A Remark on Orthonormal Bases of Continuous Functions in a Hilbert Space

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A REMARK ON ORTHONORMAL BASES
OF CONTINUOUS FUNCTIONS IN A HILBERT SPACE

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Abstract

It is shown that an orthonormal set of continuous functions on a finite interval can always be completed by the addition of continuous functions if it is a finite set but cannot always be so completed if it is an infinite set.

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A Remark on Orthonormal Bases of Continuous Functions in a Hilbert Space

The following problem has been proposed: let $H$ be the Hilbert space of square integrable functions on a finite interval $I$ and let $(\varphi_i)$ be an orthonormal set of continuous functions in $H$ — when can $(\varphi_i)$ be extended to a complete orthonormal set of continuous functions? This problem occurs under certain circumstances when approximating white noise by sums \[ \sum \theta_i \varphi_i(t) \] where the $\theta_i$ are independent Gaussian random variables.

In this note we prove that $(\varphi_i)$ can be so extended if it is a finite set and present an example to show that $(\varphi_i)$ cannot always be extended if it is infinite. Both the proof and the example apply equally well if continuity is replaced by $n$-times differentiability.

**Theorem**  
If $\varphi_1, \ldots, \varphi_n$ is an orthonormal set of continuous functions then continuous functions $\varphi_{n+1,} \varphi_{n+2,} \ldots$ can be found such that $\varphi_i, \ i = 1, 2, \ldots$ is a complete orthonormal set.

**Proof**  
Let $(\psi_i)$ be a complete orthonormal set of continuous functions (e.g., the trigonometric functions). Set

\[ \varphi_{n+1} = \alpha_1(\psi_1 - \sum_{j=1}^{n} (\psi_1, \varphi_j)\varphi_j) \]

where $\alpha_1$ is 0 if $\psi_1$ is a linear combination of the $\varphi_i$'s and is chosen to normalize $\varphi_{n+1}$ otherwise. Continuing in this way, i.e., setting
\[ \varphi_{n+k+1} = \alpha_{k+1} (\psi_{k+1} - \sum_{j=1}^{n+k} (\psi_{k+1}^{j}, \varphi_j) \varphi_j), \]

deleting the \( \varphi_{n+k} \) with \( \alpha_k = 0 \) and then renumbering gives the desired sequence.

We will need the following lemma in constructing the example.

**Lemma**  On any finite interval \( I \) with end point \( a \) there exists a complete orthonormal set \( \varphi_0, \varphi_1, \ldots \) satisfying

1. \( \varphi_0 = 1 \)
2. \( \varphi_1 \) is continuous
3. \( \varphi_i(a) = 0, \text{ if } i > 0 \).

**Proof**  There exist continuous functions \( \psi_0 \) such that \( \varphi_0, \psi_1, \psi_2, \ldots \) is a complete orthonormal set. Let \( \eta \) be a continuous function with \( \eta(a) = 0 \) and \( (\varphi_0, \eta) = 1 \). Given any continuous function \( \xi \) and any \( \epsilon > 0 \) we can, by modifying \( \xi \) in a sufficiently small neighborhood of \( a \), construct a continuous function \( \xi' \) with \( \xi'(a) = 0 \) and \( \| \xi - \xi' \| < \epsilon \). Then \( \xi'' = \xi' - (\varphi_0, \xi') \eta \) is continuous and satisfies

1. \( \xi''(a) = 0 \)
2. \( (\xi'', \varphi_0) = 0 \)
3. \( \| \xi - \xi'' \| \leq \| \xi - \xi' \| + |(\varphi_0, \xi')| \| \eta \| \)

\[ \leq \epsilon (1 + \| \eta \| ) + (\varphi_0, \xi') \| \eta \|. \]

In particular taking \( \xi = \psi_1 \) so that the last term vanishes and choosing \( \epsilon = 2^{-k}/(1 + \| \eta \| ) \) we can construct a continuous function \( \psi_{i,k} \) with
(iv) $\psi_{i,k}(a) = 0$
(v) $(\psi_{i,k}, \phi_0) = 0$
(vi) $\|\psi_i - \psi_{i,k}\| \leq 2^{-k}.$

Now when we apply the Gram-Schmidt procedure to the sequence $\phi_0, \psi_1, \psi_2, \ldots$ none of the linear combinations after the first involve $\phi_0$ by (v) so they all vanish at $a$. The resulting orthonormal sequence approximates the $\psi_i$'s arbitrarily closely, hence is complete, and hence satisfies the requirements of the lemma.

We will now construct a complete orthonormal set $\psi_1, \ldots$ with $\psi_1$ discontinuous and all the $\psi_i$, $i > 1$ continuous. This if we take $\psi_2, \psi_3, \ldots$ for our orthonormal set of continuous functions it can only be completed by adjoining the discontinuous function $\psi_1$ or $-\psi_1$. We divide the interval $I$ into subintervals $I_1$ and $I_2$ at the point $a$ and choose complete orthonormal sets $\phi_0, \phi_1, \ldots$ in $I_1$ according to the previous lemma. Then the set $\psi_1 = \phi_0 - \phi_0^2, \psi_2 = \phi_0 + \phi_0^2, \phi_1, \phi_1^2, \phi_2, \phi_2^2, \ldots$ is complete and orthonormal and has $\psi_1$ as its only discontinuous member.

In the above example $(\phi_1)_{\perp}$, the orthogonal complement of the given set of $\phi$'s was finite dimensional. An example with infinite dimensional $(\phi_1)_{\perp}$ can be constructed by breaking $I$ into three intervals, say $I_1 = [0, a], I_2 = [a, b]$ and $I_3 = [b, 1]$. We will need the following modification of the above lemma. There exists a complete orthonormal set $\phi_1^2, \phi_2^2, \ldots$ on $[a, b]$ in which each $\phi_i$ is a continuous function with $\phi_i(a) = \phi_i(b) = 0$. 


Proof

The proof is similar to that of the preceding lemma except that

(iv) is changed to

(iv') \[ \psi_{i,k}(a) = \psi_{i,k}(b) = 0 \]

and (v) is dropped.

Now we take sets \((\varphi_1^1, \varphi_1^1, \ldots), (\varphi_1^2, \varphi_2^2, \ldots)\) and \((\varphi_2^3, \varphi_2^3, \ldots)\) such that \((\varphi_1^1)\) is a complete orthonormal set in \(I_j\), \(\varphi_1^1 = 1\) on \(I_1\), \(\varphi_2^3 = 1\) on \(I_3\), and \(\varphi_2^1(a) = \varphi_2^2(a) = \varphi_2^2(b) = \varphi_2^3(b) = 0\) for \(i \geq 1\). We take for our set of continuous orthonormal functions the union of the sets \((\varphi_1^1, \varphi_2^2, \ldots)\) and \((\varphi_1^2, \varphi_2^2, \ldots)\). The orthogonal compliment of this set is all functions of the form \(c\varphi_1^1 + f\) where \(f\) vanishes outside \(I_3\). No matter how the set is completed it will contain at least one function of the above form with \(c \neq 0\) and hence with a discontinuity at \(a\).
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