2 of 3

MEMORANDUM
RM-4009-PR
APRIL 1964

20-p $0.50

SOLVABLE NUCLEAR WAR MODELS

Norman C. Dalkey

DDC
DDC-IRA B

PREPARED FOR:

UNITED STATES AIR FORCE PROJECT RAND

The RAND Corporation
SANTA MONICA • CALIFORNIA

ERRATA SHEET RM-4009-PR

SOLVABLE NUCLEAR WAR MODELS
Norman C. Dalkey

On Page 13, Change equations (3) and (4) to

(3) $\bar{M} = M - \bar{N}vy$ and

(4) $\bar{N} = N - \bar{M}ux.$

MEMORANDUM
RM-4009-PR
APRIL 1964

SOLVABLE NUCLEAR WAR MODELS

Norman C. Dalkey

This research is sponsored by the United States Air Force under Project RAND—contract No. AF 49(638)-700 monitored by the Directorate of Development Planning, Deputy Chief of Staff, Research and Development, Hq USAF. Views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of the United States Air Force.

DDC AVAILABILITY NOTICE
Qualified requesters may obtain copies of this report from the Defense Documentation Center (DDC).

The RAND Corporation
1700 MAIN ST • SANTA MONICA • CALIFORNIA

PREFACE

This Memorandum is an offshoot of a continuing project concerned with developing decision models for strategic planning and command and control. The present report deals with a highly aggregated, two-sided war game, with a payoff function which attempts to take into account the increasing concern of the participants as a critical level of damage is approached.

The contents of this paper were presented at the Symposium on Performance Measurements for Command and Control held at MITRE, 12-14 January 1964.

SUMMARY

The report deals with an aggregated, two-sided war game, one of several designed to study the use of abstract models for strategic planning. A payoff function for the war game is defined making use of an assumption of increasing concern as a critical level of damage is approached. A very simple, one-weapon version of the central nuclear war game has an analytic solution, indicating the existence of a stable (equilibrium) point in pure strategies if each side has a nonnegligible counterforce capability.

ACKNOWLEDGEMENTS

David Langfield has made major contributions to the form of the model and its solution. Margaret Ryan analyzed most of the data and programmed the one-weapon model for the 7090.

SOLVABLE NUCLEAR WAR MODELS

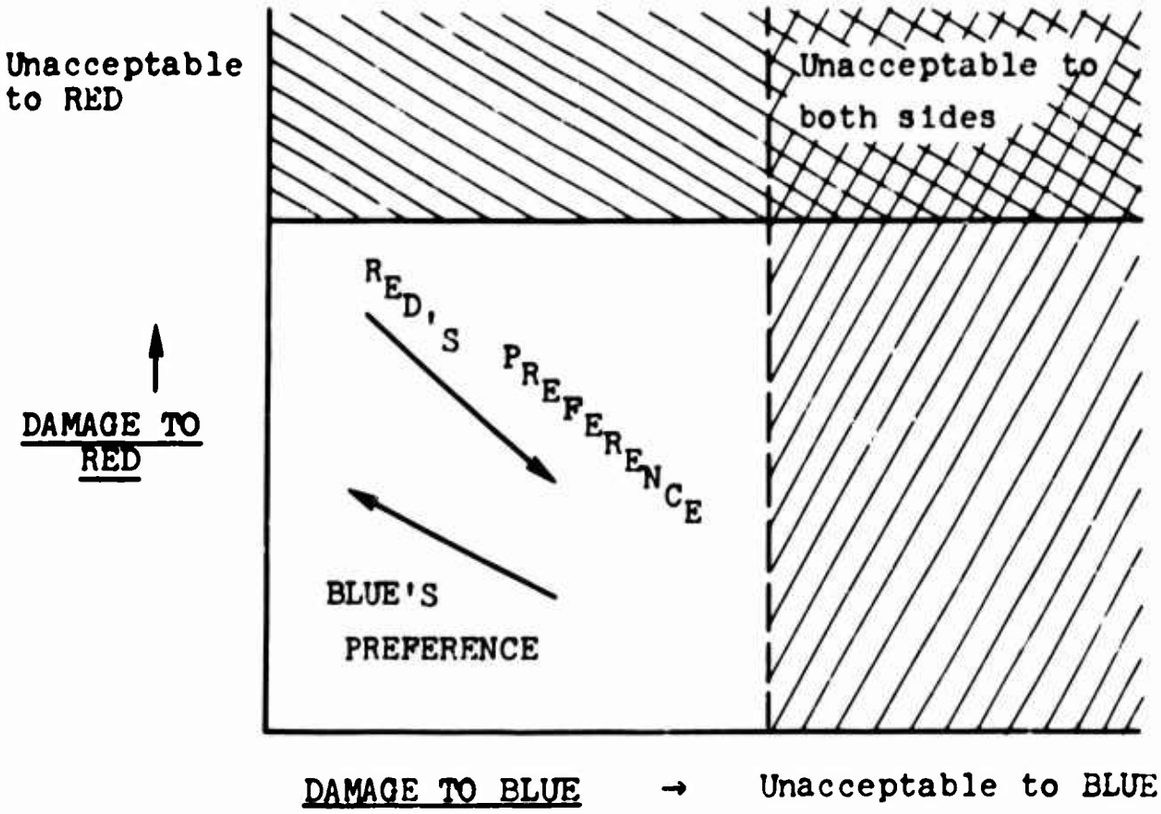
1. A PAYOFF FUNCTION FOR NUCLEAR WAR

A basic desirable quality of an abstract war game is that it be rapidly computable. Thus, at worst, a large number of cases can be examined to give some insight into the influence of various parameters. At best, an analytic solution may be found which will allow precise computation of preferred strategies, and enable a wide range of sensitivity studies. Before a solution can be sought, however, it is necessary to specify a payoff function.

Defining a payoff function for abstract games involving central nuclear war has been frustrating mainly because certain possible outcomes look completely unacceptable to both sides, and no solution has been found that excludes these "irrational" outcomes. The payoff is highly nonzero sum, and a form of very powerful cooperation would have to be postulated to define a "solution." This is an awkward assumption for a situation as noncooperative as nuclear war.

The problem can be represented in a utility space where damage to the value targets of one side is measured along the ordinate and damage to the other along the abscissa.

A standard assumption is that each side recognizes some damage level it considers "unacceptable." The definition of "unacceptable" is vague, and may range from losses considered sufficient to deter a nation from initiating a nuclear conflict to a level of damage beyond which the nation will not be "viable"—i.e., will not be able to recuperate to pre-war levels within some reasonable time. In the present analysis we are not concerned with deterrence, but rather with the critical level once deterrence has failed.



BASIC PAYOFF SPACE

Fig. 1

Within the region of "acceptable" outcomes, each side presumably will prefer greater damage to the other and less to itself, as indicated by the arrows in Fig. 1. Within the unacceptable area, preferences on either side will probably be very weak—"it doesn't matter."

This payoff function leads to a well-known dilemma—at least in a simplified analysis—as soon as both sides have the absolute ability to inflict unacceptable damage on the other.

Table 1

Blue \ Red	CF	CC
CF	O O	— W
CC	W —	— —

Table 1 is a simplified representation of the game that ensues where neither side has a sufficient counterforce capability to prevent the other side from inflicting unacceptable losses. The "payoff" to Blue is represented by the upper figure in each box, the payoff to Red in the lower.

If both sides go counterforce (assuming rough symmetry), then neither gains any particular advantage.

If one side elects counter-value (CC), then the side that elects counterforce (CF) receives unacceptable damage, represented by a negative infinity in the matrix, and the other, being relatively undamaged, receives 1 (representing "win"). If both sides elect counter-value, then both sides receive unacceptable damage, the double negative infinity in the lower right box.

From Blue's point of view, if Red elects CF, then Blue would elect CC. If Red elects CC, then Blue has no preference between CF and CC, and he might as well elect CC. In short, CC dominates CF—it is at least as good as CF no matter what Red does and is better if Red elects CF. The situation is symmetric for Red, and hence, both sides would presumably elect CC, and both receive unacceptable damage.

This simple analysis has been used to "demonstrate" that a central nuclear war is "irrational" for both sides and therefore probably won't occur. It can also be used, unfortunately, to "demonstrate" that central nuclear war, if it occurs, will most likely be catastrophic for both sides.

2. THE ASSUMPTION OF INCREASING CONCERN

One element that is missing in the analysis is the fact that, if we take the notion of critical damage seriously, then, as the level of damage approaches the critical value for a given side, the significance of additional damage should go up sharply. The amount of damage to Red that would be required to compensate for a given amount of damage to Blue should accelerate.

The lines of equal preference for Blue should curve upward, and theoretically become asymptotic to the critical level as illustrated in Fig. 2.

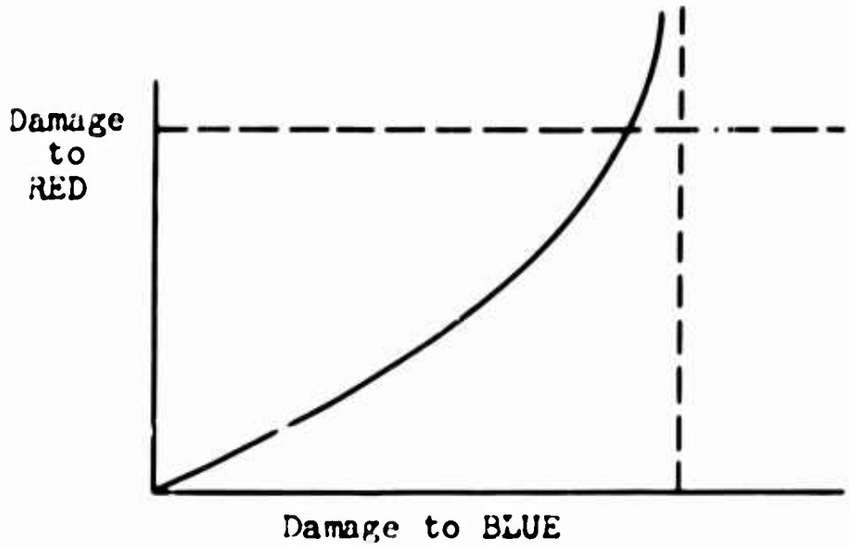


Fig. 2
Assumption of Increasing Concern

A simple expression which defines a payoff function with the required property is

$$P_B = D_R - D_B - \frac{A}{C_B - D_B} , \tag{1}$$

where P_B is the payoff to Blue, D_R is damage to Red value targets, D_B damage to Blue value targets, C_B is Blue's critical level, and A is a scaling constant. An equivalent formulation for Red is obtained by interchanging subscripts. Equation (1) is, of course, only one out of an infinite number of possible functions with the same general properties. For the purpose of investigating the consequences of this "assumption of increasing concern," (1) has at least the advantage of simplicity.

Another possible interpretation of the assumption of increasing concern makes use of the fact that it is extremely difficult to know the precise numerical value of the critical point at which damage will become unacceptable. In this case, which is probably

realistic, we can assume that concern will mount as the probability of sustaining disastrous damage increases. A first, rough approximation to this interpretation would use the scaling factor, A, in the expression $\frac{A}{C_B - D_B}$ as a measure of the uncertainty of Blue concerning the location of C_B.

5. APPLICATION TO CENTRAL WAR GAME

Some basic consequences of the assumption of increasing concern can be examined by a simple war model involving one type of value target and a single exchange with one type of offensive weapon. The strongest form of nuclear dilemma results when the game is symmetric—i.e., the two sides have equal strength and attack simultaneously. For this case we can set down the interaction equations

$$(2) \qquad \overline{M}_B = \frac{M_B(1-X_{MR})}{1-X_{MB}\, X_{MR}\, U_M^{\,2}} \qquad \text{and}$$

$$(3) \qquad D_B = \overline{M}_R(1-X_{MR}) \ ,$$

where M_B is the initial Blue missile force, X_{MB} is the percentage of the missile force allocated to counterforce, U_M the counterforce effectiveness of a missile, and $\overline{M}_B$ the number of Blue missiles surviving after Red attack. The equations have been simplified to include the assumption of symmetry—i.e., $M_B = M_R$, etc. Corresponding values for Red are obtained by interchanging subscripts.

In one case examined, the counterforce effectiveness of each side was set very low, an offensive weapon on one side being given only a .20 probability of destroying an offensive weapon on the other if allocated to a counterforce mission. On the other hand, each weapon was given a probability of 1 of destroying a value target. The other initial conditions were more or less arbitrary: The critical damage level for each side was set at 100, and the offensive forces at 150. Thus, if one side allocated all his forces to counterforce, the other side could still achieve more than the critical damage level. The scaling constant was set at 1000, and a normalizing constant of 10 was added to produce roughly zero payoff for each if the two sides mutually chose pure counterforce strategies. Thus the payoff to Blue was

$$(4) \qquad P_B = D_R - D_B - \frac{1000}{100-D_B} + 10 .$$

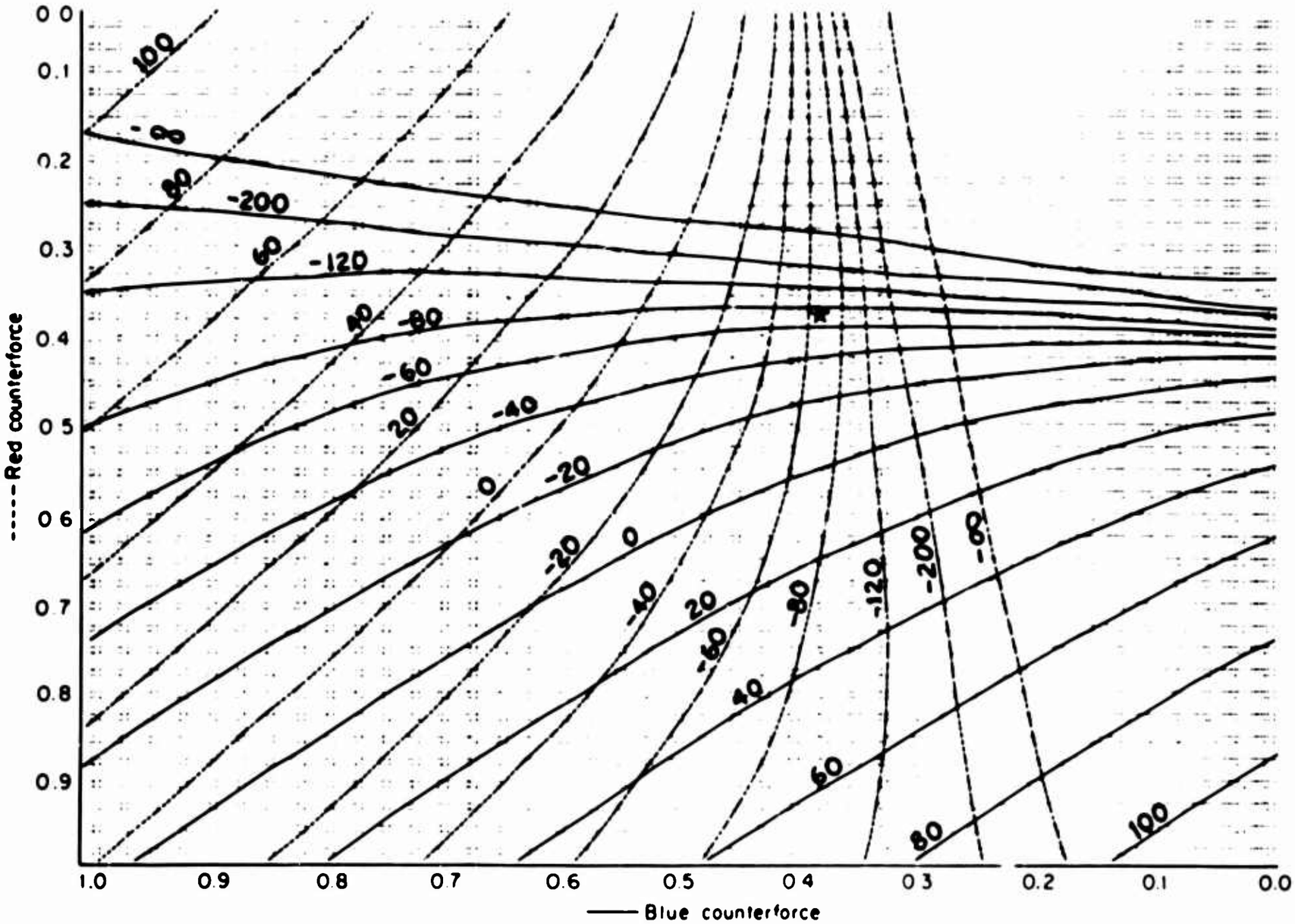
The payoff matrix for this case is shown in Fig. 3, with payoff to Blue in solid lines and payoff to Red in dotted lines. Inspection indicates that there exists a unique (strong) equilibrium point* at the pair of allocations indicated by the star, namely XMR = XMB = .375.

If either player allocates .375 of his forces to counterforce, then the other will have a maximum payoff at the corresponding allocation of .375. The damage in number of targets to each side for this pair of strategies is 87—well below the critical value.

A similar game was computed in which both sides had a considerable counterforce capability, UM = .4. The payoff matrix is shown in Fig. 3. As before, a strong equilibrium point exists, in this case with considerably less damage to each player. In addition, the degree of

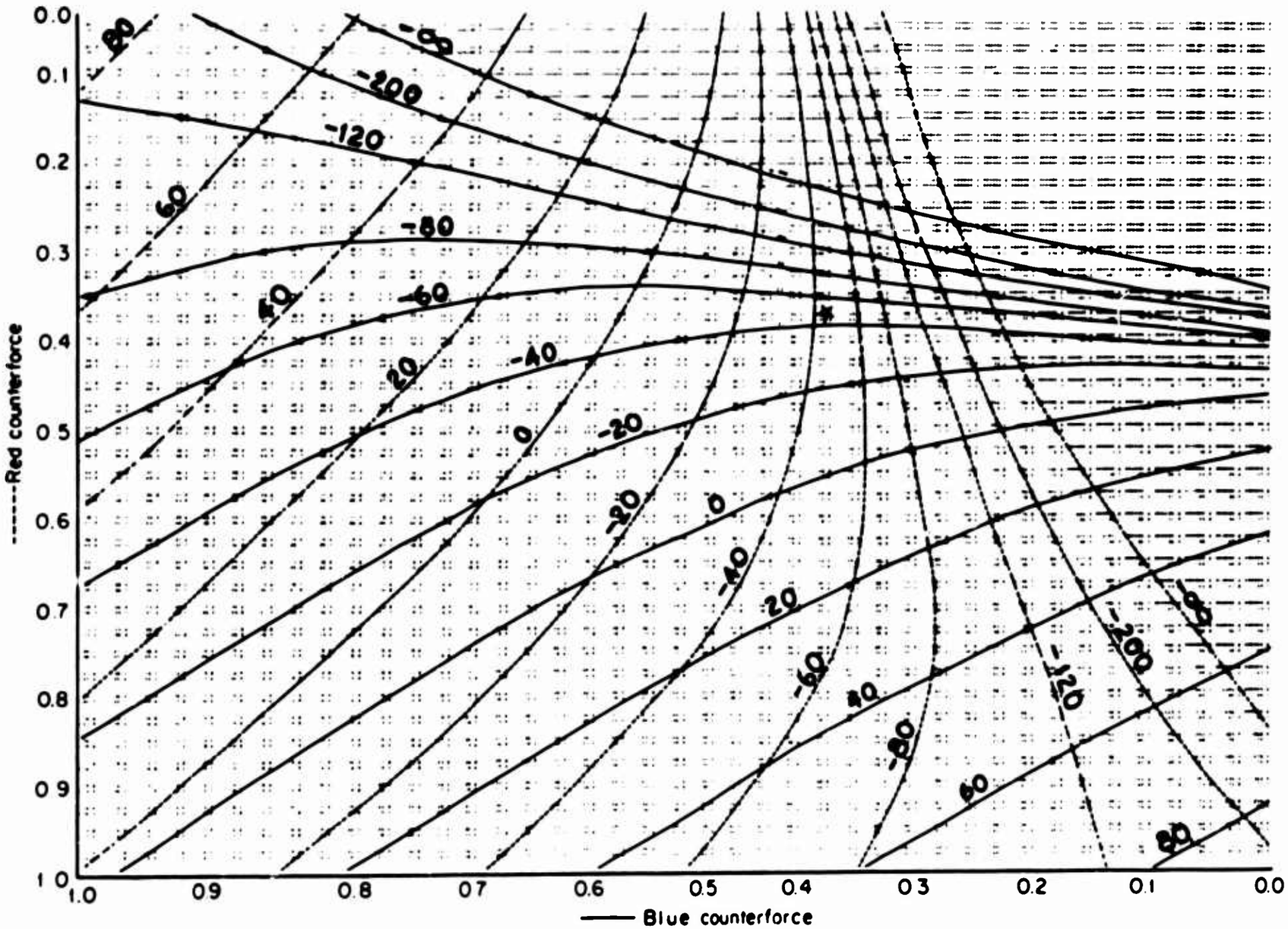
---

*A strong equilibrium point is the analogue for non-cooperative, non-zero sum games of the saddle point for zero-sum games.



Symmetric central war game

Fig. 3



Symmetric central war game

Fig. 4

stability has increased, in the sense that the amount each player loses if he plays non-optimally is much greater than for UM = .2.

If the game is played with a payoff without the increasing concern factor (which, incidentally, makes the game zero sum), there is either no equilibrium point or only one, which gives unacceptable damage to both sides.

In the appendix, an analytic solution for the simple game defined by (2), (3) and (4) is given, as well for the slightly more complex non-symmetric one-weapon game. The general features of the solution are similar to those in the special cases described above.

4. DISCUSSION

The most interesting outcome of the analysis is that, given the assumption of increasing concern, a stable non-cooperative "solution" to a central nuclear war game can exist if each side has a nonvanishing counterforce capability. The same comment would apply to a defense capability if it were assumed that this capability directly subtracts from the size of the counter-value attack (e.g., if the game consists of a budgetary allocation between offense and defense on each side with a fixed budget).

It might be noted that much of the analysis is still valid for a quite different kind of game; namely, that in which both sides engage first in a counterforce exchange, retaining part of their forces for subsequent counter-value threats. The effectiveness of the counterforce attack would go up for both sides, because of a reduction in empty hole targets, but otherwise the measured damage in the game could be taken to be the damage potential of remaining forces. With the assumption of increasing concern, each side would, above all, be interested in reducing his opponent's damage potential to a level below that needed

for inflicting a critical amount of damage, but at the same time maintaining his own potential as high as possible. The equilibrium point analysis would still be valid for selecting the preferred allocation to counterforce.

5. THE ASSUMPTION OF DECREASING CONCERN

There is enough literature on national military policy to suggest the rough reality of the notions of critical damage and relative tradeoffs of damage at levels far away from the critical. However, whether the assumption of increasing concern can be maintained (particularly for Soviet values) is not demonstrable from the literature. In fact, in some cases it is possible to arrive at exactly the contrary interpretation—i.e., that there is operative a doctrine of decreasing concern.

Verbally, this doctrine makes as much sense as the doctrine of increasing concern. It would say, for example, that as the level of critical damage is approached, one or two more value targets make little or no difference; you are almost dead. However, if we look at the consequences of the assumption of decreasing concern, there may be some doubts of its rationality.

The assumption of decreasing concern would produce indifference curves and reference directions in the mutual damage space as illustrated in Fig. 5.

The assumption has the consequence that from a point A, both sides gain by moving to a point of higher mutual damage, B. and, in fact, both sides gain by an outcome where each is damaged beyond the critical level. The verbal reasonableness of the assumption of decreasing concern appears to break down at this point. Part of the explanation for the apparent paradox here is that the assumption of decreasing concern would appear reasonable in a noncompetitive situation, e.g., an overwhelming natural disaster. It is not reasonable for a situation of armed conflict.

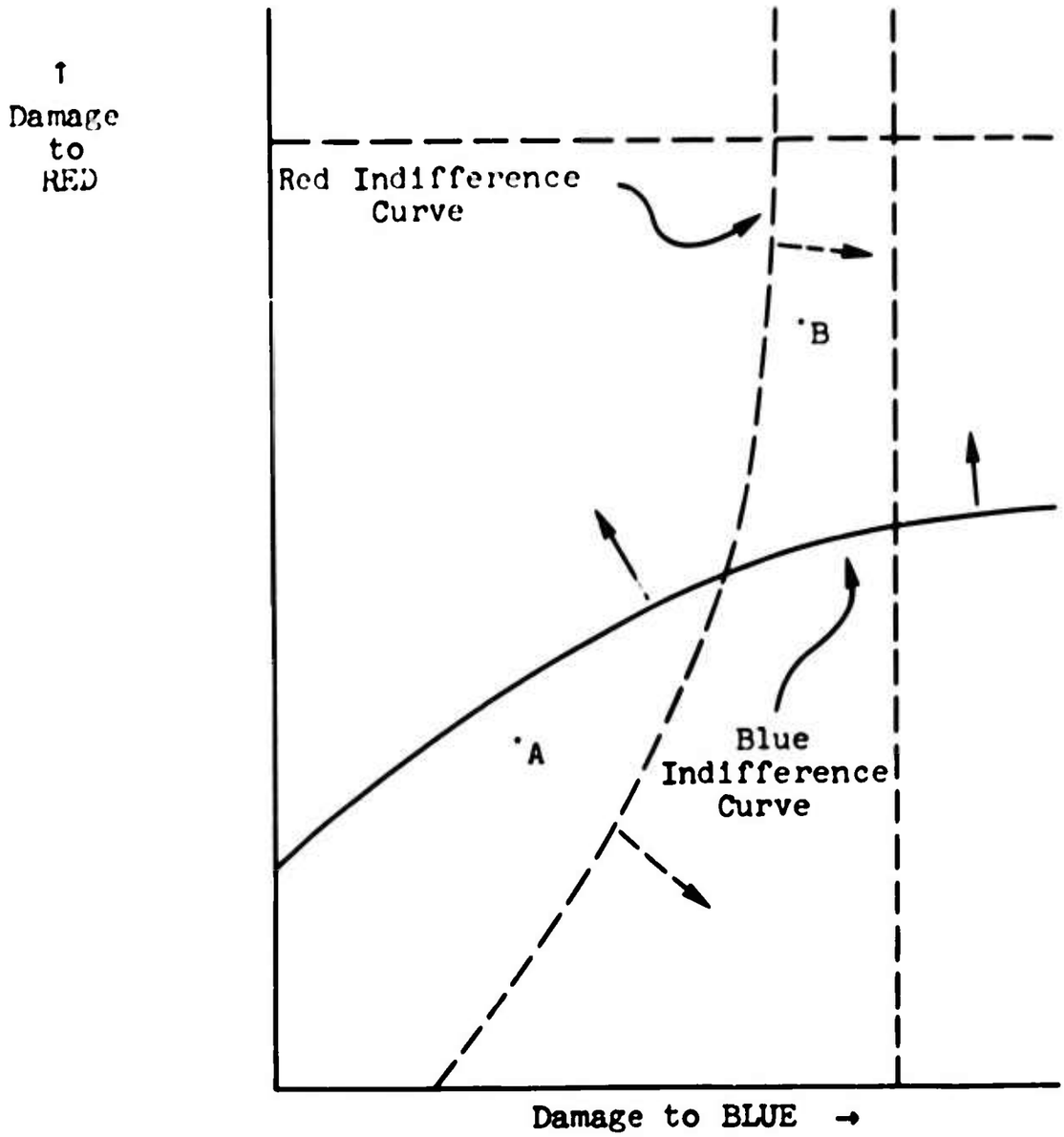


Fig. 5
Assumption of Decreasing Concern

6. MATHEMATICAL APPENDIX

6.1 Elements of the One-Weapon Model

Blue	Red	
M	N	Number of offensive weapons
u	v	Counterforce effectiveness
x	y	Proportion of forces allocated to counterforce
C_B	C_R	Critical level of damage
A	B	Sensitivity

To be complete, a counter-value effectiveness should be listed. For simplicity this is assumed to be 1.0.

The payoff is defined as

$$(1) \qquad P_B = D_R - D_B - \frac{A}{\lceil C_B - D_B \rceil} \qquad \text{and}$$

$$(2) \qquad P_R = D_B - D_R - \frac{B}{\lceil C_R - D_R \rceil} \quad ,$$

where D_R and D_B is the damage to value targets for Red and Blue respectively. The corner quotes indicate that the expression $C_B - D_B$ is considered zero if it becomes negative.

6.2. Derivation of Interaction Equations

Let $\overline{M}$ and $\overline{N}$ indicate the number of Blue and Red weapons surviving the counterforce exchange. We have

$$(3) \qquad \overline{M} = M - \overline{N}ux \qquad \text{and}$$

$$(4) \qquad \overline{N} = N - \overline{M}vy \ .$$

This is a pair of simultaneous equations in two unknowns which can be solved for $\bar{M}$ and $\bar{N}$ as follows:

$$(5)^{*} \qquad \bar{M} = \frac{M-Nvy}{1-uvxy} \qquad \text{and}$$

$$(6)^{*} \qquad \bar{N} = \frac{N-Mux}{1-uvxy} \; .$$

We then have

$$(7) \qquad D_B = \bar{N}(1-y) \qquad \text{and}$$

$$(8) \qquad D_R = \bar{M}(1-x) \; .$$

Introducing (5), (6), (7), (8) in (1) and (2) we obtain

$$(9) \quad P_B = \frac{M-Nvy}{1-uvxy}(1-x) - \frac{N-Mux}{1-uvxy}(1-y) - \frac{A}{C_B - \frac{N-Mux}{1-uvxy}(1-y)} \qquad \text{and}$$

$$(10) \quad P_R = \frac{N-Mux}{1-uvxy}(1-y) - \frac{M-Nvy}{1-uvxy}(1-x) - \frac{B}{C_R - \frac{M-Nvy}{1-uvxy}(1-x)} \; .$$

6.3. Derivation of Solution

The notion of solution to be applied is that of an equilibrium point in pure strategies—that is, a point (x^*,y^*) such that $P_B(x^*,y^*) \geq P_B(x,y^*)$ for all x, and

---

*Strictly speaking, these equations should include operators which prevent $\bar{M}$ and $\bar{N}$ from becoming negative.

$P_R(x^*,y^*) \geq P_R(x^*,y)$ for all y. P_B and P_R are continuous outside the regions where they are infinite. We look for a maximum of P_B as a function of x and a maximum of P_R as a function of y. This will furnish a pair of equations which, if they have a solution in the region of interest, and the maxima are not local, define an equilibrium point.

Taking the partial derivatives of P_B with respect to x and P_R with respect to y, and simplifying, we obtain

$$(11) \quad \frac{\partial P_B}{\partial x} = (1-u+uy(1-v))[C_B(1-uvxy) - (N-Mux)(1-y)]^2 - Au(1-y)(1-uvxy)^2 \quad \text{and}$$

$$(12) \quad \frac{\partial P_R}{\partial y} = (1-v+vx(1-u))[C_R(1-uvxy) - (M-Nvy)(1-x)]^2 - Bv(1-x)(1-uvxy)^2 .$$

At a maximum, $\frac{\partial P_B}{\partial x} = 0$, and $\frac{\partial P_R}{\partial y} = 0$.

In the symmetric case, where M = N, u = v, and A = B, it is clear that x = y. For this case, (11) and (12) are identical, and reduce to

$$(13) \quad (1-u)[C(1+ux) - M(1-x)]^2 - Au(1-x)(1+ux) = 0 .$$

The solution to (13) for x gives the symmetric equilibrium point. In the symmetric case, as long as $0 \leq u \leq 1$, there is no problem of $\overline{M}$ or $\overline{N}$ becoming negative.