NOTE ON REPEATED SELECTION IN THE NORMAL CASE

Technical Report No. 19

Department of Navy
Office of Naval Research

Contract No. Monr-409(39)
Project No. (NR 042-212)

BIOMETRICS UNIT
DEPARTMENT OF PLANT BREEDING

NEW YORK STATE COLLEGE OF AGRICULTURE

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ITHACA, NEW YORK

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BU-166-M            D. S. Robson            April, 1964

ABSTRACT

A $k$-cycle selection model is specified by a $(k+1)$-variate normal distri-
bution of the variables $X,Y_1=X+\epsilon_1,\ldots,Y_k=X+\epsilon_k$ with selection at the $i^{th}$ stage
removing a fraction

$$P_i = P(Y_i > y_i \mid Y_1 > y_1,\ldots,Y_{i-1} > y_{i-1})$$

The distribution of $X$ in this selected fraction is then convolved with the
$N(0,\epsilon_{i+1}^2)$ distribution of $\epsilon_{i+1}$ to form the distribution of $Y_{i+1}$. An expres-
sion is given for the characteristic function of $X$ in the $k^{th}$ selected
fraction.

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Selection for a quantitative trait often continues for several cycles, as in the successive annual screening of a plant population in the process of developing new varieties. With plant selection, as with most other selection problems, the trait $x$ being selected for cannot be measured without error, and actual selections are based on the observation $y_i = x + e_i$ in the $i^{th}$ cycle of the process. We shall assume here that the error chance variable $e_i$ is $N(0, \sigma^2_i)$ (normally distributed with mean 0 and variance $\sigma^2_i$) and that the error $e_i$ attaching to $x$ in the $i^{th}$ stage is independent of the error $e_j$ attaching to that same $x$ (or any other $x$) in the $j^{th}$ stage. Further, we suppose that in the unselected population the chance variable $x$ is $N(\mu, \sigma^2)$, so that $y_i = x + e_i$ is $N(\mu, \sigma^2 + e^2_i)$.

The population available at the $k^{th}$ stage is assumed to be of infinite size, and selection consists of removing the upper fraction $P_k$ of the available $y$-population for further selection at stage $k+1$. The fraction of the original population available for selection at stage $k+1$ is therefore $P_1P_2\cdots P_k$, and our concern here shall lie with the distribution of $x$ in this remaining fraction. These fractions are defined by

\[ P_1 = P(y_1 > y_1) \]
\[ P_1P_2 = P_1P(y_2 > y_2 | y_1 > y_1) \]
\[ P_1P_2\cdots P_k = P_1P_2\cdots P_{k-1}P(y_k > y_k | y_1 > y_1, y_2 > y_2, \ldots, y_{k-1} > y_{k-1}) \]

and our results are based upon the observation that this remaining fraction is
simply the tail probability in a k-variate normal distribution,

\[ P_{1}P_{2} \cdots P_{k} = P(Y_{1} > y_{1}, Y_{2} > y_{2}, \ldots, Y_{k} > y_{k}) \]

Since the joint distribution of \( X, Y_{1}, Y_{2}, \ldots, Y_{k} \) is the \((k+1)\)-variate normal distribution with mean \( \mu \) and covariance matrix

\[
\Lambda = \begin{bmatrix}
\sigma_{1}^{2} & \sigma_{1}^{2} & \cdots & \sigma_{1}^{2} \\
\sigma_{2}^{2} & \sigma_{2}^{2} & \cdots & \sigma_{2}^{2} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{k}^{2} & \sigma_{k}^{2} & \cdots & \sigma_{k}^{2}
\end{bmatrix} = [\sigma_{ij}]
\]

then the distribution of \( x \) for fixed values of \( Y_{1}, \ldots, Y_{k} \) is normal with mean

\[
E(X|Y_{1}, \ldots, Y_{k}) = \mu - \frac{\Lambda_{i1}}{\Lambda_{00}} (y_{1} - \mu_{1}) - \cdots - \frac{\Lambda_{ik}}{\Lambda_{00}} (y_{k} - \mu_{k})
\]

and

\[
\text{var}(X|Y_{1}, \ldots, Y_{k}) = \frac{\Lambda}{\Lambda_{00}}
\]

where \( \Lambda \) is the determinant of \( \Lambda \) and \( \Lambda_{ij} \) is the cofactor of the \( ij^{th} \) element of \( \Lambda \).

The joint distribution of \( Y_{1}, \ldots, Y_{k} \) is normal with mean \( \xi = (\xi_{1}, \ldots, \xi_{k}) \) and covariance matrix \( \Lambda_{00} \). Using the expansion

\[
\Lambda = \sigma_{00}^{2} \Lambda_{00} - \sum_{i,j=1}^{k} \sigma_{0i}\sigma_{0j} \Lambda_{00.ij}
\]

where \( \Lambda_{00.ij} \) is the cofactor of \( \sigma_{ij} \) in \( \Lambda_{00} \), we may then express the conditional
moment generating function of $X$ as

$$E(t^X | y_1, \ldots, y_k)$$

$$= e^{t\xi + \frac{t^2}{2}(\sigma_{oo} - \frac{1}{\lambda_{oo}} \sum_{i,j=1}^{k} \sigma_{io} \sigma_{oj} \Lambda_{oo} \cdot ij)} + \frac{t}{\lambda_{oo}} \sum_{i,j=1}^{k} \sigma_{io} (y_j - \xi) \Lambda_{oo} \cdot ij$$

and then

$$E(t^X | y_1 > y_1, \ldots, y_k > y_k) = e^{t\xi + \frac{t^2}{2} \sigma_{oo} (p_1^{1} p_2^{2} \ldots p_k^{k}) - 1 \frac{1}{(2\pi)^{k/2} \sigma_{oo}^{k/2}}}$$

$$\int e^{-\frac{1}{2\lambda_{oo}} \sum_{i,j=1}^{k} [(u_i - \xi)(u_j - \xi) - 2t\sigma_{io}(u_j - \xi) + t^2 \sigma_{io} \sigma_{oj} \Lambda_{oo} \cdot ij]} \, du_1 \ldots du_k$$

The exponent in the integral reduces to

$$\sum_{i,j=1}^{k} (u_i - \xi - \sigma_{io} t)(u_j - \xi - \sigma_{oj} t) \Lambda_{oo} \cdot ij$$

hence, transforming to the standard normal $z_1 = (y_i - \xi)/\sigma_{ii}$, we obtain

$$E(t^X | \frac{Y_{1-i} - \xi}{\omega_1} > z_1, \ldots, \frac{Y_{k-i} - \xi}{\omega_k} > z_k)$$

$$= e^{t\xi + \frac{t^2}{2} \sigma^2 \frac{1}{\omega_1} \prod_{i=1}^{k} \frac{1}{\sigma_{ii}}} - \frac{1}{2\pi} \sum_{i,j=1}^{k} R_{oo} \cdot ij \cdot v_i v_j \int e^{-\frac{1}{2\sqrt{\pi}} \sum_{i=1}^{k} \sigma_{oo} \cdot ij} \, dv_1 \ldots dv_k$$
where

\[
R \equiv \begin{pmatrix}
1 & \frac{\sigma^2}{\omega_1 \omega_2} & \cdots & \frac{\sigma^2}{\omega_1 \omega_k} \\
\frac{\sigma^2}{\omega_1 \omega_2} & 1 & \cdots & \frac{\sigma^2}{\omega_2 \omega_k} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\sigma^2}{\omega_1 \omega_k} & \frac{\sigma^2}{\omega_2 \omega_k} & \cdots & 1
\end{pmatrix}
\]

The mean value of X in this selected fraction of the population is obtained by differentiating once with respect to t, first writing

\[
E(e^{tx} | \frac{Y_1 - \xi}{\omega_1} > z_1, \cdots, \frac{Y_k - \xi}{\omega_k} > z_k)
\]

\[
= \varphi_X(t)P_{R \cdot 0}(v_1 > z_1 - \frac{\sigma^2}{\omega_1} t, \cdots, v_k > z_k - \frac{\sigma^2}{\omega_k} t)/P_1 P_2 \cdots P_k
\]

so that the derivative becomes

\[
\frac{1}{P_1 P_2 \cdots P_k} \{ \varphi'_X(t)P_{R \cdot 0}(v_1 > z_1 - \frac{\sigma^2}{\omega_1} t, \cdots, v_k > z_k - \frac{\sigma^2}{\omega_k} t) \}
\]

\[
+ \frac{\sigma^2}{\sqrt{2\pi}} \sum_{j=1}^{k} \frac{1}{\omega_j} e^{-\frac{1}{2}(z_j - \frac{\sigma^2}{\omega_j} t)^2}
\]

\[
P_{R \cdot 0}(v_1 > z_1 - \frac{\sigma^2}{\omega_1} t, \cdots, v_{j-1} > z_{j-1} - \frac{\sigma^2}{\omega_{j-1}} t, v_j > z_j - \frac{\sigma^2}{\omega_j} t, v_{j+1} > z_{j+1})
- \frac{\sigma^2}{\omega_{j+1}} t, \ldots, v_k > z_k - \frac{\sigma^2}{\omega_k} t | z_j \}

Setting \( t=0 \), we obtain the mean value

\[ \xi + P \frac{P \cdots P_k}{1 P_2 \cdots P_k} \sum_{j=1}^{k} \frac{1}{\omega_j \sqrt{2\pi}} e^{\frac{-z_j^2}{2}} \int_{-\infty}^{z_j} \frac{1}{\omega_j \sqrt{2\pi}} e^{\frac{-u^2}{2}} du \]

or

\[ \xi + P \frac{P \cdots P_k}{1 P_2 \cdots P_k} \sum_{j=1}^{k} \frac{1}{\omega_j \sqrt{2\pi}} e^{\frac{-z_j^2}{2}} \int_{-\infty}^{z_j} \frac{1}{\omega_j \sqrt{2\pi}} e^{\frac{-u^2}{2}} du \int_{-\infty}^{u} \frac{1}{\omega_j \sqrt{2\pi}} e^{\frac{-v^2}{2}} dv \]

\[ - \frac{\sigma^2}{\omega_{j-1}\omega_j} z_j, u_{j+1} > z_{j+1} - \frac{\sigma^2}{\omega_j\omega_{j+1}} z_j, \ldots, u_k > z_k - \frac{\sigma^2}{\omega_k \omega_k} z_j \]

where
\[ R(\omega) = \begin{vmatrix} 1 - \frac{\sigma^4}{\omega_1 \omega_j} & \cdots & \frac{\sigma^2}{\omega_1 \omega_{j-1}} (1 - \frac{\sigma^2}{\omega_j}) & \frac{\sigma^2}{\omega_1 \omega_{j+1}} (1 - \frac{\sigma^2}{\omega_j}) & \cdots & \frac{\sigma^2}{\omega_1 \omega_k} (1 - \frac{\sigma^2}{\omega_j}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\sigma^2}{\omega_1 \omega_{j-1}} (1 - \frac{\sigma^2}{\omega_j}) & \cdots & 1 - \frac{\sigma^2}{\omega_{j-1} \omega_j} & \frac{\sigma^2}{\omega_{j-1} \omega_{j+1}} (1 - \frac{\sigma^2}{\omega_j}) & \cdots & \frac{\sigma^2}{\omega_{j-1} \omega_k} (1 - \frac{\sigma^2}{\omega_j}) \\ \frac{\sigma^2}{\omega_1 \omega_{j+1}} (1 - \frac{\sigma^2}{\omega_j}) & \cdots & \frac{\sigma^2}{\omega_{j-1} \omega_{j+1}} (1 - \frac{\sigma^2}{\omega_j}) & 1 - \frac{\sigma^4}{\omega_{j+1} \omega_{j+1}} & \cdots & \frac{\sigma^2}{\omega_{j+1} \omega_k} (1 - \frac{\sigma^2}{\omega_j}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\sigma^2}{\omega_1 \omega_k} (1 - \frac{\sigma^2}{\omega_j}) & \cdots & \frac{\sigma^2}{\omega_{j-1} \omega_k} (1 - \frac{\sigma^2}{\omega_j}) & \frac{\sigma^2}{\omega_{j+1} \omega_k} (1 - \frac{\sigma^2}{\omega_j}) & \cdots & 1 - \frac{\sigma^4}{\omega_{j+1} \omega_{j+1}} \end{vmatrix} \]
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