Technical Note

Solution of the Matrix Equation

\[ \frac{d}{dt} X(t) = A(t)X(t) + X(t)B(t) + U(t) \]

M. Athans

29 June 1965

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Lincoln Laboratory

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Lexington, Massachusetts
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SOLUTION OF THE MATRIX EQUATION

\[ \frac{d}{dt} X(t) = A(t) X(t) + X(t) B(t) + U(t) \]

MICHAEL ATHANS*

DEPARTMENT OF ELECTRICAL ENGINEERING
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

*Consultant, M.I.T. Lincoln Laboratory

TECHNICAL NOTE 1965-26

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LEXINGTON MASSACHUSETTS
ABSTRACT

The purpose of this note is to state the solution to the inhomogeneous matrix differential equation

\[ \frac{d}{dt} X(t) = A(t) X(t) + X(t) B(t) + U(t). \]

Accepted for the Air Force
Stanley J. Wisniewski
Lt Colonel, USAF
Chief, Lincoln Laboratory Office
I. TERMINOLOGY

Suppose that we are given the time varying $n \times n$ matrices $A(t), B(t), U(t)$. We shall assume that

a. the elements of $A(t)$ and $B(t)$ are continuous functions of the time $t$

b. the elements of $U(t)$ are piecewise continuous functions of $t$.

We shall seek the solution of the matrix differential equation

$$\frac{d}{dt} X(t) = A(t) X(t) + X(t) B(t) + U(t)$$

(1)

subject to the initial condition

$$X(t_0) = X_0$$

(2)

where $X(t)$ is an $n \times n$ matrix.

II. THE HOMOGENEOUS CASE

Bellman in Reference [1] (page 175) considers the homogeneous equation

$$\frac{d}{dt} X(t) = A(t) X(t) + X(t) B(t)$$

(3)

subject to the initial condition

$$X(t_0) = X_0$$

(4)

His result is that the solution of (3) is given by the relation

$$X(t) = \Phi(t; t_0) X_0 \Psi(t; t_0)$$

(5)

where $\Phi(t; t_0)$ is a nonsingular fundamental matrix which satisfies the differential equation
\[
\frac{d}{dt} \Phi(t; t_0) = A(t) \Phi(t; t_0); \quad \Phi(t_0; t_0) = 1 \tag{6}
\]

and where \(\Psi(t; t_0)\) is a nonsingular fundamental matrix which satisfies the differential equation

\[
\frac{d}{dt} \Psi(t; t_0) = \Psi(t; t_0) B(t); \quad \Psi(t_0; t_0) = 1 \tag{7}
\]

III. THE INHOMOGENEOUS CASE

We claim that the solution of the differential equation

\[
\frac{d}{dt} X(t) = A(t) X(t) + X(t) B(t) + U(t) \tag{8}
\]

with \(X(t_0) = X_0\) is given by

\[
X(t) = \Phi(t; t_0) \left[ X_0 + \int_{t_0}^{t} \Phi^{-1}(\tau; t_0) U(\tau) \Psi^{-1}(\tau; t_0) d\tau \right] \Psi(t; t_0) \tag{9}
\]

To see this, differentiate (9) with respect to \(t\) to obtain (we use dots to indicate differentiation) the relations

\[
\dot{X}(t) = \dot{\Phi}(t; t_0) \left[ X_0 + \int_{t_0}^{t} \Phi^{-1}(\tau; t_0) U(\tau) \Psi^{-1}(\tau; t_0) d\tau \right] \Psi(t; t_0)
+ \Phi(t; t_0) \Phi^{-1}(t; t_0) U(t) \Psi^{-1}(t; t_0) \Psi(t; t_0)
+ \Phi(t; t_0) \left[ X_0 + \int_{t_0}^{t} \Phi^{-1}(\tau; t_0) U(\tau) \Psi^{-1}(\tau; t_0) d\tau \right] \dot{\Psi}(t; t_0)
= A(t) \Phi(t; t_0) \left[ X_0 + \int_{t_0}^{t} \Phi^{-1}(\tau; t_0) U(\tau) \Psi^{-1}(\tau; t_0) d\tau \right] \Psi(t; t_0)
+ \Phi(t; t_0) \left[ X_0 + \int_{t_0}^{t} \Phi^{-1}(\tau; t_0) U(\tau) \Psi^{-1}(\tau; t_0) d\tau \right] \Psi(t; t_0) B(t) + U(t)
\]

\[
\frac{d}{dt} X(t) = A(t) X(t) + X(t) B(t) + U(t) \tag{10}
\]
\[ X(t) = A(t) X(t) + X(t) B(t) + U(t) \]  

(11)

IV. TIME INVARIANT CASE

If \( A \) and \( B \) are constant matrices, then

\[ \Phi(t; t_0) = e^{A(t-t_0)} \]

(12)

\[ \psi(t; t_0) = e^{B(t-t_0)} \]

(13)

and, so, the solution reduces to

\[
X(t) = e^{A(t-t_0)} \left[ X_0 + \int_{t_0}^{t} e^{-A(\tau-t_0)} U(\tau) e^{-B(\tau-t_0)} d\tau \right] e^{B(t-t_0)}
\]

(14)

REFERENCE

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### 3. REPORT TITLE
Solution of the Matrix Equation $\frac{d}{dt} X(t) = A(t) X(t) + X(t) B(t) + U(t)$

### 13. ABSTRACT
The purpose of this note is to state the solution to the inhomogeneous matrix differential equation $\frac{d}{dt} X(t) = A(t) X(t) + X(t) B(t) + U(t)$.

### 14. KEY WORDS
- matrix algebra
- differential equations
- functions