NOTES ON MATRIX THEORY - VI

Richard Bellman, Irving Glicksberg
and Oliver Gross

P-594
10 November 1954

Approved for OTS release

COPY 1 OF 1

HARD COPY $1.00
MICROFICHE $0.50

DDC
AUG 27 1964

The RAND Corporation
1730 MAIN ST. SANTA MONICA, CALIFORNIA
SUMMARY

We derive an identity in matrix theory which yields a number of interesting inequalities.
NOTES ON MATRIX THEORY – VI

by

Richard Bellman, Irving Glicksberg
and Oliver Gross

1. Introduction

In two recent notes, [1], [2], we have shown how various results relating to the determinant of a positive definite matrix, could be derived from the well-known identity

\[ \frac{c_n}{|A|^{1/2}} = \int_{-\infty}^{\infty} e^{-(x,Ax)} \, dV_n \]

where the integration is over all \( x \) and \( c_n = \sqrt{n!} \), a constant depending only upon the dimension \( n \).

If we define, for \( k = 1,2,\ldots,n \),

\[ |A|_k = \prod_{i=1}^{k} \lambda_i \]

where \( \lambda_i \), \( i = 1,2,\ldots,n \), are the characteristic roots of \( A \), a positive definite matrix, arranged in increasing order of magnitude, it was shown by Ky Fan, [3], [4], that

\[ |A^\lambda + B^\mu|_k \geq |A|_k^{\lambda} |B|_k^{\mu}, \quad \lambda,\mu \geq 0, \quad \lambda + \mu = 1. \]

This result, together with some additional results was recently obtained, in a different way, by Oppenheim, [5].

The purpose of the present note is to establish an identity for

\[ |A|_k \]

similar to (1). This identity may then be utilized to derive (3)
in the same manner that (1) was used to obtain the particular
case \( k = n \) of (3), cf. \(|1|\), namely by a direct application of
Holder's inequality.

The identity is

**Theorem.**

(4) \[
\frac{\sqrt{\pi}^{-k}}{|A|^{1/2}} = \text{Max} \int \frac{e^{-|x| \cdot A\cdot x}}{L} \, dV_k,
\]

where \( L \) is a \( k \)-dimensional subspace of the \( N \)-dimensional space.

**Proof:** It is easy to see that we may begin with the case where \( A \)
is already in diagonal form, \((x, Ax) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \ldots + \lambda_n x_n^2\), since
an orthogonal transformation leaves the formulation of the problem
invariant. Hence we must show that

(5) \[
\frac{\sqrt{\pi}^{-k}}{(\lambda_1, \lambda_2, \ldots, \lambda_k)^{1/2}} = \text{Max} \int \frac{e^{-\sum \lambda_i x_i^2}}{L} \, dV_k.
\]

Define \( V_\alpha (\rho) \) to be the volume contained in the \( k \)-dimensional
region

(6) \[
\lambda_1 x_1^2 + \lambda_2 x_2^2 + \ldots + \lambda_n x_n^2 \leq \rho,
\]

\((x, \alpha_i) = 0, i = 1, 2, \ldots, (n-k)\).

Then

(7) \[
V_\alpha (\rho) = \rho^{k/2} V_\alpha (1)
\]

and we have
(8) 
\[ \int_{(x, \alpha_1) = 0} e^{-\lambda_1 x^1 - \lambda_2 x^2 - \cdots - \lambda_n x^n} dV_k = \int_0^\infty e^{-\rho} dV_\alpha(\rho) \]
\[ = [ (k) \int_0^\infty e^{-\rho^{k/2}} \rho^{k/2-1} d\rho ] V_\alpha(1). \]

To complete the proof we must show that the maximum of \( V_\alpha(1) \)
is attained when the relations \((x, \alpha_1) = 0\) are \(x_{k+1} = x_{k+2} = \cdots = x_n = 0\).

This, however, is an immediate consequence of the formula for the
volume of an ellipsoid in terms of the characteristic values of
the associated symmetric matrix, and the separation theorem for
characteristic roots furnished by the min–max characterization of
the characteristic roots.

**BIBLIOGRAPHY**


