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Technical Report 372

Measurement of Properties of Spread Channels by the Two-Frequency Method with Application to Radar Astronomy

T. Hagfors

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MEASUREMENT OF PROPERTIES OF SPREAD CHANNELS BY THE TWO-FREQUENCY METHOD WITH APPLICATION TO RADAR ASTRONOMY

T. HAGFORS

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ABSTRACT

A particular observation scheme based on the two-frequency correlation function approach for the measurement of time-varying spread channels having time-stationary Gaussian statistical properties is analyzed. It is shown that the scheme works well for underspread as well as overspread channels. The method also appears to be useful in cases where there is correlation between the signals arriving with different delays. Expressions are derived for the variances of the determined correlation and target-scattering functions derived by double Fourier transformation of the observed correlation function.

Accepted for the Air Force
Stanley J. Wisniewski
Lt Colonel, USAF
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MEASUREMENT OF PROPERTIES OF SPREAD CHANNELS BY THE TWO-FREQUENCY METHOD WITH APPLICATION TO RADAR ASTRONOMY

I. INTRODUCTION

In many communication channels, even in the absence of additive noise, a signal applied to the channel might be seriously distorted. This distortion arises because the channel destroys the original phase and amplitude relationships between the various components of the input signal. If a detailed knowledge of the phase and amplitude characteristics of the channel exists, the distortion may be removed by appropriate filtering. In addition to the distortion, there will also be a certain amount of time variation in many channels. The application of a single sine wave to the channel will therefore result in an output signal of finite bandwidth. Because of these properties, it is most convenient to describe such a channel by certain statistical properties. In this report we shall assume throughout that these statistical properties are stationary in time. As a result, the various different frequency components in the output will be completely uncorrelated when the input waveform is a single sine wave. Similarly, it will be assumed in the analysis, wherever convenient, that the statistical properties of the channel are largely independent of the radio frequency of observation. This is equivalent to assuming that the signals scattered at various delays are uncorrelated, and this is often a very good approximation. However, such an assumption is not necessary for the application of the measuring scheme.

There is considerable literature on the description and measurement of such doubly spread channels. Green[1] describes such a channel in terms of a target-scattering function which describes the way in which the power incident is distributed in delay and doppler displacement. Kailath[2] characterized this type channel in terms of a tap-gain correlation function which, loosely stated, describes the autocorrelation function of the time-varying impulse response as a function of delay. Hagfors[3] describes the channel in terms of a two-dimensional correlation function describing the loss of correlation between two time-varying frequency responses with a shift in frequency Δf and a shift in time T. An instructive survey of the various descriptions has recently been given by Gallager.[4]

Several methods have been proposed for the measurement of the various functions describing the statistical properties of the spread channel. Kailath[5] proposes to use a noise waveform at the input and compute the fourth-order moments at the output to derive the tap-gain correlation function. This method appears to work both when the product of doppler spread B and delay spread L is greater than and less than unity. Green[1] and Hagfors[3] both describe methods of measurement that only work well for underspread channels, i.e., when $BL < 1$; however, they did not consider the presence of additive noise. Price[6] and Levin[7] dealt with the problem of additive noise in the estimation of the statistical properties of the channel.

In this report, we analyze a particular measuring scheme based on a two-frequency transmission when additive noise is present. It will be shown that the two-frequency method of observation, apparently contrary to common belief,[5] can be made to work satisfactorily both in the overspread and underspread situations. A similar discussion has recently been carried out for the case of pseudo-random or a chirp input to the random channel.[4]

II. DESCRIPTION OF MEASUREMENT SCHEME

The basic observation scheme is shown in block diagram form in Fig. 1. The transmitter emits two sine waves of equal strength at frequencies $f_0 \pm \Delta f/2$. The modulation frequency Δf must be variable and it is highly desirable that Δf is very accurately known, particularly when there are large delays in the channel such as in planetary and lunar radar astronomy. It is convenient to choose Δf as an integral multiple of some smallest frequency separation Δf_0, and let this be derived from a frequency standard. The necessity of this requirement will be briefly explained in more detail below.

The received signal will consist of two spectra which, in the absence of mean doppler displacement which we will ignore in this report, will be centered on the two transmitted frequencies $f_0 \pm \Delta f/2$. If a mean doppler displacement is present we shall assume that the local oscillators

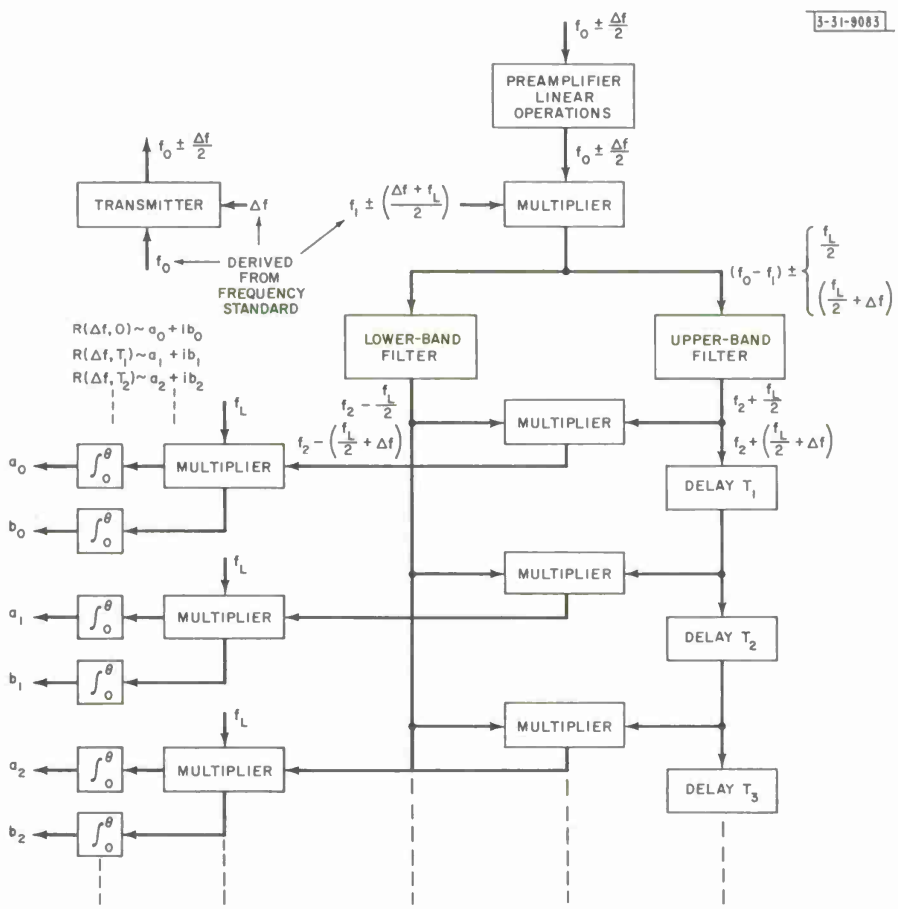

Fig. 1. Block diagram of two-frequency measurement scheme.

of the receiving system are programmed to correct for it. When the channel is "underspread," nearly all the significant measurements can be made without the two spectra overlapping; in the "overspread" situation this will not be the case.[1] The received signal is multiplied by a distorted replica of the transmitted signal, the distortion amounting to a change in the modulation frequency by an amount f_L, so that the modulation frequency of the multiplying signal is $f_L + \Delta f$. Also, the center frequency of the multiplying signal is different from f_0 by an amount $f_2 = f_0 - f_1$. The offset frequency f_L must be chosen so that it is greater than the maximum bandwidth B_m of the response of the time-varying channel when a single sine wave is applied at the input. The IF frequency f_2 must be greater than the sum of $f_L/2$, the maximum frequency separation Δf, and half the maximum bandwidth in order to prevent "foldover" (see Fig. 2). Following the multiplier the signal is split into two parts by upper- and lower-band filters. In the strictly underspread situation, these two filters should be constructed so that only the signals centered on $f_2 + f_L/2$ and $f_2 - f_L/2$ are passed, respectively. In the overspread situation such a discrimination will no longer be possible, and we must allow the filters also to pass some of the signals centered on $f_2 \pm [(f_L/2) + \Delta f]$. This will tend to introduce a certain amount of "self-noise" into the system.

The upper-band signal is passed into a delay line with discrete delays 0, T_1, T_2, etc., corresponding to the time shifts in the two-dimensional correlation function. These time shifts should be chosen to be integral multiples of f_L^{-1} for convenience. The delayed and undelayed signals are multiplied together and the amplitude and phase of the component in the output corresponding to the frequency f_L are determined by means of the operations shown in Fig. 1. The integration time Θ should be chosen as an integral multiple of the period Δf_0^{-1} of the lowest modulation frequency for reasons that will become apparent from the analysis that follows.

The pairs of numbers, or the set of complex numbers, resulting from the measurement are shown below to result in a determination of the complex correlation functions with frequency shift Δf and with time shifts 0, T_1, T_2, T_3, etc.

Let us now briefly return to the reason why the modulation frequency must be so accurately known. Consider as an example the situation encountered in radar astronomy. The transmission of the two sine waves corresponds to measuring one particular Fourier component of the target-scattering function, giving power as a function of range. If this function is to be recovered completely we must know both amplitude and phase of the Fourier components for different Δf.

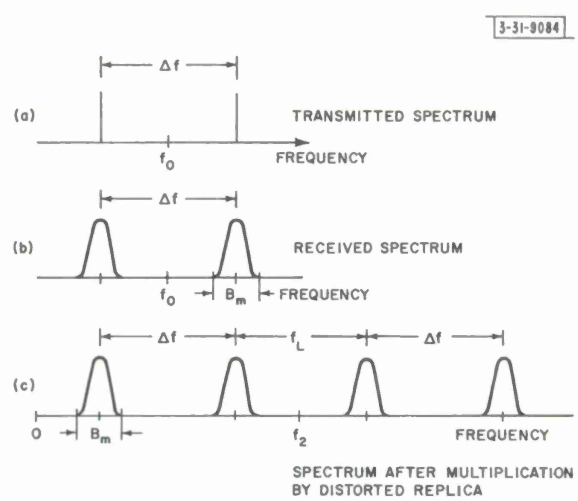

Fig. 2. Various frequency spectra in the processing scheme.

The phase and amplitude relationships must be well defined at the target. In radar astronomy and in other applications there will, however, be a large number of modulation cycles between the transmitter and the target. A very slight inaccuracy in the modulation frequency might, therefore, result in a large phase inaccuracy at the target.

The sets of results obtained with a number of different frequency separations Δf can, if desired, be used to compute the target-scattering function by a double Fourier transform[3] (see Sec. III-E below).

III. ANALYSIS OF MEASURING SCHEME

In this section it will first be shown how the mean values of the output of the processing scheme shown in Fig. 1 are related to the two-frequency correlation function both for underspread and overspread channels, first without additive thermal noise, then with thermal noise included. In the subsequent sections we will repeat the analysis under the same situations for the variance of the output so that confidence limits can be placed on the mean values determined.

Before starting on this program it is worthwhile to define the complex two-frequency correlation function in terms of a time-varying frequency response. Suppose the complex amplitude of a pure sine wave at the channel input is e_{in} and that the frequency is f_0. The complex amplitude at the output is then time varying because of the channel fluctuation and can be expressed as follows:

$$e_{out}(t) = H(f_0, t)\, e_{in} \quad . \tag{1}$$

The ratio of mean power at the output to input power becomes

$$\frac{P_{out}}{P_{in}} = \langle |H(f_0, t)|^2 \rangle_{avg} \quad , \tag{2}$$

where the averaging is considered an ensemble averaging, in general. The (unnormalized) two-frequency correlation function is now defined as

$$R(\Delta f, T) = \langle H^*(f_0 - \tfrac{\Delta f}{2}, t)\, H(f_0 + \tfrac{\Delta f}{2}, t + T) \rangle_{avg} \quad , \tag{3}$$

where we have assumed that the channel is stationary both in time and frequency. In case the channel is not stationary in frequency, the correlation function will also depend on f_0. The transmission loss is obtained from Eq. (3) by substituting $\Delta f = 0$ and $T = 0$. We assume throughout this report that $H(f_0, t)$ is a Gaussian process.

A. Evaluation of Mean, No Noise

The signal at the input of the receiver will have the complex amplitude referred to the center frequency f_0 given by

$$F_{in}(t) = H(f_0 - \tfrac{\Delta f}{2}, t)\, e^{-i\pi\Delta f t} + H(f_0 + \tfrac{\Delta f}{2}, t)\, e^{i\pi\Delta f t} \quad , \tag{4}$$

provided the transmitted amplitudes are equal to unity. The complex amplitude at the output of the first multiplier, this time referred to the intermediate frequency $f_2 = f_0 - f_2$, becomes (apart from a constant of proportionality)

$$F_2(t) = [H(f_0 - \frac{\Delta f}{2}, t) + H(f_0 + \frac{\Delta f}{2}, t)\, e^{2\pi i \Delta f t}]\, e^{\pi i f_L t}$$

$$+ [H(f_0 + \frac{\Delta f}{2}, t) + H(f_0 - \frac{\Delta f}{2}, t)\, e^{-2\pi i \Delta f t}]\, e^{-\pi i f_L t} \quad . \tag{5}$$

The upper- and lower-band signals appearing separately above can always be separated in the two filters following the first multiplier, leaving two signals with amplitudes referred to frequencies $f_2 + f_L/2$ and $f_2 - f_L/2$, respectively:

$$F_2^+(t) = [H(f_0 - \frac{\Delta f}{2}, t) + H(f_0 + \frac{\Delta f}{2}, t)\, e^{2\pi i \Delta f t}]$$

$$F_2^-(t) = [H(f_0 + \frac{\Delta f}{2}, t) + H(f_0 - \frac{\Delta f}{2}, t)\, e^{-2\pi i \Delta f t}] \quad . \tag{6}$$

We next filter out as much of the undesired signal as possible by assuming the maximum bandwidth of the time-varying response to be B_m, so that the following representation applies.

$$H(f_0 \pm \frac{\Delta f}{2}, t) = \int_{-\infty}^{\infty} H(f_0 \pm \frac{\Delta f}{2}, \nu)\, e^{2\pi i \nu t}\, d\nu$$

$$= \int_{-(B_m/2)}^{B_m/2} H(f_0 \pm \frac{\Delta f}{2}, \nu)\, e^{2\pi i \nu t}\, d\nu \quad . \tag{7}$$

By including rectangular filters in the lower- and upper-band filters of bandwidth equal to B_m, the desired signal is passed undistorted and a certain fraction of the undesired signal is rejected. The results of passing the signals [Eq. (6)] through such filters centered on $f_2 \pm f_L/2$ are:

$$F_3^+(t) = H(f_0 - \frac{\Delta f}{2}, t) + \begin{cases} \displaystyle\int_{-(B_m/2)+\Delta f}^{B_m/2} H(f_0 + \frac{\Delta f}{2}, \nu - \Delta f)\, e^{2\pi i t \nu}\, d\nu & \Delta f < B_m \\ 0 & \Delta f > B_m \end{cases} \quad , \tag{8a}$$

$$F_3^-(t) = H(f_0 + \frac{\Delta f}{2}, t) + \begin{cases} \displaystyle\int_{-(B_m/2)}^{(B_m/2)-\Delta f} H(f_0 - \frac{\Delta f}{2}, \nu + \Delta f)\, e^{2\pi i t \nu}\, d\nu & \Delta f < B_m \\ 0 & \Delta f > B_m \end{cases} \quad . \tag{8b}$$

The second multiplication with subsequent coherent detection of the component at frequency f_L is represented mathematically as:

$$V(T_\ell) = \int_0^{\Theta} F_3^{+*}(t)\, F_3^-(t + T_\ell)\, dt \quad , \tag{9}$$

or explicitly,

$$V(T_\ell) = \int_0^{\Theta} dt \left[H^*(f_0 - \frac{\Delta f}{2}, t) + D(B_m - \Delta f) \int_{-(B_m/2)+\Delta f}^{B_m/2} H^*(f_0 + \frac{\Delta f}{2}, \nu - \Delta f)\, e^{-2\pi i \nu t}\, d\nu \right]$$

$$\times \left[H(f_0 + \frac{\Delta f}{2}, t + T_\ell) + D(B_m - \Delta f) \int_{-(B_m/2)}^{(B_m/2)-\Delta f} H(f_0 - \frac{\Delta f}{2}, \nu + \Delta f)\, e^{2\pi i \nu (t + T_\ell)}\, d\nu \right] \quad , \tag{10}$$

where D(x) is a function that is zero when the argument is negative, and unity when it is positive. The mean value of the output becomes

$$\langle V(T_\ell)\rangle_{avg} = \Theta \langle H^*(f_0 - \frac{\Delta f}{2}, t)\, H(f_0 + \frac{\Delta f}{2}, t + T_\ell)\rangle_{avg} = \Theta R(\Delta f, T_\ell) \quad . \tag{11}$$

The other three terms will vanish exactly if the integration time Θ is chosen as an integral multiple of Δf_0^{-1}, as explained in Sec. II. This can be seen from the following development.

$$\left\langle \int_0^\Theta H^*(f_0 - \frac{\Delta f}{2}, t) \int_{-(B_m/2)}^{(B_m/2)-\Delta f} H(f_0 - \frac{\Delta f}{2}, \nu + \Delta f)\, e^{2\pi i \nu (t+T_\ell)}\, d\nu\, dt \right\rangle_{avg}$$

$$= \int_0^\Theta dt \int_{-\infty}^{\infty} d\nu' \int_{-(B_m/2)}^{(B_m/2)-\Delta f} \langle H^*(f_0 - \frac{\Delta f}{2}, \nu')\, H(f_0 - \frac{\Delta f}{2}, \nu + \Delta f)\rangle_{avg}\, e^{-2\pi i[\nu'-\nu)t-\nu T_\ell]}\, d\nu$$

$$= \int_{-(B_m/2)}^{(B_m/2)-\Delta f} d\nu\, \langle |H(f_0 - \frac{\Delta f}{2}, \nu + \Delta f)|^2\rangle_{avg}\, e^{2\pi i \nu T_\ell} \int_0^\Theta e^{-2\pi i \Delta f t}\, dt$$

$$= 0 \quad , \tag{12}$$

and the latter of these two integrals obviously must vanish exactly under the conditions imposed on Θ. The product of the two integrals in Eq. (10) will also vanish for the same reason. There is only one exception to this, namely, when $\Delta f = 0$ all four terms in Eq. (10) contribute an equal amount. Hence, we conclude,

$$\langle V(\Delta f, T_\ell)\rangle_{avg} = \left\langle V\left(m\Delta f_0, \frac{\ell}{f_L}\right)\right\rangle_{avg} = \begin{cases} \Theta R\left(m\Delta f_0, \frac{\ell}{f_L}\right) & m \geqslant 1 \ , \ \ell = 0, 1, \ldots \\ 4\Theta R\left(0, \frac{\ell}{f_L}\right) & m = 0 \ , \ \ell = 0, 1, \ldots \end{cases} \quad . \tag{13}$$

These results apply whether or not the channel is overspread.

B. Evaluation of Mean with Noise Present

When additive noise is present at the receiver input there will be an additional term in the input signal.

$$F_{in}(t) = H(f_0 - \frac{\Delta f}{2}, t)\, e^{-\pi i \Delta f t} + H(f_0 + \frac{\Delta f}{2}, t)\, e^{\pi i \Delta f t} + n(t) \quad . \tag{14}$$

Here, n(t) is a noise signal which might be expressed as

$$n(t) = \int_{f_0-(\Delta f_{max}/2)}^{f_0+(\Delta f_{max}/2)} N(f)\, e^{2\pi i f t}\, df \quad , \tag{15}$$

where Δf_{max} signifies a largest modulation frequency to be used. In the output of the processor certain new noise terms will appear. The terms containing cross products of noise and signal will not contribute to the mean, because noise and signal are uncorrelated. The noise-noise term, however, might contribute to the mean output. To see this we may argue as follows. The noise associated with $F_3^+(t)$ becomes

$$F_3^+(t)_n = \int_{-(\Delta f/2)-(B_m/2)}^{-(\Delta f/2)+(B_m/2)} N(\nu + f_0)\, e^{2\pi i\nu t}\, d\nu \tag{16a}$$

and, similarly, for $F_3^-(t)$,

$$F_3^-(t)_n = \int_{(\Delta f/2)-(B_m/2)}^{(\Delta f/2)+(B_m/2)} N(\nu + f_0)\, e^{2\pi i\nu t}\, d\nu \quad . \tag{16b}$$

The noise contribution to the output signal $V(T_\ell)$, i.e., $V_n(T_\ell)$, becomes

$$V_n(T_\ell) = \int_0^\Theta dt \int_{-(\Delta f/2)-(B_m/2)}^{-(\Delta f/2)+(B_m/2)} d\nu\; N^*(\nu + f_0)\, e^{-2\pi i\nu t}$$

$$\times \int_{(\Delta f/2)-(B_m/2)}^{(\Delta f/2)+(B_m/2)} d\nu' N(\nu' + f_0)\, e^{2\pi i\nu'(t+T_\ell)} \quad . \tag{17}$$

Taking the mean of this we find

$$\langle V_n(T_\ell)\rangle_{avg} = \int_0^\Theta dt \int_{-(\Delta f/2)-(B_m/2)}^{-(\Delta f/2)+(B_m/2)} d\nu\; \langle |N(\nu + f_0)|^2\rangle\, e^{-2\pi i\nu t}$$

$$\times \int_{(\Delta f/2)-(B_m/2)}^{(\Delta f/2)+(B_m/2)} d\nu'\, e^{2\pi i\nu'(t+T_\ell)}\, \delta(\nu' - \nu)$$

$$= \int_0^\Theta dt \int_{-(\Delta f/2)-(B_m/2)}^{-(\Delta f/2)+(B_m/2)} \langle |N(\nu + f_0)|^2\rangle\, d\nu\; e^{2\pi i\nu T_\ell}$$

$$\times \{D\,[\nu - \tfrac{1}{2}(\Delta f - B_m)] - D\,[\nu - \tfrac{1}{2}(\Delta f + B_m)]\}$$

$$= (B_m - \Delta f)\, \Theta N_o D(B_m - \Delta f)\, \frac{\sin \pi\, T_\ell(B_m - \Delta f)}{\pi T_\ell(B_m - \Delta f)} \quad ,$$

where N_o is the noise power per unit bandwidth, and $D(x)$ has the same significance as in Eq. (10). Hence, the output of the processor with noise present becomes

$$\left\langle V\left(m\Delta f_0, \frac{\ell}{f_L}\right)\right\rangle_{avg} = \Theta\left[R\left(m\Delta f_0, \frac{\ell}{f_L}\right) + N_o(B_m - m\Delta f_0)\, \frac{\sin \pi \frac{\ell}{f_L}(B_m - m\Delta f_0)}{\pi \frac{\ell}{f_L}(B_m - m\Delta f_0)}\, D(B_m - m\Delta f_0)\right]$$

$$m \geqslant 1 \quad . \tag{18}$$

We conclude that the processor gives an output proportional to the desired correlation function whenever the two lines can be completely separated. When the two lines overlap there will be a noise bias in addition (see Fig. 3); hence, this bias must be established before the correlation function can be determined.

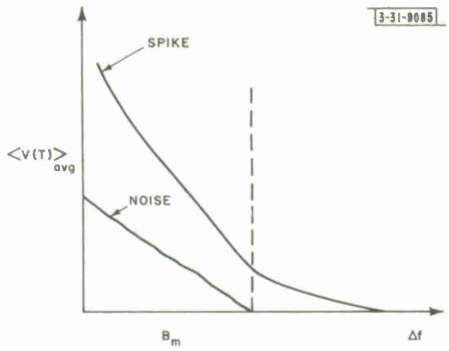

Fig. 3. Output of the processor shown in Fig. 1 when noise is present.

C. Evaluation of Variance, No Noise

The variance of the output may be computed directly from the expression [Eq. (9)]:

$$\langle |V(T_\ell)|^2\rangle_{avg} = \int_0^\Theta \int_0^\Theta dt\, dt'\, \langle F_3^{+*}(t)\, F_3^-(t + T_\ell)\, F_3^+(t')\, F_3^{-*}(t' + T_\ell)\rangle_{avg} \quad . \tag{19}$$

Now, since we are assuming a Gaussian process, the integrand can be expanded in terms of second-order moments as follows.

$$\begin{aligned}
&\langle F_3^{+*}(t)\, F_3^-(t + T_\ell)\, F_3^+(t')\, F_3^{-*}(t' + T_\ell)\rangle_{avg} \\
&= \langle F_3^{+*}(t)\, F_3^-(t + T_\ell)\rangle_{avg}\, \langle F_3^+(t')\, F_3^{-*}(t' + T_\ell)\rangle_{avg} \\
&\quad + \langle F_3^{+*}(t)\, F_3^+(t')\rangle_{avg}\, \langle F_3^-(t + T_\ell)\, F_3^{-*}(t' + T_\ell)\rangle_{avg} \\
&\quad + \langle F_3^{+*}(t)\, F_3^{-*}(t' + T_\ell)\rangle_{avg}\, \langle F_3^-(t + T_\ell)\, F_3^+(t')\rangle_{avg} \quad .
\end{aligned} \tag{20}$$

The first term on the right-hand side in Eq. (20) can be seen to correspond simply to the square of the mean value which has already been determined. The last term in Eq. (20) is completely negligible because it involves mean values of products of frequency responses, either both unconjugated or both conjugated. Only the middle term will therefore contribute to the variance. In order to evaluate this term we write

$$F_3^+(t) = H(f_0 - \tfrac{\Delta f}{2}, t) + D(B_m - \Delta f) \int_{-(B_m/2)+\Delta f}^{B_m/2} H(f_0 + \tfrac{\Delta f}{2}, \nu - \Delta f)\, e^{2\pi i \nu t}\, d\nu \quad ,$$

$$F_3^-(t) = H(f_0 + \tfrac{\Delta f}{2}, t) + D(B_m - \Delta f) \int_{-(B_m/2)}^{(B_m/2)-\Delta f} H(f_0 - \tfrac{\Delta f}{2}, \nu + \Delta f)\, e^{2\pi i \nu t}\, d\nu \quad , \tag{21}$$

where the factor $D(B_m - \Delta f)$ is unity whenever $B_m > \Delta f$, and zero otherwise. We have:

$$\langle F_3^{+*}(t)\, F_3^{+}(t')\rangle_{avg} = R(0, t'-t) + D(B_m - \Delta f) \int_{-(B_m/2)}^{B_m/2} d\nu' \int_{-(B_m/2)+\Delta f}^{B_m/2} d\nu$$

$$\times\, e^{-2\pi i(\nu' t - \nu t')} \langle H^*(f_0 - \tfrac{\Delta f}{2}, \nu')\, H(f_0 + \tfrac{\Delta f}{2}, \nu - \Delta f)\rangle_{avg}$$

$$+ D(B_m - \Delta f) \int_{-(B_m/2)}^{B_m/2} d\nu' \int_{-(B_m/2)+\Delta f}^{B_m/2} d\nu\, e^{2\pi i(\nu' t' - \nu t)}$$

$$\times \langle H(f_0 - \tfrac{\Delta f}{2}, \nu')\, H^*(f_0 + \tfrac{\Delta f}{2}, \nu - \Delta f\rangle_{avg} + D(B_m - \Delta f) \iint_{-(B_m/2)+\Delta f}^{B_m/2} d\nu\, d\nu'$$

$$\times \langle H^*(f_0 + \tfrac{\Delta f}{2}, \nu - \Delta f)\, H(f_0 + \tfrac{\Delta f}{2}, \nu' - \Delta f)\rangle_{avg}\, e^{2\pi i(\nu' t' - \nu t)} \quad . \tag{22}$$

Because of the assumed time stationarity of the statistical properties we must have

$$\langle H^*(f_0 - \tfrac{\Delta f}{2}, \nu)\, H(f_0 + \tfrac{\Delta f}{2}, \nu')\rangle_{avg} = \langle H^*(f_0 - \tfrac{\Delta f}{2}, \nu)\, H(f_0 + \tfrac{\Delta f}{2}, \nu)\rangle_{avg}\, \delta(\nu' - \nu)$$

Using this we obtain

$$\langle F_3^{+*}(t)\, F_3^{+}(t')\rangle_{avg} = R(0, t'-t) + D(B_m - \Delta f) \int_{-(B_m/2)+\Delta f}^{B_m/2} d\nu\, e^{2\pi i\nu(t'-t)}$$

$$\times \langle H^*(f_0 - \tfrac{\Delta f}{2}, \nu - \Delta f)\, H(f_0 + \tfrac{\Delta f}{2}, \nu - \Delta f\rangle_{avg}\, e^{2\pi i \Delta f t} + D(B_m - \Delta f)\, e^{-2\pi i \Delta f t'}$$

$$\times \int_{-(B_m/2)+\Delta f}^{B_m/2} e^{2\pi i\nu(t'-t)} \langle H^*(f_0 + \tfrac{\Delta f}{2}, \nu - \Delta f)\, H(f_0 - \tfrac{\Delta f}{2}, \nu - \Delta f)\rangle_{avg}\, d\nu$$

$$+ D(B_m - \Delta f) \int_{-(B_m/2)+\Delta f}^{B_m/2} d\nu\, e^{2\pi i\nu(t'-t)} \langle H^*(f_0, \nu - \Delta f)\, H(f_0, \nu - \Delta f)\rangle_{avg} \quad . \tag{22a}$$

The other moment which must be evaluated is

$$\langle F_3^{-}(t + T_\ell)\, F_3^{-*}(t' + T_\ell)\rangle = \langle F_3^{-}(t)\, F_3^{-*}(t')\rangle_{avg} \tag{23}$$

The last equation follows from the time stationarity of the random process. The average in Eq. (23) can be obtained from Eq. (22) by changing the sign on Δf in all arguments of H, by interchanging the roles of t and t', and by changing the limits of integration. One then obtains

$$\langle F_3^{-}(t)\, F_3^{-*}(t')\rangle_{avg} = R(0, t-t') + D(B_m - \Delta f)\, e^{-2\pi i \Delta f t'} \int_{-(B_m/2)}^{(B_m/2)-\Delta f} d\nu\, e^{-2\pi i\nu(t'-t)}$$

$$\times \langle H^*(f_0 + \tfrac{\Delta f}{2}, \nu + \Delta f)\, H(f_0 - \tfrac{\Delta f}{2}, \nu + \Delta f)\rangle_{avg} + D(B_m - \Delta f)\, e^{2\pi i \Delta f t} \int_{-(B_m/2)}^{(B_m/2)-\Delta f} d\nu$$

$$\times\, e^{-2\pi i\nu(t'-t)} \langle H^*(f_0 - \tfrac{\Delta f}{2}, \nu + \Delta f)\, H(f_0 + \tfrac{\Delta f}{2}, \nu + \Delta f)\rangle_{avg} + D(B_m - \Delta f) \int_{-(B_m/2)}^{(B_m/2)-\Delta f} d\nu$$

$$\times\, e^{-2\pi i\nu(t'-t)} \langle |H(f_0, \nu + \Delta f)|^2\rangle_{avg} \quad . \tag{24}$$

Of the 16 terms resulting from the multiplication of Eq. (22) by Eq. (24), only those terms which depend on the combination $t - t'$ will contribute in the integrals over t and t'. The terms depending explicitly on either t or t' will all vanish. The contributing terms will be the following six:

$$
\begin{aligned}
&\text{(I)} \quad |R(0, t-t')|^2 + \\
&\text{(II)} \quad + R(0, t'-t) \cdot D(B_m - \Delta f) \int_{-(B_m/2)}^{(B_m/2)-\Delta f} d\nu \; e^{-2\pi i(t'-t)\nu} \; \langle |H(f_0, \nu + \Delta f)|^2 \rangle_{avg} + \\
&\text{(III)} \quad + R(0, t-t') \; D(B_m - \Delta f) \int_{-(B_m/2)+\Delta f}^{B_m/2} d\nu \; e^{2\pi i(t'-t)\nu} \; \langle |H(f_0, \nu - \Delta f)|^2 \rangle_{avg} + \\
&\text{(IV)} \quad + D(B_m - \Delta f) \; e^{-2\pi i \Delta f(t'-t)} \int_{-(B_m/2)+\Delta f}^{B_m/2} d\nu \; e^{2\pi i \nu (t'-t)} \\
&\qquad \times \langle H^*(f_0 - \tfrac{\Delta f}{2}, \nu - \Delta f) \; H(f_0 + \tfrac{\Delta f}{2}, \nu - \Delta f) \rangle_{avg} \\
&\qquad \times \int_{-(B_m/2)}^{(B_m/2)-\Delta f} d\nu \; e^{-2\pi i \nu (t'-t)} \; \langle H^*(f_0 + \tfrac{\Delta f}{2}, \nu + \Delta f) \; H(f_0 - \tfrac{\Delta f}{2}, \nu + \Delta f) \rangle_{avg} + \\
&\text{(V)} \quad + D(B_m - \Delta f) \; e^{-2\pi i \Delta f(t'-t)} \int_{-(B_m/2)+\Delta f}^{B_m/2} d\nu \; e^{2\pi i \nu (t'-t)} \\
&\qquad \times \langle H^*(f_0 + \tfrac{\Delta f}{2}, \nu - \Delta f) \; H(f_0 - \tfrac{\Delta f}{2}, \nu - \Delta f) \rangle_{avg} \\
&\qquad \times \int_{-(B_m/2)}^{(B_m/2)-\Delta f} d\nu \; e^{-2\pi i \nu (t'-t)} \; \langle H^*(f_0 - \tfrac{\Delta f}{2}, \nu + \Delta f) \; H(f_0 + \tfrac{\Delta f}{2}, \nu + \Delta f) \rangle_{avg} + \\
&\text{(VI)} \quad + D(B_m - \Delta f) \int_{-(B_m/2)}^{(B_m/2)-\Delta f} d\nu \; e^{-2\pi i(t'-t)\nu} \; \langle |H(f_0, \nu + \Delta f)|^2 \rangle_{avg} \\
&\qquad \times \int_{-(B_m/2)+\Delta f}^{B_m/2} e^{2\pi i(t'-t)\nu} \; \langle |H(f_0, \nu - \Delta f)|^2 \rangle_{avg} \, d\nu \quad . \qquad (25)
\end{aligned}
$$

The interpretation and discussion of such a complicated expression is rather difficult, and it simplifies matters greatly if we can make a specific assumption about the channel properties to insert into the general terms in (25). As a specific example we choose

$$
\langle H^*(f_0 - \tfrac{\Delta f}{2}, \nu) \; H(f_0 + \tfrac{\Delta f}{2}, \nu) \rangle_{avg} = \begin{cases} \dfrac{\alpha}{1 + 2\pi i \tau_o \Delta f} & \text{if } |\nu| < \dfrac{B_m}{2} \\ 0 & \text{otherwise} \end{cases} \qquad (26)
$$

This corresponds to assuming an exponential power-vs-range variation with power falling off to e^{-1} at $\tau = \tau_o$. For $R(\Delta f, T)$, we obtain

$$
R(\Delta f, T) = \frac{\alpha}{1 + 2\pi i \tau_o \Delta f} \int_{-(B_m/2)}^{B_m/2} e^{2\pi i \nu T} \, d\nu = \frac{\alpha B_m}{1 + 2\pi i \tau_o \Delta f} \, \frac{\sin \pi T B_m}{\pi T B_m} \quad . \qquad (27)
$$

The contribution to the variance caused by the terms in (25) in this case is found by substitution of Eqs. (26) or (27) and integration over t and t' between the limits 0 and Θ. On the assumption that $\Theta \gg B_m^{-1}$, the contributions become

$$\text{Term (I):} \qquad \Theta B_m \alpha^2 \quad ;$$

$$\text{Terms (II) \& (III):} \quad \begin{cases} 2\Theta(B_m - \Delta f)\,\alpha^2 & \text{if } |\Delta f| < B_m \\ 0 & \text{otherwise} \end{cases} ;$$

$$\text{Terms (IV) \& (V):} \quad \begin{cases} 2\Theta(B_m - \Delta f)\,\alpha^2 \dfrac{1}{1 + (2\pi\tau_o\Delta f)^2} & \text{if } |\Delta f| < B_m \\ 0 & \text{otherwise} \end{cases} ;$$

$$\text{Term (VI):} \quad \begin{cases} \Theta(B_m - \Delta f)\,\alpha^2 & \text{if } |\Delta f| < B_m \\ 0 & \text{otherwise} \end{cases} . \tag{28}$$

Half this variance corresponds to the real and half to the imaginary part of $R(\Delta f, T)$, so we might define the relative uncertainty of the two components as follows ($\Delta f > 0$).

$$\frac{\delta(\mathrm{Re}\,V)}{\langle \mathrm{Re}\,V\rangle_{avg}} = \frac{\left(1/2\left\{1 + (2\pi\tau_o\Delta f)^2 + 2D(B_m - \Delta f)\left(1 - \frac{\Delta f}{B_m}\right)[3 + (2\pi\tau_o\Delta f)^2]\right\}\right)^{1/2}}{\sqrt{B_m\Theta}\left(\frac{\sin\pi TB_m}{\pi TB_m}\right)}$$

$$\frac{\delta(\mathrm{Im}\,V)}{\langle \mathrm{Im}\,V\rangle_{avg}} = \frac{\left(1/2\left\{1 + (2\pi\tau_o\Delta f)^2 + 2D(B_m - \Delta f)\left(1 - \frac{\Delta f}{B_m}\right)[3 + (2\pi\tau_o\Delta f)^2]\right\}\right)^{1/2}}{\sqrt{B_m\Theta}\left(\frac{\sin\pi TB_m}{\pi TB_m}\right)(2\pi\tau_o\Delta f)} . \tag{29}$$

For the singular case of $\Delta f = 0$, one obtains

$$\frac{\delta(\mathrm{Re}\,V)}{\langle \mathrm{Re}\,V\rangle_{avg}} = \frac{\sqrt{7/2}}{4\sqrt{B_m\Theta}\left(\frac{\sin\pi TB_m}{\pi TB_m}\right)} . \tag{30}$$

The relative uncertainty in the imaginary part is infinite at $\Delta f = 0$ because the mean vanishes at $\Delta f = 0$.

D. Evaluation of Variance with Noise Present

The evaluation of the variance proceeds along the same lines as before except that $F_3(t)$ has an additional noise term as follows.

$$F_3^+(t) = H(f_0 - \frac{\Delta f}{2}, t) + D(B_m - \Delta f) \int_{-(B_m/2)+\Delta f}^{B_m/2} H(f_0 + \frac{\Delta f}{2}, \nu - \Delta f)\, e^{2\pi i \nu t}\, d\nu$$

$$+ \int_{-(\Delta f/2)-(B_m/2)}^{-(\Delta f/2)+(B_m/2)} N(\nu + f_0)\, e^{2\pi i \nu t}\, d\nu \quad ,$$

$$F_3^-(t) = H(f_0 + \frac{\Delta f}{2}, t) + D(B_m - \Delta f) \int_{-(B_m/2)}^{(B_m/2)-\Delta f} H(f_0 - \frac{\Delta f}{2}, \nu + \Delta f)\, e^{2\pi i \nu t}\, d\nu$$

$$+ \int_{(\Delta f/2)-(B_m/2)}^{(\Delta f/2)+(B_m/2)} N(\nu + f_0)\, e^{2\pi i \nu t}\, d\nu \quad . \tag{31}$$

For the same reasons as before it will only be the middle term on the right-hand side of Eq. (20) which contributes. In the expression corresponding to Eq. (22) will be the following additional term due to the noise:

$$N_o \frac{\sin \pi B_m (t - t')}{\pi (t - t')} = \langle F_n^{+*}(t)\, F_n^+(t') \rangle_{avg} \quad . \tag{32a}$$

Similarly, there will be an identical contribution to Eq. (24), namely,

$$N_o \frac{\sin \pi B_m (t - t')}{\pi (t - t')} = \langle F_n^{-*}(t')\, F_n^-(t) \rangle_{avg} \quad . \tag{32b}$$

There will now be the following five terms in expression (25), in addition to those already present:

$$N_o^2 \left[\frac{\sin \pi B_m (t - t')}{\pi (t - t')} \right]^2 + N_o \frac{\sin \pi B_m (t - t')}{\pi (t - t')} \Bigg\{ R(0, t - t') + R(0, t' - t)$$

$$+ D(B_m - \Delta f) \int_{-(B_m/2)}^{(B_m/2)-\Delta f} d\nu\, e^{-2\pi i (t'-t)\nu} \langle | H(f_0, \nu + \Delta f) |^2 \rangle_{avg}$$

$$+ D(B_m - \Delta f) \int_{-(B_m/2)+\Delta f}^{B_m/2} d\nu\, e^{2\pi i (t'-t)\nu} \langle | H(f_0, \nu - \Delta f) |^2 \rangle_{avg} \Bigg\} \quad . \tag{33}$$

With the assumption [Eq. (26)] about the channel properties, these terms contribute the following amount to the variance after the double integration over t and t':

$$B_m N_o^2 \Theta + 2 B_m \alpha N_o \Theta + D(B_m - \Delta f)\, 2(B_m - \Delta f)\, \alpha N_o \Theta \quad .$$

For the total variance in the presence of noise one therefore obtains:

$$\Theta B_m \left\{ \alpha^2 + N_o^2 + 2\alpha N_o + D(B_m - \Delta f) \left(1 - \frac{\Delta f}{B_m}\right) \left[3\alpha^2 + \frac{2\alpha^2}{1 + (2\pi \tau_o \Delta f)^2} + 2\alpha N_o \right] \right\} \quad . \tag{34}$$

The signal-to-noise ratio, defined as the square of the mean (excluding the noise bias which is assumed to be taken out), divided by the variance just found becomes:

(1) For the real part,

$$\frac{(\text{mean signal})^2}{\text{noise (= variance)}} = \frac{\Theta}{2B_m}\left(\frac{P_s}{N_o}\right)^2\left(\frac{1}{1+(2\pi\tau_o\Delta f)^2}\right)^2\left(\frac{\sin\pi TB_m}{\pi TB_m}\right)$$

$$\times\left\{1+\frac{1}{B_m^2}\left(\frac{P_s}{2N_o}\right)^2+\frac{P_s}{B_mN_o}+2D(B_m-\Delta f)\left(1-\frac{\Delta f}{B_m}\right)\right.$$

$$\left.\times\left[\left(\frac{P_s}{2N_o}\right)^2\frac{1}{B_m^2}\frac{3+(2\pi\tau_o\Delta f)^2}{1+(2\pi\tau_o\Delta f)^2}+\frac{P_s}{2N_oB_m}\right]\right\}^{-1} , \tag{35}$$

where $2\alpha B_m = P_s$ = total received signal power.

(2) For the imaginary part, the corresponding result is obtained by multiplication of Eq. (35) by $(2\pi\tau_o\Delta f)^2$.

When the signal power is much less than the noise power within the bandwidth B_m, the result [Eq. (35)] simplifies quite considerably to give

$$\frac{(\text{mean signal})^2}{\text{noise}} = \frac{\Theta}{2B_m}\left(\frac{P_s}{N_o}\right)^2\left(\frac{1}{1+(2\pi\tau_o\Delta f)^2}\right)^2\left(\frac{\sin\pi TB_m}{\pi TB_m}\right) . \tag{36}$$

It is interesting to observe that this closely resembles the signal-to-noise ratio of the output of Price's optimum processor in the case where the signal waveform spans many coherence intervals (Price[8]). A naive interpretation of the result is

$$\frac{(\text{signal})^2}{\text{noise}} = \frac{1}{2}\,\frac{(\text{signal energy})\times(\text{signal-power density})}{(\text{noise-power density})^2}$$

$$= \frac{1}{2}\,B_m\Theta\left(\frac{\text{signal-power density}}{\text{noise-power density}}\right)^2 . \tag{37}$$

E. Uncertainties in Target-Scattering Function

It is not always sufficient to be able to discuss the variance of the outcome of the experimental scheme shown in Fig. 1. In order to compare matters with the results of pulse experiments designed to measure the target-scattering function $\sigma(\tau, f_D)$ directly, it is necessary to find the uncertainties arising in the determination of $\sigma(\tau, f_D)$ from the correlation function measurements. Suppose, therefore, that we have gone through the correlation function measurements and have determined a set of complex numbers:

$$V(m\Delta f_0, \ell T_o)\begin{cases} m = 0, 1, \ldots, M \\ \ell = 0, 1, \ldots, L , \quad T_o = \dfrac{\ell}{f_L} \end{cases} , \tag{38}$$

and that the integration times $\Theta = \Theta_\ell$. Our estimate of the target-scattering function obviously would take the form

$$\sigma(\tau, f_D)_{est} \sim \sum_{\ell=-L}^{L}\frac{1}{\Theta(m)}\sum_{m=-M}^{M} V(m\Delta f_0, \ell T_o)\, e^{2\pi i(m\Delta f_0\tau+\ell T_o f_D)} . \tag{39}$$

We shall not be concerned with scale factors here. A target-scattering function determined in this manner will deviate from the true scattering function for two reasons. There will be a truncation error because the summations are carried to finite limits; this indicates a range resolution of the order $\Delta\tau = \ell/M\Delta f_0$ and a frequency resolution $\Delta f_D = \ell/LT_o = f_L/L$. Also, there will be a systematic error due to the finite sampling on the correlation function. Here we shall not consider these systematic errors. Another source of errors is caused by the fact that the correlation function is subjected to statistical fluctuations due to additive noise or self-noise. We shall be concerned with these latter type errors. In other words, we would like to study the quantity

$$\Delta\sigma(\tau, f_D) \sim \sum_{-L}^{L} \sum_{-M}^{M} \frac{1}{\Theta(m)} \{V(m\Delta f_0, \ell T_o) - \langle V(m\Delta f_0, \ell T_o)\rangle_{avg}\} e^{2\pi i(m\Delta f_0 \tau + \ell T_o f_D)} \quad . \tag{40}$$

Hence we shall attempt to determine $\langle \Delta\sigma(\tau, f_D)^2\rangle_{avg}$. It is found that

$$\langle \Delta\sigma(\tau, f_D)^2\rangle_{avg} \sim \sum_{m=-M}^{M} \frac{1}{\Theta(m)^2} \sum_{\ell} \sum_{n} \{\langle V(m\Delta f_0, \ell T_o)\, V^*(m\Delta f_0, nT_o)\rangle_{avg}$$

$$- \langle V(m\Delta f_0, \ell T_o)\rangle_{avg} \langle V^*(m\Delta f_0, nT_o)\rangle_{avg}\} e^{2\pi i T_o f_D(\ell - n)} \tag{41}$$

Assuming here, as throughout, that the channel and the noise are both Gaussian, there will only be the terms depending on

$$\langle F_{3m}^{+*}(t)\, F_{3m}^{+}(t')\rangle_{avg} \langle F_{3m}^{-}(t+T)\, F_{3m}^{-*}(t'+T')\rangle_{avg} \quad ,$$

which will contribute to the entire triple sum in Eq. (41). The first of the two factors above will be identical to the one determined previously [see Eqs. (22) and (32a)]:

$$\langle F_{3m}^{+*}(t)\, F_{3m}^{+}(t')\rangle_{avg} = R(0, t'-t) + e^{2\pi i t \Delta f_0 m} D(B_m - m\Delta f_0)$$

$$\times \int_{-(B_m/2)+\ell\Delta f_0}^{B_m/2} d\nu\, e^{2\pi i \nu(t'-t)}$$

$$\times \left\langle H^*\left(f_0 - \frac{m\Delta f_0}{2}, \nu - m\Delta f_0\right) H\left(f_0 + \frac{m\Delta f_0}{2}, \nu - m\Delta f_0\right)\right\rangle_{avg}$$

$$+ e^{-2\pi i t' \Delta f_0 m} D(B_m - m\Delta f_0) \int_{-(B_m/2)+\ell\Delta f_0}^{B_m/2} d\nu\, e^{2\pi i \nu(t'-t)}$$

$$\times \left\langle H^*\left(f_0 + \frac{m\Delta f_0}{2}\right) H\left(f_0 - \frac{m\Delta f_0}{2}, \nu - m\Delta f_0\right)\right\rangle_{avg}$$

$$+ D(B_m - m\Delta f_0) \int_{-(B_m/2)+\ell\Delta f_0}^{B_m/2} d\nu\, e^{2\pi i \nu(t'-t)}$$

$$\times \langle H^*(f_0, \nu - m\Delta f_0)\, H(f_0, \nu - m\Delta f_0)\rangle_{avg} + N_o \frac{\sin \pi B_m(t-t')}{\pi(t-t')} \quad . \tag{42}$$

The second factor takes on a slightly different form than before. Putting $T' - T = \delta T = (\ell - n)\, T_o$, one obtains

$$\langle F_{3m}^{-}(t)\, F_{3m}^{-*}(t' - \delta T)\rangle_{avg} = R(0, t - t' - \delta T) + e^{-2\pi i(t'+\delta T)\Delta f_0 m}\, D(B_m - m\Delta f_0)$$

$$\times \int_{-(B_m/2)}^{(B_m/2)-m\Delta f_0} d\nu \left\langle H^*\left(f_0 + \frac{m\Delta f_0}{2}, \nu + m\Delta f_0\right) H\left(f_0 - \frac{m\Delta f_0}{2}, \nu + m\Delta f_0\right)\right\rangle_{avg}$$

$$\times e^{2\pi i\nu(t'-t+\delta T)} + e^{2\pi i t\Delta f_0 m}\, D(B_m - m\Delta f_0) \int_{-(B_m/2)}^{(B_m/2)-m\Delta f_0} d\nu$$

$$\times \left\langle H^*\left(f_0 - \frac{m\Delta f_0}{2}, \nu + m\Delta f_0\right) H\left(f_0 + \frac{m\Delta f_0}{2}, \nu + m\Delta f_0\right)\right\rangle_{avg} e^{-2\pi i\nu(t'+\delta T-t)}$$

$$+ D(B_m - m\Delta f_0) \int_{-(B_m/2)}^{(B_m/2)-m\Delta f_0} d\nu\, \langle |H(f_0, \nu + m\Delta f_0)|^2\rangle_{avg}\, e^{-2\pi i\nu(t'+\delta T-t)}$$

$$+ N_o \frac{\sin \pi B_m(t - t' - \delta T)}{\pi(t - t' - \delta T)} \quad . \tag{43}$$

In the subsequent multiplication and integration over t and t', only those terms will contribute which depend on t and t' through the combination $t - t'$. This leaves the following eleven terms (note, we put $t' - t = \Delta t$):

(I) $R(0, \Delta t)\, R^*(0, \Delta t + \delta T) +$

(II) $+ R(0, \Delta t)\, D(B_m - m\Delta f_0) \int_{-(B_m/2)}^{(B_m/2)-m\Delta f_0} R(0, \nu + m\Delta f_0)\, e^{-2\pi i\nu(\Delta t+\delta T)}\, d\nu +$

(III) $+ R^*(0, \Delta t + \delta T)\, D(B_m - m\Delta f_0) \int_{-(B_m/2)+m\Delta f_0}^{B_m/2} R(0, \nu - m\Delta f_0)\, e^{2\pi i\nu\Delta t}\, d\nu +$

(IV) $+ D(B_m - m\Delta f_0)\, e^{-2\pi i m\Delta f_0(\Delta t+\delta T)} \int_{-(B_m/2)+m\Delta f_0}^{B_m/2} R(m\Delta f_0, \nu - m\Delta f_0)\, e^{2\pi i\nu\Delta t}\, d\nu$

$\times \int_{-(B_m/2)}^{(B_m/2)-m\Delta f_0} R^*(m\Delta f_0, \nu + m\Delta f_0)\, e^{2\pi i\nu(\Delta t+\delta T)}\, d\nu +$

(V) $+ D(B_m - m\Delta f_0)\, e^{-2\pi i m\Delta f_0\Delta t} \int_{-(B_m/2)+m\Delta f_0}^{B_m/2} R^*(m\Delta f_0, \nu - m\Delta f_0)\, e^{2\pi i\nu\Delta t}\, d\nu$

$\times \int_{-(B_m/2)}^{(B_m/2)-m\Delta f_0} R(m\Delta f_0, \nu + m\Delta f_0)\, e^{-2\pi i\nu(\Delta t+\delta T)}\, d\nu +$

$$\text{(VI)} \quad + D(B_m - m\Delta f_0) \int_{-(B_m/2)+m\Delta f_0}^{B_m/2} R(0, \nu - m\Delta f_0)\, e^{2\pi i\nu\Delta t}\, d\nu$$

$$\times \int_{-(B_m/2)}^{(B_m/2)-m\Delta f_0} R(0, \nu + m\Delta f_0)\, e^{-2\pi i\nu(\Delta t+\delta T)}\, d\nu \; +$$

$$\text{(VII)} \quad + N_o \frac{\sin \pi B_m \Delta t}{\pi \Delta t} R^*(0, \Delta t + \delta T) \; +$$

$$\text{(VIII)} \quad + N_o \frac{\sin \pi B_m (\Delta t + \delta T)}{\pi(\Delta t + \delta T)} R(0, \Delta t) \; +$$

$$\text{(IX)} \quad + D(B_m - m\Delta f_0) \left\{ N_o \frac{\sin \pi B_m \Delta t}{\pi \Delta t} \int_{-(B_m/2)}^{(B_m/2)-m\Delta f_0} R(0, \nu + m\Delta f_0)\, e^{-2\pi i\nu(\Delta t+\delta T)}\, d\nu \; + \right.$$

$$\left. \text{(X)} \quad + N_o \frac{\sin \pi B_m (\Delta t + \delta T)}{\pi(\Delta t + \delta T)} \int_{-(B_m/2)+m\Delta f_0}^{B_m/2} R(0, \nu - m\Delta f_0)\, e^{2\pi i\nu\Delta t}\, d\nu \right\} \; +$$

$$\text{(XI)} \quad + N_o^2 \frac{\sin \pi B_m \Delta t \cdot \sin \pi B_m (\Delta t + \delta T)}{\pi \Delta t \cdot \pi(\Delta t + \delta T)} \quad . \tag{44}$$

Again we make the assumption that the observation time spent at each frequency separation is large enough so that the integration over Δt can be carried between infinite limits. Also, in order to obtain a reasonably simple result suitable for discussion we evaluate all the above integrals on the assumption that the channel properties are of the form of Eqs. (26) or (27). It is found that:

Terms (I) + (VII) + (VIII) + (XI)

$$= \Theta \frac{\sin \pi B_m \delta T}{\pi \delta T} (\alpha^2 + 2\alpha N_o + N_o^2) \quad ;$$

Terms (II) + (III) + (VI) + (IX) + (X)

$$= \Theta D(B_m - m\Delta f_0) \frac{\sin \pi (B_m - m\Delta f_0)\, \delta T}{\pi \delta T} (3\alpha^2 + 2\alpha N_o) \quad ;$$

Terms (IV) + (V)

$$= 2\Theta D(B_m - m\Delta f_0) \frac{\sin \pi (B_m - m\Delta f_0)\, \delta T}{\pi \delta T} \cdot \frac{\alpha^2}{1 + (2\pi\tau_o \Delta f)^2} \quad . \tag{45}$$

When this is substituted back into the expression for the variance, we obtain

$$\langle \Delta\sigma(\tau, f_D)^2 \rangle_{avg} \sim \sum_{-M}^{M} \frac{1}{\Theta(m)} \sum_{\ell} \sum_{n} \left\{ \frac{\sin \pi B_m T_o(\ell - n)}{\pi T_o(\ell - n)} (\alpha + N_o)^2 + D(B_m - m\Delta f_0) \right.$$

$$\left. \times \frac{\sin \pi (B_m - m\Delta f_0)\, T_o(\ell - n)}{\pi T_o(\ell - n)} (3\alpha^2 + 2\alpha N_o + \frac{2\alpha^2}{1 + (2\pi\tau_o \Delta f_0 \ell)^2} \right\} e^{2\pi i T_o f_D (\ell - n)} \quad . \tag{46}$$

Certain conclusions can be drawn about integration times $\Theta(m)$ from this expression. In the general case, the double sum over ℓ and n is a function of both m and f_D; let this function be denoted by $S(m, f_D)$. Hence,

$$\langle \Delta\sigma(\tau, f_D)^2 \rangle_{avg} \sim \sum_{-M}^{M} \frac{S(m, f_D)}{\Theta(m)} \quad . \tag{47}$$

If we were to ask which set of $\Theta(m)$ would lead to an optimization of the output signal-to-noise ratio for a given point (τ, f_D) and for a given total observation time $\Theta_o = \sum_m \Theta(m)$, we would obtain

$$\Theta(m) \sim \sqrt{S(m, f_D)} \quad . \tag{48}$$

However, if the channel is distinctly underspread, or if the noise-power density N_o is considerably greater than the signal-power density, then $S(m, f_D)$ will be essentially independent of m and we conclude that equal times should be spent at each frequency separation.

IV. DISCUSSION OF APPLICATION TO RADAR ASTRONOMY

Let us finally very briefly consider an example of the determination of the target-scattering function in radar astronomy by this method. For simplicity we consider the particular case of $\alpha < N_o$, i.e., where the energy density of the received signal is less than the noise-energy density. This is a situation most frequently encountered in radar astronomy except in cases of lunar and Venus reflections by the very largest radar astronomy installations. It is also a situation on which extensive theoretical considerations regarding optimum detection have been based.[8] In the variance terms in Eq. (46), we then only need include the term involving N_o^2. First we settle on a depth range to be investigated; ideally, this depth should be equal to the radius of the target. In practice, however, we might specify the depth range somewhat smaller than this because the echo toward the limbs of the object is frequently too weak to be detected anyway. We let the depth range be Δf_0^{-1}; this specifies the smallest frequency separation we want to use in the experiment. Similarly, the smallest time shift T_o must be chosen about equal to B_m^{-1}. The next question is concerned with the number of resolution cells we want within the specified ambiguity range. Suppose we desire 2M resolution cells in depth and 2L in frequency. With the flat noise spectrum assumed in the above calculations we obtain

$$\langle \Delta\sigma(\tau, f_D)^2 \rangle_{avg} = N_o^2 \frac{2M+1}{\Theta} \sum_{\ell=-L}^{L} \sum_{n=-L}^{L} \frac{\sin \pi(\ell - n)}{T_o \pi(\ell - n)} e^{2\pi i T_o f_D (\ell - n)} \tag{49}$$

If L is at least moderately large, the double sum over ℓ and n will be approximately $(2L + 1)\, T_o^{-1}$ when $|f_D| < B_m/2$, and zero otherwise. Hence, within the bandwidth containing the signal we have

$$\langle \Delta\sigma(\tau, f_D)^2 \rangle_{avg} = N_o^2 (2L + 1)\,(2M + L)\, \frac{B_m}{\Theta} \quad . \tag{50}$$

On the assumption of an exponential target-scattering function, the mean value of the signal will be

$$\langle \sigma \rangle_{avg} = \frac{\alpha}{T_o} \frac{1}{\Delta f_0 \tau_o} e^{-\tau/\tau_o} \quad \text{when } |f_D| < \frac{B_m}{2} \quad . \tag{51}$$

Now, $\Delta f_0 \tau_o$ is some numerical factor depending on the ratio of the decay constant τ_o and the depth to which we want to investigate the target. Let this ratio be β^{-1}. The signal-to-noise ratio becomes

$$\left(\frac{\text{signal}}{\text{noise}}\right)^2 = \left(\frac{\alpha}{N_o}\right)^2 \frac{B_m \Theta \beta^2}{(2L+1)(2M+1)} e^{-2\tau/\tau_o} , \tag{52}$$

where $(2L + 1)(2M + 1)$ is the number of resolution cells. Finally, introducing the total observation time $\Theta_o = M\Theta$ and the total power $P_{rec} = \alpha B_m{}^2$ we obtain

$$\left(\frac{\text{signal}}{\text{noise}}\right)^2 = \left(\frac{P_s}{2N_o}\right)^2 \frac{\beta^2 \Theta_o}{M(2L+1)(2M+1) B_m} e^{-2\tau/\tau_o} . \tag{53}$$

V. CONCLUSIONS

We have shown that a measuring scheme can be devised which makes the two-frequency method of investigating statistically stationary spread channels possible both in the underspread and overspread situations. In case the channel has correlated taps, the statistical properties of the channel vary with center frequency f_0 so that the observation must be repeated for several center frequencies. The method described appears to be a rather flexible one in that the various channels in the observational scheme shown in Fig. 1 should be rather easy to implement and to alter at will, particularly if some of the processing is being carried out by digital techniques. It appears to be rather difficult to compare the method with other methods of obtaining target-scattering function estimates because few such estimates are available and the constraints on the measuring methods are often different. One might think that a direct comparison could be made of the results of Gallager's chirp experiment[4] and the present results. It turns out, however, that it is not at all obvious how such a comparison is to be carried out because in the present report we measure $R(\Delta f, T)$ for a particular Δf with variable T, whereas the chirp method measures $R(\alpha T, T)$ where α is a constant determining the rate of change of the frequency in the chirp. We believe, however, that the method described might be advantageous in cases where there is a peak-power constraint on the transmissions that can be used. A disadvantage under these circumstances would be the fact that the average transmitted power is only equal to half the peak power.

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REFERENCES

1. P.E. Green, Jr., "Radar Astronomy Measurement Techniques," Technical Report 282, Lincoln Laboratory, M.I.T. (12 December 1962), DDC 400563.
2. T. Kailath, "Sampling Models for Linear Time-Variant Filters," Technical Report 352, Research Laboratory of Electronics, M.I.T. (June 1959).
3. T. Hagfors, "Some Properties of Radio Waves Reflected from the Moon and Their Relation to the Lunar Surface," J. Geophys. Res. 66, 777-785 (1961).
4. R.G. Gallager, "Characterization and Measurement of Time- and Frequency-Spread Channels," Technical Report 352, Lincoln Laboratory, M.I.T. (30 April 1964), DDC 443715.
5. T. Kailath, "Measurements on Time-Variant Communication Channels," Trans. IRE, PGIT IT-8, S229 (1962).
6. R. Price, "Maximum-Likelihood Estimation of the Correlation Function of a Threshold Signal, and Its Application to the Measurement of the Target Scattering Function in Radar Astronomy," Group Report 34G-4, Lincoln Laboratory, M.I.T. (25 May 1962), DDC 276903, H-403.
7. M.J. Levin, "Estimation of the Second-Order Statistics of Randomly Time-Varying Linear Systems," Group Report 34G-7, Lincoln Laboratory, M.I.T. (2 November 1962), DDC 289607, H-445.
8. R. Price, Chapter VI in Radar Astronomy, J.V. Evans and T. Hagfors, eds. (McGraw-Hill, New York) (in preparation).

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13. ABSTRACT

A particular observation scheme based on the two-frequency correlation function approach for the measurement of time-varying spread channels having time-stationary Gaussian statistical properties is analyzed. It is shown that the scheme works well for underspread as well as overspread channels. The method also appears to be useful in cases where there is correlation between the signals arriving with different delays. Expressions are derived for the variances of the determined correlation and target-scattering functions derived by double Fourier transformation of the observed correlation function.

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