INVERSE PROBLEMS IN RADIATIVE TRANSFER: I AYERED MEDIA

R. E. Bellman, H. H. Kagiwada, R. E. Kalaba and S. Ueno

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INVERSE PROBLEMS
IN RADIATIVE TRANSFER:
LAYERED MEDIA

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A basic problem in radiative transfer is the estimation of the physical parameters of a scattering and absorbing medium based on measurements of the diffusely reflected light. In this series of papers we shall show how such tasks may be viewed as nonlinear boundary-value problems which may be solved computationally using the technique of quasilinearization.

In this Memorandum we consider a stratified medium consisting of two layers. Our aim is to determine the optical depth and the albedo for single scattering of each layer based on measurements of the angular dependence of the light diffusely reflected from the slab. The results of some numerical experiments are presented, and FORTRAN IV listings are provided.

The results will be of interest to meteorologists, experimental physicists, and control engineers.
SUMMARY

Consider a slab which consists of two layers which absorb and scatter light. First the basic differential-integral equation for the intensity of the diffusely reflected light is given, the source being uniform parallel rays. Then we show how to determine the nature of each layer using measurements of the light diffusely reflected from the slab. We view this as a nonlinear boundary-value problem and show that it may be resolved computationally using the technique of quasilinearization. Numerical examples and FORTRAN programs are provided.
I. INTRODUCTION

Problems of radiative transfer in planetary and stellar atmospheres have been extensively treated by many investigators.\(^{(1,2,3)}\) Much of this work has dealt with what we may call direct problems, i.e., the determination of the intensities of radiation produced by certain light incident on a medium with known scattering and absorbing properties. Recently, though, as a growing list of papers attests, there has been increasing interest in inverse problems. In these, the aim is to deduce the nature of the medium (and often the source) on the basis of measurements of the radiation field produced by incident radiation.\(^{(4,5,6)}\)

With this Memorandum we initiate the study of inverse problems in radiative transfer via the method of quasilinearization.\(^{(7,8)}\) We wish to show that such problems can be viewed as multipoint boundary-value problems for large systems of nonlinear ordinary differential equations. These equations can then be resolved numerically using current digital computers and the method of quasilinearization.

We consider a slab which consists of two layers of isotropically scattering material of unknown optical thickness and albedo for single scattering. Given the measured intensity of the light diffusely reflected from the slab, we wish to determine the optical thickness of the two layers and their albedos for single scattering.

In subsequent papers we shall consider more complex processes involving anisotropic scattering, time-dependence, and noncoherent scattering.
II. BASIC EQUATIONS

We consider an inhomogeneous, plane-parallel, non-emitting and isotropically scattering atmosphere of finite optical thickness $\tau_1$ whose optical properties depend only upon $\tau$, the optical height above the bottom ($0 \leq \tau \leq \tau_1$). Let parallel rays of light of net flux $\pi$ per unit area normal to their direction of propagation be incident on the upper surface in the direction characterized by the number $\mu_o$ ($0 < \mu_o < 1$), where $\mu_o$ is the cosine of the angle measured from the normal to the surface. The bottom surface is a completely absorbing barrier, so that no light is reflected from it. This is not an essential requirement. See Fig. 1.

Fig. 1—Incident and reflected rays for a slab of thickness $\tau_1$
The intensity of the diffusely reflected light in the direction \( \cos^{-1} \mu \) is \( r(\mu, \mu_0, \tau_1) \). We define a related function \( R(\mu, \mu_0, \tau_1) \), symmetric in \( \mu \) and \( \mu_0 \), by writing
\[
r(\mu, \mu_0, \tau_1) = \frac{R(\mu, \mu_0, \tau_1)}{4 \mu}.
\] (1)

The function \( R \) satisfies the integro-differential equation (1,9)
\[
\frac{\partial R(\mu, \mu_0, \tau_1)}{\partial \tau_1} = -\left( \frac{1}{\mu} + \frac{1}{\mu_0} \right) R(\mu, \mu_0, \tau_1) + \lambda(\tau_1) \left[ 1 + \frac{1}{2} \int_0^1 R(\mu, \mu', \tau_1) \frac{du'}{\mu'} \right] \left[ 1 + \frac{1}{2} \int_0^1 R(\mu', \mu_0, \tau_1) \frac{du'}{\mu'} \right]
\] (2)

where \( \lambda(\tau_1) \) is the albedo for single scattering, and \( R \) is subject to the condition
\[
R(\mu, \mu_0, \Omega) = 0.
\] (3)

This equation is obtained by means of the theory of invariant imbedding. A discrete version of Eqs. (2) and (3) is obtained by replacing the integrals by finite sums using a Gaussian quadrature formula of order \( N \). Let us introduce new functions \( R_{ij}(\tau_1) \),
\[
R_{ij}(\tau_1) = R(\mu_i, \mu_j, \tau_1); \quad i,j = 1,2,\ldots,N,
\] (4)

where \( \{\mu_i\} \) is the set of \( N \) roots of the shifted Legendre polynomial \( P^*_N(\mu) = P_N(1-2\mu) \). The values of \( \mu_i \) and the corresponding Christoffel weights are tabulated in Ref. 10 for \( N = 3,4,\ldots,15 \). The functions \( R_{ij}(\tau_1) \) satisfy the system of ordinary nonlinear differential equation.
\[
\frac{dR_{ij}(\tau_i)}{d\tau_i} = - \left( \frac{1}{\mu_i} + \frac{1}{\mu_j} \right) R_{ij}(\tau_i) \\
+ \lambda(\tau_i) \left[ 1 + \frac{1}{2} \sum_{k=1}^{N} R_{ik}(\tau_i) \frac{w_k}{\mu_k} \right] \left[ 1 + \frac{1}{2} \sum_{k=1}^{N} R_{kj}(\tau_i) \frac{w_k}{\mu_k} \right],
\]  

(5)

with the initial conditions

\[
R_{ij}(0) = 0,
\]  

(6)

for \( i = 1, 2, \ldots, N; j = 1, 2, \ldots, N. \)
III. AN INVERSE PROBLEM

Let us now consider a medium composed of two layers, with albedo for single scattering \( \lambda_1 \) in the lower layer and albedo \( \lambda_2 \) in the upper layer \( (\lambda_1 \neq \lambda_2) \). The total optical thickness of this slab is \( T \) and the thickness of the lower layer is \( c \). See Fig. 2. We wish to determine \( c \), \( T \), and the parameters \( \lambda_1 \) and \( \lambda_2 \).

\[
\begin{align*}
\mu_i & \quad \mu_j \\
\lambda_2 \quad \text{(Layer 2)} \quad & \quad \lambda_1 \quad \text{(Layer 1)} \\
\text{Layer 1} & \quad \text{Layer 2} \\
T-c & \quad T_1 + T \\
c & \quad T-c
\end{align*}
\]

Fig. 2—A stratified medium

Let us assume that we have obtained \( N^2 \) (noisy) measurements of the diffusely reflected light, \( b_{ij} \), where \( b_{ij} \) is the intensity of the light diffusely reflected in the direction \( \mu_i \) caused by incident parallel rays of net flux \( \pi \) in the direction \( \mu_j \). The constants which characterize this medium, \( \lambda_1, \lambda_2, c, \) and \( T \) are to be determined so that the theoretical diffuse reflection pattern produced by using the estimated values in the differential Eqs. (5) will agree as closely
as possible, in the least squares sense, with the observed field 
\( \{b_{ij}\} \), i.e., we wish to minimize the expression

\[
\sum_{i=1}^{N} \sum_{j=1}^{N} \left[ r_{ij}(T) - b_{ij} \right]^2.
\]

(7)

Let us assume that the albedo function has the form

\[
\lambda(T) = a + b \tanh 10(T - c)
\]

(8)

so that

\[
\lambda_1 = a - b \quad \text{in Layer 1},
\]

\[
\lambda_2 = a + b \quad \text{in Layer 2},
\]

(9)

and \( c \) is the position of the boundary between Layer 1 and Layer 2.

See Fig. 3.

The "observations" \( \{b_{ij}\} \) are produced computationally with the use of Eq. (8) for the albedo function, where we set

\[
a = 0.5, \quad b = 0.1, \quad c = 0.5,
\]

(10)

and with the use of the differential Eqs. (5), integrating out to a thickness

\[
T = 1.0.
\]

The observations which this produces are given in Table 1.
Fig. 3—The albedo function

![Diagram of the albedo function with layers](image)

Table 1

THE MEASUREMENTS \(b_{ij}\)

<table>
<thead>
<tr>
<th></th>
<th>(j = 1)</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<td>1</td>
<td>0.079914</td>
<td>0.028164</td>
<td>0.014304</td>
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<td>0.006707</td>
<td>0.005515</td>
<td>0.004970</td>
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<td>0.143038</td>
<td>0.091522</td>
<td>0.058437</td>
<td>0.040826</td>
<td>0.031405</td>
<td>0.026378</td>
<td>0.023989</td>
</tr>
<tr>
<td>3</td>
<td>0.167000</td>
<td>0.134331</td>
<td>0.099653</td>
<td>0.075106</td>
<td>0.060044</td>
<td>0.051445</td>
<td>0.047248</td>
</tr>
<tr>
<td>4</td>
<td>0.178898</td>
<td>0.157955</td>
<td>0.126408</td>
<td>0.099392</td>
<td>0.081253</td>
<td>0.070435</td>
<td>0.065042</td>
</tr>
<tr>
<td>5</td>
<td>0.185284</td>
<td>0.170817</td>
<td>0.142072</td>
<td>0.114229</td>
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<td>0.082423</td>
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</tr>
<tr>
<td>6</td>
<td>0.188723</td>
<td>0.177733</td>
<td>0.150791</td>
<td>0.122665</td>
<td>0.102104</td>
<td>0.089349</td>
<td>0.082870</td>
</tr>
<tr>
<td>7</td>
<td>0.190354</td>
<td>0.180898</td>
<td>0.154995</td>
<td>0.126773</td>
<td>0.105829</td>
<td>0.092748</td>
<td>0.086083</td>
</tr>
</tbody>
</table>
IV. NUMERICAL SOLUTION VIA QUASILINEARIZATION

Let us consider the problem posed above in more general and flexible terms. An R-dimensional column vector $x(t)$ is a solution of the differential equation

$$\dot{x} = f(x), \quad 0 \leq t \leq T.$$  

(11)

The value of $T$ is assumed known. This again represents no essential restriction as we shall see. We wish to determine the initial vector $c$,

$$x(0) = c,$$  

(12)

in such a manner that we minimize the quadratic form

$$(x(T) - b, x(T) - b) = Q,$$  

(13)

where $x(T)$ is the solution of Eq. (1) for $t = T$. The first $s$ components of $c$ are required to be zero and the remaining ones are variables to be determined.

Our computational formalism proceeds as follows: First an initial approximation to the desired initial vector is selected. We then proceed inductively. Suppose that we have obtained a $k^{th}$ approximation to $c$, which we denote by $c^k$, and a $k^{th}$ approximation to the solution function $x(t)$, which we denote by $x^k(t)$. Note that the superscripts refer to the order of the approximation and not to the components of $c$ and $x$. We obtain the vectors $c^{k+1}$ and $x^{k+1}(t)$ in this manner. The vector $x^{k+1}(t)$ is a solution of the linear system

$$x^{k+1} = f(x^k) + J(x^k)(x^{k+1} - x^k),$$  

(14)
where $J(x^k)$ is the Jacobian matrix with elements

$$J_{ij} = \frac{\partial f_i}{\partial x_j}.$$  \hspace{1cm} (15)

We produce a particular solution $p(t)$ of the system (14) by numerically integrating the system

$$\dot{p} = f(x^k) + J(x^k) (p - x^k),$$  \hspace{1cm} (16)

where $p(t)$ is subject to the initial condition

$$p(0) = 0.$$  \hspace{1cm} (17)

Then we produce numerically $R$ independent solutions of the homogeneous system

$$\dot{h}_i = J(x^k) h_i,$$  \hspace{1cm} (18)

where the vector $h_i(t)$ is subject to the initial condition

$$(j \text{ component of } h_i(0)) = \delta_{ij}, \quad i = s + 1, \ldots, R,$$  \hspace{1cm} (19)

$\delta_{ij}$ being the Kronecker delta function. These integrations are readily carried out on the interval $0 \leq t \leq T$, since complete sets of initial conditions are given. From general theory we know that the vector $x^{k+1}$ is representable in the form

$$x^{k+1}(t) = p(t) + \sum_{i=s+1}^{R} m_i h_i(t),$$  \hspace{1cm} (20)

where the numbers $m_i$ are arbitrary constants. In view of the way we have chosen the initial conditions in Eqs. (17) and (19), we see that
m_t represents the initial value of the \( i^{th} \) component of the vector \( x^{k+1} \). In particular, when the integrations are completed, we shall have, in numerical form, the values of the vectors \( p(T), h_{s+1}(T), \ldots, h_R(T) \).

We now wish to minimize the form

\[
\left( p(T) + \sum_{i=s+1}^{R} m_i h_i(T) - b, p(T) + \sum_{i=s+1}^{R} m_i h_i(T) - b \right)
\]

\[
= F( m_{s+1}, m_{s+2}, \ldots, m_R )
\]

over the values of \( m_i, i = s+1, s+2, \ldots, R \), where \( b \) is known experimentally and \( p(T), h_{s+1}(T), h_{s+2}(T), \ldots, h_R(T) \) are known computationally. This minimization is done by solving, numerically, the linear algebraic system of equations

\[
\frac{\partial F}{\partial m_i} = 0, \quad i = s+1, s+2, \ldots, R.
\]

The values of \( m_{s+1}, m_{s+2}, \ldots, m_R \) so obtained are the values of the last \( R-s \) components of the vector \( c^{k+1} \), and \( x^{k+1}(t) \) can be determined from Eq. (20) using the values of \( m_{s+1}, m_{s+2}, \ldots, m_R \) and the stored values of \( p(t) \) and \( h_i(t), i = s+1, s+2, \ldots, R \). The entire process is then repeated to obtain the \((k+2)\)nd approximations.

To simplify the procedure of finding the \( m \)'s, let us observe that the relation

\[
x^{k+1}(T) = b
\]

can be written in the form
If we let $H$ be the matrix whose $i^{th}$ column is $h_i(T)$, $i = s+1, s+2, \ldots, R$, $y$ be the column vector $b - p(T)$, and $m$ be the column vector whose $i^{th}$ component is $m_i$, then the above relation can be rewritten

$$Hm = y,$$  

where $H$ is a matrix with $R$ rows and $R-s$ columns, and $m$ and $y$ are $R-s$ dimensional column vectors. According to the theory of the method of least squares the vector $m$ which minimizes the form

$$(Hm - y, Hm - y) = 0$$

is

$$m = (H^tH)^{-1}H^ty,$$

where $H^t$ is the transpose of $H$.

A few general comments on the procedure are now in order. The selection of a good initial approximation is important, since in this case the method is rapidly convergent, with the number of correct digits approximately doubling with each additional step; if, however, the initial approximation is too poor, the method may be divergent. At each step we must integrate $R(R-s+1)$ first order differential equations with given initial conditions to produce the $R-s$ homogeneous solutions and the one particular solution. In addition we must solve a system of $R-s$ linear algebraic equations. This may be a sizable computing load, and the solving of the linear algebraic equations can be a source of great difficulty.
V. NUMERICAL EXPERIMENTS TO DETERMINE c,
THE THICKNESS OF THE LOWER LAYER

In the first series of experiments we wish to determine only the thickness of the lower layer, assuming that all of the other parameters of the medium are known. We use a seven-point Gaussian quadrature, so that \( N = 7 \). We consider \( c \) to be a function of \( \tau_1 \) for which \( \frac{dc}{d\tau_1} = 0 \). Following the method prescribed in the preceding section, we obtain \( N^2 = 49 \) linear differential equations for \( R_{i,j}^{k+1} \) and one equation for \( c^{k+1} \). Thus there are 50 linear differential equations

\[
\frac{dR_{i,j}^{k+1}}{d\tau_1} = \left\{ -\left( \frac{1}{\mu_1} + \frac{1}{\mu_j} \right) R_{i,j}^k + \lambda(c^k) \left( 1 + \frac{1}{2} \sum_{L=1}^{N} R_{i,L}^k \frac{W_L}{\mu_L} \right) \right. \times \left. \left( 1 + \frac{1}{2} \sum_{L=1}^{N} R_{j,L}^k \frac{W_L}{\mu_L} \right) \right\} + \left\{ -\left( \frac{1}{\mu_1} + \frac{1}{\mu_j} \right) \left( R_{i,j}^{k+1} - R_{i,j}^k \right) \right. \\
+ \frac{1}{2} \lambda(c^k) \left[ \left( 1 + \frac{1}{2} \sum_{L=1}^{N} R_{i,L}^k \frac{W_L}{\mu_L} \right) \times \sum_{L=1}^{N} \left( R_{j,L}^k - R_{j,L}^k \right) \frac{W_L}{\mu_L} \right] \\
+ \left( 1 + \frac{1}{2} \sum_{L=1}^{N} R_{i,L}^k \frac{W_L}{\mu_L} \right) \times \sum_{L=1}^{N} \left( R_{i,L}^{k+1} - R_{i,L}^k \right) \frac{W_L}{\mu_L} \right\} \\
+ \left\{ (c^{k+1} - c^k) \left( 1 + \frac{1}{2} \sum_{L=1}^{N} R_{i,L}^k \frac{W_L}{\mu_L} \right) \left( 1 + \frac{1}{2} \sum_{L=1}^{N} R_{j,L}^k \frac{W_L}{\mu_L} \right) \right. \times \left. \left( -10 b \ \text{sech}^2 10 (\tau_1 - c^k) \right) \right\},
\]

\[
\frac{dc^{k+1}}{d\tau_1} = 0,
\]
where
\[ \lambda(c^k) = a + b \tanh 10(\tau_1 - c^k) \]

One may reduce the number of differential equations in the system (28) by using the symmetry property
\[ R_{ij}^{k+1}(\tau_1) = R_{ji}^{k+1}(\tau_1), \quad i = 1, 2, \ldots, N, \]
\[ j = 1, 2, \ldots, N. \]  \hspace{1cm} (29)

Thus instead of \( N^2 + 1 = 50 \) differential equations for \( R_{ij}^{k+1} \) and \( c^{k+1} \), we have \( (N(N+1)/2) + 1 = 29 \) differential equations plus the finite Eqs. (29) to fill in the missing values of \( R \). However, we will still speak of the matrix \( \{R_{ij}^{k+1}\} \) as having 7 rows and 7 columns.

Now let the 50-dimensional vector \( x^{k+1}(\tau_1) \) represent the 49 elements \( R_{ij}^{k+1}(\tau_1) \) taken in some order,
\[ x^{k+1}_i(\tau_1) = R_{ij}^{k+1}(\tau_1) \]  \hspace{1cm} (30)
for \( i = 1, 2, \ldots, 49 \) as \( i = 1, 2, \ldots, 7, \) and \( j = 1, 2, \ldots, 7 \), and
\[ x^{k+1}_{50}(\tau_1) = c^{k+1}(\tau_1). \]  \hspace{1cm} (31)

We express \( x^{k+1}(\tau_1) \) as a sum of a particular solution \( p(\tau_1) \) and a homogeneous solution \( h(\tau_1) \) of the Eqs. (28),
\[ x^{k+1}(\tau_1) = p(\tau_1) + m h(\tau_1). \]  \hspace{1cm} (32)

We require that the multiplier \( m \) be chosen to minimize the expression
\[
\sum_{l=1}^{49} \left[ p_L(1) + m h_L(1) - b_L \right]^2, \quad (33)
\]

where the singly subscripted set \( \{b_L\} \) \((l = 1, 2, \ldots, 49)\) represents the doubly subscripted set of constants \( \{b_{i,j}\}\) \(i = 1, 2, \ldots, 7, j = 1, 2, \ldots, 7\) taken in the same order as the transformation (30). The value of \( m \) which minimizes (33) is that for which

\[
m = \frac{\sum_{l=1}^{49} h_L(1)[b_L - p_L(1)]}{\sum_{l=1}^{49} [h_L(1)]^2}, \quad (34)
\]

as we see from a simple differentiation. With the proper choice of initial values \( p(0) \) and \( h(0) \), this constant \( m \) is the initial value

\[
m = c_k^{+1}(0)
\]

and hence gives directly the thickness of Layer 1.

The results of three experiments with different initial guesses of the thickness \( c \) are shown in Table 2. The initial approximation is generated in each run by integrating the nonlinear Eqs. (5) with the value of \( c \) listed as approximation zero.

Table 2

<table>
<thead>
<tr>
<th>Approximation</th>
<th>Run 1</th>
<th>Run 2</th>
<th>Run 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.2</td>
<td>0.8</td>
<td>0.0</td>
</tr>
<tr>
<td>1</td>
<td>0.62</td>
<td>0.57</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.5187</td>
<td>0.5024</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.500089</td>
<td>0.499970</td>
<td>0.499991</td>
</tr>
<tr>
<td>4</td>
<td>0.499990</td>
<td>0.499991</td>
<td></td>
</tr>
<tr>
<td>True Value</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>
Even with 60 per cent errors in the initial approximations, the value of c is determined to one part in fifty thousand or 0.002 per cent in four iterations. The time required on the IBM 7044 is 1½ minutes per run, using an integration step size of 0.01 and the Adams-Moulton method. Making use of the symmetry of R, 58 linear differential equations have to be integrated for the particular and homogeneous solutions for each approximation. In Run 3 the solution diverges because the initial guess is not good enough.
VI. NUMERICAL EXPERIMENTS TO DETERMINE T,
THE OPTICAL THICKNESS

Next we consider the estimation of the total optical thickness \( T \) when all of the other system constants are known and 49 measurements \( b \) are given. The formulation of the linear boundary-value problem proceeds as for the previous case with one major difference. The terminal boundary for the interval of integration is unknown. In order that the interval be fixed, we define a new independent variable \( \sigma \),

\[ \sigma T = \tau_1, \]

with \( 0 \leq \sigma \leq 1 \) as the range of integration. The constant \( T \) is a solution of the equation \( \frac{dT}{d\sigma} = 0 \). Then one may obtain the linear equations which correspond to (28), with independent variable \( \sigma \) instead of \( \tau_1 \).

Three trials are made to determine \( T \), with initial guesses \( T^0 = 0.9, 1.5 \) and 0.5 respectively. Recall that the true value is \( T = 1.0 \). Within four minutes of computing time, four iterations are carried out per trial, and the results obtained are correct to one part in 100,000.
VII. NUMERICAL EXPERIMENTS TO DETERMINE $\lambda_1$, $\lambda_2$, and $c$

THE ALBEDOS AND OPTICAL THICKNESS

In the final experiment, on the basis of the 49 observations, we try to determine $\lambda_1$, $\lambda_2$, and $c$ assuming $T = 1.0$ is known. Again we use $N = 7$. This time there are three homogeneous solutions and a particular solution to compute, each with $N(N+1)/2 + 3 = 31$ components, so that $4 \times 31 = 124$ linear differential equations are to be integrated during each stage of the successive approximation calculations. A standard (Gaussian elimination) matrix inversion procedure is used to invert the $3 \times 3$ matrix of the linear algebraic system. The results in Table 3 are obtained in three iterations which consume 2 minutes of computing time on the IBM 7044 machine.

<table>
<thead>
<tr>
<th>Approximation</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$c$</th>
</tr>
</thead>
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<td>0</td>
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<td>0.4</td>
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<td>0.6052</td>
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<tr>
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<td>0.599994</td>
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</tr>
<tr>
<td>True Values</td>
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<td>0.6</td>
<td>0.5</td>
</tr>
</tbody>
</table>
VIII. DISCUSSION

The general conceptual and computational approach to inverse problems which we have discussed here is by no means limited to the simple physical model considered. At the expense of additional computing time we may treat atmospheres having anisotropic scattering, time-dependent sources, and so on. We may also wish to consider wave rather than particle processes. Applications to orbit determination, (11) system identification, (12,13) cardiology (14) and other areas (15) have been made. Of particular importance is the question of the effect of errors in the observations on the accuracy of the estimates of the parameters. In this connection it appears that the use of the min-max criterion rather than that of least squares may be efficacious. These and other matters will be discussed in later papers.
APPENDIX

This Appendix lists the FORTRAN IV programs which were written for (I) The Estimation of $c$, (II) The Estimation of $T$, and (III) The Estimation of $\lambda_1$, $\lambda_2$, and $c$. The major difference in the three programs lies in the subroutine DAUX, in which the right-hand sides of the differential equations being integrated must be evaluated in the same order as the dependent variables themselves and both sets stored in the array called "T". $T(2)$ contains the current value of the independent variable, $T(3)$ contains the integration step length, the next $M$ locations from $T(4)$ through $T(M+3)$ contain the $M$ variables which are being integrated, followed by the $M$ derivatives in locations $T(M+4)$ through $T(2M+3)$. The integration routine used (RAND 7044 Library Routine Number W031) is called upon with the statements CALL INTS ($T$, $M$, 2, 0, 0, 0, 0, 0, 0) and CALL INTM, and these in turn call upon subroutine DAUX.

In program (III), the matrix inversion and linear algebraic equation solving subroutine MATINV (RAND Library Routine Number W019) is used. The matrix of coefficients in the program is $\text{EMAT}$, the number of unknowns is three, and the right-hand vector is $FVEC$. 
$1$BTIC RTIMV

PROGRAM FOR THE ESTIMATION OF \( c \), THE THICKNESS OF THE LOWER LAYER

COMMON \( N, RT(7), WT(7), WR(7), AR(7, 7), NPPNT, M1MAX, KMAX, DELTA, XTAU \),
1 ZERLAM, XLAM(2), B2(7, 7), R2(7, 7), IFLAG, R(28, 101), T(1491), SIG,
2 P(28, 101), H(28, 3, 101), PTAU, PLAM(2), HTAU(3), HLAM(2, 3), P2(7, 7),
3 H2(7, 7, 3), CONST(3), NEQ

PHASE I

1 READ1000, N
PRINT899
PRINT900, N
READ1001, (RT(I), I=1, N)
PRINT901, (RT(I), I=1, N)
READ1001, (WT(I), I=1, N)
PRINT901, (WT(I), I=1, N)
DO 2 I=1, N
WR(I)=WT(I)/RT(I)
DO 2 J=1, N
2 AR(I, J)= 1.0/RT(I) + 1.0/RT(J)

899 FORMAT(1H146X36HRADIATIVE TRANSFER - INVERSE PROBLEM /)
1000 FORMAT(6I12)
900 FORMAT(6I120)
1001 FORMAT(6E12.8)
901 FORMAT(6E20.8)
READ1000, NPRNT, M1MAX, KMAX
PRINT900, NPRNT, M1MAX, KMAX
READ1001, DELTA
PRINT901, DELTA
READ1001, XTAU, ZERLAM, XLAM(1), XLAM(2)
PRINT902
PRINT903, XTAU, ZERLAM, XLAM(1), XLAM(2)

902 FORMAT(1H123HPHASE I = TRUE SOLUTION /)
903 FORMAT(1H130HAPPROXIMATION ITERATIONS

1 1X111THICKNESS =, F10.4 /
2 1X11HALBEDO(X) =, 20HA + 8*TANH(10*(X-C)) //
3 1X3HA =, E16.8, 10X3HB =, E16.8, 10X3HC =, E16.8 //
CALL NONLIN
DO 3 I=1, N
DO 3 J=1, N
3 B2(I, J)=R2(I, J)

PHASE II

4 READ1001, XTAU, ZERLAM, XLAM(1), XLAM(2)
K=C
PRINT904, K
PRINT903, XTAU, ZERLAM, XLAM(1), XLAM(2)

CALL NONLIN

904 FORMAT(1H1 13HAPPROXIMATION, I3/)

QUASILINEARIZATION ITERATIONS
DO 5 K1=1,KMAX
PRINT904,K1
CALL PANDH
CALL LINEAR
CONTINUE

READ1000,IGO
GO TO (1,4),IGO
END

SUBROUTINE DAUX
DIMENSION V2(7,7),X(3),F(7),G(7)
COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
1 ZERLAM,XLAM(2),B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
2 P(28,101),H(28,3,101),PTAU,PLAM(2),HTAU(3),HLAM(2,3),P2(7,7),
3 H2(7,7,3),CONST(3),NEQ
GO TO (1,2),IFLAG

C

CNONLINEAR

L=3
DO 4 I=1,N
L=L+1
4 V2(I,J)=T(L)
DO 5 I=1,N
DO 5 J=1,N
5 V2(I,J)=V2(J,I)
L=L+1
VLAM2=T(L)
SIG=T(2)
Y=XTAU*SIG
X(1)=ZERLAM
X(2)=XLAM(1)
X(3)=VLAM2
CALL ALBEDO(Y,X,Z)
ZLAMDA=Z

DO 6 I=1,N
F(I)=0.0
DO 7 K=1,N
7 F(I)=F(I) + WR(K)*V2(I,K)
6 F(I)=0.5*F(I) + 1.0

DO 8 I=1,N
DO 8 J=1,I
L=L+1
DR=-AR(I,J)*V2(I,J) + ZLAMDA*F(I)*F(J)
8 T(L)=DR
DO 9 I=1,1
L=L+1
9 T(L)=0.0
C RETURN
C C LINEAR
C 2 SIG = T(2)
Y = XTAU * SIG
X(1) = ZERLAM
X(2) = XLAM(1)
X(3) = XLAM(2)
CALL ALBEDO(Y, X, Z)
ZLAMDA = 2
C DO 16 I = 1,N
F(I) = 0.0
DO 17 K = 1,N
17 F(I) = F(I) + W(R(K)) * R2(I, K)
16 F(I) = 0.5 * F(I) + 1.0
C C P'S
C
L = 3
DO 14 I = 1,N
DO 14 J = 1,I
L = L + 1
14 V2(I, J) = T(L)
DO 15 I = 1,N
DO 15 J = 1,I
15 V2(I, J) = V2(J, I)
L = L + 1
VLAM2 = T(L)
C DO 10 I = 1,N
G(I) = 0.0
DO 10 K = 1,N
10 G(I) = G(I) + (V2(I, K) - R2(I, K)) * W(R(K))
ARG = 10.0 * C * (Y - XLAM(2))
XTANX = 10.0 * XLAM(1) * (1.0 - (TANH(ARG)) ** 2)
M = 3 * NEG
DO 12 I = 1,N
DO 12 J = 1,I
FIJ = F(I) * F(J)
CAPF = -AR(I, J) * R2(I, J) + ZLAMDA * FIJ
T1 = -AR(I, J) * (V2(I, J) - R2(I, J))
T2 = 0.5 * ZLAMDA * (F(I) * G(J) + F(J) * G(I))
T3 = CAPF
T4 = (V2AM2 - XLAM(2)) * XTANX * FIJ
M = M + 1
12 T(M) = T1 + T2 + T3 + T4
DO 19 I = 1,1
M = M + 1
19 T(M) = 0.0
C C H'S
C
DO 100 K=1,N
DO 24 I=1,N
DO 24 J=1,I
L=L+1
24 V2(I,J)=T(L)
DO 25 I=1,N
DO 25 J=1,N
V2(I,J)=V2(J,I)
L=L+1
VLAM2=T(L)
DO 20 I=1,N
G(I)=0.0
DO 20 J=1,N
G(I)=G(I) + V2(I,J)*WR(J)
DO 22 I=1,N
DO 22 J=1,I
FIJ=F(I) + V2(I,J)*WR(J)
T1=-AR(I,J)*V2(I,J)
T2=0.5*ZLAMDA*(F(I)*G(J)+F(J)*G(I))
T3=0.0
T4=VLAM2*XTANX*FIJ
M=M+1
C 22 T(M)=T1+T2+T3+T4
C 100 CONTINUE
RETURN
END

NONLIN
SUBROUTINE NONLIN
COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
ZERLAM,XLAM(2),B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
P(28,101),H(28,101),PTAU,PLAM(2),HTAU(3),HLAM(2,3),P2(7,7),
CONST(3),NEQ
C
IFLAG=1
T(2)=0.0
T(3)=DELTA
M=1
L1=0
L3=3
DO 1 I=1,N
DO 1 J=1,I
L1=L1+1
L3=L3+1
R2(I,J)=0.0
R(L1,M)=R2(I,J)
1 T(L3)=R2(I,J)
L3=L3+1
2  T(L3) = XLAM(2)
C
NEQ = (N*(N+1))/2 + 1
CALL INTS(T, NEQ, 2, 0, 0, 0, 0, 0)
C
SIG = T(2)
CALL OUTPUT
C
DO 5 M1 = 1, M1MAX
DO 4 M2 = 1, NPRNT
CALL INTM
M = M + 1
L1 = 0
L3 = 3
DO 3 I = 1, N
DO 3 J = 1, I
L1 = L1 + 1
L3 = L3 + 1
R2(I, J) = T(L3)
3  R(L1, M) = R2(I, J)
4  SIG = T(2)
5  CALL OUTPUT
C
RETURN
END

SUBROUTINE PANDH
COMMON N, RT(7), WT(7), WR(7), XR(7), NPRT, MAX, KMAX, DELTA, TAU, ZERLAM, XLAM(2), B2(7, 7), R2(7, 7), IFLAG, R(28, 101), T(1491), SIG, P(28, 101), H(28, 101), PTAU, PLAM(2), HTAU(3), HLAM(2, 3), P2(7, 7), H2(7, 7, 3), CONST(3), NEQ
IFLAG = 2
T(2) = 0.0
T(3) = DELTA
M = 1
C
L1 = 0
L3 = 3
DO 1 I = 1, N
DO 1 J = 1, I
L1 = L1 + 1
L3 = L3 + 1
P(L1, M) = 0.0
1  T(L3) = P(L1, M)
L3 = L3 + 1
PLAM(2) = 0.0
2  T(L3) = PLAM(2)
C
DO 7 K = 1, I
L1 = 0
DO 5 I = 1, N
DO 3 J = 1, I
C
L1 = L1 + 1
L3 = L3 + 1
H(L1, K+M) = 0.0
T(L3) = H(L1, K+M)
C
L3 = L3 + 1
HLAM(2, K) = 1.0
T(L3) = HLAM(2, K)
C
L = 0
DO 8 I = 1, N
DO 8 J = 1, I
L = L + 1
8 R2(I, J) = R(L, M)
DO 9 I = 1, N
DO 9 J = I, N
9 R2(I, J) = R2(J, I)
C
NEQ = 2*((N*(N+1))/2 + 1)
CALL INTS(T, NEQ, 2, 0, 0, 0, 0, 0)
LMAX = (N*(N+1))/2
PRINT52*(P(L, M), H(L, 1+M)*L=1, LMAX)
52 FORMAT(1HOF9.4, 5E20.8/<10X5E20.8))
C
DO 51 M1 = 1, M1MAX
DO 50 M2 = 1, NPRNT
CALL INTM
M = M + 1
CPREV, APPROX, R(I, J)
L1 = 0
DO 10 I = 1, N
DO 10 J = 1, I
L1 = L1 + 1
10 R2(I, J) = R(L1, M)
DO 11 I = 1, N
DO 11 J = I, N
11 R2(I, J) = R2(J, I)
L1 = 0
L3 = 3
DO 12 I = 1, N
DO 12 J = 1, I
L1 = L1 + 1
L3 = L3 + 1
12 P(L1, M) = T(L3)
L3 = L3 + 1
DO 13 K = 1, 1
L1 = 0
DO 14 I = 1, N
DO 14 J = 1, I
L1 = L1 + 1
L3 = L3 + 1
14 H(L1, K+M) = T(L3)
13 L3 = L3 + 1
50 CONTINUE
51 PRINT52*(P(L, M), H(L, 1+M)*L=1, LMAX)
SUBROUTINE LINEAR
DIMENSION CHK1(3)
DIMENSION A(49,3),B(49),EMAT(55,55),P(50),INDEX(50,2)
1*PIVOT(50),FVEC(50,1)
COMMON N*RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1 MAX,KMAX,DELTAP,TAU,
1 ZERLAM, XLAM(2),B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
2 P(28,101),H(28,3,101),PTAU,PTAM(2),HTAU(2),HLAM(2,3),P2(7,7),
3 H2(7,7,3),CONST(3),NEQ

CBOUNDARY CONDITIONS
MLAST=NPRNT*M1 MAX+1
DO 1 K=1,1
L=0
DO 2 I=1,N
DO 2 J=1,1
L=L+1
2 H2(I,J,K)=H(L,K,MLAST)
DO 1 I=1,N
DO 1 J=1,N
1 H2(I,J,K)=H2(J,I,K)
L=L
DO 3 I=1,N
DO 3 J=1,1
L=L+1
3 P2(I,J)=P(L,MLAST)
DO 4 I=1,N
DO 4 J=1,N
4 P2(I,J)=P2(J,I)

CLEAST SQUARES
DO 5 K=1,1
L=L
DO 5 I=1,N
DO 5 J=1,N
L=L+1
5 A(L,K)=H2(I,J,K)
L=L
DO 6 I=1,N
DO 6 J=1,N
L=L+1
6 B(L)=B2(I,J)-P2(I,J)

C
LMAX=N**2
PRINT60
60 FORMAT(1HO)
DO 61 L=1,LMAX
PRINT82,*A(L,K),K=1,1,B(L)
C
DO 8 I=1,1
DO 7 J=1,1
SUM=0.0
DO 9 L=1,LMAX
9 SUM=SUM+A(L,I)*A(L,J)
7 EMAT(I,J)=SUM
SUM=0.0
DO 10 L=1,LMAX
10 SUM=SUM + A(L,I)*B(L)
8 FVEC(I+1)=SUM
C
PRINT60
DO 81 I=1,1
81 PRINT82*(EMAT(I,J),J=1,1)*FVEC(I+1)
82 FORMAT(10X6E20.8)
C
FVEC(I+1)=FVEC(I+1)/EMAT(I+1)
C
DO 11 I=1,1
11 CONST(I)=FVEC(I+1)
C
XLAM(2)=CONST(1)
PRINT903*XTAU*ZERLAM,XLAM(1),XLAM(2)
903 FORMAT(10H0/
1 1X1HTHICKNESS =, F10.4 /
2 1X1HALBEDO(X) =, 20HA + B*TANH(10*(X-C)) //
3 1X3HA =, E16.8, 10X3HB =, E16.8, 10X3HC =, E16.8 //)
C
CNEW APPROXIMATION
C
M=1
L=0
DO 12 I=1,N
DO 12 J=1,I
L=L+1
SUM=P(L,M)
DO 13 K=1,1
13 SUM=SUM + CONST(K)*H(L,K,M)
12 R(L,M)=SUM
L=0
DO 14 I=1,N
DO 14 J=1,I
L=L+1
14 R2(I,J)=R(L,M)
SIG=0.0
CALL OUTPUT
C
DO 50 M1=1,M1MAX
DO 50 M2=1,NPRINT
M=M+1
L=0
DO 15 I=1,N
DO 15 J=1,I
L=L+1
SUM=P(L,M)
DO 16 K=1,1
16 SUM=SUM + CONST(K)*H(L,K,M)
15 R4(L,M)=SUM
L=0
DO 17 I=1,N
DO 17 J=1,I
L = L + 1
17  R2(I,J) = R(L,M)
18  SIG = SIG + DELTA
50  CALL OUTPUT

C
RETURN
END

$IBFC$ OUTPUT
SUBROUTINE OUTPUT
DIMENSION X(3)
COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
1 ZERLAM,XLAM(2),B2(7,7),R2(7,7),IFLAG,P(28,101),T(1491),SIG,
2 P(28,101),H(28,3,101),PTAU,PLAM(2),HTAU(3),HLAM(2,3),P2(7,7),
3 H2(7,7,3),CONST(3),NEQ
DO 1 I=1,N
DO 1 J=1,N
1 R2(I,J) = R2(J,I)
Y = XTAU*SIG
X(1) = ZERLAM
X(2) = XLAM(1)
X(3) = XLAM(2)
CALL ALBEDO(Y,X,Z)
PRINT 100, SIG, Y, Z
100 FORMAT(1HO 7H$SIGMA = F6.2 4H$HTAU = F6.2 4H$ALBEDO = F6.2/
DO 2 J=1,N
2 PRINT 101, J, (R2(I,J), I=1,N)
101 FORMAT(11O 7F10.6)
RETURN
END

$IBFC$ ALBEDO
SUBROUTINE ALBEDO(Y,X,Z)
DIMENSION X(3)
COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
1 ZERLAM,XLAM(2),B2(7,7),R2(7,7),IFLAG,P(28,101),T(1491),SIG,
2 P(28,101),H(28,3,101),PTAU,PLAM(2),HTAU(3),HLAM(2,3),P2(7,7),
3 H2(7,7,3),CONST(3),NEQ
ARG = 10.0*(Y-X(3))
Z = X(1) + X(2)*TANH(ARG)
RETURN
END
II. PROGRAM FOR THE ESTIMATION OF T, THE THICKNESS OF THE MEDIUM

$SIBFTC Restinv$

COMMON N, RT(I), WT(I), WR(I), AR (I), NPRNT, M1MAX, KMAX, DELTA, XTAU,
1 ZERLAM, XLAM(2), B2(I), R2(I), IFLAG, R(28, 101), T(1491), SIG,
2 P(28, 101), H(28, 3, 101), PTAU, PLAM(2), HTAU(3), HLAM(2, 3), P2(7, 7),
3 H2(7, 7, 3), CONS(3), NEQ

C PHASE I

1 READ1000, N
PRINT899
PRINT900, N
READ1001, (RT(I), I=1, N)
PRINT901, (RT(I), I=1, N)
READ101, (WT(I), I=1, N)
PRINT901, (WT(I), I=1, N)
DO 2 I=1, N
WR(I) = WT(I)/RT(I)
DO 2 J=1, N
2 AR(I, J) = 1.0/RT(I) + 1.0/RT(J)

C

899 FORMAT(1H146X'PHASE I - TRUE SOLUTION / )
1000 FORMAT(6I12)
900 FORMAT(6I20)
1001 FORMAT(6E12.8)
901 FORMAT(6E20.8)
READ1000, NPRNT, M1MAX, KMAX
PRINT900, NPRNT, M1MAX, KMAX
READ10601, DELTA
PRINT901, DELTA
READ1001, XTAU, ZERLAM, XLAM(1), XLAM(2)
PRINT902
PRINT903, XTAU, ZERLAM, XLAM(1), XLAM(2)
902 FORMAT(1H1233PHASE I - TRUE SOLUTION / )
903 FORMAT(1H0/)
1 1X11THICKNESS =, F10.4 /
2 1X11HALBEOO(X) =, 20HA + B*TANH(10*(X-C)) //
3 1X3HA =, E16.8, 10X3HB =, E16.8, 10X3HC =, E16.8 /)
CALL NONLIN
DO 3 I=1, N
DC 3 J=1, N
3 B2(I, J)=R2(I, J)

C PHASE II

4 READ1001, XTAU, ZERLAM, XLAM(1), XLAM(2)
K=0
PRINT904, K
PRINT903, XTAU, ZERLAM, XLAM(1), XLAM(2)

C CALL NONLIN

904 FORMAT(1H1 13TH APPROXIMATION, 13/ )
C QUASILINEARIZATION ITERATIONS
DC 5 K1=1,KMAX
PRINT904,K1
CALL PANDH
CALL LINEAR
5 CONTINUE

C
READ1000,IGO
GO TO (1*4),IGO
END
$SUBROUTINE DAUX
SUBROUTINE DAUX
DIMENSION V2(7,7),X(3),F(7),G(7)
COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTAXTAU,
1 ZERLAM,XLAM(2),B2(7,7),P7(7),IFLAG,P(28+101),T(1491),SIG,
2 P2(28+101),H(28,3+101),PTAU,PLAM(2),HTAU(3),HLAM(2,3),P2(7,71),
3 H2(7,7,3),CONST(3),NEQ
GO TO (1*2),IFLAG
C
CNO\LINEAR
C
4 L=3
DO 4 I=1,N
DO 4 J=1*I
L=L+1
4 V2(I,J)=T(L)
DO 5 I=1,N
DO 5 J=1,N
5 V2(I,J)=V2(J,I)
L=L+1
VT(I,J)=T(L)
SIG=T(2)
Y=VTAU+SIG
X(1)=ZERLAM
X(2)=XLAM(1)
X(3)=XLAM(2)
CALL ALBEDO1,Y,X,2)
ZLAMDA=Z
C
DO 6 I=1,N
F(I)=0.0
DO 7 K=1,N
7 F(I)=F(I)+WR(K)*V2(I,K)
6 F(I)=0.5*F(I)+1.0
C
DO 8 I=1,N
DO 8 J=1,I
L=L+1
DR=AR(I,J)*V2(I,J)+ZLAMDA*X(I)*F(J)
8 T(L)=DR*VTAU
DO 9 I=1,N
L=L+1
9 T(L)=0.0
RETURN

C
C CLINEAR
C

2  SIG=T(I,2)
  Y=XTAU*SIG
  X(1)=ZERLAM
  X(2)=XLAM(I)
  X(3)=XLAM(2)
  CALL ALBEDO(1,*,X,Z)
  ZLAMDA=Z

C
DO 16 I=1,N
  F(I)=0.0
DO 17 K=1,N
  F(I)=F(I) + WR(K)*R2(I,K)
16  F(I)=0.5*F(I) + 1.0

C CP'S
C
L=3
DO 14 I=1,N
  L=L+1
14  V2(I,J)=T(L)
DO 15 J=1,N
DO 15 J=1,N
  L=L+1
  VTAU=T(L)

C
DO 10 I=1,N
  G(I)=0.0
DO 10 K=1,N
10  G(I)=G(I) + (V2(I,K)-R2(I,K))*WR(K)
  ARG=10.0*(Y-XLAM(2));
  PARTL=10.0*SIG*XLAM(1)*(1.0-(TANH(ARG))**2)
  M=3+NEO
DO 12 I=1,N
DO 12 J=1,I
  F1=JF(I,J)
  CAPF=-AR(I,J)*R2(I,J)+ZLAMDA*F1
  T1=-XTAU*AR(I,J)*R2(I,J)
  T2=XTAU*0.5*ZLAMDA*(F(I)*G(J)*F(J)*G(J))
  T3=XTAU*CAPF
  T4=(VTAU-XTAU)*(CAPF+XTAU*F1)*PARTL
  M=M+1
12  T(M)=T1+T2+T3+T4
DO 19 I=1,1
19  T(M)=0.0

C CH'S
C
DO 100 K=1,N
C
DO 24 I=1,N
DO 24 J=1,N
L=L+1
24 V2(I,J)=T(L)
DO 25 I=1,N
DO 25 J=1,N
25 V2(I,J)=V2(I,J)+V2(I,J)*WR(J)
C
DO 20 I=1,N
G(I)=0.0
DO 20 J=1,N
20 G(I,J)=G(I)+V2(I,J)*WR(J)
C
DO 22 I=1,N
DO 22 J=1,N
FIJ=F(I)*F(J)
CAPF=-AR(I,J)*R2(I,J)+ZLAMDA*FIJ
T1=-XTAU*AR(I,J)*V2(I,J)
T2=XTAU*0.5*ZLAMDA*(F(I)*G(J)+F(J)*G(I))
T3=0.0
T4=XTAU*(CAPF+XTAU*FIJ*PARTL)
M=M+1
22 T(M)=T1+T2+T3+T4
C
DO 29 I=1,N
M=M+1
29 T(M)=0.0
100 CONTINUE
RETURN
END
$IFTC

SUBROUTINE NONLIN
COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
1 ZERLA^,XLAM(2),B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
2 P(28,101),H(28,3,101),PTAU,PLAM(2),HTAU(3),HLAM(2,3),P2(7,7),
3 H2(7,3),CONST(3),NEQ
C
NONLINEAR D.E. FOR TRUE SOLUTION OR FOR INITIAL APPROX.
C
IFLAG=1
T(2)=0.0
T(3)=DELTA
M=1
L1=0
L3=3
DO 1 I=1,N
DO 1 J=1,N
L1=L1+1
L3=L3+1
R2(I,J)=0.0
R(L1,M)=R2(I,J)
1 T(L3)=R2(I,J)
SUBROUTINE LINEAR

DIMENSION CHKI(3)
DIMENSION A(49,3),B(49),EMAT(50,50),P(50),INDEX(50,2)
1,I,PIVOT(50),FVEC(50),1)
COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
1 ZERLAM,XLAM(2),B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
2 P(28,101),H(28,3,101),PTAU,PLAM(2),HTAU(3),HLAM(2,3),P2(7,7),
3 H2(7,7,3),CONST(3),NEQ

BOUNDARY CONDITIONS

MLAST=NPRNT*M1MAX + 1
DO 1 K=1,N
1 L=0
DO 2 I=1,N
1 L=L+1
2 H2(I,J,K)=H(L,K,MLAST)
DO 1 I=1,N
DO 1 J=1,N
1 H2(I,J,K)=H2(J,I,K)
L=0
DO 3 I=1,N
3 L=L+1
3 P2(I,J)=P(L,MLAST)
DO 4 I=1,N
DO 4 J=1,N
4 P2(I,J)=P2(J,I)
CLEAST SQUARES
DO 5 K=1,1
L=0
DO 5 1=1,N
DO 5 J=1,N
L=L+1
5 A(L,K)=H2(I,J,K)
L=0
DO 6 I=1,N
DO 6 J=1,N
L=L+1
6 B(L)=B2(I,J) - P2(I,J)

C
LMAX=N**2
PRINT60
60 FORMAT(1HO)
DO 61 L=1,LMAX
61 PRINT82,A(L,K),K=1,1),B(L)

C
DO 8 I=1,1
DO 7 J=1,1
SUM=0.0
DO 9 L=1,LMAX
SUM=SUM + A(L,I)*A(L,J)
9 EMAT(I,J)=SUM
SUM=0.0
DO 10 L=1,LMAX
10 SUM=SUM + A(L,I)*B(L)
8 FVEC(I,1)=SUM

C
PRINT60
DO 81 I=1,1
81 PRINT82,EMAT(I,J),J=1,1),FVEC(I,1)
82 FORMAT(10X6E20.8)

C
FVEC(1,1)=FVEC(1,1)/EMAT(1,1)

C
DO 11 I=1,1
11 CONST(I)=FVEC(I,1)

C
XTAU=CONST(1)
PRINT903,XTAU,ZERLAM,XLAM(1),XLAM(2)
903 FORMAT(1HO/
1 1X11THICKNESS =, E16.8 /
2 1X11HALBEDO(X) =, 20HA + B*TANH(10*(X-C)) //
3 1X3HA =, E16.8, 10X3HB =, E16.8, 10X3HC =, E16.8 //)

C
CNEW APPROXIMATION
C
M=1
L=0
DO 12 I=1,N
DO 12 J=1,I
L=L+1
SUM=P(L,M)
DO 13 K=1,1
13 \texttt{SUM} = \texttt{SUM} + \texttt{CONST(K)} \times \texttt{H(L,K,M)} \\
12 \texttt{R(L,M)} = \texttt{SUM} \\
\quad L = 0 \\
\quad \texttt{DO 14} \ I = 1 \text{ to } N \\
\quad \texttt{DO 14} \ J = 1 \text{ to } I \\
\quad L = L + 1 \\
14 \quad \texttt{R2(I,J)} = \texttt{R(L,M)} \\
\quad \texttt{SIG} = 0.0 \\
\quad \text{CALL OUTPUT} \\
C \\
\quad \texttt{DO 50} \ M1 = 1 \text{ to } M1 \text{MAX} \\
\quad \texttt{DO 18} \ M2 = 1 \text{ to } NPRNT \\
\quad \texttt{M} = M + 1 \\
\quad L = 0 \\
\quad \texttt{DO 15} \ I = 1 \text{ to } N \\
\quad \texttt{DO 15} \ J = 1 \text{ to } I \\
\quad L = L + 1 \\
15 \quad \texttt{SUM} = \texttt{P(L,M)} \\
\quad \texttt{DO 16} \ K = 1 \text{ to } I \\
\quad \texttt{SUM} = \texttt{SUM} + \texttt{CONST(K)} \times \texttt{H(L,K,M)} \\
16 \quad \texttt{R(L,M)} = \texttt{SUM} \\
\quad \texttt{L} = 0 \\
\quad \texttt{DO 17} \ I = 1 \text{ to } N \\
\quad \texttt{DO 17} \ J = 1 \text{ to } I \\
\quad L = L + 1 \\
17 \quad \texttt{R2(I,J)} = \texttt{R(L,M)} \\
18 \quad \texttt{SIG} = \texttt{SIG} + \texttt{DELTA} \\
50 \quad \text{CALL OUTPUT} \\
C \\
\quad \text{RETURN} \\
\quad \text{END} \\
$\$IBFTC \ PANDH \ LIST \ SUBROUTINE \ PANDH \ COMMON \ N\texttt{RT}(7),W\texttt{T}(7),W\texttt{R}(7),A\texttt{R}(7,7),N\texttt{PRNT},M\texttt{MAX},K\texttt{MAX},\texttt{DELTA},\texttt{XTAU}, \\
1 \texttt{ZERLAM},\texttt{XLAM}(2),B2(7,7),R2(7,7),\texttt{IFLAG},\texttt{R}(28,101),T(1491),\texttt{SIG}, \\
2 \texttt{P}(28,101),\texttt{H}(28,3,101),\texttt{PTAU},\texttt{PLAM}(2),\texttt{HTAU}(3),\texttt{HLAM}(2,3),\texttt{P2}(7,7), \\
3 \texttt{H2}(7,7,3),\texttt{CONST(3)},\texttt{NEO} \\
\quad \texttt{IFLAG} = 2 \\
\quad \texttt{T}(2) = 0.0 \\
\quad \texttt{T}(3) = \texttt{DELTA} \\
\quad \texttt{M} = 1 \\
C \ P'S \ C \\
\quad \texttt{L1} = 0 \\
\quad \texttt{L3} = 3 \\
\quad \texttt{DO 1} \ I = 1 \text{ to } N \\
\quad \texttt{DO 1} \ J = 1 \text{ to } I \\
\quad \texttt{L1} = \texttt{L1} + 1 \\
\quad \texttt{L2} = \texttt{L3} + 1 \\
\quad \texttt{P(L1,M)} = 0.0 \\
1 \quad \texttt{T(L3)} = \texttt{P(L1,M)} \\
\quad \texttt{L3} = \texttt{L3} + 1 \\
\quad \texttt{PTAU} = 0.0 \\
2 \quad \texttt{T(L3)} = \texttt{PTAU} \\
C
C H+S
C
C DO 7 K=1,1
L1=0
DO 3 I=1,N
DO 3 J=1,I
L1=L1+1
L3=L3+1
H(L1,K,M)=0.0
3 T(L3)=H(L1,K,M)
C
L3=L3+1
6 HTAU(K)=1.0
7 T(L3)=HTAU(K)
C
L=0
DO 8 I=1,N
DO 8 J=1,I
L=L+1
8 R2(I,J)=R(L,M)
DO 9 I=1,N
DO 9 J=I,N
9 R2(I,J)=R2(J,I)
C
NEG=2*((N*(N+1))/2 + 1)
CALL INTS(T,NEG,0,0,0,0,0,0)
LMAX=(N*(N+1))/2
C
DO 51 M1=1,M1MAX
DO 50 M2=1,NPRNT
CALL INTM
M=M+1
CPREV APPROX R(I,J)
L1=0
DO 10 I=1,N
DO 10 J=1,I
L1=L1+1
10 R2(I,J)=R(L1,M)
DO 11 I=1,N
DO 11 J=I,N
11 R2(I,J)=R2(J,I)
L1=0
L3=3
DO 12 I=1,N
DO 12 J=1,I
L1=L1+1
L3=L3+1
12 P(L1,M)=T(L3)
L3=L3+1
DO 13 K=1,1
L1=0
DO 14 I=1,N
DO 14 J=I,I
L1=L1+1
L3=L3+1
14 M(L1,K,M)=T(L2)
15 L2=L2+1
50 CONTINUE
51 CONTINUE
RETURN
END
$IBF2C OUTPUT
SUBROUTINE OUTPUT
DIMENSION X(3)
COMMON N,RT(7),WT(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
1 ZERLAM,XLAM(2),B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
2 P(28,101),H(28,3,101),PTAU,PLAM(2),HTAU(3),HLAM(2,3),P2(7,7),
3 H2(7,7,3),CONST(3),NEQ
DO 1 I=1,N
DO 1 J=1,N
1 R2(I,J)=R2(J,I)
Y=XTAU*SIG
X(1)=ZERLAM
X(2)=XLAM(1)
X(3)=XLAM(2)
CALL ALBEDO(Y,X,Z)
PRINT100, SIG,Y,Z
100 FORMAT(1HO 7HSIGMA =F6.2, 4X5HTAU =, F6.2, 4X8HALBEDO =, F6.2/)
DO 2 J=1,N
2 PRINT101,J,(R2(I,J),I=1,N)
101 FORMAT(I10, 7F10.6)
RETURN
END
$IBF2C ALBEDO
SUBROUTINE ALBEDO(Y,X,Z)
DIMENSION X(3)
COMMON N,RT(7),WT(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
1 ZERLAM,XLAM(2),B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
2 P(28,101),H(28,3,101),PTAU,PLAM(2),HTAU(3),HLAM(2,3),P2(7,7),
3 H2(7,7,3),CONST(3),NEQ
ARG=10.0*(Y-X(3))
Z=X(1) + X(2)*TANH(ARG)
RETURN
END
III. PROGRAM FOR THE ESTIMATION OF $\lambda_1$, $\lambda_2$, AND $c$, THE TWO ALBEDOS AND THE THICKNESS OF THE LOWER LAYER

$IBFTC \ RTINV$

COMMON N, RT(7), WT(7), WR(7), AR(7, 7), NPRNT, M1MAX, KMAX, DELTA, XTAU,
X1AM(3), B2(7, 7), R2(7, 7), IFLAG, 9(28, 101), T(1491), SIG,
P(28, 101), H(28, 3, 101), PLAM(3), HLAM(3, 3), P2(7, 7),
H2(7, 7, 3), CONST(3), NEQ

PHASE I

1 READ1000, N
PRINT899
PRINT900, N
READ1001, (RT(I), I = 1, N)
PRINT901, (RT(I), I = 1, N)
READ1001, (WT(I), I = 1, N)
PRINT901, (WT(I), I = 1, N)
DO 2 I = 1, N
WR(I) = WT(I) / RT(I)
DO 2 J = 1, N
2 AR(I, J) = 1.0 / RT(I) + 1.0 / RT(J)

C
899 FORMAT(1H146X36HRADIATIVE TRANSFER - INVERSE PROBLEM /
1000 FORMAT(6I12)
900 FORMAT(6I120)
1001 FORMAT(6F12.8)
901 FORMAT(6F20.8)
READ1000, NPRNT, M1MAX, KMAX
PRINT900, NPRNT, M1MAX, KMAX
READ1001, DELTA
PRINT901, DELTA
READ1001, XTAU, (X1AM(I), I = 1, 3)
PRINT902
PRINT903, XTAU, (X1AM(I), I = 1, 3)

902 FORMAT(1H1234PHASE I - TRUE SOLUTION /
903 FORMAT(1HC/ 1 1X11THICKNESS = * F10.4 /
2 1X11HALBEDO(X) = 20HA + B*TANH(10*(X-C)) //
3 1X3HA = E16.8, 10X3HB = E16.8, 10X3HC = E16.8 //)
CALL NONLIN
DO 3 I = 1, N
DO 3 J = 1, N
3 B2(I, J) = R2(I, J)

C

PHASE II

4 READ1001, XTAU, (X1AM(I), I = 1, 3)
K = 0
PRINT904, K
PRINT903, XTAU, (X1AM(I), I = 1, 3)

C
CALL NONLIN
C
904 FORMAT(1H1 13HAPPROXIMATION, I3/ )
C
C QUASILINEARIZATION ITERATIONS
DO 5 K1=1,KMAX
PRINT904,K1
CALL PANOH
CALL LINEAR
5 CONTINUE
C
C
READ1UG0,IGO
GO TO (1,4),IGO
END
$IBFTC DAUX LIST
SUBROUTINE DAUX
DIMENSION V2(7,7),X(3),F(7),G(7)
1 *VLAM(3)
COMMON N,RT(7),WT(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTAXTAU,
1 XLAM(3), B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
2 P(28,101),H(28,3,101),PLAM(3),HLAM(3,3),P2(7,7),
3 H2(7,7,3),CONST(3),NEQ
GO TO (1,2)*IFLAG
C
CNONLINEAR
C
1 L=3
DO 4 I=1,N
DO 4 J=1,I
L=L+1
4 V2(I,J)=T(L)
DO 5 I=1,N
DO 5 J=1,N
5 V2(I,J)=V2(J,I)
DO 51 I=1,3
L=L+1
51 VLAM(I)=T(L)
SIG=T(2)
Y=XTAU*SIG
DO 52 I=1,3
52 X(I)=VLAM(I)
CALL ALBEDO(Y,X,Z)
ZLAMDA=Z
C
DO 6 I=1,N
F(I)=.0*C
DO 7 K=1,N
7 F(I)=F(I) + WR(K)*V2(I,K)
6 F(I)=C*.5*F(I) + 1.0
C
DO 8 I=1,N
DO 8 J=1,I
L=L+1
8 T(L)=DR
DR=-AR(I,J)*V2(I,J) + ZLAMDA*F(I)*F(J)
8 T(L)=DR
DO 9 I=1,3
9 L=L+1
40

T(L)=C·0
RETURN
C
C
CLINEAR
C

SIG=T(2)
Y=XTAN*SIG
DO 21 I=1·N
X(I)=X(LAM(I))
CALL ALBEEDO(Y·X·Z)
ZLAMDA=Z
C
DO 16 I=1·N
F(I)=0·0
DO 17 K=1·N
17 F(I)=F(I) + WR(K)*R2(I·K)
16 F(I)=0·5·F(I) + 1·0
C
C
CP'S
C
L=3
DO 14 I=1·N
DO 14 J=1·I
L=L+1
14 V2(I·J)=T(L)
DO 15 I=1·N
DO 15 J=I·N
15 V2(I·J)=V2(J·I)
DO 18 I=1·3
L=L+1
18 VLAM(I)=T(L)
C
DO 10 I=1·N
G(I)=0·0
DO 10 K=1·N
10 G(I)=G(I) + (V2(I·K)-R2(I·K))*WR(K)
ARG=10·0·(Y-XLAM(3))
TARG=TANH(ARG)
XTANX=-10·0·XLAM(2)*(1·0·TARG**2)
M=3+NEQ
DO 12 I=1·N
DO 12 J=1·I
FIJ=F(I)*F(J)
CAPF=-AR(I·J)*R2(I·J) + ZLAMDA*FIJ
T1=CAPF
T2=-AR(I·J)*(V2(I·J)-R2(I·J))
1 + 0·5·ZLAMDA*(F(I)*G(J) + F(J)*G(I))
T3=(VLAM(1)-XLAM(1))*FIJ
T4=(VLAM(2)-XLAM(2))*TARG*FIJ
T5=(VLAM(3)-XLAM(3))*XTANX*FIJ
M=M+1
12 T(M)=T1+T2+T3+T4+T5
DO 19 I=1·3
M=M+1
19  \( T(M) = 0.0 \)

CH* S

20  \( G(I) = G(I) + V2(I,J) \times WR(J) \)

C

21  \( T(M) = T1 + T2 + T3 + T4 + T5 \)

C

22  \( T(M) = 0.0 \)

100  CONTINUE

RETURN

FND

$IBFTC NONLIN

SURROUNNonLIN

COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
1 XLAM(3), B2(7,7),R2(7,7),IFLAG,P(28,101),T(1491),SIG,
2 H(28,101),P(28,101),PLAM(3),HLAM(3,3),P2(7,7),
3 H2(7,3),CONST(3)*NEQ

C  

C  

IFLAG=1
T(2)=0.0
T(3)=DELTA
M=1
L1=0
L3=3
DO 1 I=1,N
DO 1 J=1,I
L1 = L1 + 1
L3 = L3 + 1

2(I,J) = 0.0
R(L1,M) = R2(I,J)

T(L3) = R2(I,J)
DO 2 I = 1..3
L3 = L3 + 1

2 T(L3) = XLAM(I)

NEQ = (N*(N+1))/2 + 3
CALL INTS(T, NEQ, 2, 0, 0, 0, 0, 0, 0)

SIG = T(2)
CALL OUTPUT

DO 5 M1 = 1, M1 MAX
DO 4 M2 = 1, NPRNT
CALL INTM M = M + 1
L1 = 0
L3 = 3
DO 3 I = 1, N
DO 3 J = 1, I
L1 = L1 + 1
L3 = L3 + 1
R2(I, J) = T(L3)
3 R(L1, M) = R2(I, J)
4 SIG = T(2)
5 CALL OUTPUT

RETURN
END

SUBROUTINE PANDH
COMMON NRT(7), W7(7), WR(7), AR(7, 7), NPRNT, M1 MAX, KMAX, DELTA, XTAU,
1 XLAM(3), E2(7,7), R2(7,7), IFLAG, R(28,101), T(1491), SIG,
2 F(28,101), H(28,3,1C1), PLAM(3), HLA(3,3), P2(7, 7),
3 H2(7,7,3), CONST(?), NEQ

IFLAG = 2
T(2) = 0.0
T(3) = DELTA
M = 1

L1 = 0
L2 = 3
DO 1 I = 1, N
DO 1 J = 1, I
L1 = L1 + 1
L3 = L3 + 1
P(L1, M) = 0.0
1 T(L3) = P(L1, M)
DO 2 I = 1, 3
L3 = L3 + 1
FLAM(I) = 0.0
2 T(L3) = PLAM(I)

C PLAM

C

DO 7 K = 1, 3
L1 = 0
DO 3 I = 1, N
DO 3 J = 1, I
L1 = L1 + 1
L3 = L3 + 1
H(L1, K, M) = 0.0
3 T(L3) = H(L1, K, M)

C

DO 7 I = 1, 3
L3 = L3 + 1
HLAM(I, K) = 0.0
IF (I - K) > 6, 7
6 HLAM(I, K) = 1.0
7 T(L3) = HLAM(I, K)

C

L = 0
DO 8 I = 1, N
DO 8 J = 1, I
L = L + 1
8 R2(I, J) = R(L, M)
DO 9 I = 1, N
DO 9 J = 1, N
9 R2(I, J) = R2(J, I)

C

NEO = 4 * (N * (N + 1)) / 2 + 3
CALL INTS(T, NEO, 2, 0, 0, 0, 0, 0, 0)
LMAX = (N * (N + 1)) / 2
PRINT52, T(2), (P(L, M) * H(L, 1, M)), L = 1, LMAX
52 FORMAT(1H0, 9, 4, 5E20.8, /10X, 5E20.8)

C

DO 51 M1 = 1, MMAX
DO 50 M2 = 1, NPRNT
CALL INTM
M = M + 1
CPrev.Approx. R(I, J)
L1 = 0
DO 10 I = 1, N
DO 10 J = 1, I
L1 = L1 + 1
10 R2(I, J) = R(L1, M)
DO 11 I = 1, N
DO 11 J = 1, N
11 R2(I, J) = R2(J, I)
L1 = 0
L3 = 3
DO 12 I = 1, N
DO 12 J = 1, I
L1 = L1 + 1
L3 = L3 + 1
12 P(L1, M) = T(L3)
L3 = L3 + 3
DO 13 K = 1, 3
L1 = 0
DO 14 I = 1, N
DO 14 J = 1, I
L1 = L1 + 1
L3 = L3 + 1
14 H(L1, K, M) = T(L3)
13 L3 = L3 + 3
50 CONTINUE
51 PRINT 52, T(2) * (P(L, M) * H(L, I, M) * L = L + LMAX)
RETURN
END

SUBROUTINE LINEAR
DIMENSION CHKI(3)
DIMENSION A(49, 3), B(49), EMA1(50, 50), PIVOT(50), INDEX(50, 2)
1, 1 PIVOT(50), FVEC(50, 1)
COMMON N, RT(7), WT(7), WR(7), AR(7, 7), NPRNT, M1MAX, KMAX, DELTA, XTAU,
1 XLM(3), B2(7, 7), R2(7, 7), IFLAG, P(28, 101), T(1491, 1), SIG,
2 P(28, 101), H(28, 3, 101), PLAM(3), HLAM(3, 3), P2(7, 7),
3 H2(7, 7, 3), CONST(3), N
C BOUNDARY CONDITIONS
MLAST = NPRNT * M1MAX + 1
DO 1 K = 1, 3
L = 0
DO 2 I = 1, N
DO 2 J = 1, I
L = L + 1
2 H2(I, J, K) = H(L, K, MLAST)
DO 1 I = 1, N
DO 1 J = 1, N
1 H2(I, J, K) = H2(J, I, K)
L = 0
DO 3 I = 1, N
DO 3 J = 1, I
L = L + 1
3 P2(I, J) = P(L, MLAST)
DO 4 I = 1, N
DO 4 J = 1, N
4 P2(I, J) = P2(J, I)
CLEAST SQUARES
DO 5 K = 1, 3
L = 0
DO 5 I = 1, N
DO 5 J = 1, N
L = L + 1
5 A(L, K) = H2(I, J, K)
L = 0
DO 6 I = 1, N
DO 6 J = 1, N
L = L + 1
6 B(L) = B2(I, J) - P2(I, J)
C LMAX = N ** 2
```
C
PPRINT60
60 FORMAT(1HO)
DO 61 L=1,LMAX
61 PRINT82,(A(L,K)+K=1,3)*B(L)
C
DO 8 I=1,3
DO 7 J=1,3
SUM=0.0
DO 9 L=1,LMAX
9 SUM=SUM + A(L,I)*A(L,J)
7 EMAT(I,J)=SUM
SUM=0.0
DO 10 L=1,LMAX
10 SUM=SUM + A(L,I)*B(L)
8 FVEC(I,1)=SUM
C
PRINT60
DO 81 I=1,3
81 PRINT82,EMAT(I,J),J=1,3,FVEC(I,1)
82 FORMAT(10X6E20.8)
C
CALL MATINV(EMAT,3,FVEC,1,DETERM,PIVOT,INDEX,IPIVOT)
C
DO 11 I=1,3
11 CONST(I)=FVEC(I,1)
C
DO 20 I=1,3
20 XLAM(I)=CONST(I)
PRINT903,XTAU,(XLAM(I),I=1,3)
903 FORMAT(1HO/
1 1X11HTHICKNESS =, E16.8 /
2 1X11HALBEDO(X) =, 20HA + 8*TANH(10*(X-C)) //
3 1X3HA =, E16.8, 10X3HB =, E16.8, 10X3HC =, E16.8 //)
C
CNEW APPROXIMATION
C
M=1
L=0
DO 12 I=1,N
DO 12 J=1,I
L=L+1
SUM=P(L,M)
DO 13 K=1,3
13 SUM =SUM + CONST(K)*H(L,K,M)
12 R(L,M)=SUM
L=0
DO 14 I=1,N
DO 14 J=1,I
L=L+1
14 R2(I,J)=R(L,M)
SIG=0.0
CALL OUTPUT
C
DO 50 M1=1,M1MAX
DO 50 M2=1,NPRNT
```
M=M+1
L=0
DO 15 I=1,N
DO 15 J=1,I
L=L+1
SUM=P(L,M)
DO 16 K=1,3
16 SUM=SUM + CONST(K)*H(L,K,M)
15 R(L,M)=SUM
L=0
DO 17 I=1,N
DO 17 J=1,I
L=L+1
17 R2(I,J)=R(L,M)
18 SIG=SIG + DELTA
50 CALL OUTPUT
C
RETURN
END
$IBFTC OUTPUT
SUBROUTINE OUTPUT
DIMENSION X(3)
COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,MAX,MMAX,DELTA,XTAU,
1 XLAM(3), B2(7,7), R2(7,7), IFLAG,R(28,101),T(1491),SIG,
2 P(28,101),H(28,3,101),PLAM(3),HLAM(3,3),P2(7,7),
3 H2(7,7,3),CONST(3),NEQ
DO 1 I=1,N
DO 1 J=1,N
1 R2(I,J)=R2(J,I)
Y=XTAU*SIG
DO 3 I=1,3
3 X(I)=XLAM(I)
CALL ALBEDO(Y,X,Z)
PRINT100, SIG,Y,Z
100 FORMAT(I0,7HSIGMA =,F6.2, 4X5HTAU =, F6.2, 4X8MALBEDO =,F6.2/)
DO 2 J=1,N
2 PRINT101,J,(R2(I,J)),I=1,N
101 FORMAT(I10,7F10.6)
RETURN
END
$IBFTC ALBEDO
SUBROUTINE ALBEDO(Y,X,Z)
DIMENSION X(3)
COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,MAX,MMAX,DELTA,XTAU,
1 XLAM(3), B2(7,7), R2(7,7), IFLAG,R(28,101),T(1491),SIG,
2 P(28,101),H(28,3,101),PLAM(3),HLAM(3,3),P2(7,7),
3 H2(7,7,3),CONST(3),NEQ
ARG=10*0*(Y-X(3))
Z=X(1) + X(2)*TANH(ARG)
RETURN
END
REFERENCES


