
(5) \quad \beta = \arcsin \left[2b (\sin^2 \frac{d}{2}/cT_1)\right]
\]
\[
b = \text{semiminor axis of ellipsoid}, \quad c = \text{chord length}, \quad T_1 = (1 - e^2 \cos^2 \theta_1)^{1/2}.
\]

The maximum separation of chord and arc becomes
\[
(6) \quad H = (a^2/c) (1 - \cos \frac{d}{2}) [4 \sin^2 \frac{d}{2} (\cos^2 \frac{d}{2}/c^2 - M) - e^2 Q^2]^{1/2}/\cos \frac{d}{2}
\]
\[
a = \text{the semimajor axis of ellipsoid}, \quad c = \text{chord length}, \quad M = e^2 \sin \theta_1 \sin \theta_2,
\]
\[
Q = \sin \theta_1 - \sin \theta_2, \quad e = \text{eccentricity of the spheroid}.
\]

The geographic coordinates of the point where maximum separation of chord and arc occurs
\[
(7) \quad \tan \lambda = (\cos \theta_2 \sin \Delta \lambda)/(\cos \theta_1 + \cos \theta_2 \cos \Delta \lambda)
\]
\[
\tan \phi = R/(0.99660925) \sqrt{4 \cos^2 \frac{d}{2} - R^2}
\]
\[
where R = \sin \theta_1 + \sin \theta_2.
\]

Figure 23, shows the above formulae arranged in a computing form and the computations done over the line MOSCOW TO CAPE OF GOOD HOPE. See line No. 12, TABLE 1, and Figure 17.
### Computations: Geodetic Distance and Azimuths, Normal Section Azimuths, Chord, Angle of Dip, Maximum Separation, Geographic Coordinates of Point of Maximum Separation

Clarke 1866 Ellipsoid: \( a = 6,378,206.4 \text{ meters}, \ b = 6,356,583.8 \text{ meters}, \ e^2 = 6.7686580 \times 10^{-3} \)

\[ f/2 = 1.69503765 \times 10^{-3}, \ f/8 = 4.237594 \times 10^{-4}, \ \text{1 radian} = 206,264.8062 \text{ seconds} \]

<table>
<thead>
<tr>
<th>( \phi_1 )</th>
<th>( \phi_2 )</th>
<th>1 (A) MOSCOW</th>
<th>2 (B) CAPE OF GOOD HOPE</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tan \phi_1 )</td>
<td>( 1.468 )</td>
<td>945</td>
<td>19,500</td>
</tr>
<tr>
<td>( \tan \phi_2 )</td>
<td>( 0.0472 )</td>
<td>8.4157</td>
<td></td>
</tr>
<tr>
<td>( \tan \phi )</td>
<td>( 0.0472 )</td>
<td>8.4157</td>
<td></td>
</tr>
<tr>
<td>( \cos \phi_1 )</td>
<td>( 0.9534 )</td>
<td>0.0350</td>
<td></td>
</tr>
<tr>
<td>( \cos \phi_2 )</td>
<td>( 0.9999 )</td>
<td>0.0094</td>
<td></td>
</tr>
</tbody>
</table>

\( \cos^2 \theta_1 + \cos^2 \theta_2 + \cos^2 \theta_2 = 1.489 \times 10^{-3} \times 0.9809 A = 164.1^\circ B = 10.1^\circ \)

\( \cot A = (\cos \theta_1 \tan \theta_2 - \sin \theta_1 \cos \Delta \Lambda - \sin \theta_2 \cos \Delta \Lambda) \sin \Delta \Lambda = 1.541,188,566 \times 10^{-3} \)

\( \cot B = (\cos \theta_2 \tan \theta_1 - \sin \theta_1 \cos \Delta \Lambda - \sin \theta_2 \cos \Delta \Lambda) \sin \Delta \Lambda = 5.22 \times 10^{-3} \times 188,566 \times 10^{-3} \)

\( \sin d = \cos \theta_1 \sin \Delta \Lambda \sin \theta_2 = \cos \theta_2 \cos \Delta \Lambda \sin \Lambda \sin \Lambda = 0.99985202 \times 10^{-3} d = 0.9985149 \) \( d \) (radians) = 1.3782000 \times 10^{-3}

\( \sin \theta_1 = 3.121683 \times 10^{-3} \times 1.62 \times 10^{-3} \sin \theta_2 = 3.121683 \times 10^{-3} \times 1.62 \times 10^{-3} \)

\( R = \sin \theta_1 + \sin \theta_2 = 0.26881529 \cos \frac{1}{2} \delta = 2.00 \times 10^{-3} \)

\( \delta d = -(f/8)(VQ^2/\sin^2 \frac{1}{2} d + V2/\cos^2 \frac{1}{2} d) = 4.5 \times 10^{-3} \)

\( T = (f/2) d''/\sin d. \ 5.8 \times 10^{-3} \ \text{Si A''} = T \cos \theta_2 \sin 2B + 153.0 \times 10^{-3} \) meters

\[ (1) \quad S = a(d_1 + \delta d_1) \times 10^{-3}, \text{ Si B} = 0.968 \times 10^{-3} \]

\[ (2) \text{ Geodetic} \left\{ \begin{array}{l}
g^{2AB} = 180 - A + \delta A \times 48 \quad 10.519 \\
g^{2BA} = 180 + B + \delta B \times 32.109 \\
\end{array} \right. \]

\[ (3) \text{ Azimuths} \left\{ \begin{array}{l}
a^{2AB} = \text{arc cot } [-c(\text{cot} A/T_1 + c^2 \text{Q} \cos \theta_2)/(\text{cot} \Delta \Lambda T_1 \cos \theta_2)] \\
a^{2BA} = \text{arc cot } [(\text{cot} B/T_2 + c^2 \text{Q} \cos \theta_2)/(\text{cot} \Delta \Lambda T_2 \cos \theta_2)] \\
\end{array} \right. \]

\[ (4) \text{ Chord: } c = a(\sin^2 \frac{1}{2} d - c^2 Q^2)^{1/2} \times 9.68 \times 10^{-3} \times 422.411 \]

\[ (5) \beta = \text{arc sin } [2b (\sin^2 d/2)/c T_1] \]

\[ (6) \text{ Maximum separation of chord} - \text{arc} H = (a/c) (1 - \cos \theta_2) [4 \sin^2 d/2 (\cos^2 d - 2 - \cos^2 Q^2) + 0.58] \]

\[ (7) \tan \lambda = (\cos \theta_2 \sin \Delta \Lambda)/(\cos \theta_1 + \cos \theta_2 \cos \Delta \Lambda) \]

\[ \tan \phi = R/(0.99660992) \sqrt{4 \cos^2 \frac{1}{2} d - R^2} \]

Andoyer-Lambert Approximation (Parametric latitude)

Figure 23.
Note in Figure 23 that two values of longitude are given, $\lambda$ and $A_\theta$. $\lambda$ is the longitude associated with the point where maximum separation of chord and arc occurs but corresponding to the rectangular coordinate system as defined in say Figure 14. $A_\theta$ is the true geodetic longitude of the same point and is of course obtained by adding $\lambda$ to $\lambda$, since $\lambda$ is negative.

While a continuous system based on either the great elliptic section as depicted by Figure 17, or the Forsyth-Andoyer-Lambert approximation, Figure 23, will provide all the information as indicated and accurate enough for hyperbolic electronic systems and any present operational requirements, the Forsyth-Andoyer-Lambert is probably to be preferred because of computational simplicity and existence of programs already operating for high speed computers. Should the need arise for accuracy of the order of 1 meter in distance and 1 second in azimuth over the ellipsoid, the extension to second order terms in the flattening provided by equations (110) or (137), will suffice.

Many formulae are available for geodetic lines and differential corrections are available for transforming elements such as geodetic azimuths to normal section azimuths, etc. [24]. Usually these are complicated, involve tabular material for a particular spheroid of reference, require extensive root computation, and accuracy depends on line length. For instance, Bomford says Rudoe's formulae for the reverse problem, are "Unnecessarily complex for general use," [21], page 108. Also they give "Normal Section" distances -- not geodetic. The difference between the geodesic and the Normal Section distance is of 4th order in the eccentricity of the meridian ellipse [24].

Finally this investigation has raised the question as to whether either Andoyer or Lambert should share any credit for the first order approximation formula in terms of parametric latitude recognizable intact in Forsyth's 1895 paper. While Forsyth had an erroneous second order term to the same expansion in terms of geodetic latitude, his first order term was correct and he thus had both so-called Andoyer-Lambert formulae. Gougenheim apparently had in 1950 the first correct expansion in print in terms of geodetic latitude which included the second order terms in the flattening.

REFERENCES (Distance Investigation)

[23] Sodano, E.M., General non-iterative solution of the inverse and direct geodetic problems, GIMRADA, Fort Belvoir, April 1963; also published as GIMRADA Research Note 11.
APPENDIX 1

Example of
Computations of Loran Lines
of Position (Plane Approximation)
Intersections of Loran Lines of Position
(Plane Approximation)

P. D. Thomas, Mathematician

Consider the hyperbolic system as shown in Figure 24. The hyperbolic locus with foci \( F, F' \) has for equation

\[
(c^2 - a^2) x^2 - a^2 y^2 = a^2 (c^2 - a^2), \quad (e = \frac{c}{a} > 1)
\]  

As \( a \) varies (\( a < c \)) all the hyperbolas with the fixed foci \( F, F' \) (which are \( 2c \) apart) are generated.

The hyperbolic locus with the fixed foci \( F, F'' \) when referred to the same coordinate system as (1), has for equation

\[
Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \quad (e = d/b > 1).
\]  

where one may first compute \( r = b^2 - d^2, \mu = d \cos \alpha, \nu = d \sin \alpha, S = r - c \mu, \) and then

\[
A = \mu^2 - b^2, \quad B = 2\nu, \quad C = \nu^2 - b^2, \quad D = 2(\eta - c \alpha), \quad E = 2S \nu, \quad F = S^2 - b^2 c^2.
\]

As \( b \) varies (\( b < d \)) all the hyperbolas with the fixed foci \( F, F'' \) (which are \( 2d \) apart) are generated.

The respective pairs of constants \( c, a; d, b \) for each hyperbola are related to the fundamental constants of a Loran line by

\[
c = kB_1/2, \quad a = kV_1/2; \quad d = kB_2/2, \quad b = kV_2/2 \tag{2.1}
\]

where \( \nu_i = t_i; \) \( t_i \) is the time difference, \( \nu_i \) is the difference of light microseconds, \( B_1 \) is the length (measured in light microseconds) of the direct line (baseline) between two Loran stations. \( k \) is the length of a light microsecond in the linear units in which \( x \) and \( y \) are expressed.\(^1\)

Since five distinct points determine a conic uniquely, two conics can have at most four points in common. For the hyperbolas (1) and (2) there will always be four real points of intersection except when \( F', F, F'' \) are collinear (\( a = 0 \)) and then there will be two.

**ALGEBRAIC SOLUTIONS**

I. If equations (1) and (2) are solved simultaneously for \( x \) one obtains the quartic equation

\[
x^4 + Hx^3 + Jx^2 + Mx + N = 0 \tag{3}
\]

where one may first compute \( G = c^2 - a^2, \beta_0 = CG + \alpha a^2, \omega = F - CG, \delta = BEG, \gamma = a^2 B^2 - E^2, L = \beta_0^2 - G B^2 a^2, \) and then \( H = 2a^2 (D \beta_0 - \delta) / L, J = a^2 (a^2 D^2 + 2\beta_0 \omega + G \gamma) / L, \)

\(^1\) Loran; Pierce, McKenzie, Woodward; McGraw Hill, 1948, pages 52, 53, 174.
Figure 24. Two plane hyperbolas with a common focus.
\[ M = 2a^4(Dw + \delta)/L, \quad N = a^4(\omega^2 + GE^2)/L. \] The corresponding values of \( y \) are then given by
\[ y = \pm [G(x^2 - a^2)]^{1/2}/a. \]

Equation (3) may be solved by the standard algebraic method\(^2\) or by any of a number of numerical techniques.\(^3\)

II. Again, if equations (1) and (2) are written in the forms
\[ x^2 - Qy^2 = a^2, \quad x^2 + Uxy + Vy^2 + Wx + Ry + T = 0, \]
where \( Q = a^2/(c^2 - a^2), \quad U = B/A, \quad V = C/A, \quad W = D/A, \quad R = E/A, \quad T = F/A \)
and these forms of the equations solved simultaneously with the line of slope \( m \) through the common focus \( F(c,0) \) whose equation is \( y = m(x - c) \), one obtains the two equations:
\[
\begin{align*}
(Qm^2 - 1)x^2 - 2cQm^2x + (a^2 + c^2Qm^2) &= 0, \\
(1 + Um +Vm^2)x^2 + [W + (R - cU)m - 2cVm^2]x + (c^2Vm^2 - cRm + T) &= 0.
\end{align*}
\]

The resultant of the quadric equations (4) is the condition that they have the same solutions or correspondingly that the parameter \( m \) will be restricted to those values for the points common to the hyperbolas (1) and (2).

The resultant of the quadric equations \( a_0x^2 + a_1x + a_4 = 0, \quad b_0x^2 + b_1x + b_2 = 0 \) is given by
\[
(a_0b_2 - b_0a_2)^2 + (b_1a_2 - a_1b_2)(a_0b_1 - a_1b_0) = 0.
\]
From (4) \( a_0 = Qm^2 - 1, \quad a_1 = -2cQm^2, \quad a_2 = a^2 + c^2Qm^2, \quad b_0 = 1 + Um +Vm^2, \)
\( b_1 = [W + (R - cU)m - 2cVm^2], \quad b_2 = c^2Vm^2 - cRm + T, \) and these values placed in (5) lead to the quartic equation
\[
k_1m^4 + k_2m^3 + k_3m^2 + k_4m + k_5 = 0,
\]
where with \( G = c^2 - a^2, \quad \Omega = (a^2 + c^2) V + O \quad (c^2 - T), \quad \theta_o = R + cU, \quad \phi = c^2 + cW + T, \)
\( \eta = R - cU, \quad \xi = a^3U - cR, \quad \rho = a^2 - T, \quad \rho' = a^2 + T \) one finds:
\[
\begin{align*}
k_1 &= (GV + \phi Q)^2 - a^2\theta_0^2, \\
k_2 &= 2[\xi Q + 2\eta caV + a^2RQ \cdot (W + 2c) + c^2QU(cW + 2T)], \\
k_3 &= \xi^2 - a^2\eta^2 + 2\rho'\Omega + W[4a^2cV + 2cPQ - a^2W], \\
k_4 &= 2(\rho' \xi - a^2W\eta), \\
k_5 &= \rho'\xi^2 - a^2W^2.
\end{align*}
\]
Again the solutions of (6) may be found by well known algebraic or numerical methods. The values of \( m \) obtained are of course the slopes of the lines through \( F(c,0) \) and the points of intersection of the hyperbolas (1) and (2).

\(^2\)College Algebra, H. B. Fine, Page 486.


\(^4\)College Algebra, H. B. Fine, Page 512.
POLAR SOLUTION

The following procedure involves tables of the trigonometric functions but no root extraction. First express the equations of (1) and (2) in polar form both referred to the common focus F(c,0), and the corresponding rectangular coordinates in terms of the polar parameters. Find for equation (1)

\[ r_a = \frac{c^2 - a^2}{\pm a - c \cos \theta} \quad (c > a) \quad (\text{see equation (3) PLANE, page 37 with } R = r_a, \beta = \theta) \]

\[ x = c + r_a \cos \theta, \quad y = r_a \sin \theta \quad (7) \]

and for equation (2)

\[ r_b = \frac{(d^2 - b^2) [d \cos (\theta - a) \pm b]}{d^2 \cos^2 (\theta - a) - b^2} \quad (d > b) \]

\[ x = c + r_b \cos \theta, \quad y = r_b \sin \theta \quad (8) \]

Since (7) and (8) express the two hyperbolas in polar form with respect to the same pole F(c,0), a common focus of the two loci, it is clear (see Figure 24) that at a point of intersection P'(x,y) the two values \( r_a \) and \( r_b \) are equal to a common value \( r' \) for a common value of \( \theta \) and the distances to P' from F' and F'' are then given by \( r_1 = r' + 2a, \ r_2 = r' + 2b \).

Equating the values of \( r_a, r_b \) from (7) and (8) one obtains

\[ r' = \frac{c^2 - a^2}{\pm a - c \cos \theta} = \frac{d^2 - b^2}{d \cos (\theta - a) \pm b} \quad (9) \]

and since \( c, d, a \) are constants, (9) is a relation between the parameters \( a, b, \) and \( \theta \). That is given any two of the three the third may be found from (9).

Consider \( a \) and \( b \) given. First write (9) in the form

\[ \frac{d \cos (\theta - a) \mp b}{\pm a - c \cos \theta} = \frac{d^2 - b^2}{c^2 - a^2} = K, \text{ whence} \]

\[ (d \cos a + cK) \cos \theta + (d \sin a) \sin \theta = \pm aK \pm b. \quad (10) \]

The solution of the trigonometric equation (10) is

\[ \theta_i = \beta + \gamma_i \]

\[ \tan \beta = (d \sin a)/(d \cos a + cK) \quad (i = 1,2,3,4) \]

\[ \cos \gamma_i = (\pm aK \mp b) \sin \beta / d \sin a. \quad (11) \]

From (11) it is seen that in general there will be four angles \( (\gamma_i) \), and thus four values
of \( \theta_i \), four values of \( r_i' \) from (9) and four sets of rectangular coordinates from \( x_i = c + r_i' \cos \theta_i \), \( y_i = r_i' \sin \theta_i \) (i = 1,2,3,4) (12)

and for each point of intersection two of the additional distances

\[
  r_i = r_i' \pm 2b, \quad r_{i+4} = r_i' \pm 2a \quad (i = 1,2,3,4).
\]

(13)

A procedure for using equations (9) through (13) will be described and used for two examples. Since \( a,b,c,d,\alpha \) will be given, first compute

\[
  K = \frac{(d^2 - b^2)}{(c^2 - a^2)}, \quad \nu = d \cos \alpha, \quad \nu = \frac{d \sin \alpha}{d \cos \alpha + cK}.
\]

From \( \tan \beta \), using tables, find \( \beta \) and \( \sin \beta \). Then compute

\[
  \cos \gamma_i = (\pm aK \pm b) \sin \beta / \nu \quad (i = 1,2,3,4), \quad \theta_i = \beta + \gamma_i \quad (i = 1,2,3,4).
\]

Next compute

\[
  r_i' = \frac{c^2 - a^2}{\pm a - c \cos \theta_i} = \frac{d^2 - b^2}{d \cos (\theta_i - \alpha) \pm b} \quad i = 1,2,3,4
\]

choosing the proper value (with respect to sign) of \( \pm a, \pm b \) in each member which will make them equal and positive for each value of \( \theta_i \). Now the rectangular coordinates may be computed from \( x_i = c + r_i' \cos \theta_i, \; y_i = r_i' \sin \theta_i \). Useful checks are provided at this point by the relations

\[
  (x_i - c)^2 + y_i^2 = r_i'^2 \quad \text{and by } \Sigma x_i = -H \text{ from equation (3)}.
\]

\( H = 2a^2 (D\beta_0 - \delta)/L, \; \beta_0 = CG + Aa^2, \delta = BEG, \; \mu = 2a^2. \)

\( \cos \gamma_i = (\pm aK \pm b) (\sin \beta / \nu) = (\pm 1 \pm 1) (0.27059805) \pm (0.54119610), \; 0.
\]

\( 0 < \gamma_i < 2\pi. \)

\( \gamma_i = 57^\circ 14' \; 05^\prime 666, \; 90^\circ, \; 122^\circ 45' \; 54^\prime 334, \; 270^\circ \)

\( \theta_1 = \beta + \gamma_1, \; \theta_1 = 79^\circ 44' \; 05^\prime 666, \; \theta_2 = 112^\circ 30', \; \theta_3 = 145^\circ 15' \; 54^\prime 334, \; \theta_4 = 292^\circ 30' \)

\[
  r_i' = \frac{3}{\pm 1 - 2 \cos \theta_i} = \frac{3}{2 \cos (\theta_i - 45) \pm 1}.
\]

(Choose the proper value of \( \pm 1 \) in each member which will make them equal and positive for each value of \( \theta_i \). If this cannot be done the values of \( \theta_i \) may be in error.) The work may be arranged in table form as follows:
Table 1.

<table>
<thead>
<tr>
<th>θ_i</th>
<th>θ_i - 45</th>
<th>sin θ_i</th>
<th>cos θ_i</th>
<th>cos (θ_i - 45)</th>
<th>r_i'</th>
</tr>
</thead>
<tbody>
<tr>
<td>79 44 05.666 34 44 05.666</td>
<td>0.98399379</td>
<td>0.17820275</td>
<td>0.82179706</td>
<td>4.6613215</td>
<td></td>
</tr>
<tr>
<td>112 30</td>
<td>67 30</td>
<td>0.92387953</td>
<td>-0.38268343</td>
<td>0.38268343</td>
<td>1.6993635</td>
</tr>
<tr>
<td>145 15 54.334</td>
<td>100 15 54.334</td>
<td>0.56978031</td>
<td>-0.82179706</td>
<td>-0.17820275</td>
<td>4.6613215</td>
</tr>
<tr>
<td>292 30</td>
<td>247 30</td>
<td>-0.92387953</td>
<td>0.38268343</td>
<td>-0.38268343</td>
<td>12.785918</td>
</tr>
</tbody>
</table>

x_i = 2 + r_i' cos θ_i
y_i = r_i' sin θ_i
r_i = r_i' ± 2
r_i + 4 = r_i' ± 2

Checks were computed but are not shown here. Figure 25 shows the results of Table 1 graphically.

Example 2. Let c = 3, a = d = 2, b = 1, α = 30°. sin α = ½, cos α = √3/2
K = 0.6, tan β = 1/(√3 + 1.8) = 1/(3.5320508) = 0.28312164, ν = 1, μ = √3.
β = 15° 48' 28° 676. sin β = 0.27241402, cos γ_i = (± 1.2 ± 1) / 2 (0.54482804)

cos γ_i = ± 0.59931084, ± 0.054482804

γ_i = 53° 10' 46° 00' 86° 52' 36° 550, 126° 49' 14° 000, 273° 07' 23° 450

θ_i = β + γ_i, θ_i = 68° 59' 14° 676, θ_2 = 102° 41' 05° 226, θ_3 = 142° 37' 42° 676

θ_4 = 288° 55' 52° 126. r_i' = 5 ± 2 - 3 cos θ_i = 3 / 2 cos (θ_i - 30) ± 1. The work is arranged in the following table:
Table 2

<table>
<thead>
<tr>
<th>$\theta_i$</th>
<th>$\theta_i - 30$</th>
<th>$\sin \theta_i$</th>
<th>$\cos \theta_i$</th>
<th>$\cos (\theta_i - 30)$</th>
<th>$r_i'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0° 59' 14.676</td>
<td>88° 59' 14.676</td>
<td>0.93350166</td>
<td>0.35857308</td>
<td>0.77728423</td>
<td>5.40961166</td>
</tr>
<tr>
<td>102° 41' 05.226</td>
<td>72° 41' 05.226</td>
<td>0.97559289</td>
<td>-0.21958714</td>
<td>0.29762840</td>
<td>1.88057496</td>
</tr>
<tr>
<td>142° 37' 42.676</td>
<td>112° 37' 42.676</td>
<td>0.60698032</td>
<td>-0.79471687</td>
<td>-0.38475484</td>
<td>13.015729</td>
</tr>
<tr>
<td>288° 55' 52.126</td>
<td>258° 55' 52.126</td>
<td>-0.94590914</td>
<td>0.32443167</td>
<td>-0.19198850</td>
<td>4.86994806</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x_i = 3 + r_i' \cos \theta_i$</th>
<th>$y_i = r_i' \sin \theta_i$</th>
<th>$r_i = r_i' \pm 2$</th>
<th>$r_i + 4 = r_i' \pm 4$</th>
<th>$\tan \theta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.93974111</td>
<td>5.04988146</td>
<td>4.30961166</td>
<td>9.40961166</td>
<td>2.60337906</td>
</tr>
<tr>
<td>2.58704999</td>
<td>1.83467556</td>
<td>2.88057499</td>
<td>5.88057499</td>
<td>-4.86994806</td>
</tr>
<tr>
<td>-7.34381941</td>
<td>7.90091356</td>
<td>15.015729</td>
<td>9.015729</td>
<td>-7.6376927</td>
</tr>
<tr>
<td>4.57996538</td>
<td>-4.60652838</td>
<td>6.86994806</td>
<td>8.86994806</td>
<td>-2.9155822</td>
</tr>
</tbody>
</table>

Checks of the computations of Table 2 were made as follows:

1. Using $(x_i - 3)^2 + y_i^2 = r_i^2$ and values from Table 2:

<table>
<thead>
<tr>
<th>$(x_i - 3)^2$</th>
<th>$y_i^2$</th>
<th>$(x_i - 3)^2 + y_i^2$</th>
<th>$r_i^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.762 59557</td>
<td>25.501 30276</td>
<td>29.263 89831</td>
<td>29.263 89831</td>
</tr>
<tr>
<td>0.170 52777</td>
<td>3.366 03441</td>
<td>3.536 56218</td>
<td>3.536 56218</td>
</tr>
<tr>
<td>106.994 59999</td>
<td>26.414 60341</td>
<td>169.409 20140</td>
<td>169.409 20140</td>
</tr>
<tr>
<td>2.496 29060</td>
<td>21.220 10372</td>
<td>23.716 39432</td>
<td>23.716 39410</td>
</tr>
</tbody>
</table>

2. From the formulas of (2) and (3) find $A = 2$, $B = 2\sqrt{3}$, $C = 0$, $D = -6(\sqrt{3} + 2)$, $E = -6(\sqrt{3} + 1)$, $F = 9(2\sqrt{3} + 3)$, $\delta = \text{BEG} = -60(\sqrt{3} + 3)$, $\beta_0 = a^2A + CG = 8$, $L = \beta_0^2 - a^2GB^2 = -11 \times 2^4$ $H = -2^4[48(\sqrt{3} + 2) + 60(\sqrt{3} + 3)] / 11 \times 2^4$ $= \pi (2/11)[26.1961524] = -4.76293680$.

From Table 2, $\Sigma x_i = 4.76293700 = -H = 4.76293680$. Again computing $N$ from equations (3), find $N = -429.826515$. From Table 2 find $\Pi x_i = -429.826494$ and $\Pi x_i = N$.

3. From equation (6), compute the quantities:

$U = B/A = \sqrt{3}$, $V = C/A = 0$, $W = D/A = -3(\sqrt{3} + 2)$, $R = E/A = -3(\sqrt{3} + 1)$, $T = F/A = 9(2\sqrt{3} + 3)/2$, $\phi = c^2 + cW + T = 9/2$, $\theta_0 = R + cU = -3$, $\rho = a^2 + T = \frac{1}{2}(2+3)$ $Q = a^2/(c^2 - a^2) = 4/5$, $k_1 = (GV + \phi)Q - a^2q^2 = -2^43^2/5^4$, $k_2 = \rho^2 - a^2W^2 + (1189 + 684\sqrt{3})/2^3$.

Now from equation (6), $\Pi m_i = \Pi \tan \theta_i = k_2/k_1 = -5^2(1189 + 684\sqrt{3})/2^33^2 = 25.756540$.

Now forming $\Pi \tan \theta_i$ from the values in Table 2, find $\Pi \tan \theta_i = -25.756539$. 

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Figure (26) depicts the solution graphically.

**SUMMARY REMARKS (Plane Approximation)**

While the formulas (9) through (13) are convenient for hand computing, since no root extraction is involved, the use of trigonometric tables may make it unsuitable for larger machine coding and computation, and it may be better to use the algebraic solution, equation (3). If the algebraic solution is to be used, the number of significant figures to be retained in the coefficients of the resulting quartic, equation (3), will have to be considered relative to the number of significant figures required in the rectangular coordinates of the intersections points.

If solutions only above the base line, $F' F''$, are desired (see Figure 24), then in the trigonometric solution, equations (9) – (13), $\theta$ should be limited to $\pi > \theta > a$.

Note that the parameters $a$ and $b$ of the two families of confocal hyperbolas are related to the fundamental constants of a Loran line by the relations (2.1).
Figure 25. Intersection of plane hyperbolas. Example 1.
APPENDIX 2

Computations
Using Andoyer-Lambert
First Order Formulae Without Conversion
to Parametric Latitude
DISTANCE COMPUTING FORM, ANDOYER-LAMBERT APPROXIMATION
(No conversion to parametric latitudes)
Clarke Spheroid 1866 \(a = 6,378,206.4\) meters
\(f/2 = 0.00169503765\), \(f/4 = 0.000847518825\)
1 radian = 206,264.8062 seconds

| \(\phi_1\) | 40 30 37.757 | 1. | Original \(\lambda_1\) | 17 19 43.280 |
| \(\phi_2\) | 40 00 00.000 | 2. | Terminus \(\lambda_2\) | 18 00 00.000 |
| \(\sin \phi_1\) | 0.64958723 | 2. West of 1. | \(\Delta \lambda = \lambda_2 - \lambda_1\) | 40 16.120 |
| \(\cos \phi_1\) | 0.76028707 | \(\sin \phi_2\) | 0.64278761 | \(\sin \Delta \lambda\) | 0.01171632 |
| \(\tan \phi_1\) | 0.85439731 | \(\cos \phi_2\) | 0.76604444 | \(\cos \Delta \lambda\) | 0.9993136 |
| \(\tan \phi_2\) | 0.83909963 | \(\cos d = \sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2 \cos \Delta \lambda\) | \(-0.11558604\) | \(\cot A = \frac{M}{\sin \Delta \lambda}\) | \(-0.98898047\) |
| \(M = \phi_1 \tan \phi_2 - \sin \phi_1 \cos \Delta \lambda\) | \(+0.1176282\) | \(\cot B = \frac{N}{\sin \Delta \lambda}\) | \(+1.00396882\) |
| \(N = \phi_2 \tan \phi_1 - \sin \phi_2 \cos \Delta \lambda\) | \(+0.1262251\) | \(\sin d = \frac{\cos \phi_1 \sin \Delta \lambda}{\sin A}\) | \(+0.1104900\) |
| \(\frac{\sin d}{\sin B} = \frac{\cos \phi_1 \sin \Delta \lambda}{\sin A}\) | \(+0.1262251\) | \(\sin A\) | 134 40 46.816 |

\[K = (\sin \phi_1 - \sin \phi_2)^2\]
\[L = (\sin \phi_1 + \sin \phi_2)^2\]
\[\delta d = (f/4) (HK + GL)\]
\[d \text{ (radians)} = 0.01262293382\]
\[d + \delta d \text{ (rad)} = 0.01261599\]
\[2A = 269 21 33.632\]
\[2B = 89 46 22.994\]
\[\sin 2A = -0.99993749\]
\[\sin 2B = 0.99999216\]
\[U = (f/2) \cos \phi_1 \sin 2A = -9.979732265 \times 10^{-4}\]
\[V = (f/2) \cos \phi_2 \sin 2B = -9.9468111 \times 10^{-4}\]
\[VT = 9.441146 \times 10^{-4}\]
\[\delta A = VT - U = 0.00194468\]
\[\delta B = -\delta A = -0.00194468\]
\[\delta A + \delta B = 6 47.259\]
\[\delta A - \delta B = -134 40 46.816\]
\[A = 134 40 46.816\]
\[B = 89 46 22.994\]
\[\delta A = 180\]
\[\delta B = 180\]
\[a_{1-1} = 45 26 00.348\]
\[a_{2-1} = 224 59 58.759\]
\[a_{1-1} = a_{AB} = 180^\circ - A + \delta A\]
\[a_{2-1} = a_{BA} = 180^\circ + B + \delta B\]

Line No. 1 (See Tables 1,2 – pages 65,66)
DISTANCE COMPUTING FORM, ANDOYER-LAMBERT APPROXIMATION
(No conversion to parametric latitudes)
Clarke Spheroid 1866  \( a = 6,378,206.4 \) meters
\( f/2 = 0.00169503765 \),  \( f/4 = 0.000847518825 \)
1 radian = 206,264.8062 seconds

\[
\begin{align*}
\phi_1 & = 9^\circ 59' 58.349'' \\
\phi_2 & = 10^\circ 00' 00.000'' \\
\sin \phi_1 & = 0.99359255 \\
\cos \phi_1 & = 0.193481756 \\
\tan \phi_1 & = 4.98 \times 10^{-6} \\
\sin \phi_2 & = 0.9934818 \\
\cos \phi_2 & = 0.19326698 \\
\tan \phi_2 & = 4.98 \times 10^{-6} \\
\end{align*}
\]

2. West of 1.

\[
\Delta \lambda = \lambda_2 - \lambda_1 = 1.28 \text{ deg} \\
\Delta \lambda = 0.02541535 \\
\cos \Delta \lambda = 0.99967188 \\
\sin \Delta \lambda = 0.01250000 \\
\tan \Delta \lambda = 0.00169503765 \\
\end{align*}
\]

\[
\begin{align*}
\phi \text{ in } \text{radians} & = \frac{206,264.8062 \text{ seconds}}{3600} \\
1 \text{ radian} & = 206,264.8062 \text{ seconds} \\
\sin 1 \text{ radian} & = 0.0174532925 \\
\cos 1 \text{ radian} & = 0.9998476924 \\
\end{align*}
\]

\[
\begin{align*}
M & = \cos \phi_1 \tan \phi_1 - \sin \phi_1 \cos \Delta \lambda + 0.0011432 \\
N & = \cos \phi_2 \tan \phi_1 - \sin \phi_2 \cos \Delta \lambda - 0.0000388 \\
\end{align*}
\]

\[
\begin{align*}
\sin \Delta \lambda & = \frac{\cos \phi_2 \sin \phi_1 - 0.99352645}{\sin \phi_1} \\
\cos \Delta \lambda & = \frac{\cos \phi_1 \sin \phi_2 - 0.9934818}{\sin \phi_2} \\
\end{align*}
\]

\[
\begin{align*}
K & = \sin \phi_1 - \sin \phi_2 = 3.1 \times 10^{-6} \\
L & = \sin \phi_1 + \sin \phi_2 = 1.2057612 \\
\delta d & = \frac{(f/4)(HK + GL)}{1.0410 \times 10^{-6}} \\
d & = \frac{0.99352645}{\sin \phi_1} \\
\end{align*}
\]

\[
\begin{align*}
2A & = 179^\circ 29' 18.914'' \\
2B & = 180^\circ 00' 08.320'' \\
U & = \frac{1.467352 \times 10^{-5}}{1.467352 \times 10^{-5}} \\
V & = \frac{1.467352 \times 10^{-5}}{1.467352 \times 10^{-5}} \\
\end{align*}
\]

\[
\begin{align*}
\delta A & = 0.03037 \\
\delta B & = 0.03037 \\
\end{align*}
\]

\[
\begin{align*}
+ 180 & = 90^\circ 45' 39.457'' \\
+ 180 & = 90^\circ 00' 00.037'' \\
\end{align*}
\]

Line No. 2 (See Tables 1, 2 – pages 65, 66)
DISTANCE COMPUTING FORM, ANDOYER-LAMBERT APPROXIMATION
(No conversion to parametric latitudes)
Clarke Spheroid 1866  a = 6,378,206.4 meters
f/2 = 0.00169503765, f/4 = 0.000847518825
1 radian = 206,264.8062 seconds

<table>
<thead>
<tr>
<th>φ₁</th>
<th>69° 48' 05.70&quot;</th>
<th>Origin</th>
<th>λ₁</th>
<th>9° 39' 21.63&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>φ₂</td>
<td>70° 00' 00.00&quot;</td>
<td>2 West of 1.</td>
<td>λ₂</td>
<td>18° 00' 00.00&quot;</td>
</tr>
<tr>
<td>sin φ₁</td>
<td>0.938 50257</td>
<td>Δλ = λ₂ - λ₁</td>
<td>8° 22' 31.863</td>
<td></td>
</tr>
<tr>
<td>cos φ₁</td>
<td>0.345 27226</td>
<td>sin Δλ</td>
<td>0.145 650 90</td>
<td></td>
</tr>
<tr>
<td>tan φ₁</td>
<td>2.918 15324</td>
<td>cos Δλ</td>
<td>0.989 33502</td>
<td></td>
</tr>
<tr>
<td>tan φ₂</td>
<td>2.747 447 72</td>
<td>cos d = sin φ₁ sin φ₂ + cos φ₁ cos φ₂ cos Δλ</td>
<td>-0.998 73458</td>
<td></td>
</tr>
<tr>
<td>M=cos φ₁ tan φ₂ - sin φ₁ cos Δλ</td>
<td>4.050 13428</td>
<td>cot A = M</td>
<td>1.138 22992</td>
<td></td>
</tr>
<tr>
<td>N = cos φ₁ tan φ₂ - sin φ₂ cos Δλ</td>
<td>-0.000 00801</td>
<td>cot B = N</td>
<td>0.0000 5499</td>
<td></td>
</tr>
<tr>
<td>sin d = (cos φ₂ sin Δλ + 0.050 1943 sin A + 0.994 58101</td>
<td>sin B</td>
<td>A</td>
<td>82° 09' 47.599</td>
<td></td>
</tr>
<tr>
<td>K = (sin φ₁ - sin φ₂)²</td>
<td>+1.146 22 X 10^-6</td>
<td>H = (d + 3 sin d)/(1 - cos d)</td>
<td>+15.12 988826</td>
<td></td>
</tr>
<tr>
<td>L = (sin φ₁ + sin φ₂)²</td>
<td>+3.529 61177</td>
<td>G = (d - 3 sin d)/(1 + cos d)</td>
<td>-0.50 344 892</td>
<td></td>
</tr>
<tr>
<td>δd = -(f/4) (HK+GL)</td>
<td>+0.000 150197</td>
<td>s = a (d + δd)</td>
<td>321,862.777 meters</td>
<td></td>
</tr>
<tr>
<td>d (radians)</td>
<td>+0.050 629 29</td>
<td>S = 193.992 1 n.m.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>d + 8d (rad)</td>
<td>+0.050 629 29</td>
<td>T = d/sin d</td>
<td>1,000 42</td>
<td></td>
</tr>
<tr>
<td>2A</td>
<td>164° 15' 35.15&quot;</td>
<td>2B</td>
<td>180° 00' 22.68&quot;</td>
<td></td>
</tr>
<tr>
<td>sin 2A</td>
<td>+1.791 274 41</td>
<td>sin 2B</td>
<td>-0.000 10998</td>
<td></td>
</tr>
<tr>
<td>U = (f/2) cos² φ₁ sin 2A</td>
<td>+5.4169 X 10^-5</td>
<td>V = (f/2) cos² φ₂ sin 2B</td>
<td>-2.18 X 10^-5</td>
<td></td>
</tr>
<tr>
<td>VT = -2.182 X 10^-8</td>
<td>+5.4839 X 10^-5</td>
<td>UT</td>
<td>+5.4840 X 10^-5</td>
<td></td>
</tr>
<tr>
<td>δA = VT - U</td>
<td>-5.4839 X 10^-5</td>
<td>δB = UT + V</td>
<td>-5.4862 X 10^-5</td>
<td></td>
</tr>
<tr>
<td>+ A = 80° 11' 11.51</td>
<td>+δB</td>
<td>+01° 11.51</td>
<td></td>
<td></td>
</tr>
<tr>
<td>+180°</td>
<td>97°</td>
<td>+180°</td>
<td>27°</td>
<td></td>
</tr>
<tr>
<td>a₁-₁</td>
<td>97° 52' 01.12&quot;</td>
<td>a₂-₁</td>
<td>27° 00' 00.02&quot;</td>
<td></td>
</tr>
<tr>
<td>a₁-₁ = a AB = 180° - A + δA</td>
<td>a₂-₁ = a BA = 180° + B + δB</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Line No. 3 (See Tables 1, 2 - pages 65,66)
COMPUTING FORM, ANDOYER-LAMBERT
(No conversion to parametric latitudes)
Clarke Spheroid, 1866  \( a = 6,378,206.4 \) meters
\( f/2 = 0.00169503765, f/4 = 0.000847518825 \)
1 radian = 206,264.8062 seconds

\[
\phi_1 \quad 13 \quad 04 \quad 12.564 \quad 1. \quad \text{Origin} \quad \lambda_1 \quad 14 \quad 51 \quad 13.283
\]

\[
\phi_2 \quad 10 \quad 00 \quad 00.000 \quad 2. \quad \text{Terminus} \quad \lambda_2 \quad 18 \quad 00 \quad 00.000
\]

\[
\sin \phi_1 \quad -1.173 \quad 64.819 \quad 2. \quad \text{West of 1.} \quad \Delta \lambda = \lambda_2 - \lambda_1 \quad 3 \quad 08 \quad 46.717
\]

\[
\cos \phi_2 \quad -984.80775 \quad \sin \phi_1 \quad 0.226 \quad 14.397 \quad \sin \Delta \lambda \quad 0.054 \quad 88.588
\]

\[
\cos^2 \phi_2 \quad -969.81630 \quad \cos \phi_1 \quad -974.09339 \quad \cos \Delta \lambda \quad 0.993 \quad 492.63
\]

\[
\cos^2 \phi_1 \quad -948.85891 \quad \cos d = \sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2 \cos \Delta \lambda \quad 0.997 \quad 11869
\]

\[
K = (\sin \phi_1 - \sin \phi_2)^2 \quad +.00273581
\]

\[
L = (\sin \phi_1 + \sin \phi_2)^2 \quad +.16983376
\]

\[
H = (d + 3 \sin d)/(1 - \cos d) \quad +105.33468
\]

\[
G = (d - 3 \sin d)/(1 + \cos d) \quad -.07593015
\]

\[
\delta d = -f(HK + GL)/4 \quad -2.35734 \times 10^{-4}
\]

\[
R = \sin \Delta \lambda / \sin d \quad .72354184
\]

\[
T = d / \sin d \quad 1.000 \quad 9616
\]

\[
\sin A = R \cos \phi_2 \quad .71254961
\]

\[
\sin B = R \cos \phi_1 \quad .704 \quad 79769
\]

\[
A \quad 134 \quad 33 \quad 26.138
\]

\[
2A \quad 269 \quad 06 \quad 52.276
\]

\[
\sin 2A \quad -.99988058
\]

\[
B \quad 44 \quad 48 \quad 47.926
\]

\[
2B \quad 89 \quad 37 \quad 35.062
\]

\[
\sin 2B \quad +.999 \quad 97874
\]

\[
U = (f/2) \cos \phi_2 \sin 2A \quad +.0016081595
\]

\[
V = (f/2) \cos \phi_2 \sin 2B \quad +.001643891
\]

\[
U \quad \text{(rad)} \quad -.0016081595
\]

\[
V \quad \text{(rad)} \quad +.001643891
\]

\[
UT \quad -001609706
\]

\[
\delta A = VT - U \quad +11 \quad 11.110
\]

\[
\delta B = UT + V \quad +11 \quad 11.108
\]

\[
\alpha_{AB} = 180^\circ - A + \delta A \quad 45 \quad 31 \quad 44.912
\]

\[
\alpha_{BA} = 180^\circ + B + \delta B \quad 224 \quad 59 \quad 33.629
\]

Line No. 4 (See Tables 1, 2 - pages 65, 66)
### Computing Form, Andoyer-Lambert

*(No conversion to parametric latitudes)*

**Clarke Spheroid, 1866**

- Parameter $a = 6,378,206.4$ meters
- Parameter $f/2 = 0.00169503765$, $f/4 = 0.000847518825$
- $1$ radian $= 206,264.8062$ seconds

#### Conversion Details

- Clarke Spheroid, 1866
- $a = 6,378,206.4$ meters
- $f/2 = 0.00169503765$, $f/4 = 0.000847518825$
- $1$ radian $= 206,264.8062$ seconds

<table>
<thead>
<tr>
<th>$\phi_1$</th>
<th>73° 35' 09.206&quot;</th>
<th>Origin</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_2$</td>
<td>70° 00' 00.000&quot;</td>
<td>Terminus</td>
</tr>
<tr>
<td>$\sin \phi_2$</td>
<td>0.93969262</td>
<td>2. West of 1.</td>
</tr>
<tr>
<td>$\cos \phi_2$</td>
<td>0.34202014</td>
<td>$\sin \phi_4$</td>
</tr>
<tr>
<td>$\cos^2 \phi_2$</td>
<td>0.11697778</td>
<td>$\cos \phi_4$</td>
</tr>
<tr>
<td>$\cos^4 \phi_2$</td>
<td>0.07985016</td>
<td>$\cos d = \sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2 \cos \Delta \lambda$</td>
</tr>
</tbody>
</table>
| $K = (\sin \phi_1 - \sin \phi_2)^2$ | 0.00038272 | $d$ | 8° 45' 59.408"
| $L = (\sin \phi_1 + \sin \phi_2)^2$ | 3.60596184 | $d$ (radians) | 0.10064445 |
| $H = (d + 3 \sin d)/(1 - \cos d)$ | 0.4794541793 | $\sin d$ | 0.10047463 |
| $G = (d - 3 \sin d)/(1 + \cos d)$ | 0.10044369 | $s = a(d + \delta d)$ | 643.728.709 meters |
| $\delta d = -f(HK + GL)/4$ | 0.00028184 | $s$ | 347.5867 n.m. |
| $R = \sin \Delta \lambda / \sin d$ | 2.501643125 | $T = d / \sin d$ | 1.0016902 |
| $\sin A = R \cos \phi_2$ | 0.85557813 | $\sin B = R \cos \phi_1$ | 0.70688025 |
| 1A | 12° 10' 34.813" | 2A | 242° 21' 09.626 |
| 2A | 12° 10' 34.813" | 2B | 89° 57' 47.860 |
| $\sin 2A$ | -0.885 82060 | $\sin 2B$ | +0.99999980 |
| $U = (f/2) \cos^3 \phi_1 \sin 2A$ | | $V = (f/2) \cos^3 \phi_2 \sin 2B$ | |
| $U$ (rad) | -1.19895 × 10⁻⁴ | $V$ (rad) | +1.98282 × 10⁻⁴ |
| $U$ | | $V$ | |
| $VT$ | +1.98617 × 10⁻⁴ | UT | -1.20098 × 10⁻⁴ |
| $\delta A = VT - U + = 0° 01' 05.698$ | | $\delta B = -UT + V - = 0° 05' 671$ |
| $a_{AB} = 180° - A + \delta A$ | 58° 50' 30.886 | $a_{BA} = 180° + B + \delta B$ | 224° 59' 57.601 |

**Line No. 5** (See Tables 1, 2 - pages 65, 66)
DISTANCE COMPUTING FORM, ANDOYER-LAMBERT APPROXIMATION
(No conversion to parametric latitudes)
Clarke Spheroid 1866 a = 6,378,206.4 meters
t/2 = 0.00169503765, t/4 = 0.000847518825
1 radian = 206,264.8062 seconds

<table>
<thead>
<tr>
<th>Φ</th>
<th>39 37 06.013</th>
<th>Origin</th>
<th>λ</th>
<th>8 36 13.296</th>
</tr>
</thead>
<tbody>
<tr>
<td>φ1</td>
<td>39 37 06.013</td>
<td>1.</td>
<td></td>
<td>8 36 13.296</td>
</tr>
<tr>
<td>φ2</td>
<td>40 00 00.000</td>
<td>2.</td>
<td></td>
<td>18 00 00.000</td>
</tr>
<tr>
<td>Sin φ1</td>
<td>0.637 627 09</td>
<td>2. West of 1.</td>
<td>Δλ = λ2 - λ1 = 9 23 16.734</td>
<td></td>
</tr>
<tr>
<td>Cos φ1</td>
<td>0.770 50 935</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tan φ1</td>
<td>0.827 816 05</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tan φ2</td>
<td>0.839 099 63</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M = cos φ1 tan φ2 - sin φ1 cos Δλ = + 0.017 23.525</td>
<td>cot λ = M / sin Δλ = + 1.105 04.06</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N = cos φ1 tan φ1 - sin φ2 cos Δλ = - 0.000 3450</td>
<td>cot B = N / sin Δλ = - 0.002 1150</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sin d = cos φ1 sin Δλ = 0.125 6194</td>
<td>sin A = 0.994 16595</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>= cos φ2 sin Δλ = 0.125 6194 sin B = 0.999 99998</td>
<td>A = 83 58 09.894</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>K = (sin φ1 - sin φ2)² + 2. 016 1384 110-5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>L = (sin φ1 + sin φ2)² = 1.639 57 88</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>δd = (f/4) (HK + GL) = + 0.000 17366</td>
<td>s = a (d + δd) = 104 664.697 meters</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d (radians) = + 1.125 98 180</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d + δd (rad) = + 1.125 158 46</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2A = 167 56 19.748</td>
<td>2B = 180 01 29.250</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sin 2A = + 1.208 95 605</td>
<td>Sin 2B = - 000 42.300</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>U = (f/2) cos² φ1 sin 2A = + 0.002 01166</td>
<td>V = (f/2) cos² φ2 sin 2B = - 4.21 × 10⁻⁷</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>VT = - 4.22 × 10⁻⁹</td>
<td>UT = + 0.000 210 723</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>δA = VT - U = - 000 105 58</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>+ δA = 0 03.137</td>
<td>+ δB = 0 43.55</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A = 83 58 09.874</td>
<td>B = 90 00 43.65</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>+ 180 = 86 11 56.189</td>
<td>+ 180 = 370 00 00.093</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a₁₂ = a₂₁ = 180° - A + δA</td>
<td>a₁₁ = a₂₂ = 180° + B + δB</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Line No. 6 (See Tables 1,2 - pages 65,66)
DISTANCE COMPUTING FORM, ANDOYER-LAMBERT APPROXIMATION

(No conversion to parametric latitudes)

Clarke Spheroid 1866

\[ a = 6,378,206.4 \text{ meters} \]

\[ f/2 = 0.00169503765 \]

\[ f/4 = 0.000847518825 \]

1 radian = 206,264.8062 seconds

\[ \phi_1 = 44^\circ 54' 28.507" \]

\[ \phi_2 = 40^\circ 00' 00.000" \]

\[ \sin \phi_1 = 0.705 969 46 \]

\[ \cos \phi_1 = 0.708 245 38 \]

\[ \tan \phi_1 = 0.996 799 09 \]

\[ \tan \phi_2 = -839 099 63 \]

\[ \cos \phi = 0.992 045 91 \]

\[ \cos \phi = 0.992 050 04 \]

\[ \Delta \lambda = \lambda_2 - \lambda_1 = 7.12 \quad 16.117 \]

\[ \sin \Delta \lambda = 0.125 110.25 \]

\[ \cos \Delta \lambda = 0.999 778 \]

\[ \sin \Delta \lambda = 0.999 780 \]

\[ \sin d = \sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2 \cos \Delta \lambda \]

\[ \cos \phi_1 \tan \phi_2 - \sin \phi_1 \cos \Delta \lambda \]

\[ M = \cos \phi_1 \tan \phi_2 - \sin \phi_1 \cos \Delta \lambda \]

\[ N = \cos \phi_2 \tan \phi_1 - \sin \phi_2 \cos \Delta \lambda \]

\[ \sin d = \frac{\cos \phi_1 \sin \Delta \lambda}{\sin A} \cdot 125 844 04 \]

\[ \sin B = \frac{0.763 406 87}{A} \]

\[ M = \cos \phi_1 \tan \phi_2 - \sin \phi_1 \cos \Delta \lambda \]

\[ N = \cos \phi_2 \tan \phi_1 - \sin \phi_2 \cos \Delta \lambda \]

\[ \cot A = \frac{\sin \Delta \lambda}{\sin \Delta \lambda} \]

\[ \cot B = \frac{N}{\sin \Delta \lambda} \]

\[ \sin d = \frac{\cos \phi_1 \sin \Delta \lambda}{\sin A} \cdot 125 844 04 \]

\[ \sin B = \frac{0.763 406 87}{A} \]

\[ K = (\sin \phi_1 - \sin \phi_2)^2 \]

\[ L = (\sin \phi_1 + \sin \phi_2)^2 \]

\[ \delta d = \frac{(f/4)(K + L)}{\sin \Delta \lambda} \]

\[ d = \frac{126 178 58}{\sin \Delta \lambda} \]

\[ d + \delta d = \frac{126 158 62}{\sin \Delta \lambda} \]

\[ 2A = 260^\circ 28' 08.632" \]

\[ 2B = 9^\circ 47' 20.492" \]

\[ \sin 2A = -0.986 196 63 \]

\[ \sin 2B = -0.999 993 22 \]

\[ U = (f/2) \cos \phi_1 \sin 2A = -9.375 065 10 \]

\[ V = (f/2) \cos \phi_2 \sin 2B = -9.424 066 21 \]

\[ UT = 8.409 356 \times 10^{-4} \]

\[ \delta B = \frac{-18.354 417 88 \times 10^{-4}}{+6' \quad 18.552'} \]

\[ + \delta A = 6' \quad 18.617' \]

\[ \delta A = 6' \quad 18.617' \]

\[ + \delta B = 6' \quad 18.552' \]

\[ + 180^\circ \]

\[ + 180^\circ \]

\[ a_{1-2} = a_{AB} = 180^\circ - A + \delta A \]

\[ a_{2-1} = a_{BA} = 180^\circ + B + \delta B \]

Line No. 7 (See Tables 1, 2 - pages 65, 66)

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DISTANCE COMPUTING FORM, ANDOYER-LAMBERT APPROXIMATION

(No conversion to parametric latitudes)
Clarke Spheroid 1866 $a = 6,378,206.4$ meters
$f/2 = 0.00169503765$, $f/4 = 0.000847518825$
1 radian = 206,264.8062 seconds

<table>
<thead>
<tr>
<th>$\phi_1$</th>
<th>$\phi_2$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\Delta \lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>28° 12' 03.567'E</td>
<td>18° 00' 00.000W</td>
<td></td>
<td></td>
<td>46° 42' 03.567'W</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\sin \phi_1$</th>
<th>$\cos \phi_1$</th>
<th>$\sin \phi_2$</th>
<th>$\cos \phi_2$</th>
<th>$\tan \phi_1$</th>
<th>$\tan \phi_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.990 326 920</td>
<td>0.342 979 875</td>
<td>0.439 692 629</td>
<td>0.342 020 142</td>
<td>2.749 994 722</td>
<td>1.947 992 755</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\sin d$</th>
<th>$\cos \phi_1 \sin \Delta \lambda$</th>
<th>$\cos \phi_2 \sin \Delta \lambda$</th>
<th>$\frac{\sin \lambda_1 - \sin \lambda_2}{\sin \Delta \lambda}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.999 988 800</td>
<td>0.248 917 300</td>
<td>0.706 96 556</td>
<td>0.44 59 18 810</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>K</th>
<th>L</th>
<th>$\delta d$</th>
<th>d + $\delta d$</th>
<th>2A</th>
<th>2B</th>
<th>V</th>
<th>U</th>
</tr>
</thead>
<tbody>
<tr>
<td>$9,384 603 4 \times 10^{-4}$</td>
<td>$3,648 794 6 \times 10^{-4}$</td>
<td>0.000 5 2 5 5 1 2</td>
<td>$-251 56 2 0 7 6$</td>
<td>$-180° 10° 30.5 0 6$</td>
<td>$-189° 5 8 32.6 2 0$</td>
<td>$-180° 2 9 5.8 35 4$</td>
<td>$-180° 5 9 5 4.8 9 4$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>UT</th>
<th>$\Delta A$</th>
<th>$\Delta A$</th>
<th>$\Delta A$</th>
<th>$\Delta A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1.982 1 0 2 5 1 0 4 \times 10^{-4}$</td>
<td>$-3.095 4 8 6 0 \times 10^{-7}$</td>
<td>$+1.988 9 2 3 2 2 \times 10^{-4}$</td>
<td>$+1.024$</td>
<td>$+1.04$</td>
</tr>
</tbody>
</table>

Line No. 8 (See Tables 1, 2 - pages 65, 66)
COMPUTING FORM, ANDOYER-LAMBERT
(No conversion to parametric latitudes)
Clarke Spheroid, 1866  a = 6,378,206.4 meters
f/2 = 0.00169503765, f/4 = 0.000847518825
1 radian = 206,264.8062 seconds

| Φ₁ | 27° 49' 42.30" | Origin |
| Φ₂ | 40° 00' 00.00" | Terminus |
| Sin Φ₁ | 0.642 78761 | 2. West of 1. |
| Cos Φ₁ | 0.776 04444 | Sin Φ₁ | 0.466 82458 |
| Cos² Φ₁ | 0.586 24008 | Cos Φ₁ | 0.89434 944 |
| Cos² Φ₁ | 0.78074 922 | Cos d = Sin Φ₁ Sin Φ₂ + Cos Φ₁ Cos Φ₂ Cos ΔL
K = (Sin Φ₁ - Sin Φ₂)² | 0.03096 2988 |
L = (Sin Φ₁ + Sin Φ₂)² | 1.2312 3921 |
H = (d + 3 Sin d)/(1 - Cos d) | + 10.323 8296 |
G = (d - 3 Sin d)/(1 + Cos d) | - 0.75410 8629 |
Δd = -F(Hk + Gl)/4 | + 0.005 15996 |
R = Sin Δφ/Sin d | 1.18077 3187 |
Sin A = R Cos Φ₁ | 0.866 22251 |
Sin B = R Cos Φ₂ | 0.99999 920 |
A | 60° 01' 21.339 |
2A | 120° 02' 42.678 |
Sin 2A | -0.865 63079 |
U = (f/2) Cos Φ₁ Sin 2A | \( \frac{1}{r} \) \( 56.69 \) |
V = (f/2) Cos Φ₂ Sin 2B | \( \frac{1}{r} \) \( 0.512 \) |
U (rad) | 56.69 |
V (rad) | 0.512 |
UT - | 0.512 |
UT | 1° 40.26 |
ΔA = UT - U - | 3° 57.265 |
ΔB = -UT + V - | 4° 21.388 |
α₂ₐ = 180° - A + ΔA | 1° 27' 41.236 |
α₂ₐ = 180° + B + ΔB | 2° 29' 59.576 |

Line No. 9 (See Tables 1,2 - pages 55, 66)
COMPUTING FORM, ANDOYER-LAMBERT
(No conversion to parametric latitudes)
Clarke Spheroid, 1866  \( a = 6,378,206.4 \) meters
\( f/2 = 0.00169503765, f/4 = 0.000847518825 \)
1 radian = 206,264.8062 seconds

\[ \begin{array}{c|c|c}
\phi_1 & \phi_2 & \lambda_1 \\
35^\circ 18' & 40^\circ 00' & 102^\circ 02' \\\n\hline
\sin \phi_2 & 0.64278161 & 2. West of 1. \\
\cos \phi_2 & 0.76604444 & \sin \phi_1 = 0.57803821 \\
\cos^2 \phi_2 & 0.58692408 & \cos \phi_1 = 0.91600970 \\
\hline
\cos^2 \phi_1 & 0.66587183 & \cos d = \sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2 \cos \Delta \lambda = 0.05861401 \\
K = (\sin \phi_1 - \sin \phi_2)^2 & 0.00419248 & d = 38^\circ 38' 83^\prime\prime.060 \\
L = (\sin \phi_1 + \sin \phi_2)^2 & 1.49041368 & d \text{ (radians)} = 1.51214871 \\
H = (d + 3 \sin d)/(1 - \cos d) & 4.78761188 & \sin d = 0.99828068 \\
G = (d - 3 \sin d)/(1 + \cos d) & -1.40059863 & s = a(d + \delta d) = 9,655,912.218 \text{ meters} \\
\delta d = -(H+G)/4 & -0.00175216 & s = 5213.8079 \text{ n.m.} \\
R = \sin \Delta \lambda / \sin d & -0.867 154065 & T = d / \sin d = 1.5147305 \\
\sin A = R \cos \phi_2 & + 0.66427850 & \sin B = R \cos \phi_1 & + 0.70760611 \\
2A & 83 & 83.7191 & 2B & 90 & 90.416 \\
\sin 2A & +.99307665 & \sin 2B & +.99999900 \\
U = (f/2) \cos^2 \phi_1 \sin 2A & V = (f/2) \cos^2 \phi_2 \sin 2B \\
U \text{ (rad)} & 0.00120864 & V \text{ (rad)} & 0.0009946799 \\
U & 3 & 3 & 5 & 25.169 \\
V & 5 & 10.780 & UT & -2 & 30.23 \\
\delta A = VT - U & 1 & 14.585 & \delta B = -UT + V & 25.034 \\
\alpha_{AB} = 180^\circ - A + \delta A & 138 & 23 & 42.394 & \alpha_{BA} = 180^\circ + B + \delta B & 225 & 00 & 00.674 \\
\end{array} \]

Line No. 10 (See Tables 1,2 - pages 65,66)
INVERSE COMPUTATION
(Andoyer-Lambert Formula)
Clarke 1866 Ellipsoid
40-50-6000 Line

\[ \phi_1 = 40^\circ 00' 00.0000N \]
\[ \phi_2 = 35 18 45.644N \]

1. Point of Origin
\[ \lambda_1 = 18^\circ 00' 00.0000W \]
2. Terminal Point
\[ \lambda_2 = 102 02 29.370E \]

Point 1 should be west of point 2

\[ \tan \beta = b/a \tan \phi \]
\[ \tan \phi_1 = 0.83909963 \]
\[ \tan \phi_2 = 0.70837174 \]

\[ \tan \beta_1 = 0.83625502 \]
\[ \tan \beta_2 = 0.70597031 \]

\[ \cot A = \frac{\cos \beta_1 \tan \beta_2 - \sin \beta_1 \cos \Delta \lambda}{\sin \Delta \lambda} \]
\[ \cot B = \frac{\cos \beta_2 \tan \beta_1 - \sin \beta_2 \cos \Delta \lambda}{\sin \Delta \lambda} \]

<table>
<thead>
<tr>
<th>cot A</th>
<th>angle</th>
<th>sin</th>
<th>cos</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.99659760</td>
<td>45° 05'</td>
<td>51°495</td>
<td>0.70831073</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>cot B</th>
<th>angle</th>
<th>sin</th>
<th>cos</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.89069853</td>
<td>41 41</td>
<td>29.068</td>
<td>0.66511838</td>
</tr>
</tbody>
</table>

\[ \sin \sigma = \frac{\cos \beta_1 \sin \Delta \lambda}{\sin B} = \frac{\cos \beta_2 \sin \Delta \lambda}{\sin A} \]

\[ \cos \sigma = \sin \beta_1 \sin \beta_2 + \cos \beta_1 \cos \beta_2 \cos \Delta \lambda \]

\[ M = (\sin \beta_1 + \sin \beta_2)^2 \]
\[ N = (\sin \beta_1 - \sin \beta_2)^2 \]

\[ U = \frac{\sigma - \sin \sigma}{1 + \cos \sigma} \]
\[ V = \frac{\sigma + \sin \sigma}{1 - \cos \sigma} \]

\[ s = a \sigma - H (MU + NV) \]
\[ a \sigma = 9659955.089 \]

\[ \delta A'' = - \cos ^2 \beta_1 \sin B \cos \left( \frac{t_0''}{\sin \sigma} \right) \]
\[ \delta B'' = - \cos ^2 \beta_1 \sin A \cos \left( \frac{t_0''}{\sin \sigma} \right) \]

\[ A = 45° 05' 51°495 \]
\[ A_1 = 44 59 59.902 \]

\[ \alpha_1 = 180^\circ + A_1 224^\circ 59' 59\cdot902 \]

Line No. 10 as computed by ACIC, converting to parametric latitude.

(From Page 39 of the ACIC Technical Report No. 80 — August 1957)
| \( \phi_1 \) | 18° 29' 57.900' | Origin   | \( \lambda_1 \) | 67° 07' 30.300' |
| \( \phi_2 \) | 43° 03' 19.600' | Terminus | \( \lambda_2 \) | 115° 32' 54.700' |
| \( \sin \phi_2 \) | 0.822 705 76' | 2. West of 1. | \( \Delta \lambda = \lambda_2 - \lambda_1 \) | 48° 45' 24.400' |
| \( \cos \phi_2 \) | 0.533 912 83' | \( \sin \phi_1 \) | \( \sin \Delta \lambda \) | 0.951 919 80' |
| \( \cos^2 \phi_2 \) | 0.899 328 87' | \( \cos \phi_1 \) | 0.848 326 88' \( \cos \Delta \lambda \) | 0.659 356 87' |
| \( \cos^2 \phi_1 \) | 0.935 532 02' | \( \cos \Delta \lambda \) | \( \sin \Delta \lambda \) | \( \cos \Delta \lambda \) | 0.693 442 06' |
| \( K = (\sin \phi_1 - \sin \phi_2)^2 \) | 0.133 532 502' | d | 17° 40' 00.199' |
| \( L = (\sin \phi_1 + \sin \phi_2)^2 \) | 1.000 016 52' | d (radians) | 931.941114' |
| \( H = (d + 3 \sin d)/(1 - \cos d) \) | +9.338 805 75' | \( \sin d \) | 0.739 2401 |
| \( G = (d - 3 \sin d)/(1 + \cos d) \) | -0.828 100 908' | \( \sin \Delta \lambda \) | \( \sin \Delta \lambda \) | \( \sin \Delta \lambda \) |
| \( \delta d = -d(\sin \Delta \lambda + \sin \Delta \lambda) \) | -0.549 349 74' \( \times 10^{-4} \) | \( \delta \) | -1.125 -400 59' |
| \( R = \sin \Delta \lambda \sin \Delta \lambda \) | 1.077 149 76' | \( T = d \sin \Delta \lambda \) | 1.125 -400 59' |
| \( \sin A = R \cos \phi_2 \) | 0.943 224 61' | \( \sin B = R \cos \phi_1 \) | 0.964 590 46' |
| A | 48° 00' 24.196' | B | 105° 17' 34.164' |
| 2A | 96° 00' 48.392' | 2B | 210° 35' 08.328' |
| \( \sin 2A \) | 0.994 499 04' | \( \sin 2B \) | -0.518 825 97' |
| \( U = (f/2) \cos^2 \phi_1 \sin 2A \) | \( V = (f/2) \cos^2 \phi_2 \sin 2B \) | \( U \) | 1.575 999 21 \( \times 10^{-3} \) |
| \( U \) (rad) | 1.575 999 21 \( \times 10^{-3} \) | \( V \) (rad) | -1.604885 -2 \( \times 10^{-4} \) |
| \( UT = UT - U \) | -5.182 234 \( \times 10^{-4} \) | \( \delta A = UT + V \) | 1.706 106 \( \times 10^{-3} \) |
| \( \delta A = UT + V \) | 1.706 106 \( \times 10^{-3} \) | \( \delta B = -UT + V \) | 7° 36.892 |
| \( \beta = 180° - \beta \) | 180° - A + \( \delta A \) | 180° - B + \( \delta B \) | 2° 10' 02.272 |

Line No. 11 (See Tables 1, 2 - pages 65, 66)
DISTANCE COMPUTING FORM, ANDOYER-LAMBERT APPROXIMATION
(No conversion to parametric latitudes)
Clarke Spheroid 1866 a = 6,378,206.4 meters
f/2 = 0.00169503765, f/4 = 0.000847518825
1 radian = 206,264.8062 seconds

<table>
<thead>
<tr>
<th>φ₁</th>
<th>35°</th>
<th>45'</th>
<th>19.5&quot; (N)</th>
<th>Moscow</th>
<th>λ₁</th>
<th>-37° 34' 15.450(E)</th>
</tr>
</thead>
<tbody>
<tr>
<td>φ₂</td>
<td>-33°</td>
<td>56'</td>
<td>03.5&quot;</td>
<td>Cape of Good Hope</td>
<td>λ₂</td>
<td>-18° 28' 41.400(E)</td>
</tr>
</tbody>
</table>

Δλ = λ₂ - λ₁ = 19° 05' 34.050

| φ₁ | 35.622 | 642.95 | | cos φ₁ | 0.562 | 226.78 | sin φ₂ | -0.558 | 241.99 | sin Δλ | 3.27 | 09901 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| tan φ₁ | 1.468 | 995.22 | cos φ₂ | 0.824 | 678.19 | cos Δλ | 0.944 | 99007 |
| tan φ₂ | -0.69284 | 15.7 | cos d = sin φ₁ sin φ₂ + cos φ₁ cos φ₂ cos Δλ | 0.6026 | 782 |

M = cos φ₁ tan φ₂ - sin φ₁ cos Δλ - 1.159 | 795.25 | cot A = \( \frac{M}{\sin \Delta \lambda} \) = -3.545 | 20119 |

N = cos φ₂ tan φ₁ - sin φ₂ cos Δλ - 1.746 | 326.49 | cot B = \( \frac{N}{\sin \Delta \lambda} \) = 5.338 | 83129 |

sin d = \( \frac{\cos φ₁ \sin Δλ}{\sin B} \) = 0.99979459 sin A = 0.02044267 sin B = 0.918410519

K = (sin φ₁ - sin φ₂)² = 1.917 | 906.29 |

L = (sin φ₁ + sin φ₂)² = 0.072 | 039.01 |

δ₀ = (f/4) (HK + GL) = 0.007 | 225.61 |

sin 2A = 0.532 | 9 | 59.048 |

sin 2A = -0.532 | 50.250 |

U = (f/2) cos²φ₁ sin 2A = -2.804 | 548.10 |

VT = 0.72023 | 15 | 10 | - 4 |

δ₁ = VT - U = 0.534 | 78 | 10 | - 4 |

δ₂ = δ₁ + 3 | 16.46 |

- A = -16 | 15 | 59.624 |

\( a₁ = 15° 48' 16.939 \) = 180° - A + δ₁

\( a₂ = 190° 39' 31.445 \) = 180° + B + δ₂

Line No. 12 (See Tables 1, 2 - pages 65, 66)
APPENDIX 3

Computations
Using Forsyth-Andoyer-Lambert Type
Second Order Formulae
Without Conversion to Parametric Latitude
DISTANCE COMPUTING FORM — ANDOYER-LAMBERT
TYPE APPROXIMATION WITH SECOND ORDER TERMS

(No conversion to parametric latitudes)
Clarke Spheroid 1866, a = 6,378,206.4 meters

\[ f/2 = 0.00169503765, \quad f/4 = 0.000847518825, \quad f^2/128 = 0.0897860195 \times 10^{-6} \]

\[ \phi_1 = 40^\circ 30' 18.75^\prime, \quad \phi_1 = 6^\circ 00' 00.00^\prime, \quad \phi_2 = 6^\circ 00' 00.00^\prime, \quad \phi_2 = 40^\circ 16.92^\prime \]

\[ \sin \phi_1 = 0.649, \quad \sin \phi_2 = 0.642 \]
\[ \cos \phi_1 = 0.760, \quad \cos \phi_2 = 0.766 \]
\[ \tan \phi_1 = 0.839, \quad \tan \phi_2 = 0.969 \]

\[ \cos \Delta = \lambda - \lambda_1 \]
\[ \delta u = V - U = 0.00199444 \]
\[ \delta v = -U + V = 0.00199444 \]

\[ M = \cos \phi_1 \tan \phi_2 - \sin \phi_1 \cos \Delta \]
\[ N = \cos \phi_2 \tan \phi_1 - \sin \phi_2 \cos \Delta \]

\[ K_1 = -0.0028 \sin \phi_1 \sin \phi_2 \]
\[ K_2 = -0.0039 \sin \phi_1 \sin \phi_2 \]

\[ X = K_1 + K_2 + 0.00199444 \]
\[ Y = -0.00199444 \]

\[ X^2 = 1.0008982 \times 10^{-6}, \quad Y^2 = -0.00199444 \times 10^{-6} \]

\[ \delta d = (f/2) \cos^2 \phi_1 \sin \phi_2 \sin \phi_1 \]
\[ \delta d = (f/2) \cos^2 \phi_1 \sin \phi_2 \]

\[ \delta d_1 = \delta d - 0.00199444 \]
\[ \delta d_2 = \delta d + 0.00199444 \]

\[ S(\delta d_1) = a(d_1 + \delta d_1) \]
\[ S(\delta d_2) = a(d_1 + \delta d_2) \]

\[ T = d/d \sin d \]

Line No. 1, See Tables 1 and 2. True distance 80,466.490 meters.
DISTANCE COMPUTING FORM — ANDOYER-LAMBERT  
TYPE APPROXIMATION WITH SECOND ORDER TERMS  
(No conversion to parametric latitudes)  
Clarke Spheroid 1866, a = 6,378,206.4 meters

\[ f/2 = 0.00169503765, \quad f/4 = 0.000847518825, \quad f^2/128 = 0.0897860195 \times 10^{-6} \]

| \( \phi_1 \) | 9 | 59 | 46.54 | 1. | \( \lambda_1 \) | 16 | 31 | 55.87 |
| \( \phi_2 \) | 10 | 00 | 00.000 | 2. | \( \lambda_2 \) | 18 | 00 | 00.000 |
| \( \sin \phi_1 \) | 9.193 | 59.85 | 1 | 2. | \( \cos \phi_1 \) | 7.193 | 46.84 | 1.52 | 415.35 |
| \( \cos \phi_1 \) | 7.193 | 46.84 | 1. | 52 | \( \tan \phi_1 \) | 9.193 | 59.85 | 185.87 |
| \( \Delta \phi \) | 9.193 | 59.85 | 1. | 52 | \( \Delta \lambda \) | 16 | 31 | 55.87 |

Line No. 2, See Tables 1 and 2. True distance 160.932, 956 meters.
DISTANCE COMPUTING FORM — ANDOYER-LAMBERT
TYPE APPROXIMATION WITH SECOND ORDER TERMS

(No conversion to parametric latitudes)
Clarke Spheroid 1866, a = 6,378,206.4 meters
f/2 = 0.00169503765, f/4 = 0.000847518825, f^2/128 = 0.0897860195 x 10^-6

\begin{align*}
\phi_1 &= 69.48.05.001 \\
\phi_2 &= 90.00.00.000 \\
\sin \phi_1 &= 0.938502357 \\
\cos \phi_1 &= 0.34527226 \quad \sin \phi_2 = 0.93968262 \quad \sin \Delta \lambda = 0.31313131 \\
\tan \phi_1 &= 0.9176122565 \quad \tan \phi_2 = 0.240404040 \\
\tan \phi_1 \cdot \tan \phi_2 &= 0.02043826 \quad \cos \phi_1 \cdot \cos \phi_2 = 0.99929458 \\
M &= \cos \phi_1 \tan \phi_2 - \sin \phi_1 \cos \Delta \lambda \\
N &= \cos \phi_2 \tan \phi_1 - \sin \phi_2 \cos \Delta \lambda \\
\sin d &= \cos \phi_1 \sin \Delta \lambda / \sin v = \cos \phi_2 \sin \Delta \lambda / \sin u \\
\csc d &= 1 + \cos d + 1.998934158 \\
\cot d &= \frac{1 + \cos d}{1 + \cos d} \quad \sin u = 1.99058100 \\
K_1 &= (\sin \phi_1 + \sin \phi_2)^2 \quad \sin v = 1.00000000 \\
K_2 &= (\sin \phi_1 - \sin \phi_2)^2 (1 + \cos d) \\
X &= K_1 + K_2 + 1.7660444444 \\
Y &= K_1 - K_2 + 1.76380610 \\
X^2 &= 3.11891.256 \\
Y^2 &= 3.11811.1166 \\
A &= 64d_1 + 16d_2^2 \cot d + 4.024384564 \\
B &= -2D - 5.63933866 \\
C &= -2(30d_1 + 8d_2^2 \cot d + E/2) - 3.418383907 \\
BY &= 7.63611.6999 \\
EY^2 &= 3.4355432666 \\
\Sigma &= AX + BY + CX^2 + DXY + EY^2 + 4.65868810 \\
\delta d_f &= -(f/2) (X_d - 3Y \sin d) + 0.00000000 \\
\delta d_f^2 &= +(f^2/128) \Sigma + 0.00000000 \\
S(\delta d_f) &= a(d_e + \delta d_f) - 3.31.862.977.916 \quad S(\delta d_f^2) = a(d_e + \delta d_f + \delta d_p) - 3.31.865.641 \quad m \\
\end{align*}

\[ T = \frac{d}{\sin d} = 1.00042 \]

\[ \begin{align*}
2u &= 164.89 \quad 15 \quad 35.154 \\
\sin 2u &= 0.271276641 \\
U &= (f/2) \cos^2 \phi_1 \sin 2u + 5.48169 \times 10^{-5} \\
VT &= 2.182 \times 10^{-5} + 0.4839 \times 10^{-5} \\
\delta u &= VT - U - 5.4839 \times 10^{-5} \\
+ \delta u &= 0.8207 + 0.5799 \\
- u &= 11.311 \\
+ 180 &= 52.01.112 \\
a_{11} &= 52 \quad 01.112 \\
\end{align*} \]

\[ a_{12} = a_{UV} = 180^\circ - u + \delta u \]

\[ a_{12} = a_{uv} = 180^\circ + v + \delta v \]

Line No. 3, See Tables 1 and 2. True distance 321.866.796 meters.
DISTANCE COMPUTING FORM — ANDOYER-LAMBERT TYPE APPROXIMATION WITH SECOND ORDER TERMS

(No conversion to parametric latitudes)

Clarke Spheroid 1866, a = 6,378,206.4 meters

\[ f/2 = 0.00169503765, \quad f/4 = 0.000847518825, \quad f/128 = 0.089760195 \times 10^{-6} \]

| \( \phi_1 \) | 13.0 | 12.574 | 1. Origin | \( \lambda_1 \) | 14.57 | 13.283 |
| \( \phi_2 \) | 10.0 | 0.000 | 2. TERMINUS | \( \lambda_2 \) | 18.0 | 0.000 |
| \( \sin \phi_1 \) | 0.226 | 143.897 | | | |
| \( \cos \phi_1 \) | 0.974 | 0.093 | | | |
| \( \tan \phi_1 \) | 0.317 | 6.982 | | | |
| \( \sin \phi_2 \) | 0.044 | 0.053 | | | |
| \( \cos \phi_2 \) | 0.988 | 0.116 | | | |
| \( \tan \phi_2 \) | 0.276 | 3.268 | | | |
| \( \Delta \lambda = \lambda_2 - \lambda_1 \) | 3.08 | 46.919 | | | |
| \( M = \cos \phi_1 \tan \phi_2 - \sin \phi_1 \cos \Delta \lambda \) | -1.054 | 0.044 | 0.053 | \( \cot u = M/\sin \Delta \lambda \) | -0.984 | 0.6223 |
| \( N = \cos \phi_1 \tan \phi_2 - \sin \phi_2 \cos \Delta \lambda \) | 0.005 | 0.044 | 0.053 | \( \cot v = N/\sin \Delta \lambda \) | 1.006 | 546.36 |
| \( \sin d = \cos \phi_1 \sin \Delta \lambda/\sin v = \cos \phi_2 \sin \Delta \lambda/\sin u \) | 0.005 | 0.044 | 0.053 | u | 0.134 | 0.3576 |
| \( \csc d = \cot d \) | 0.005 | 0.044 | 0.053 | v | 0.134 | 0.3576 |
| \( 1 + \cos d \) | 1.997 | 11.69 | 1 - \cos d | 0.028 | 0.13 |
| \( \sin \phi_1 + \sin \phi_2 \) | 0.143 | 0.093 | | | |
| \( K_1 = (\sin \phi_1 + \sin \phi_2)^2/(1 + \cos d) \) | 0.000820189 | K_2 = (\sin \phi_1 - \sin \phi_2)^2/(1 - \cos d) | 0.856 | 0.434 | 0.23 |
| X = K_1+K_2 | 0.156 | 0.073 | Y = K_1 - K_2 | -0.174 | 0.078 |
| X^2 + 1.074 | 0.073 | Y^2 + 1.016 | 0.073 | d_r | 0.090 | 0.073 |
| A = 64\,d_r + 16d_r^2 \cot d | 1.022 | 0.073 | D = 48 \sin d + 8d_r^2 \csc d | 0.249 | 0.170 |
| B = -2d_r - 8d_r^2 \cot d | 0.997 | 0.073 | 1.51 | 0.272 |
| C = -\left(30d_r + 8d_r^2 \cot d + E/2\right) \sin d = 0.0053922 \text{ AX} = 6.283561457 | \Sigma = A + B + C + D + E/2 | 9.851 | 4.072 |
| BY | 0.480 | 0.073 | DX | -3.859 | 0.073 |
| EY^2 | 0.485 | 0.073 | D = 3.859 | 0.073 |
| \delta d_1 | -\left(4/9\right) \left(X_d - 3Y \sin d\right) = 0.0038359 | | 7.0.058 | 10.7 |
| \delta d_1^2 | + \delta d_1 + \delta d_1^2 | 0.058 | 14.0 | |
| S(\delta d_2) = a(\delta d + \delta d_2) | 0.782 | 0.54 | |
| T = d/\sin d | 1.000 | 0.228 |
| 2u | 369 \degree | 0', | 51.992 \degree | 2v | 89 \degree | 37 \degree |
| \sin 2u | -0.998 | 88056 | \sin 2v | +0.999 | 97876 |
| U = (f/2) \cos^2 \phi \sin 2u = -0.001606551 | V = (f/2) \cos^2 \phi \sin 2v | +0.001643891 |
| VT | 0.00164 | 4.730 | UT | -0.0016080971 |
| d \delta u | VT - Ut | 0.001352 | 0.241 |
| + \delta u | 11 | 10.998 |
| -u | 131 \degree | 33 \minute | 25.986 |
| a_{1-2} | +180 \degree | 45 \degree | |
| a_{2-3} = a_{uv} = 180 \degree - u + \delta u | +180 \degree | 45 \degree |
| a_{1-3} = a_{uv} = 180 \degree + v + \delta v | +180 \degree | 57 \degree | 58.449 |

Line No. 4, See Tables 1 and 2. True distance 482.258.163 meters.
### DISTANCE COMPUTING FORM — ANDOYER-LAMBERT

**TYPE APPROXIMATION WITH SECOND ORDER TERMS**

(No conversion to parametric latitudes)

Clarke Spheroid 1866, a = 6,378,206.4 meters

\( f/2 = 0.00169503765, f/4 = 0.000847518825, f^2/128 = 0.0897860195 \times 10^{-6} \)

| \( \phi_1 \) | 93 | 35 | 09.306 | 1. Origin | \( \lambda_1 \) | 3 | 16 | 35.41 |
| \( \phi_2 \) | 70 | 00 | 00.000 | 2. Terminus | \( \lambda_2 \) | 18 | 00 | 00.00 |
| sin \( \phi_1 \) | -0.959 | 24.44 | 1. west of l. | \( \Delta \lambda = \lambda_2 - \lambda_1 \) | 14.33 | 34.89 |
| cos \( \phi_1 \) | 0.282 | 57968 | \( \sin \phi_2 \) | -0.9896262 | \( \sin \Delta \) | -451 | 34.16 |
| tan \( \phi_1 \) | -0.394 | 62200 | \( \cos \phi_2 \) | -0.3420354 | \( \cos \Delta \) | -0.967 | 88.44 |
| tan \( \phi_2 \) | -2.744 | 40044 | \cos d = \sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2 \cos \Delta \) | +0.34 | 49.33 |
| M = \cos \phi_1 \tan \phi_2 - \sin \phi_1 \cos \Delta \) | -1.25 | 87537 | \( \cot u = \cos \phi_2 \) | -0.605 | 05.49 |
| N = \cos \phi_2 \tan \phi_1 - \sin \phi_2 \cos \Delta \) | +0.25 | 150207 | \( \cot v = N/\sin \Delta \) | +1.006 | 38.37 |
| sin d = \cos \phi_1 \sin \Delta /\sin v = \cos \phi_2 \sin \Delta /\sin u \) | ±1.00 | 49.45 | \( u = 1.24 \) | 34.46 |
| \( \csc d \) | +0.982 | 23010 | \( v = 1.47 \) | 59.15 |
| \( 1 + \cos d \) | +1.199 | 93963 | \( 1 - \cos d \) | \( \cos \phi_1 \) | +0.005 | 06.05 |
| \( \sin u \) | +0.855 | 59.96 |
| \( \sin (\phi_1 + \phi_2) \) | +0.605 | 94.84 |
| \( \sin (\phi_1 - \phi_2) \) | -0.003 | 37.22 | \( \sin v \) | -0.006 | 88.11 |

K_1 = (\sin (\phi_1 + \phi_2) / (1 + \cos d)) + 1.80765437

K_2 = (\sin (\phi_1 - \phi_2) / (1 - \cos d)) + 1.0255423022

X = K_1 + K_2 + 1.88306667 \quad \begin{align*}
Y &= K_1 - K_2 + 1.7320827 \quad \begin{align*}
XY &= +0.36154966
\end{align*}
\end{align*}

X^2 + 3.54c_1c_2 = +2.98921581 \quad d_1 + \delta d_1 \quad +0.10129.283

A = 64d_1 + 16d_2 \cot d + 8.006 \quad D = 48 \sin d + 8d_2 \csc d + 5.629 \quad 29.324

B = -2D -1.12851408 \quad E = 30 \sin 2d \quad +5.999 \quad 96.20 \quad \sin 2d \quad +1.19393.21

C = -(30d_1 + 8d_2 \cot d + E/2) -6.82094412 \quad AX \quad +1.5 \quad 15545.36

BY = -19.50600325 \quad \begin{align*}
CY^2 &= -24.18672878 \quad \begin{align*}
DXY &= +18.360 \quad 15847
\end{align*}
\end{align*}

\delta d_1 = -(f/4) (X_d_1 \quad -0.9Y \sin d) +0.9999 \quad 0.9999 \quad +0.979 \quad 96.0

\delta d_2 = + (f/128) \quad +0.999 \quad 96.0

S(\delta d_f) = a(d_1 + \delta d_1) \quad 643,732,340 \quad m \quad S(\delta d_f) = a(d_1 + \delta d_1 + \delta d_2) \quad 643,732,440 \quad m

| a_{uv} | 180° - \u |
| \( a_{uv} \) | 180° + \v + \delta v |

Line No. 5, See Tables 1 and 2. True distance \( 643,732,440 \) meters.
DISTANCE COMPUTING FORM, ANDOYER-LAMBERT

TYPE APPROXIMATION WITH SECOND ORDER TERMS

(No conversion to parametric latitudes)
Clarke Spheroid 1866, a = 6,378,206.4 meters
\( f/2 = 0.00169503765, f/4 = 0.000847518825, f^2/128 = 0.0897860195 \times 10^{-6} \)
1 radian = 206,264.8062 seconds

\[ \begin{array}{c|c|c|c}
\phi_1 & 9 & 55 & 09.138 \\
\phi_2 & 10 & 00 & 00.00 \\
\phi_m = \frac{1}{2}(\phi_1 + \phi_2) & 9 & 57 & 34.569 \\
\Delta\phi_m = \frac{1}{2}(\phi_2 - \phi_1) & 2 & 25.431 \\
\sin \phi_m & +0.17295377 & \sin \Delta\phi_m & +0.00070507 \\
\cos \phi_m & +0.98492994 & \cos \Delta\phi_m & +0.99999975 \\
\lambda_1 & 10 & 39 & 43.554 \\
\lambda_2 & 18 & 00 & 00.00 \\
\Delta\lambda & 7 & 20 & 16.446 \\
\Delta\lambda_m = \frac{1}{2} \Delta \lambda & 3 & 40 & 08.223 \\
\sin \Delta\lambda & +0.12772073 & \sin \Delta\lambda_m & +0.06399152 \\
K & = \sin \phi_m \cos \Delta\phi_m + 0.00069444 \\
H & = \cos^2 \phi_m - \sin^2 \phi_m = \cos^2 \phi_m - \sin^2 \Delta \phi_m + 0.97008649 \\
L & = \sin^2 \Delta \phi_m + H \sin^2 \Delta \lambda_m + 0.00397292 \\
d & + 0.1261458534 & \sin d & + 0.12581156 \\
U & = 2k^2/(1 - L) + 0.060064618 & V & = 2k^2/L + 0.000242767 \\
X & = U + V & Y & = U - V + 0.059821851 \\
A & = 4T(16 + ET/15) + 80.12738460 & C & = 2T - \frac{1}{2}(A + E) - 67.82000290 \\
\beta & = 2D & B & = -2D \\
\beta & = 2L & DXY & = 0.0218475 \\
(TX - 3Y) & -0.118997925 & \delta f & = -(f/4)(TX - 3Y) + 1.00853 \times 10^{-4} \\
T + \delta f & + 1.02075795 & S_1 & = a \sin d \left(T + \delta f\right) \\
\Sigma & = X(A + CX) + Y(B + EY) + DXY & -1.70432971 & \delta f^2 = \left(f^2/128\right) \Sigma & - 1.53 \times 10^{-7} \\
T + \delta f + \delta f^2 + 1.02075780 & S_2 & = a \sin d \left(T + \delta f + \delta f^2\right) \\
\sin (\alpha_2 + \alpha_1) = (K \sin \Delta \lambda)/L & \alpha_2 + \alpha_1 & = \alpha_2 - \alpha_1 \\
\sin (\alpha_2 - \alpha_1) = (K \sin \Delta \lambda)/(1 - L) & \alpha_2 - \alpha_1 & = 361 \ 16 \ 45.188 \\
(\delta \alpha_1 + \delta \alpha_2) = -(f/2) H \ (T + 1) \ \sin (\alpha_2 + \alpha_1) & = 7.351613 \times 10^{-5} \\
(\delta \alpha_2 - \delta \alpha_1) = -(f/2) H \ (T - 1) \ \sin (\alpha_2 - \alpha_1) & = -7.350644 \times 10^{-5} \\
\alpha_1 & 91 & 16 & 30.040 \\
\delta \alpha_1 & = -15.162 \\
\alpha_{1-2} & 91 & 16 & 14.878 \\
\alpha_{1-2} = \alpha_1 + \delta \alpha_1 \\
d = 7 \degree 13' 39.9450 \\
Line No. 6, see Tables 1 and 2. (Pages 65,66)
DISTANCE COMPUTING FORM — ANDOYER-LAMBERT
TYPE APPROXIMATION WITH SECOND ORDER TERMS
(No conversion to parametric latitudes)
Clarke Spheroid 1866, a = 6,378,206.4 meters
\( f/2 = 0.00169503765 \), \( f/4 = 0.000847518825 \), \( f^2/128 = 0.0897860195 \times 10^{-6} \)

| \( \phi_1 \) | 14° 54' 28.50" | 1. Origin | \( \lambda_1 \) | 10° 10' 43.883' |
| \( \phi_2 \) | 40° 00' 00.00" | 2. Terminus | \( \lambda_2 \) | 7° 12' 16.119' |

\[ \sin \phi_1 = \frac{1}{11.1216} \]
\[ \cos \phi_1 = \frac{7.4903}{11.1216} \]
\[ \tan \phi_1 = \frac{0.6627}{11.1216} \]

\[ \Delta \lambda = \lambda_2 - \lambda_1 \]
\[ \cos \Delta = \frac{1}{10.8920} \]
\[ \sin \Delta = \frac{0.0125}{10.8920} \]

\[ M = \cos \phi_1 \tan \phi_2 - \sin \phi_1 \cos \Delta = -1.06 \]
\[ \cot u = M/\sin \Delta = -0.89 \]

\[ N = \sin \phi_1 \tan \phi_2 - \sin \phi_2 \cos \Delta = +1.06 \]

\[ \sin d = \cos \phi_1 \sin \Delta/\sin \phi_2 \cos \Delta = \cos \phi_2 \sin \Delta/\sin \phi_1 \cos \Delta = +1.06 \]
\[ \cos d = \sin \phi_1 \sin \phi_2 \cos \Delta = \cos \phi_2 \sin \phi_1 \cos \Delta = +1.06 \]

\[ \cot v = N/\sin \Delta = -0.89 \]

\[ \sin \phi_2 = \sin \phi_1 \sin \Delta/\sin \phi_2 = \cos \phi_1 \cos \Delta = +1.06 \]
\[ \cos \phi_2 = \sin \phi_1 \cos \Delta/\sin \phi_2 = \cos \phi_1 \sin \Delta = +1.06 \]

\[ \Delta \phi = \sum (\sin \phi_1, \sin \phi_2) \]

\[ D = 48 \sin d + 8 d^2 \cos d = +1.06 \]

\[ B = -2D + 16 d + 2 \cot d = +1.06 \]

\[ C = -(3d^2 + 8 \cot d + 6 \sin d) \]

\[ \cos \phi_{12} = \cos \phi_1 \sin \Delta/\sin \phi_2 = \cos \phi_2 \sin \Delta/\sin \phi_1 = +1.06 \]

\[ \cot \phi_{12} = N/\sin \Delta = -0.89 \]

Line No. 7, See 'ables 1 and 2. True distance
\[ 804.664.771 \text{ meters.} \]
DISTANCE COMPUTING FORM — ANDOYER-LAMBERT
TYPE APPROXIMATION WITH SECOND ORDER TERMS

(No conversion to parametric latitudes)
Clarke Spheroid 1866, a = 6,378,206.4 meters
f/2 = 0.00169503765, f/4 = 0.000847518825, f^2/128 = 0.0897860195 \times 10^{-6}

\phi_1 + 96.00 26.832' 1. Origin \lambda_1 28 42 03.567 E
\phi_2 + 90.00 00.000 W
\sin \phi_1 \times 900 \times 3692
\cos \phi_1 \times 2.41 996 95 \sin \phi_2 \times 2.9369262 \sin \Delta = +22 758 62
\tan \phi_1 \times 4.012 9158 \cos \phi_2 \times +34 205 34 \cos \Delta
\tan \phi_2 \times 2.474 4074 \cos \Delta = \sin \phi_2 \sin \phi_1 + \cos \phi_1 \cos \phi_2 \cos \Delta = +29 685 2493
M = \cos \phi_1 \tan \phi_2 - \sin \phi_1 \cos \Delta = 0.011 246 9 \cot \mu = M / \sin \Delta = 0.001 645 36
N = \cos \phi_2 \tan \phi_1 - \sin \phi_2 \Delta \cot \mu = N / \sin \Delta = 7 100 050 491
\sin d = \cos \phi_2 \sin \Delta / \sin v = \cos \phi_1 \sin \Delta / \sin \mu + 2.48 919 30 u + 90 05 15.823
\csc d + 4.010 9855 \cot d + 3.890 9493 \sin d = \cos \phi_2 \sin \Delta / \sin v = \cos \phi_1 \sin \Delta / \sin \mu + 2.48 919 30 v + 90 05 15.823
1 + \cos d + 1 968 52 495 -1 \cos d + 1.091 405 25 \sin u = +29 685 2493
(sin \phi_1 + \sin \phi_2) \times 5.44 1146 \times 180 \sin \phi_1 - \sin \phi_2) \times 3.34 403 10 \sin v = +70 965 356
K_1 = (\sin \phi_1 + \sin \phi_2)^2 / (1 + \cos d) + 1.85 85 31 \times 180 K_2 = (\sin \phi_1 - \sin \phi_2)^2 / (1 - \cos d) + 0.29 81 583
X = K_1 + K_2 + 1.85 85 31 \times 180 Y = - K_1 - K_2 + 1.85 85 31 \times 180 XY = +3.45 65 871
X^2 + 3.55 949 63 Y^2 + 3.32 492 38 \sin d_r = 2 51 62 22 8 \sin d_2 = 0.63 28 35 14
A = 64 d_r + 16 d_2 \cot d_r + 20 39 10 22 \sin 2d_r + 8 d_2 \csc d_r + 4.89 71 15
B = -2D - 17 96 38 430 E = 30 \sin 2d_r + 14 34 49 53 \sin 2d_r + 48 14 54 13
C = -30 d_r + 8 d_2 \cot d_r + E / 2 -16 74 92 10 3 AX = 37 29 15 72 2
BY = 70 99 30 28 CX^2 = 8 79 39 18 12 7 DXY = + 48 00 06 17
EY^2 + 48 04 86 68 \Sigma = AX + BY + CX^2 + DXY + EY^2 + 13 45 80 19
\delta d_f = -(f / 4) (X_d - 3 Y \sin d) + 0.005 2 55 12 \delta d_f^2 = + (f / 128) \Sigma + 12 10 02 21 X + 10 - 6
\delta d_r + \delta d_f = + 3 23 14 75 88 d_r + \delta d_f = + 2 55 31 68 5
S(\delta d_r) = a(d_r + \delta d_r) \times 1 60 9 31 69 \times 10^{-3} m S(\delta d_f) = a(d_r + \delta d_f + \delta d_f) \times 1 60 9 32 9 043 \times 10^{-3} m
T = d / \sin d = + 1 00 76 65 64 / 7
2u + 180 10 37 486
sin 2u - 0.003 09 01
U = (f / 2) \cos^2 \phi_1 \sin 2u + -2.06 2 94 91 \times 10^{-4} V = (f / 2) \cos^2 \phi_1 \sin 2v + 1.88 28 91 25 \times 10^{-4}
VT = + 30 38 19 X - 10 - 4 \delta v = - UT + V + 1.88 28 91 25 X - 10 - 4
\delta u + \delta v + 41.396 + 41.396 + 18 053 + 18 753 + 18 053 + 18 753
+ 180 89 05 22 643 + 180 89 05 22 643
\alpha_{u_1} = \alpha_{u_1} = 180^0 - u + \delta u
\alpha_{v_1} = \alpha_{v_1} = 180^0 + v + \delta v

Line No. 8, See Tables 1 and 2. True distance, 1 609 329.060 meters.
DISTANCE COMPUTING FORM — ANDOYER-LAMBERT
TYPE APPROXIMATION WITH SECOND ORDER TERMS

(No conversion to parametric latitudes)
Clarke Spheroid 1866, a = 6,378,206.4 meters

\[ f/2 = 0.00169503765, f/4 = 0.000847518825, f^2/128 = 0.0897860195 \times 10^{-6} \]

\[
\phi_1 = 27 \ 49 \ 42.130 \ \text{1. Origin.} \quad \lambda_1 = 32 \ 54 \ 12.999 \ E
\]
\[
\phi_2 = 40 \ 00 \ 00.000 \ \text{2. Terminator} \quad \lambda = 18 \ 00 \ 00.000 \ W
\]
\[
\sin \phi_1 = 0.666 \ 92458 \ \text{2. west of 1.} \quad \Delta \lambda = \lambda - \lambda_1 = 50 \ 54 \ 12.999
\]
\[
\cos \phi_1 = 0.740 \ 28916 \sin \Delta \lambda = 2.776 \ 08614
\]
\[
\tan \phi_1 = 0.907 \ 89714 \cos \phi_1 = -0.460 \ 44444 \cos \Delta \lambda = 6.630 \ 62694
\]
\[
\tan \phi_2 = 0.689 \ 94963 \ \cos \Delta = \sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2 \cos \Delta \lambda = 0.727 \ 28811
\]
\[
M = \cos \phi_1 \tan \phi - \sin \phi_1 \cos \Delta \lambda = 0.000 \ 984 \ 888 \ \cot \mu = M / \sin \Delta \lambda = 7.576 \ 82459
\]
\[
N = \cos \phi_1 \sin \phi_2 \cos \Delta \lambda = -0.000 \ 984 \ 888 \ \cot \mu = N / \sin \Delta \lambda = 7.576 \ 82459
\]
\[
\sin d = \cos \phi_1 \sin \Delta / \sin \mu = \cos \phi_2 \sin \Delta / \sin \mu = 0.666 \ 35229 \ u = 0.00 \ 01 \ 32.319
\]
\[
\csc d = 1.457 \ 02018 \ \cot d = 1.059 \ 69346 \ v = 0.04 \ 31.888
\]
\[
1 + \cos d = 1.797 \ 28881 \ 1 \cos d = 1.797 \ 28881 \ \sin u = 0.566 \ 22517
\]
\[
(\sin \phi_1 + \sin \phi_2)^2 = 0.132 \ 39211 \ (\sin \phi_1 - \sin \phi_2)^2 = 0.036 \ 96299 \ \sin v = 0.999 \ 99919
\]
\[
K_1 = (\sin \phi_1 + \sin \phi_2)/(1 + \cos d) = 0.048 \ 86852 \ K_2 = (\sin \phi_1 - \sin \phi_2)/(1 - \cos d) = 0.048 \ 53736
\]
\[
X = K_1 + K_2 = 1.826 \ 35371 \ Y = K_1 - K_2 = 1.549 \ 28929 \ XY = 1.495 \ 21843
\]
\[
X^2 + Y^2 = 682 \ 86045 \ Y^2 = 682 \ 86045 \ d = 0.056 \ 43398 \ d = 0.592 \ 19238
\]
\[
A = 64d + 16d^2 \ \text{cot} d = 58.113 \ 16.031 \ D = 48 \ \sin d + 8d^2 \ \text{csc} d = 37.531 \ 4887
\]
\[
B = -2d = -9.042 \ 9917 \ E = 30 \ \sin 2d = 29.949 \ 6783 \ \text{sin} 2d = 29.949 \ 3256
\]
\[
C = -(30d + 8d^2) \ \text{cot}(d + 6d^2) = -24.978 \ 554 \ \text{AX} = 148.022 \ 0341
\]
\[
BY = 44.916 \ 69970 \ CX^2 = 29.034 \ 23950 \ DXY = 18.581 \ 25731
\]
\[
\text{EY}^2 = 10.555 \ 88700 \ \Sigma = AX + BY + CX^2 + DXY + EY^2 = 48.553 \ 85672
\]
\[
\delta d_f = -(f/4)(X_d - 3Y \ \sin d) = 0.000 \ 558 \ 996 \ \delta d_f^2 = (f^2/128) \delta d_f^2 = 0.000 \ 000 \ 8011
\]
\[
\delta d_f - f^2/128 \delta d_f = 0.256 \ 944 \ 904 \ \delta d_f^2 + \delta d_f^2 + \delta d_f^2 = 0.56 \ 850 \ 275
\]
\[
\delta ddp = d + \delta d_f + \delta d_f^2 = 0.56 \ 850 \ 275
\]
\[
S(dddp) = a(d_f + \delta d_f + \delta d_f^2) = 0.825 \ 985. \ 088 \ m
\]

\[ T = d / \sin d = 1.102 \ 1357 \ 04 \]
\[ a = 180 \ 08 \ 43.516 \]

\[ \sin 2u = 0.566 \ 22517 \ \sin 2v = 0.000 \ 558 \ 996 \]
\[ U = (f/2) \ \cos^2 \phi_1 \ \sin 2u = 0.004 \ 9583022 \ V = (f/2) \ \cos^2 \phi_2 \ \sin 2v = -2.524 \ 59 \times 10^{-6}
\]
\[ V = 0.004 \ 9583022 \]
\[ V = 0.004 \ 9583022 \]
\[ \delta u = UT - U = -0.001 \ 5 \ 4 \ 967 \]
\[ + \delta u = -3.57 \ 267 \]
\[ -u = 60 \ 01 \ 21.339 \]
\[ +180 \ 00 \ 00 \ 00.88 \]
\[ a_{uv} = 180^\circ - u + \delta u \]

Line No. 9, See Tables 1 and 2. True distance = 4.829 \ 984.247 \ meters.
DISTANCE COMPUTING FORM — ANDOYER-LAMBERT
TYPE APPROXIMATION WITH SECOND ORDER TERMS

(No conversion to parametric latitudes)

Clarke Spheroid 1866, a = 6,378,206.4 meters

\( f/2 = 0.00169503765, \ f/4 = 0.000847518825, \ f^2/128 = 0.0897860195 \times 10^{-6} \)

\[
\begin{align*}
\phi_1 & = 35^\circ 18' 45.644^\prime \text{ N.} & \phi_2 & = 18^\circ 45.644^\prime \text{ N.} \\
\sin \phi_1 & = 0.5821 & \sin \phi_2 & = 0.0990 \\
\cos \phi_1 & = 0.8116 & \cos \phi_2 & = 0.6426 \\
\tan \phi_1 & = 0.728 & \tan \phi_2 & = 0.0990 \\
\end{align*}
\]

\[
\begin{align*}
\cos \Delta \lambda & = \lambda_2 - \lambda_1 = 2^\circ 02' 19.270 \\
\sin \Delta \lambda & = 0.0897860195 \times 10^{-6} \\
\tan \Delta \lambda & = 0.0897860195 \\
\cos \Delta \lambda & = 0.8116 \\
\sin \Delta \lambda & = 0.0990 \\
\end{align*}
\]

\[
\begin{align*}
\Delta = & 0.000847518825 \times 10^{-6} \times 128 = 0.0897860195 \\
\delta \phi & = \phi_2 - \phi_1 \\
\delta \lambda & = \lambda_2 - \lambda_1 \\
\end{align*}
\]

\[
\begin{align*}
\text{Dist} & = \Delta + \delta \phi + \delta \lambda \\
\end{align*}
\]

Line No.10, See Tables 1 and 2. True distance 9,655.969.751 meters.
DISTANCE COMPUTING FORM — ANDOYER-LAMBERT

TYPE APPROXIMATION WITH SECOND ORDER TERMS

(No conversion to parametric latitudes)

Clarke Spheroid 1866, a = 6,378,206.4 meters

\[ f/2 = 0.00169503765, \quad f/4 = 0.000847518825, \quad f^2/128 = 0.0897860195 \times 10^{-6} \]

\[
\begin{align*}
\phi_1 & = 2 \quad 55 \quad 12.425(\text{N}) \quad \text{ORIGIN} \quad \lambda_1 \quad 70 \quad 50 \quad 0.869E \\
\phi_2 & = 70 \quad 00 \quad 00.000(\text{N}) \quad \text{TERMINUS} \quad \lambda_2 \quad 18 \quad 00 \quad 00.000W
\end{align*}
\]

\[
\sin \phi_1 = 0.950 \quad 967.83 \quad 2, \text{ west of } 1.
\]

\[
\begin{align*}
\cos \phi_1 & = 0.998 \quad 700.29 \quad \sin \phi_2 = 0.7896262 \quad \sin \Delta & = 0.999 \quad 78.318 \quad \\
\tan \phi_1 & = 0.351 \quad 0.34/16 \quad \cos \phi_2 = 0.34002014 \quad \cos \Delta & = 0.33719 \\
\tan \phi_2 & = 3.747 \quad 4.972 \quad \cos d = \sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2 \cos \Delta \quad 0.5484028
\end{align*}
\]

\[
\begin{align*}
M &= \cos \phi_1 \tan \phi_2 - \sin \phi_1 \cos \Delta \quad + 65.59781 \quad \cot u = M/\sin \Delta \quad + 2.9434373 \, 3 \\
N &= \cos \phi_1 \tan \phi_1 - \sin \phi_1 \cos \Delta \quad - 0.01656.321
\end{align*}
\]

\[
\begin{align*}
\sin d &= \cos \phi_1 \sin \Delta / \sin v = \cos \phi_1 \sin \Delta / \sin u \quad + 49.9495711 \\
\csc d &= +1.00150916 \quad \cot d = +0.5492347 \quad v &= 90 \quad 05 \quad 41.649 \\
1 + \cos d &= +0.05484028 \quad 1 - \cos d = +0.745 \quad 15.323 \quad \sin u = +0.354 \quad 42.78
\end{align*}
\]

\[
\begin{align*}
(\sin \phi_1 + \sin \phi_2)^2 + 0.781408372(\sin \phi_1 - \sin \phi_2)^2 &= 0.89837152
\end{align*}
\]

\[
\begin{align*}
K_1 &= (\sin \phi_1 + \sin \phi_2)^2 (1 - \cos d) + 0.350803 \quad K_2 &= (\sin \phi_1 - \sin \phi_2)^2 (1 - \cos d) + 0.3565.8988 \\
X &= K_1 + K_2 \quad + 104.85056 \quad Y = K_1 - K_2 \quad + 0.4975085 \quad XY = +162.3877
\end{align*}
\]

\[
\begin{align*}
E^2 &= +0.5680819.37 \quad CX^2 &= -150.1434892 \quad DXY &+ 16.09783494 \\
\delta d &= -(f/4)(X_d - 3Y \sin d) = +0.99284936 \quad \delta d^2 &= +0.00000.209668
\end{align*}
\]

\[
\begin{align*}
d &= +1.57389925 \quad d &+ \delta d &= +1.57389925 \\
S(\delta d_p) &= a(d_r + \delta d) \quad 9.655 \quad 963 \quad 633 \quad m \quad S(\delta d_p) &= a(d_r + \delta d + \delta d) \quad 9.655 \quad 977 \quad 008 \quad m
\end{align*}
\]

\[
\begin{align*}
T &= d/\sin d = +1.51821.276
\end{align*}
\]

\[
\begin{align*}
\text{Line No.11, See Tables 1 and 2. True distance} &= 9.655 \quad 997.149 \quad \text{meters.}
\end{align*}
\]

136
DISTANCE COMPUTING FORM, ANDOYER-LAMBERT TYPE APPROXIMATION
WITH SECOND ORDER TERMS
(No conversion to parametric latitudes)
Clarke Spheroid 1866, a = 6,378,206.4 meters
\[ f/2 = 0.00169503765, \frac{f}{4} = 0.000847518825, f^2/128 = 0.0897860195 \times 10^{-6} \]
1 radian = 206,264.8062 seconds

\[ \phi_1 = 70 \quad \phi_2 = 69.46 \quad \phi_m = 69.53 \quad \Delta \phi_m = 0.57 \]
\[ \lambda_1 = 100.99 \quad \lambda_2 = 101.00 \]
\[ \phi_1 = 70 \quad \phi_2 = 69.46 \quad \phi_m = 69.53 \quad \Delta \phi_m = 0.57 \]
\[ \sin \phi_1 = 0.93982474 \quad \sin \Delta \phi_m = -0.00194756 \]
\[ \cos \phi_1 = -0.34384960 \quad \cos \Delta \phi_m = -0.98994810 \]
\[ k = \sin \phi_m \cos \Delta \phi_m = 0.939022956 \]
\[ H = \cos^2 \Delta \phi_m - \sin^2 \phi_m = \sin^2 \Delta \phi_m + 0.189287245 \]
\[ d + 0.93982474 \]
\[ U = 2k^2/(1-b) = 1.167412.109 \]
\[ V = 2k^3/3 \]
\[ X = U + V \quad Y = U - V \quad X = 3.133 \quad Y = 3.133 \]

\[ A = 4[16T + (E/15)T^2] = 80.00040344 \quad D = 8(6 + T^2) = 56.02349808 \]
\[ B = -2D \quad C = -2T - \frac{1}{2}(A + E) = 67.900 \quad 1.1652 \]
\[ AX = 141.1450081384 \quad BY = -197.9894918897 \quad CX^2 = 2.12x12.06558681 \]
\[ DXY = -105.0631497599 \quad EY^2 = 126.5128891 \quad \delta_f = -(f/4)(TX - 3Y) = -002.99292497 \]
\[ T + \delta_f = 1.004459909 \quad S_1 = a \sin d (T + \delta_f) = 95.9995 \quad 255 m \]
\[ T + \delta_f + \delta_f^2 = +10.044682.48 \quad S_2 = a \sin d (T + \delta_f + \delta_f^2) = 600,000.135 m \]

\[ \sin (\alpha_2 + \alpha_1) = K \sin (\Delta \lambda) = 0.01224926 \quad a_2 + a_1 = 355 \quad 16 \quad 56.699 \]
\[ \sin (\alpha_2 - \alpha_1) = (K \sin (\Delta \lambda)/(1-L)) = 253.923575 \quad a_2 - a_1 = 165 \quad 17 \quad 59.341 \]
\[ \delta_f = 0.3298925 \times 10^{-4} \quad \delta_1 = 0.3306336 \times 10^{-4} \]
\[ \delta_2 = 0.32914 \times 10^{-4} \]
\[ a_1 = 260 \quad 17 \quad 09.790 \quad a_2 = 94 \quad 59 \quad 53.139 \]
\[ \delta_1 = 0 \quad 00 \quad 06.820 \quad \delta_2 = 0 \quad 00 \quad 06.289 \]
\[ a_{1,2} = a_1 + \delta_1 \quad a_{2,1} = a_2 + \delta_2 \]
\[ d = 5 \quad 32 \quad 25.447 \quad \text{True distance} = 600,000.00 \]

True Azimuths

| Line No. 12 | 260 | 17 | 09.79 | 95 | 00 | 00.00 |
DISTANCE COMPUTING FORM, ANDOYER-LAMBERT TYPE APPROXIMATION
WITH SECOND ORDER TERMS

(No conversion to parametric latitudes)

Clarke Spheroid 1866, \( a = 6,378,206.4 \) meters

\( f/2 = 0.00169503765, f/4 = 0.000847518825, f^2/128 = 0.0897860195 \times 10^{-6} \)

1 radian = 206,264.8062 seconds

<table>
<thead>
<tr>
<th>( \phi_1 )</th>
<th>( \phi_2 )</th>
<th>( \lambda_1 )</th>
<th>( \lambda_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>60° 00′ 00.00″</td>
<td>54° 18′ 59.31′′</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\( \phi_m = \frac{1}{2}(\phi_1 + \phi_2) = 59° 09′ 19.65′′ \), Always west of 1.

\( \Delta \phi_m = \frac{1}{2}(\phi_2 - \phi_1) = 2° 50′ 30.93′′ \)

\( \sin \phi_m = 0.840 \), \( \sin \Delta \phi_m = -0.049 \), \( \sin d = 0.184 \). \( \sin \Delta \lambda = 0.10 \). \( \sin \Delta \lambda = 0.35 \)

\( \Delta \lambda = \lambda_2 - \lambda_1 = 10°.3710.192 \)

\( \Delta \lambda_m = \frac{1}{2} \Delta \lambda = 5° 18′ 35.086′′ \)

\( \cos \phi_m = 0.542 \), \( \cos \Delta \phi_m = 0.998 \), \( \cos \Delta \phi_m = 0.53996 \)

\( k = \sin \phi_m \cos \Delta \phi_m = 0.839 \), \( K = \sin \Delta \phi_m \cos \phi_m = 0.0268 \)

\( H = \cos^2 \phi_m - \sin^2 \phi_m = \cos^2 \phi_m - \sin^2 \Delta \phi_m = 0.29165 \)

\( L = \sin^2 \Delta \phi_m + H \sin^2 \Delta \lambda_m = 0.004 \), \( \sin d = 1 - 2L + 0.990 \), \( \sin d = 1.2 \)

\( d + 140 \), \( 90.8\times 45.7 \sin d + 140 \), \( 90.8\times 44.24 \)

\( T = d \sin d + 0.2833 \)

\( U = 2k^2/(1 - L) = 1.4552 \), \( V = 2k^2/L = 2.9175 \), \( 6.48 \)

\( X = U + V = 1.3072 \), \( Y = 1.262 \), \( 3925.39 \)

\( E = 60 \cos d = 0.405 \), \( 39.112 \)

\( A = 4(16T + (E/15)T^2) = 80.168 \), \( 9630 \)

\( B = -2D = -112.105 \), \( 315.02 \)

\( C = 2T - \frac{1}{2}(A + E) = 67.775 \), \( 512.35 \)

\( AX + 136.8375 \), \( 618.4392 \)

\( BY = 125 \), \( 958.4324 \)

\( CX = -197.5055 \), \( 9405 \)

\( DXY + 10.951042195 \), \( 74.912162 \)

\( EY^2 + 0.914245 \)

\( T + \delta_T = 1.004721986 \), \( S_1 = \sin d (T + \delta_T) = 0.29000 \), \( 559 \)

\( \delta_T = + (f^2/128) (AX + BY + CX^2 + DXY + EY^2) = -0.390205 \times 10^{-6} \)

\( T + \delta_T = 1.004721986 \), \( S_2 = \sin d (T + \delta_T + \delta_s) = 900.000 \), \( 3.38 \)

\( \sin (a_2 - a_1) = (K \sin \Delta \lambda)/L = 0.99986398 \), \( a_1 + a_1 = 270° \), \( 56° 43.42′′ \)

\( \sin (a_2 - a_1) = (K \sin \Delta \lambda)/(1 - L) = 0.155 \), \( 411.89′′ \) \( a_1 - a_2 = 171° \), \( 63° 3.66′′ \)

\( \frac{1}{2}(\delta a_2 - \delta a_1) = (-f/2) H (T + 1) \sin (a_1 + a_2) = 0.00254833X \times 10^{-1} \), \( \delta a_1 = 19.9048 \times 10^{-1} \)

\( \frac{1}{2}(\delta a_2 - \delta a_1) = (-f/2) H (T - 1) \sin (a_1 - a_2) = 0.00254833X \times 10^{-1} \), \( \delta a_2 = 29.99 \times 10^{-1} \)

\( a_1 = 49° 15′ 49.84′′ \), \( a_2 = 231° 00′ 55.53′′ \)

\( a_1 = 49° 15′ 49.84′′ \), \( a_2 = 231° 00′ 55.53′′ \)

\( a_{1+2} = +_t a_1 + \delta a_1 \), \( a_{1+2} = +_t a_1 + \delta a_2 \)

\( d = 0° 54′ 41.12′′ \), \( d = 0° 54′ 41.12′′ \)

True Azimuths \( 50° 00′ 00″ \), \( 30° 00′ 00″ \)

Line No. 13

True distance \( 900.000 \), \( 3.38 \) meters
DISTANCE COMPUTING FORM, ANDOYER-LAMBERT TYPE APPROXIMATION
WITH SECOND ORDER TERMS

(No conversion to parametric latitudes)

Clarke Spheroid 1866, a = 6,378,206.4 meters
f/2 = 0.00169503765, f/4 = 0.000847518825, f²/128 = 0.0897860195 × 10⁻⁶
1 radian = 206,264.8062 seconds

| φ₁ | 19 51 31.492 | 1. ORIGIN | λ₁ |        |
| φ₂ | 35 12 0.231 | 2. TERMINUS | λ₂ |        |
| φₘ = 1/2(φ₂ + φ₁) | 22 31 42.332 | 2. Always west of 1. | Δλ = λ₂ - λ₁ | 7 25 26.297 |
| Δφₘ = 1/2(φ₂ - φ₁) | 7 40 15.899 | Δλₘ = Δλ/2 | 3 19 63.38 |
| sin φₘ | 3.85 144.13 | sin Δφₘ | 0.46 60.231 | sin Δλ | 7.132 049.81 |
| cos φₘ | 6.27 180.62 | cos Δφₘ | 0.48 91.832 | cos Δλₘ | 0.56 192.57 |
| k = sin φₘ cos Δφₘ | 7.38 274.92 | K = sin Δφₘ cos φₘ | 0.43 405.634 |
| H = cos²Δφₘ - sin²φₘ = cos²φₘ - sin²Δφₘ | 7.85 103.47 | 1 - L | 0.99 409.34 |
| L = sin²Δφₘ + H sin Δλ | 1.05 80.45 | cos d = 1 - 2L | 0.88 19 909 |
| d = 1.53 780.3447 | sin d + 1.53 784.96 | T = d/sin d | 1.00 395 22.34 |
| U = 2k²/(1 - L) | 7.28 062.41 | V = 2k²/L | 1.62 8 |
| X = U + V | 7.92 279.33 | Y = U - V | -3.13 331 44 | X | -3.07 596.220 |
| Y² | + 85.540.547 | Y² | + 111 109.584 | E = 60 cos d | 0.59 291 454.00 |

A = 4[16T + (E/15)T²] + 80 - 18975.264 |
B = -2D - 112.121 731.70 |
C = 2T - 1/2(A + E) - 0.7 32 705 78.46 |
AX | + 73.498 202 843 | BY | + 37.395 384 25 | CX² | - 57.677 65 3580 |

DXY | - 17.245 817 878 | EY² | + 6.579 271 65 |

δ₁ = -((f/4) (T - 3Y)) | -0.06 526 902 |

T + δ₁ | 1.00 2 319.553 | S₁ = a sin d (T + δ₁) | 999 240.671 |

T + δ₂ | 1.00 2 323 454 | S₂ = a sin d (T + δ₁ + δ₂) | 999 251.446 |

sin (a₂ + a₁) = (K sin Δλ)/L | + 96.692 5.9 |

sin (a₂ - a₁) = (K sin Δλ)/(1 - L) | + 105 85 9.09 |

δ₂ = -((f/2) II (T + 1) sin (a₂ + a₁) - 2.28 689 18 X₁₀⁻³) | -3 δₐ₂ | -2.28 689.23X 10⁻³ |

δ₂ = -((f/2) II (T - 1) (sin a₁ - a₀) - 1.00 2 899 5 X 10⁻³) | -3 δₐ₁ | -2.28 689 93X 10⁻³ |

a₁ | 128 42 43.200 |

δ₀₂ = 0 | 9 34.50 |

a₂ = + a₁ + δ₁ |

δ₀₂ = 9 | 34.645 |

a₂ = + a₁ + δ₂ |

δ₀₂ = | 30.5 8 18.73 |

True distance | 999 251.25 |

True Azimuths | 128 33 08.34 |

Line No. 14
DISTANCE COMPUTING FORM, ANDOYER-LAMBERT TYPE APPROXIMATION
WITH SECOND ORDER TERMS

(No conversion to parametric latitudes)

Clarke Spheroid 1866, \(a = 6,378,206.4\) meters

\[
f/2 = 0.00169503765, f/4 = 0.000847518825, f'/128 = 0.0897860195 \times 10^{-6}
\]

1 radian = 206,264.8062 seconds

<table>
<thead>
<tr>
<th>(\phi_1)</th>
<th>59  30  12.0</th>
<th>(\phi_2)</th>
<th>50  00  08.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lambda_1)</td>
<td></td>
<td>(\lambda_2)</td>
<td></td>
</tr>
<tr>
<td>(\phi_m = \frac{1}{2}(\phi_2 + \phi_1) + 5445.079)</td>
<td>(\Delta \phi_m = \frac{1}{2}(\phi_2 - \phi_1) - 4)</td>
<td>(45.041)</td>
<td></td>
</tr>
<tr>
<td>(\sin \phi_m = 0.816)</td>
<td>(6.366)</td>
<td>(\sin \Delta \phi_m = -0.08282801)</td>
<td>(\sin \Delta \lambda = 0.55)</td>
</tr>
<tr>
<td>(\cos \phi_m = 0.577)</td>
<td>(0.7322)</td>
<td>(\cos \Delta \phi_m = 0.99565386)</td>
<td>(\sin \Delta \lambda_m = 0.086)</td>
</tr>
</tbody>
</table>

\(k = \sin \phi_m \cos \Delta \phi_m + 1.813857.489\) \(K = \sin \Delta \phi_m \cos \phi_m - 0.827\)

\(H = \cos^2 \phi_m \sin^2 \phi_m - \cos^2 \phi_m \sin^2 \Delta \phi_m + 0.326\) \(1.9999953\) \(1 - L = 0.990702353\)

\(L = \sin^2 \Delta \phi_m + H \sin^2 \Delta \lambda_m - 0.0092977 \times 10^{-6}\) \(\cos d = 1 - 2L + 1.981405.103\)

\(d = 0.573.146.6435\) \(\sin d = 0.19194997\)

\(U = 2K^2/(1 - L) + 1.331760.2024\) \(V = 2K^2/L + 4.19152.92.2465\)

\(X = U + V + 1.87868312.5\) \(Y = U - V + 8.45.637.2999Y\)

\(2 + 34.9481922\) \(Y^2 = 7.67.14240.90\) \(E = 60 \cos d + 0.3884.36.18\)

\(A = 4(16T + (E/15) T^3) + 80.288690.192\) \(D = 8(6 + T^2) + 456.100.5185.04\)

\(B = -2D \times 112.300.45.7008\) \(C = 2T - \frac{1}{2}(A + E) - 0.779.102.190\)

\(AX + 146.84120175\) \(BY - 94.880890334\) \(CX^2 - 225.990059237\)

\(DYX + 6.75.3540 - 0.3\) \(EY^2 + 42.1080.2008\)

\(T + \delta_1 + 1.0068.357.442\) \(S_t = a \sin (T + \delta_1) + 1.236.952.146\)

\(\delta_1^2 = -(f/4)(TX - TY) + 0.055.4453 \times 10^{-6}\)

\(T + \delta_1 + \delta_p + 1.0068.317.87\) \(S_2 = a \sin d (T + \delta_1 + \delta_p) + 1.232.649.208\)

\(\sin (a_1 + a_2) = (k \sin \Delta \lambda)/L - 0.885.4410.8\) \(a_2 + a_1 = 242.18.31.056\)

\(\sin (a_2 - a_1) = (k \sin \Delta \lambda)/(1 - L) + 1.4782.58\) \(a_1 - a_2 = 151.41.57.759\)

\(\frac{1}{2}(\delta a_1 + \delta a_2) = -(f/2)(T + 1) \sin (a_1 + a_2) + 9.8226.75\times 10^{-4}\) \(\delta a_1 + 9.8226.75\times 10^{-4}\)

\(\frac{1}{2}(\delta a_2 - \delta a_1) = -(f/2)(T - 1) \sin (a_2 - a_1) - 0.0048.85\times 10^{-4}\) \(\delta a_2 + 9.8226.75\times 10^{-4}\)

\(a_1 = 35.35.17.10.43\) \(a_2 = 201.5.10.44\)

\(\delta a_1 = 3.32.697\) \(\delta a_2 = 3.32.496\)

\(a_{2-1} = + a_1 + \delta a_1\) \(a_{2-1} = + a_2 + \delta a_2\)

\(d = 11.03.59.355\) \(True \ distance = 1.232.649.21 \ meters\)

True Azimuths \(\begin{array}{c} 25 \ 16 \ 34.25 \\ 201 \ 08 \ 33.82 \end{array}\)

Line No. 15

140
DISTANCE COMPUTING FORM, ANDOYER-LAMBERT TYPE APPROXIMATION
WITH SECOND ORDER TERMS
(No conversion to parametric latitudes)
Clarke Spheroid 1866, a = 6,378,206.4 meters
f/2 = 0.00169503765, f/4 = 0.000847518825, f^2/128 = 0.0897660195 x 10^-6
1 radian = 206,264.8062 seconds

\[ \phi_1 \quad 8 \quad 57.35.0 \quad 1. \quad \text{PANAMA} \quad \lambda_1 \quad 29.34.24.0 \]
\[ \phi_2 \quad 21.36.16.0 \quad 2. \quad \text{HAWAII} \quad \lambda_2 \quad 158.01.33.0 \]
\[ \phi_m = \frac{1}{2} (\phi_2 + \phi_1) \quad 15.12.15.5 \quad 2. \text{Always west of 1.} \]
\[ \Delta \phi_m = \frac{1}{2} (\phi_2 - \phi_1) \quad 13.52.0 \quad \Delta \lambda = \lambda_2 - \lambda_1 \quad 28.27.09.0 \]
\[ \sin \phi_m \quad +.26236.120 \quad \sin \Delta \phi_m \quad +.109561.93 \quad \sin \Delta \lambda \quad +.909.25.08 \]
\[ \cos \phi_m \quad +.944.99.79 \quad \cos \Delta \phi_m \quad +.94.092.90 \quad \sin \Delta \lambda \quad +.632.38.28 \]
\[ k = \sin \phi_m \cos \Delta \phi_m \quad +.30712.512 \quad K = \sin \Delta \phi_m \cos \phi_m \quad +.107.23.2.96 \]
\[ \text{H} = \cos^2 \Delta \phi_m \sin^2 \phi_m = \cos^2 \phi_m \sin^2 \Delta \phi_m \quad +.91.43.9.30 \quad \text{1-L} \quad +.60.52.8.30 \]
\[ \text{L} = \sin^2 \phi_m \sin^2 \Delta \phi_m \quad +.939.42.2.12 \quad \cos \Delta m \quad +.632.38.28 \]
\[ d = +.13094.2875 \quad \sin d + +.905.51.19 \quad \text{T} = \text{d/sin d} \quad +.136.76.3.82 \]
\[ \text{U} = \text{2}k/(1-L) \quad +.219.02.4.83 \quad \text{V} = 2k/L \quad +.05.79.8.46 \]
\[ \text{X} = \text{U} + V = +.274.88.6.625 \quad \text{Y} = \text{U} - V = +.164.32.9.14 \]
\[ \text{X}^2 = +.306.11.3.09 \quad \text{Y}^2 = +.326.05.2.92 \quad \text{E} = 60 \quad \cos d \quad +.14.46.33.9.6 \]
\[ \text{A} = 4 \left[ 16T^2 + (E/15)T^3 \right] \quad +.94.74.5.60 \quad \text{D} = 8(6+T^2) \quad +.62.96.4.25.3 \]
\[ \text{B} = -2D - 1.6 \quad -1.72.38.50.6 \quad \text{C} = 2T - \frac{1}{4}(A + E) \quad +.51.38.9.12 \quad \text{4.56} \]
\[ \text{AX} = +.26.23.71.0 \quad \text{BY} = -2.30.76.91.6 \quad \text{CX} = +.3.976.6.88 \quad \text{58} \]
\[ \text{DXY} = +.181.45.72.81 \quad \text{HY} = +.736.12.99 \quad \text{EY} = +.90.27.8.10 \quad \text{5} \]
\[ \text{T} + \delta T = +.136.76.2.95 \quad \text{S} = \text{a sin d (T} + \delta T) \quad +.846.6.24.0 \]
\[ \text{X} = \text{a sin d (T} + \delta T) \quad +.306.11.3.09 \quad \text{S} = \text{a sin d (T} + \delta T) \quad +.846.6.24.0 \]
\[ \text{sin (a} + a_1) = (K \sin \Delta m)/L \quad +.200.41.001 \quad \text{a} + a_1 \quad +.25.41.19.97 \]
\[ \text{sin (a} - a_1) = (K \sin \Delta m)/(1-L) +.411.64.22 \quad \text{a} - a_1 \quad - +.15.31.66 \]
\[ \frac{1}{2} \delta a_1 + \delta a_2 = -\frac{1}{2} \sin (a_1 + a_2) = -000.99.78.98 \quad \delta a_1 = -761.93.19.3 \]
\[ \frac{1}{2} (\delta a_2 - \delta a_1) = -\frac{1}{2} \sin (a_1 - a_2) = -000.23.87.78 \quad \delta a_2 = -1.233.68.29 \]
\[ a_1 = +.109.57.4.0 \quad a_2 = +.25.41.19.97 \]
\[ \delta a_1 = \text{+} +.37.14.6 \quad \delta a_2 = +.265.37.10.7 \]
\[ a_{1-2} = +.109.57.4.0 \quad a_{2-1} = +.265.37.10.7 \]
\[ \text{True Azimuths} \quad +.109.57.4.0 \quad +.265.37.10.7 \text{ meters} \]

Line No. 16

141
**DISTANCE COMPUTING FORM, ANDOYER-LAMBERT TYPE APPROXIMATION WITH SECOND ORDER TERMS**

(No conversion to parametric latitudes)

Clarke Spheroid 1866, \(a = 6,378,206.4\) meters

\(f/2 = 0.00169503765\), \(f/4 = 0.000847518825\), \(f^2/128 = 0.0897860195 \times 10^{-6}\)

\(1\) radian = 206,264.8062 seconds

<table>
<thead>
<tr>
<th>Line No.</th>
<th>True Azimuths</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>15 48 17.074</td>
</tr>
</tbody>
</table>

True distance = 10,102.069.06 meters
The principal objective of this study was an evaluation of the formulas basic to the geodetic inverse solution for distance computations used by the U. S. Naval Oceanographic Office in loran-type charting. The adequacy of the formulas for past requirements was verified but, in anticipation of future requirements, they were modified to give geodesic distances and azimuths less than a meter and a second respectively.

During the study, associated geometrical configurations were developed or studied: latitudes associated with the auxiliary sphere-spheroid configuration; a spherical rectangular coordinate system on the auxiliary sphere with hyperbolic loci referenced to it; and geometrical quantities associated with arc distance, such as chord length, dip of the chord, maximum separation of chord and arc, and the geographical position of the point of maximum separation. The formulas with their derivations are presented. (U)
14. KEY WORDS

Geodetic distance inverse solution
Andoyer-Lambert formulae generalization
Forsyth method for geodesics (Corrected)
Geodetic formulae (latitude, distance, azimuths)
Geodetic approximations (spheroid)

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2. Loran
3. Distance Computation
4. Sphere-Spheroidal Geometry

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Author: P. D. Thomas
TR-182