The attenuation of acoustic waves in a two-phase medium

by

Joseph C. F. Chow
Condensed Report on

THE ATTENUATION OF ACOUSTIC WAVES
IN A TWO-PHASE MEDIUM

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SUMMARY

A study is made of the attenuation of acoustic waves by a suspension of fluid droplets in a fluid medium. Special attention is given to the case of small droplets for which the effect of the surface tension is not negligible. Both the droplets and the surrounding fluid medium are considered to be viscous and thermal conducting. The droplets are allowed to execute large translational motion and to undergo a small deformation from a spherical shape. It is shown that the result of Epstein and Carhart on the attenuation of sound waves in a gas with the suspension of liquid droplets is applicable even when the displacement of the droplet is large compared to its radius. The effect of surface tension is to increase sound attenuation in two-phase medium by increasing the thermal dissipation. This effect is important in the suspension of gaseous bubbles in liquid for small droplets and is negligible in the case of a gaseous medium containing liquid droplets. The explicit forms for attenuation, the drag force on droplets, and the heat transfer rate between phases are given for the case which is applicable to a gas containing liquid and solid droplets. The expression for the attenuation which is applicable to the suspension of gaseous bubbles in a liquid is also given and is found to be completely dominated by the thermal dissipation.
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SYMBOLS

\( \alpha_0 \)  
\( \bar{A} \)  
\( c \)  
\( C_v, C_p \)  
\( e \)  
\( f \)  
\( H_n, J_n \)  
\( k, k_1, k_2, k \)  
\( n \)  
\( N = \frac{4}{3} + \frac{\eta}{\mu} \)  
\( p \)  
\( P_n, P_{nm} \)  
\( r_0 \)  
\( r \)  
\( R_1, R_2 \)  
\( T \)  
\( u \)  
\( u_p \)  
\( x_i \)  
\( \alpha \)  
\( \alpha_v \)  

sound speed of undisturbed medium  
viscous wave potential  
surface tension per unit length  
specific heat at constant volume and constant pressure  
internal energy per unit mass  
frequency of the incident wave  
Hankel and Bessel functions  
wave numbers of acoustic, thermal and viscous waves  
number of droplets per unit volume  
pressure  
Legendre and associated Legendre functions  
radius of the droplet  
radial coordinate  
principal radii of curvature at a given point of the droplet surface  
temperature  
fluid velocity  
velocity of the mass center of the droplet  
rectangular cartesian coordinate  
attenuation coefficient  
coefficient of volume expansion
\( \gamma = \frac{C_p}{C_v} \) specific heat ratio

\( \delta_p, \delta_g \) displacements of the droplet and gaseous medium

\( \eta \) coefficient of dilatational viscosity

\( \theta \) polar angle

\( \kappa \) thermal conductivity

\( \lambda_1 \) acoustic wavelength

\( \mu, \nu \) dynamic and kinematic viscosities

\( \xi \) radial displacement of a point on the surface of the droplet

\( \Pi_{ij} = -P \delta_{ij} + \zeta_{ij} \) stress tensor

\( \rho_0 \) density

\( \Phi(\omega) \) incident wave potential

\( \Phi_1, \Phi_2 \) acoustic and thermal potentials

\( \varphi \) azimuthal angle

\( \psi \) dissipation function

\( \omega = 2\pi f \) angular frequency

\( \Omega = \frac{k}{\rho c_p} \) thermal diffusivity

\( \text{Pr} = \frac{\nu}{\alpha} \) Prandtl number

\( (\quad)_0 \) equilibrium quantities

\( R(\quad) \) real part

\( (\quad)_{\text{av}} \) time average

\( (\quad)' \) variables inside the droplet

\( \text{KC} \) kilocycle per second
I. INTRODUCTION

The attenuation of acoustic waves in two-phase medium has attracted attention repeatedly since the publication of Sewell's paper(1)*. Until present time, the analytical treatment(1), (2), (3) is limited to the cases where the displacement of the droplet is small compared with the size of the droplet and the effect of the surface tension can be neglected. These assumptions were consistently mentioned as the possible causes of the discrepancies between the existing theories and the experimental results(4), (5), (6). In view of this, a generalized theory will be presented by introducing the surface tension effect and allowing the free movement of the droplets by formulating the basic equations with respect to a moving coordinate system fixed in the droplet. Both media are considered to be viscous and heat conducting and a small deformation of the droplet is allowed. Attention will be confined to the cases where the incident wavelength is much greater than the size of the droplet and there is no interaction between the droplets (low volume concentration).

The attenuation coefficient, drag force on the droplet, heat transfer rate between phases, and the ratio of the droplet displacement to the particle displacement of the surrounding fluid medium are obtained. The explicit expressions of these quantities are given for the case applicable to the suspension of liquid droplets in a gas and the comparison is made with Epstein and Carhart's theory. Also, the expression for the attenuation which is applicable to a liquid containing gas bubbles is obtained.

* Numbers in parentheses refer to References at the end of the paper.
11. FORMULATION OF THE PROBLEM

Consider a train of acoustic waves described by a potential
\[ \Phi_{\omega} = \exp \left[ i (k_x x - \omega t) \right] \] impinging upon a spherical droplet. The linearized equations governing the motion of a viscous and heat conducting fluid in a fixed cartesian coordinate system assume the form:

\[ \frac{\partial p}{\partial t} + \rho_0 \frac{\partial u_i}{\partial x_i} = 0 \]  
\[ \frac{\partial u_i}{\partial t} = \frac{1}{\rho_0} \frac{\partial p}{\partial x_j} \]  
\[ \frac{\partial e}{\partial t} = -p \frac{\partial u_i}{\partial x_i} + \kappa \frac{\partial^2 T}{\partial x_i \partial x_i} \]

Instead of referring these equations to axes fixed in space, they shall be referred to axes originating at the center of the droplet and moving with the velocity \( U(x,t) \) in the \( x_3 \)-direction. This transformation enables one to write the boundary conditions in a simpler form in the case of large oscillatory motion of the droplet when a spherical coordinate system is used. Retaining the same set of dependent variables, one finds that the form of the governing equations and the incident potential \( \Phi_{\omega} \) are unchanged by referring to the moving coordinate system. In this transformation the terms containing space and time derivatives remain the same because of the linearization. However, it should be noted that \( U_i \) still denotes the fluid velocity as observed in the coordinate system fixed in space.

The boundary conditions to be satisfied on the surface of the droplet are the equality of the normal and tangential velocity components, the temperature,
the heat flux, and the normal and tangential stresses with due account of the
effect of surface tension. For a small droplet, the effect of surface tension
is to limit the deformation of the droplet to small derivation from spherical
shape. One may thus evaluate the boundary conditions at $\gamma = \gamma_0$. Let the
superscript prime refer to the variables inside the droplet and $\xi (\theta, \tau)$ be the
radial displacement of a point on the surface of the droplet. The boundary
conditions are:

$$u_r = u_r'$$  \hspace{1cm} (4)
$$u_\theta = u_\theta'$$  \hspace{1cm} (5)
$$T = T'$$  \hspace{1cm} (6)

$$\kappa \frac{\partial T}{\partial r} = \kappa' \frac{\partial T'}{\partial r}$$  \hspace{1cm} (7)

$$\Pi_r = \Pi_r'$$  \hspace{1cm} (8)

$$\Pi_{\theta r} = \Pi_{\theta r}' + \frac{\xi}{\gamma_0} \left[ 2 - 2 \xi - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial \xi}{\partial \theta}) \right]$$  \hspace{1cm} (9)

the second term in the right hand side of Eq. 9 is the surface force, $C\left(\frac{1}{R_i} + \frac{1}{R_e}\right)$. It is obtained by assuming the shape of the droplet is slightly different from
the spherical one ($\frac{\xi}{\gamma_0} \ll 1$). A workable form of Eq. 9 can be obtained by
taking the derivative with respect to time and setting $\frac{\partial \xi}{\partial \tau} = u_r|_{\tau=\tau_0} - u_r \cos \theta$

The mathematical forms of the basic equations and initial conditions are
same as Epstein and Carhart ($^3$) although the independent variables have a
different physical meaning. Furthermore, except for $\frac{\xi}{\gamma_0}$ term in Eq. 9, the
boundary conditions are also identical to Epstein and Carhart ($^3$). Indeed, the
The solution of our boundary value problem can be expressed in terms of three potentials, namely acoustic $\phi_1$, thermal $\phi_2$, and viscous $A$. Once these three potentials are known, the various physical quantities can be calculated.

\[ \nabla \cdot \mathbf{u} = - \nabla \phi + \nabla \times A \quad A_r = A_\theta = 0 \]  
\[ u_r = -\frac{\hat{\rho} \phi}{r} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A) \]  
\[ u_\theta = -\frac{1}{r} \frac{\hat{\rho} \phi}{r} - \frac{1}{r} \frac{\partial}{\partial r} (r A) \]  
\[ T = \alpha_1 (\phi_1 + \phi_2) + \alpha_2 \phi_2 \]  
\[ \phi = -i \omega \rho_c \left[ \xi_1 (\phi_1 + \phi_2) + \xi_2 \phi_2 \right] \]  
\[ \Pi_{r_0} = \mu \left\{ -\frac{\partial}{\partial \theta} \left[ \frac{1}{r} \frac{\partial \phi}{\partial r} - \frac{\phi}{r^2} \right] - \left( \frac{\partial^2 A}{\partial r^2} - \frac{2 A}{r^2} \right) \right. \]  
\[ \left. + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left[ \frac{\sin \theta}{\sin \theta} \frac{\partial}{\partial \theta} (A \sin \theta) \right] \right\} \]  
\[ \Pi_{r_l} = \mu_k \left[ \beta_1 (\phi_1 + \phi_2) + \beta_2 \phi_2 \right] \]  
\[ + z \mu \left[ -\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \left( -\frac{A}{r^2} + \frac{1}{r} \frac{\partial A}{\partial r} \right) \right) \right] \]  
\[ u_\varphi = \Pi_{r_\varphi} = 0 \]

where $\phi = \phi_1 + \phi_2$ and $\alpha_1, \alpha_2, \beta_1, \beta_2, \xi_1, \xi_2$ are functions of the properties of the two-phase medium and the frequency of the incident wave (see Eq. 18). Since the main interest is the attenuation due to the presence of
small droplets, it is assumed that the attenuation of the incident wave by viscosity and heat conduction is small in the fluid medium in the absence of droplets. In other words, the study is limited to the case where the amplitude of the incident wave decreases relatively little over the region occupied by the droplets \( l_i = \frac{2a_0}{\omega} \left( \frac{N \nu \omega}{a_0^2} + \frac{\nu \omega}{a_0^2} (\epsilon - 1) \right)^{-1} \gg 2 \). This means that parameters \( \frac{N \nu \omega}{a_0^2} \approx \frac{\nu \omega}{a_0^2} \ll 1 \). In such cases, \( \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \text{ and } \gamma_2 \) are given approximately as follows:

\[
\begin{align*}
\alpha_1 & = - \frac{i \omega \nu}{a_0} - \frac{i \omega}{a_0 \alpha_0} (\epsilon - 1), \\
\alpha_2 & = -(\alpha_1 \alpha_2)^{-1} \\
\beta_1 & = 1 + i \frac{N \nu \omega}{a_0^2}, \\
\beta_2 & = 1 - \frac{\omega}{N \nu}, \\
\gamma_1 & = 1 + \frac{i N \nu \omega}{a_0^2}, \\
\gamma_2 & = 1 - N \nu.
\end{align*}
\]

The equations governing \( \phi_1, \phi_2, \) and \( A \) are,

\[
\begin{align*}
(\nabla^2 + K_1^2) \phi_1 &= 0 \quad (19) \\
(\nabla^2 + K_2^2) \phi_2 &= 0 \quad (20) \\
(\nabla^2 + K^2) A &= 0 \quad (21)
\end{align*}
\]

where

\[
\begin{align*}
K_1^2 &= \frac{\omega^2}{a_0^2} \left[ 1 + i \left( \frac{N \nu \omega}{a_0^2} + \frac{\nu \omega}{a_0^2} (\epsilon - 1) \right) \right], \\
K_2^2 &= \frac{\omega^2}{a_0^2} \left[ 1 + i \left( \frac{N \nu \omega}{a_0^2} + \frac{\nu \omega}{a_0^2} (\epsilon - 1) \right) \right], \\
K^2 &= \frac{\omega^2}{a_0^2} \left[ 1 + i \left( \frac{N \nu \omega}{a_0^2} + \frac{\nu \omega}{a_0^2} (\epsilon - 1) \right) \right], \\
K &= (1 + i) \frac{\omega^2}{2 \nu} \frac{1}{2}, \\
K &= (1 + i) \frac{\omega^2}{2 \nu} \frac{1}{2},
\end{align*}
\]

and the solution of the system satisfying the initial and boundary conditions (including the surface tension) can be written in series form:

\[
\begin{align*}
\phi_1 &= \sum_{n=0}^{\infty} i^n (2n + 1) J_n (k_1 r) R_n (\cos \theta) \exp (-i \omega t) \quad (22) \\
\phi_2 &= \sum_{n=0}^{\infty} i^n (2n + 1) H_n (k_1 r) R_n (\cos \theta) \exp (-i \omega t) \quad (23)
\end{align*}
\]
where \( J_n, H_n, P_n, \) and \( P_n^i \) are Bessel, Hankel, Legendre and associated Legendre (order 1) functions respectively. The coefficients \( B_n, C_n, D_n, B'_n, C'_n \) and \( D'_n \) are determined from the six boundary conditions given by Eqs. 4 to 9.

The values of these coefficients for \( n = 0 \) and \( n = 1 \) are given in Appendix. It is sufficient to consider first two terms of the expansion because of the rapid convergence of the series for \( \frac{r_0}{\lambda_1} \ll 1 \). If \( a_i \) denotes \( \frac{2\pi f r_0}{\lambda_1} \) \((-r_0 \kappa_i \ll 1\)) each term in the series after \( n = 1 \) is an order \( a_i^2 \) times smaller than the preceding. In terms of the small parameter \( a_i \), the relative order of magnitude of the coefficients are

\[
\begin{align*}
B_0 & \sim a_i^3, \\
B'_0 & \sim \delta, \\
C_0 & \sim a_i^2, \\
C'_0 & \sim a_i^3, \\
B_1 & \sim a_i^3, \\
B'_1 & \sim \delta, \\
C_1 & \sim a_i^3, \\
C'_1 & \sim a_i^3, \\
D_1 & \sim a_i \\
D'_1 & \sim a_i \\
\end{align*}
\]

where \( \delta = \frac{c_0}{\rho_0} \). The physical interpretation of these coefficients is that for a given incident wave of unit amplitude these coefficients give the magnitude of amplitudes for reflected waves outside and transmitted waves inside the droplet.
The deformation of the droplet $\xi$ is

$$\xi = \sum_{n=0}^{\infty} A_n P_n (\cos \theta) \exp \left( -i\omega t \right)$$

(30)

where $A_n$ can be determined from the relation

$$\frac{d\xi}{dt} - u \frac{\partial \xi}{\partial x} = -u_p \cos \theta$$

Hence $n = 0$ corresponds the pulsation of the droplet and $n = 1$ gives the translational motion of the droplet as a whole. Therefore, the term due to surface tension is not present for $n = 1$. Terms with $n > 1$ correspond to the change of the shape of the droplet and describe the higher modes of deformation.

III. ATTENUATION

When a sound wave propagates through a medium, its intensity decreases in proportion to $\exp(-\alpha x)$ where $\alpha$ is the attenuation coefficient and $x$ is the distance traversed by the wave. When there are $n$ droplets per unit volume, the total energy loss per unit volume per unit time will be $n \frac{dE}{dt}$, where $\frac{dE}{dt}$ is the average time rate of the energy dissipation per single droplet. The expression of $\alpha$ is

$$\alpha = \frac{n \frac{dE}{dt}}{E_0}$$

(31)

where $E_0 = \frac{1}{2} \rho_o a_o U_o^2$, the intensity of the incident wave.

Let $\psi_\mu$ and $\psi_\kappa$ be the viscous and thermal dissipation functions per unit volume, respectively. Then the energy dissipation is

$$\frac{dE}{dt} = \int \left( \psi_\mu + \psi_\kappa \right) \omega \cdot dV$$

(32)

where $\psi_\mu = \frac{2}{3} \chi_i \tau_{ij}$ and $\psi_\kappa = \frac{k}{\rho_o} (\nabla T)^2$. Integrating Eq. 32 over a volume surrounding the droplets, we obtain

$$\frac{dE}{dt} = - 2 \pi \rho_o a_o \sum_{n=0}^{\infty} (2n+1) R \left( B_n + B_n B_n^* \right)$$

(33)
\[ \alpha = -\frac{4\pi n}{k\Omega} \sum_{n=0}^\infty (2n+1)R(B_n + B_n^*) \]  

where \( B_n^* \) is complex conjugate of \( B_n \).

Let \( \alpha_\mu \), \( \alpha_\kappa \), and \( \Delta \alpha_\kappa \) be the attenuation coefficients due to the presence of viscosity, heat conduction, and surface tension, respectively. Neglecting terms of \( \eta > 1 \) (they are small if \( \alpha_\kappa << 1 \)), the expression for \( \alpha \) can be written as the combination of these three terms.

\[ \alpha = \alpha_\mu + \alpha_\kappa + \Delta \alpha_\kappa \]  

A. Attenuation Applicable to the Suspension of Liquid Droplets in a Gas, i.e. \( \delta << 1 \), \( \epsilon << 1 \), \( \chi << 1 \).

The explicit attenuation coefficient \( \alpha_\mu \), \( \alpha_\kappa \) and \( \Delta \alpha_\kappa \) are

\[ \alpha_\mu = \frac{9}{2} \varepsilon (\frac{\nu}{\alpha_0 r^2}) \frac{16(1+y) y^4}{16 y^4 + 72 \delta y^3 + 81 \delta (1 + 2y + 2y^2)} \]  

(36)

\[ \alpha_\kappa = 3 \varepsilon (\frac{\nu}{\alpha_0 r^2}) \frac{4(1-\delta)(1 + \frac{P_r^{1/2}}{R}) P_r y^4}{4 P_r^{1/2} y^4 + 12 (\delta \frac{\nu}{\alpha_0}) P_r^{1/2} y^3 + 9 (\delta \frac{\nu}{\alpha_0})} \]  

(37)

\[ \Delta \alpha_\kappa = \alpha_\kappa \sum_{n=1}^\infty \left( \frac{2 \varepsilon}{3 \alpha_0 P_r d_0^2} \right)^n \]  

(38)

where \( P_r = \frac{c_l}{c} \), \( \delta = \frac{c_l}{c} \), \( \epsilon = \frac{\mu}{\mu_0} \), \( \chi = \frac{K}{K} \), \( \varepsilon \) - volume fraction of the droplets.

B. Attenuation Applicable to the Suspension of Gaseous Bubbles in a Liquid, i.e. \( \delta >> 1 \), \( \epsilon >> 1 \), \( \chi >> 1 \).

The explicit attenuation coefficient \( \alpha_\mu \), \( \alpha_\kappa \), and \( \Delta \alpha_\kappa \) are...
IV. GENERAL DISCUSSION OF THE RESULTS

A. Attenuation

The effect of surface tension is to alter the thermal dissipation resulting from heat conduction, while no effect on viscous dissipation is produced. This effect is found to be negligible in the case of the suspension of water droplets in air, i.e. \( \frac{c}{\rho \cdot a_0^2} \ll 1 \).

Hence, if the droplets differ only slightly from the spherical shape in the course of motion, Epstein and Carhart's results for the sound attenuation in a gas with the suspension of liquid droplets apply even if the displacement of the droplet is large compared to its radius. On the other hand, for water containing air bubbles, the presence of the surface tension is to increase the thermal attenuation by a factor of 1.5, for \( r_o = 10^{-4} \text{ cm} \) (see Fig. 1). This effect becomes less pronounced as the size of the droplets increases.

For air containing water droplets, the ratio of thermal attenuation to the viscous attenuation is
It is seen that the thermal attenuation is predominant at low frequency range while the viscous attenuation is more important at higher frequency range (see Fig. 2).

For the suspension of air bubbles in water, it is seen from Eqs. 39 and 40 that the thermal attenuation completely dominates the viscous attenuation for all values of \( \gamma_f \) and \( \gamma^{'f} \),

\[
\frac{\alpha_k}{\alpha'} \sim 10^5
\]

This attenuation is caused by thermal dissipation inside the bubbles and is approximately given by

\[
\alpha_k = 5 \varepsilon \left( \frac{\nu}{\alpha} \right) (\epsilon^{'f}-1) \delta \left( \frac{\alpha}{\alpha} \right)^2 \mu \epsilon^{'f} y^4 \left[ 1 + 1520 \mu \epsilon^{'f} y^2 \left( \frac{1}{10} \right) \right] + o(y^4) + \ldots
\]

\[
\Delta \alpha_k = \alpha_k \sum_{n=1}^{\infty} \left( \frac{2 \varepsilon}{3 \gamma \alpha^{'f} \alpha} \right)^n
\]

for \( \gamma^{'f} \ll 1 \).

B. Droplet Displacement

The equation of motion of the droplet is

\[
m_p \frac{d\mathbf{v}_p}{dt} = \int \left( \Pi_{rr} \cos \theta - \Pi_{r\theta} \sin \theta \right) dS
\]

where \( m_p \) and \( S \) are the mass and surface area of the droplet respectively. \( \Pi_{rr} \) and \( \Pi_{r\theta} \) are given by Eqs. 15 and 16.
Evaluating the integral for $n = 1$, and dropping the terms of $O(a^2)$ and higher, one obtains expressions for the drag force $F_0$ and the amplitude of the droplet velocity $u_p$:

$$F_0 = \frac{\pi \rho_0 r_0^2 \omega}{4} \left[ -a_1 + 3i a_1^2 B_1 + 6 H_1(b) D_1 \right]$$

(47)

$$u_p = 3 r_0^{-1} \left[ -a_1 + 3i a_1^2 B_1 + 6 H_1(b) D_1 \right]$$

(48)

where $b = \kappa r_0$. Eq. 48 can be reduced to the following form which is applicable to air containing water droplets.

$$u_p = \frac{-3i K_1 H_1(b)}{3 \delta H_2(b) + 2(\delta - 1) H_0(b)}$$

(49)

The above expression can also be obtained by calculating the surface velocity of the droplet.

$$u_p = (u_{r \cos \theta} - u_{r \sin \theta}) r = \frac{2i}{r_0} \left[ -\frac{q_i}{b} + B_1 H_1(a) (2 \cos \theta - \sin \theta) + D_1 b H_0(b) \sin \theta \right]$$

(50)

This expression is identical to Eq. 49 upon substitution for $B_1$ and $D_1$.

The ratio of the droplet displacement $\xi_p$ to the radius of the droplet is

$$\frac{\xi_p}{r_0} = \frac{\frac{3}{\omega} \delta H_2(b)}{\omega \delta H_2(b) + 2(\delta - 1) H_0(b)}$$

(51)

In the limit of audibility, the range of the sound intensity level is from 0 to 135 dB (decibels) at 1 kHz based on threshold sound pressure of $2 \times 10^{-4}$ dyne per sq. cm. This corresponds to sound intensity $E_s$ of $10^{-9}$ to $10^{-5}$ ergs per sec. per sq. cm.
and a velocity amplitude of $\eta \times 10^{-5}$ to 70 cm. per sec. in air. For $\omega = 1$ kc and droplet size of 2 microns ($\gamma = 6 \times 10^{-3}$), it is found that in the case of the suspension of the water droplets in air, the droplet remains relatively motionless ($\frac{5p}{p_0} < \frac{1}{5}$) at an intensity level of 52 db and $\frac{5p}{p_0} \sim 2$ at 72 db.

The ratio of the droplet displacement to that of the surrounding medium is

$$\frac{5p}{5q} = \frac{3 \delta H_2(b)}{3 \delta H_2(b) + 2 (\delta - 1) H_0(b)} \quad (52)$$

This ratio is plotted as a function $\gamma$ (see Fig. 2) for air containing water droplets. $\frac{5p}{5q}$ approaches to unity as $\gamma$ approaches to zero which agrees with our intuition that at low frequencies there is little relative motion between the droplet and the surrounding gas. It approaches $\frac{3p_0}{p_0 + 2 p_0}$ as $\gamma \rightarrow \infty$. This agrees with the classical result for a sphere set in motion by an oscillating non-viscous fluid (8).

C. Limiting Cases for Attenuation, Drag Force, and Heat Transfer Rate

1. Attenuation: In case of low frequencies ($\gamma \ll 1$), the attenuation coefficients $\alpha_\mu$ and $\alpha_\kappa$ for $\delta \ll 1$
are approximately given by

\[ \alpha_r = N \frac{g}{8} \left[ 1 - \frac{y}{\gamma} + O(y^2) + \cdots \right] \]  

(54)

\[ \alpha_k = N(\gamma - 1) \frac{P_{Tz}}{L_{z}} \left[ 1 + \frac{P_T}{L_{z}} y + O(y^2) + \cdots \right] \]  

(55)

where \( N = \frac{\rho u}{\alpha_0 r^2} \), and \( L_{z} = \frac{C_T}{C_0} \).

2. Drag Force: The force acting on the droplet is given by Eq. 47 and can be reduced to the following form which is applicable to the suspension of water droplets in air.

\[ F_D = -4\pi \rho r^2 \omega_k \frac{H_2(b)}{3\delta H_2(b) + 2(\sigma - 1)H_0(b)} \]  

(56)

For low frequencies \( (\gamma << 1) \), \( F_D \) is approximately given by

\[ F_D = 6\pi \mu r^2 \frac{(-u_m \gamma b^2)}{q \delta} + O(y^5) + \cdots \]  

(57)

On the other hand, for low frequencies

\[ \bar{u}_{\omega_i} - \bar{u}_p = -\frac{u_m \gamma b^2}{q \delta} + O(y^5) + \cdots \]  

(58)

Hence

\[ F_D = 6\pi \mu r^2 (\bar{u}_{\omega} - \bar{u}_p) + O(y^5) + \cdots \]  

(59)

It is recognized immediately that the first term of the expression for \( F_D \) is identical to Stoke's formula of a sphere moving with a velocity of \( \bar{u}_{\omega} - \bar{u}_p \) in a viscous
medium. Furthermore, the first term of the coefficient of viscous attenuation given by Eq. 54 agrees with the dissipation function calculated on the assumption of validity of Stoke's formula.

3. Heat Transfer Rate: The heat transfer rate between the droplet and the surrounding gaseous medium per unit time and per unit area is

\[ q = \frac{1}{S} \int \kappa \frac{\partial T}{\partial r} \bigg|_{r=r_0} ds \]

\[ = \kappa r_0^{-1} \left\{ \alpha_1 \left[ a_i \mathcal{J}_0 (a_i) + a_i \mathcal{H}_0 (a_i) \mathcal{B}_0 \right] + \alpha_2 a_i \mathcal{H}_0 (a_2) C_0 \right\} \]

(60)

where \( S \) is the surface area of the droplet and dot on Bessel and Hankel functions denote the differentiation with respect to their respective arguments. In case of low frequency range, \( q \) is approximately given by

\[ q = \kappa r_0^{-1} \left[ -\alpha_1 \frac{a_i^2}{3} + i \alpha_1 a_i \mathcal{H}_0 (a_i) B_0 + i \alpha_2 a_i \mathcal{H}_0 (a_2) C_0 \right. \]

\[ \left. -\alpha_1 \mathcal{H}_0 (a_i) B_0 - \alpha_2 \mathcal{H}_0 (a_2) C_0 \right] \]

(61)

Let \( T_{\infty} \) and \( T_s \) denote the temperature of the gaseous medium at infinity and the surface temperature of the droplet, respectively. Then the temperature difference \( T_{\infty} - T_s \) takes the following form

\[ T_{\infty} - T_s = -\alpha_1 \mathcal{H}_0 (a_i) B_0 - \alpha_2 \mathcal{H}_0 (a_2) C_0 \]

(62)

Then in low frequency range
The heat transfer coefficient \( h \) \( (\frac{q}{T_{\infty} - T_s}) \) corresponding to the first term of the expansion is \( \kappa \gamma^{-1} \), which is equal to Nusselt No. \( (\frac{\kappa}{\nu}) \) of 2 \( z \) agrees with the result for the heat transfer to a sphere when Reynolds No. tends to zero and heat exchange is by conduction only \((9)\). At low frequencies, it is noted from Eq. 64 that the temperature difference \( T_{\infty} - T_s \) is proportional to \( O(\omega^0) \).
APPENDIX

The assumptions made for obtaining the coefficients are (1) the acoustic damping length \( l_1 \) is large compared with the region occupied by the droplets, i.e. \( \kappa_1 = \frac{\omega}{a_0} \), \( \kappa'_1 = \frac{\omega}{a'_0} \). (2) The droplet is small compared with the acoustic wavelength \( r_0 << \lambda_1 \).

\[
\begin{align*}
\eta &= 0 \\
B_0 &= i \frac{a_0^2}{\omega} \left[ \delta (\frac{a_0^2}{\omega^2} - 1) + ia_0 \delta \left( \frac{a_0^2}{\omega^2} - 1 \right)^2 \frac{a_2 H_1(a_2)}{H_0(a_0) (1 - \chi Z)} \right] + \frac{2c}{3 r_0 \rho_0 a_0^2} \\
B_0' &= \delta \left( 1 - \frac{2c}{3 r_0 \rho_0 a_0^2} \right)^{-1} \\
C_0 &= \frac{\alpha_2}{\alpha_1} \frac{\delta \left( \frac{a_0^2}{\omega^2} - 1 \right)}{H_0(a_0) (1 - \chi Z)} \left( 1 - \frac{2c}{3 r_0 \rho_0 a_0^2} \right)^{-1} \\
C_0' &= \chi \left( \frac{\alpha_2}{\alpha_1} \right) - \frac{a_2 H_1(a_2)}{\alpha_1 J_1(a_2)} C_0 \\
\eta &= 1 \\
B_1 &= i \frac{a_0^3}{\omega} (1 - \delta) G \\
B_1' &= \frac{a_1}{a_0} \frac{G'}{G} \\
C_1 &= \frac{a_1}{\alpha_2} \frac{\left( z + 2 z - 3 \delta \left[ G' + z (1 - \chi)(1 - \delta) G \right] J_1(a_1) + \left[ \delta a_2^2 \left( 1 - (1 - \delta) G \right) \right] a_2 J_0(a_2) \right)}{2 (x - 1) J_1(a_1) H_1(a_2) + a_2 J_0(a_2) H_1(a_2) - \chi J_1(a_1) a_2 H_0(a_2)} \\
C_1' &= \frac{a_1}{\alpha_2} \frac{(1 + 2 z) \delta}{\alpha_1} \frac{G' - 3 x}{2 (x - 1) J_1(a_1) H_1(a_2) + a_2 J_0(a_2) H_1(a_2) - \chi J_1(a_1) a_2 H_0(a_2)}
\end{align*}
\]
\[ D_1 = \frac{a}{b} (\delta - 1) M \left\{ (3 \delta H_2(b) + 2 (\delta - 1) H_0(b)) M - \delta (\delta + 2) b H_1(b) J_2(b) \right\}^{-1} \]

\[ D'_1 = -a_1 (1 + \delta + \frac{a_1 \delta B_1}{a t}) + D_1 \left[ \frac{2 b H_0(b) - 6 \delta H_1(b)}{2 b' J_0(b) - 6 J_1(b)} \right] \]

where \( \delta = \frac{B_0}{C_0} \), \( \sigma = \frac{K}{\mu} \), \( \lambda = \frac{K}{\kappa} \), \( a_1 = K_1 r_0 \), \( a_2 = K_2 r_0 \),

\[ b = K r_0, \quad z = \frac{J_2(a_1)}{a_1 J_1(a_2) H_0(a_2)} \]

\[ G = \frac{H_2(b) [ b' J_1(b) - 2 (1 - \delta) J_2(b)] - \delta H_1(b) J_2(b')}{[3 \delta H_2(b) + 2 (\delta - 1) H_0(b)] M - \delta (\delta + 2) b H_1(b) J_2(b')} \]

\[ G' = 3 \delta G, \quad M = b' J_1(b') - 2 (1 - \delta) J_2(b) \]

In obtaining \( D \), and \( D'_1 \), we neglect the terms containing \( C_1 \) and \( C'_1 \), since these terms are small compared with the others.

For \( \delta \ll 1 \), \( \sigma \ll 1 \), and \( |b| \ll 2 \), the expression of \( B_1 \) and \( D_1 \) can be reduced to the following forms:

\[ B_1 = i a^2 (1 - \delta) \frac{H_2(b)}{3 \delta H_2(b) + 2 (\delta - 1) H_0(b)}, \quad D_1 = \frac{3 i B_1}{a^2 b H_2(b)} \]
REFERENCES


### TABLE I

Physical Constants of Water and Air at 20°C and Atmospheric Pressure

Used in Preparation of Figures 1 and 2 and in Equations 36, 37, 40, 41, and 52.

<table>
<thead>
<tr>
<th></th>
<th>$\alpha_0$ (cm/sec)</th>
<th>$\rho_0$ (g/cm³)</th>
<th>$\nu$ (cm²/sec)</th>
<th>$\Omega$ (cm²/sec)</th>
<th>$C_p$ (cal/°C)</th>
<th>$\zeta$</th>
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<tbody>
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<td>Water</td>
<td>1.45 x 10⁵</td>
<td>1.00</td>
<td>0.011</td>
<td>1.43 x 10⁻³</td>
<td>1.0</td>
<td>1.00³³³⁶</td>
</tr>
<tr>
<td>Air</td>
<td>3.30 x 10⁴</td>
<td>1.29 x 10⁻³</td>
<td>0.141</td>
<td>0.187</td>
<td>0.24</td>
<td>1.4</td>
</tr>
</tbody>
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FIG. 1  THEORETICAL ATTENUATION PARAMETER AND DISPLACEMENT RATIO FOR WATER DROPLETS IN AIR
FIG. 2  THEORETICAL ATTENUATION PARAMETER FOR AIR BUBBLES IN WATER