Group Report

1964-63

R. T. Prosser

An Inequality for Certain Correlation Functions

29 October 1964

Prepared under Electronic Systems Division Contract AF 19(628)-500 by

Lincoln Laboratory

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Lexington, Massachusetts
The work reported in this document was performed at Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology, with the support of the U.S. Air Force under Contract AF 19(628)-500.
AN INEQUALITY FOR CERTAIN CORRELATION FUNCTIONS

R. T. PROSSER

Group 66

GROUP REPORT 1964-63

29 OCTOBER 1964
ABSTRACT

An inequality for certain types of generalized autocorrelation functions, of interest in the study of varactor diodes, is herein established.

Accepted for the Air Force
Stanley J. Wisniewski
Lt Colonel, USAF
Chief, Lincoln Laboratory Office
An Inequality for Certain Correlation Functions

A study of the properties of varactor diodes has recently led to the following problem whose solution is set out below: Let \( q(t) \) be a continuous function, periodic with period \( 2\pi \). It is known that the auto-correlation function \( R(t) \) associated with \( q(t) \), as defined by

\[
R(t) = \frac{2\pi}{q(t-t')} q(t') \, dt'
\]

(1)

has the property that it achieves its maximum at the origin:

\[
R(t) \leq R(0).
\]

(2)

Now suppose \( U(x) \) and \( V(x) \) are continuous monotone increasing functions, defined at least on the range of \( q(t) \). Consider the modified correlation function \( S(t) \), defined by

\[
S(t) = \frac{2\pi}{U(q(t-t')) V(q(t'))} \, dt'
\]

(3)

Question: does \( S(t) \) achieve its maximum at the origin, i.e., does (2) hold for \( S(t) \)?

We shall show here that the answer is affirmative. The proof depends
on a simple inequality found in Hardy and Littlewood [1, page 261].

**Lemma** Let \( \{a_i\} \) and \( \{b_i\} \) be two finite sequences of real numbers, both arranged in decreasing order. Then for any permutations \( \pi \) and \( \sigma \) of the integers, we have

\[
\sum_{i=1}^{n} a_{\pi(i)} b_{\sigma(j)} \leq \sum_{i=1}^{n} a_i b_i \quad (4)
\]

**Proof** It suffices to consider the case where \( a_{\pi(i)} = a_i \). Then either \( b_{\sigma(j)} = b_j \), or else for some \( j \) and \( k \) we have \( j < k \) and \( b_{\sigma(j)} < b_{\sigma(k)} \). Then we have

\[
(a_j b_{\sigma(k)} + a_k b_{\sigma(j)}) - (a_j b_{\sigma(j)} + a_k b_{\sigma(k)})
\]

\[
= (a_j - a_k) (b_{\sigma(k)} - b_{\sigma(j)}) \geq 0 \quad (5)
\]

Hence we will not diminish the sum \( \sum a_i b_{\sigma(i)} \) by exchanging \( b_{\sigma(j)} \) and \( b_{\sigma(k)} \). A finite number of such exchanges leads to a new permutation \( \sigma' \) with \( b_{\sigma'(i)} = b_i \), and a sum no smaller than the original.

To show that (2) holds for \( S(t) \) defined by (3), we simply approximate the continuous function \( q(t) \) uniformly by a step function \( r(t) \) defined so that

\[
r(t) = q\left(\frac{2i\pi}{n}\right) = q_i \quad \text{when} \quad \frac{(2i - 1)\pi}{n} \leq t < \frac{(2i + 1)\pi}{n} \quad i = 0, 1, ..., n-1
\]

(6)
By choosing \( n \) sufficiently large, we can arrange so that

\[
|q(t) - r(t)| < \varepsilon \quad \text{for all } t \quad 0 \leq t < 2\pi
\]  

(7)

Since \( U \) and \( V \) are continuous, we can also arrange so that

\[
|U(q(t)) - U(r(t))| < \varepsilon
\]

\[
|V(q(t)) - V(r(t))| < \varepsilon \quad \text{for all } t, \ 0 \leq t < 2\pi
\]  

(8)

Using \( r(t) \), we define the function \( T(t) \) by

\[
T(t) = \frac{2\pi}{n} \int_0^{2\pi} U(r(t - t')) V(r(t')) \, dt
\]

\[
= \sum_{j=1}^{n} U(q_{i-j}) V(q_j) \quad \text{if } t = \frac{2i\pi}{n}
\]  

(9)

Now the inequality of the Lemma tells us that

\[
T(t) \leq T(0) \quad \text{for } t = \frac{2i\pi}{n}
\]  

(10)

On the other hand, we have
$|S(t) - T(t)| \leq \int_0^{2\pi} |U(q(t-t')) - U(r(t-t'))| \cdot |V(q(t'))| \, dt' - 2\pi$

$= 4\pi A\epsilon$ \hspace{1cm} (11)

where $A$ is bigger than the maximum value attained by $|U(q(t))|$ or $|V(q(t))|$ as $t$ ranges from 0 to $2\pi$. Combining (10) and (11), we obtain

$$S(t) = S(0) + 8\pi A\epsilon \quad \text{for} \quad t = \frac{2i\pi}{n} \hspace{1cm} (12)$$

Since this inequality must hold for all choices of $n$, we conclude that

$$S(t) \leq S(0) \quad \text{for all} \quad t, \quad 0 \leq t < 2\pi \hspace{1cm} (13)$$
DISTRIBUTION

Division 6 Office
G. P. Dinneen
W. E. Morrow, Jr.
S. Gould

Group 61
L. J. Ricardi

Group 62
P. Rosen

Group 63
H. Sherman
R. M. Lerner
W. L. Black

Group 64
P. E. Green

Group 66
R. T. Prosser
B. Reiffen
T. J. Goblick, Jr.
H. Yudkin
T. S. Pitcher
J. R. Kinney
E. Weiss
File (10)
An Inequality for Certain Correlation Functions

An inequality for certain types of generalized auto correlation functions, of interest in the study of varactor diodes, is herein established.
INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization (corporate author) issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b. Project Number: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers (either by the originator or by the sponsor), also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

   (1) "Qualified requesters may obtain copies of this report from DDC."

   (2) "Foreign announcement and dissemination of this report by DDC is not authorized."

   (3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through...

   (4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through...

   (5) "All distribution of this report is controlled. Qualified DDC users shall request through...

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring (paying for) the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as (TS), (S), (C), or (U).

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, rules, and weights is optional.