A NOTE ON PRIMITIVE MATRICES
I. N. Herstein

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A Note on Primitive Matrices

I. N. Herstein

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Suppose that $A$ is a square matrix consisting of nonnegative elements. In certain considerations it is important to know when all the elements of some power of $A$ are strictly positive. Frobenius [2] gave a very simple necessary and sufficient condition for this to happen. In this note we give a simple proof of this result. Our proof is algebraic in nature and avoids the use of the convergence of powers of a matrix.

All matrices considered here will have real elements. For two such matrices (not necessarily square) $B = (b_{ij}), C = (c_{ij})$ we define

$B \preceq C$ if $b_{ij} \leq c_{ij}$ for each $i, j$.

$B \preceq C$ if $B \preceq C$ but $B \not\preceq C$.

$B > C$ if $b_{ij} > c_{ij}$ for each $i, j$.

A square matrix $A \geq 0$ (A is then called nonnegative) is said to be indecomposable if for no permutation matrix $P$ does

$$PAP^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

where the $A_{11}$ are square submatrices.

The fundamental result about nonnegative, indecomposable matrices is due to Frobenius [2]; this, and other, results have recently been rederived and extended in a greatly simplified manner by Wielandt [3] and Debreu and Herstein [1]. It is

THEOREM. Let $A \geq 0$ be an indecomposable matrix. Then $A$ has a positive characteristic root $r$ such that

1. $r$ is a simple root.
2. to \( r \) can be associated a characteristic vector \( x > 0 \).

3. \( \text{if} \, a \, \text{an} \, \text{any other characteristic root of} \, A, \, |a| \leq r. \)

If \( A > 0 \) then 3. can be sharpened to \( |a| \leq r \) for all characteristic roots \( a \neq r \) of \( A \).

If \( A \neq 0 \) is indecomposable and if \( A \) has no characteristic root other than \( r \) of maximal absolute value then \( A \) is said to be primitive.

In this paper we prove the

THEOREM* (Probenius). Let \( A \neq 0 \). Then \( A^m > 0 \) for some integer \( m > 0 \) if and only if \( A \) is primitive.

Suppose that \( A^m > 0 \). Then \( A \) must be indecomposable; for if

\[
PAP^{-1} = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix} \quad \text{then} \quad PAP^{-1} = \begin{pmatrix} B^m & C^m \\ 0 & D^m \end{pmatrix} \text{contradicting} \ A^m > 0.
\]

Now suppose that \( r \) and \( re^{1+i\pi} \neq r \) are characteristic roots of \( A \) of maximal absolute value. Then \( A^m, A^{m+1} \) are both positive and have \( r^m, re^{im\pi} \), and \( r^{m+1}, r^{m+1}e^{(m+1)i\pi} \) respectively as roots of maximal absolute value.

Since the largest root of a positive matrix is simple and is actually greater than any other root in absolute value. We must have

\[
re^{im\pi} = r, \quad r^{m+1}e^{(m+1)i\pi} = r^{m+1}, \quad \text{whence} \ e^{i\pi} = 1, \text{a contradiction}.
\]

There remains but to show that if \( A \) is primitive then \( A^m > 0 \) for a suitable integer \( m > 0 \). This will be proved as a consequence of the following few lemmas, which by themselves are of some interest.

Lemma 1. If \( A \) is primitive then \( A^m \) is primitive for every positive integer \( m \).

Proof. Since \( r \) is a simple root of \( A \) and is the only root of \( A \) of absolute value \( r \), \( r^m \) is a simple root of \( A^m \) and is the only root of \( A^m \) of absolute value \( r^m \). So we need but show that \( A^m \) is indecomposable for every integer
m > 0. Suppose that for some s A^s is not indecomposable; we can then assume that
A^s = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}. Now \( Ax = rx \) for \( x > C \), so \( A^s x = r^s x \); partition
\( x \) according to the partitioning of \( A^s \) and we have
\[
\begin{pmatrix} B & C \\ 0 & D \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = r^s \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\]
That is \( Dx_2 = r^s x_2 \), and since \( x_2 \) is positive, \( r^s \) is a characteristic root
of \( D \). Since the transpose, \( A' \), of \( A \) is also indecomposable, we have
\( A'Y = ry \) for \( Y > 0 \). Partitioning as above we obtain that \( r^s \) is a character-
istic root of \( B' \), and so of \( B \). Being a characteristic root of both \( B \)
and \( D \), \( r^s \) must be a multiple root of \( A^s \), which is a contradiction. The
lemma is thereby proved.

Lemma 2. (Wielandt). Let \( \varepsilon \) be any positive number. Suppose \( A \neq 0 \) is an
nn matrix. Then \(( \varepsilon I + A)^{n-1} > 0 \) where \( I \) is the identity
matrix.

Proof. It clearly suffices to show that for any vector \( x, x > 0 \),
\[
( \varepsilon I + A)^{n-1} x > 0. \]
Let
\[
x_{j+1} = (\varepsilon I + A)^{-1} x.
\]
Hence a zero component can occur in \( x_{j+1} \) only where a zero component al-
ready occurred in \( x_j \). However, not every such zero component can be pre-
served in \( x_{j+1} \). For if so, by a suitable reordering of the coordinates,
\[
x_j = \begin{pmatrix} p \\ 0 \end{pmatrix}, \quad p > 0,
\]
whence \( x_{j+1} = \varepsilon \begin{pmatrix} p \\ 0 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} p \\ 0 \end{pmatrix} = \begin{pmatrix} q \\ 0 \end{pmatrix},
\]
from which it follows that \( a_{21}p > 0 \). This together with \( p > 0 \) forces
\( a_{21} = 0 \), violating the indecomposability of \( A \). So each application of
\( \varepsilon I + A \) to \( x \) decreases the number of zero coordinates by at least one.
Hence \(( \varepsilon I + A)^{n-1} x > 0 \).

As an easy consequence of Lemma 2 we obtain
Lemma 3. If $A = (a_{ij})$ is indecomposable and $a_{ii} > 0$ for each $i$ then $A^{n-1} > 0$.

For let $\epsilon$ be chosen satisfying $0 < \epsilon < \min a_{ii}$. Then $A = \epsilon I + B$ where $B > 0$ is indecomposable. Lemma 2 then yields $A^{n-1} > 0$.

Let $A^m = (a_{ij}^{(m)})$. Then we have

Lemma 4. Let $A > 0$ be indecomposable. Then for any $i,j$ we can find an $m = \gamma(i,j) > 0$ so that $a_{ij}^{(m)} > 0$.

Proof. Consider first the case $i \neq j$. Since

$$(I+\epsilon A)^{n-1} = A^{n-1} + \left(\begin{array}{c} \epsilon A^{n-2} \\ \vdots \\ \epsilon A \end{array}\right) I > 0$$

by Lemma 2, $a_{ij}^{(m)} > 0$ for some $m \leq n-1$. Now suppose $i = j$. Since $A$ is indecomposable, no column of zeros can occur in $A$. So there is a $k$ with $a_{ki} > 0$. If $k = i$ then $a_{ii}^{(m)} > 0$ for all $m$ trivially. If, on the other hand, $k \neq i$, then $a_{ik}^{(m)} > 0$ for some $m$, and since $\sum a_{ik}^{(m+1)} = \sum a_{ik}^{(m)} a_{ij} > 0$ for $m > 0$ the lemma is proved.

We are now in position to complete the proof of Theorem*. Let $A$ be primitive. Pick $m_1$ so that in $A^{m_1}$, $a_{ij}^{(1)} > 0$. Let $A_1 = A^{m_1} = (a_{ij}^{(1)})$.

By Lemma 1 $A_1$ is primitive, so there is an $m_2$ such that in $A_2 = A^{m_2}$, $a_{22}^{(1)} > 0$. Since $a_{11}^{(1)} = a_{11} > 0$, $a_{11}^{(1)} > 0$. Let $A_2 = A_1^{m_2}$. Continuing in this way we arrive at an $A_n = A^{m_1 m_2 \ldots}$ which is primitive and whose diagonal elements are all positive. By Lemma 3 $A^n > 0$ for some $t$, hence $A^m > 0$ for some suitably chosen integer $m$.

Cowles Commission for Research in Economics

and The University of Chicago
FOOTNOTES

1. This paper is a result of the work being done at the Cowles Commission for Research in Economics on the "Theory of Resource Allocation" under sub-contract to the RAND Corporation.

2. Numbers in square brackets refer to the bibliography at the end of this paper.

BIBLIOGRAPHY

