AD NUMBER
AD-486 295

NEW LIMITATION CHANGE

TO DISTRIBUTION STATEMENT - A
Approved for public release;
distribution is unlimited

LIMITATION CODE: 1

FROM No Prior DoD Distr Scy Cntrl St'mt Assigned

AUTHORITY
AFFDL, Oct 12, 1970.

19990303142

THIS PAGE IS UNCLASSIFIED
MATRIX ANALYSIS METHODS FOR ANISOTROPIC INELASTIC STRUCTURES

W. R. JENSEN
W. E. FALBY
N. PRINCE

GRUMMAN AIRCRAFT ENGINEERING CORPORATION

TECHNICAL REPORT AFFDL-TR-65-220

APRIL 1966

AIR FORCE FLIGHT DYNAMICS LABORATORY
RESEARCH AND TECHNOLOGY DIVISION
AIR FORCE SYSTEMS COMMAND
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

This document is subject to special export controls and each transmittal to foreign governments or foreign nationals may be made only with prior approval of AFFDL (FDTR), Wright-Patterson Air Force Base, Ohio 45433.
NOTICES

When Government drawings, specifications, or other data are used for any purpose other than in connection with a definitely related Government procurement operation, the United States Government thereby incurs no responsibility nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data, is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use, or sell any patented invention that may in any way be related thereto.

Copies of this report should not be returned to the Research and Technology Division unless return is required by security considerations, contractual obligations, or notice on a specific document.
MATRIX ANALYSIS METHODS FOR ANISOTROPIC INELASTIC STRUCTURES

W. R. JENSEN
W. E. FALBY
N. PRINCE

This document is subject to special export controls and each transmittal to foreign governments or foreign nationals may be made only with prior approval of AFFDL (FDTR), Wright-Patterson Air Force Base, Ohio 45433.
FOREWORD

This report, prepared by the Grumman Aircraft Engineering Corporation, Bethpage, New York, covers work performed under Air Force Contract AF 33(615)-2260. The contract was sponsored by the Air Force Flight Dynamics Laboratory of the Research and Technology Division, Air Force Systems Command, Wright-Patterson Air Force Base, Ohio. It was accomplished under Project No. 1467, "Structural Analysis Methods," Task 146701, "Stress-Strain Analysis Methods for Structures Exposed to Creep Environment." The work was administered by Mr. Laszlo Berke, PDDTH, Project Engineer. The report covers work conducted from January 1965 to October 1965. The manuscript was released by the authors in February 1966 for publication as an RTD Technical Report.

The investigation was supervised by Dr. Warner Lansing, Chief of Structural Mechanics. The computer programming was carried out by Mr. Albert Davidson. Some preliminary elastic analysis data were automatically generated by the ASTRAL, Automated STRuctural AnaLysis program developed by Mr. Philip Mason.

The work of Dr. T. J. Mentel on the evaluation of procedures for carrying out inelastic analyses, and on the formulation of the biaxial stress procedure is also acknowledged. His "Instability Analysis of the Constant Stress and Constant Strain Methods" is reproduced from Reference 6 in Appendix III.

Acknowledgements are made to Drs. Gabriel Isakson and Harry Armen of the Grumman Research Department for their suggestions and review of certain portions of this work.

All computations were carried out at the Grumman Computing Center.

This technical report has been reviewed and is approved.

FRANCIS J. JANIK, JR.
Chief, Theoretical Mechanics Branch
Structures Division
ABSTRACT

Most aerospace structural materials exhibit some degree of anisotropic strain hardening. During the past few years, several methods have appeared in the literature for introducing inelastic isotropic material behavior effects into existing matrix analysis procedures using the incremental theory of plasticity. A review is presented of these methods and a step-by-step routine known as the "constant strain" method is selected for the development of an anisotropic inelastic procedure.

A simple truss with one redundant is used to indicate the basic ideas of the approach. Then the procedure is generalized to the more important case of biaxially stressed structures. Nodal stresses are evaluated step-wise for increasing load through the use of an influence coefficient equation. The inelastic (plastic and creep) strains at one load level are used as initial strains at the subsequent level to account for nonlinear effects. The anisotropic behavior is considered by using a proposed extension of Hu's strain hardening theory.

Several analyses of an aluminum alloy (2024-T4) shear lag structure, which has been tested previously for the Air Force, are carried out, first assuming isotropic and then anisotropic material properties. The correlation between test results and those predicted by isotropic theory is reasonably good. The anisotropic analysis gives predicted results which are in slightly more consistent agreement with the test data.

The procedure is also modified to give an isotropic deformation theory solution, which produces numerical results in a much shorter computer time than required for the incremental theory solution. In the case of the shear lag structure investigated, the results by the two theories are in very close agreement.

Creep test results of an 1100-F aluminum shear lag structure are also available. An analysis of this structure by the proposed incremental method is carried out and its predictions too are in reasonably good agreement with the test data. The 1100-F material is very nearly isotropic and no testing of structures exhibiting anisotropic creep is known to have been performed. Hence the anisotropic creep capability of the proposed method cannot be checked out against tests at this time. A sample calculation is nevertheless carried out for a hypothetical material having this characteristic.

The approach presented, which is simple in concept and execution, is found to be a reasonably good phenomenological model of an exceedingly complex physical problem. The accompanying digital computer program is believed to be very versatile, and well suited for the inclusion of any other types of material nonlinearity that may be of interest.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>SECTION</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>I INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II INELASTIC MATRIX METHODS</td>
<td>3</td>
</tr>
<tr>
<td>A. Formulation</td>
<td>3</td>
</tr>
<tr>
<td>B. Example Problem</td>
<td>4</td>
</tr>
<tr>
<td>C. Step-by-Step Methods</td>
<td>6</td>
</tr>
<tr>
<td>D. Constant Stress Method</td>
<td>6</td>
</tr>
<tr>
<td>E. Constant Strain Method</td>
<td>7</td>
</tr>
<tr>
<td>III ISOTROPIC ELASTIC-PLASTIC ANALYSES</td>
<td>8</td>
</tr>
<tr>
<td>A. Biaxial Theory</td>
<td>8</td>
</tr>
<tr>
<td>B. Determination of Calculation Step Size</td>
<td>11</td>
</tr>
<tr>
<td>C. Description of Shear Lag Structure</td>
<td>11</td>
</tr>
<tr>
<td>D. Elastic Shear Lag Structure Analysis</td>
<td>12</td>
</tr>
<tr>
<td>E. Flow Theory Shear Lag Structure Analysis</td>
<td>12</td>
</tr>
<tr>
<td>F. Deformation Theory Analysis</td>
<td>13</td>
</tr>
<tr>
<td>IV ISOTROPIC ELASTIC-PLASTIC-CREEP ANALYSIS</td>
<td>15</td>
</tr>
<tr>
<td>A. Introduction to Creep Theory</td>
<td>15</td>
</tr>
<tr>
<td>B. Creep Theory Details</td>
<td>15</td>
</tr>
<tr>
<td>C. Description of Structure and Tests</td>
<td>16</td>
</tr>
<tr>
<td>D. Results of Creep Shear Lag Analysis</td>
<td>17</td>
</tr>
<tr>
<td>V ANISOTROPIC ELASTIC-PLASTIC ANALYSIS</td>
<td>19</td>
</tr>
<tr>
<td>A. Anisotropy in Structures</td>
<td>19</td>
</tr>
<tr>
<td>B. Hu's Strain Hardening Theory</td>
<td>19</td>
</tr>
<tr>
<td>C. Extension of Hu's Theory</td>
<td>22</td>
</tr>
<tr>
<td>D. Anisotropic Theory Details</td>
<td>23</td>
</tr>
<tr>
<td>E. Rotation of Axes of Anisotropy</td>
<td>24</td>
</tr>
<tr>
<td>F. Anisotropic Analysis of Shear Lag Structure</td>
<td>25</td>
</tr>
<tr>
<td>G. Discussion of Results</td>
<td>27</td>
</tr>
<tr>
<td>VI ANISOTROPIC ELASTIC-PLASTIC-CREEP ANALYSIS</td>
<td>29</td>
</tr>
<tr>
<td>A. Anisotropic Creep Theory</td>
<td>29</td>
</tr>
<tr>
<td>B. Sample Problem</td>
<td>29</td>
</tr>
<tr>
<td>VII CONCLUSIONS</td>
<td>31</td>
</tr>
<tr>
<td>VIII FIGURES</td>
<td>33</td>
</tr>
</tbody>
</table>
TABLE OF CONTENTS (continued)

APPENDIX

<table>
<thead>
<tr>
<th>PAGE</th>
<th>I</th>
<th>ELASTIC ANALYSIS OF SHEAR LAG STRUCTURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>61</td>
<td>A.</td>
<td>Idea1ization of Shear Lag Structure</td>
</tr>
<tr>
<td></td>
<td>B.</td>
<td>Force Method</td>
</tr>
<tr>
<td></td>
<td>C.</td>
<td>Stiffness Method</td>
</tr>
<tr>
<td></td>
<td>D.</td>
<td>Comparison of Elastic Results and</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Perspective</td>
</tr>
<tr>
<td>63</td>
<td>II</td>
<td>POISSON'S RATIO EFFECT</td>
</tr>
<tr>
<td>65</td>
<td>III</td>
<td>INSTABILITY ANALYSIS OF THE CONSTANT STRESS AND CONSTANT STRAIN METHODS</td>
</tr>
<tr>
<td>72</td>
<td>IV</td>
<td>ROTATION OF AXES OF ANISOTROPY</td>
</tr>
<tr>
<td>75</td>
<td>V</td>
<td>INELASTIC MATRIX COMPUTER PROGRAM</td>
</tr>
<tr>
<td></td>
<td>A.</td>
<td>Program Description</td>
</tr>
<tr>
<td></td>
<td>B.</td>
<td>Symbols and Format of Data Cards</td>
</tr>
<tr>
<td></td>
<td>C.</td>
<td>Anisotropic Parameter Matrices</td>
</tr>
<tr>
<td></td>
<td>D.</td>
<td>Restart Procedure</td>
</tr>
<tr>
<td></td>
<td>E.</td>
<td>Time Estimates</td>
</tr>
<tr>
<td></td>
<td>F.</td>
<td>Sample Data Form</td>
</tr>
<tr>
<td></td>
<td>G.</td>
<td>Flow Charts</td>
</tr>
<tr>
<td></td>
<td>H.</td>
<td>Fortran Program</td>
</tr>
<tr>
<td>115</td>
<td>VI</td>
<td>STRESS DISTRIBUTIONS DUE TO UNIT INITIAL STRESSES</td>
</tr>
<tr>
<td></td>
<td>A.</td>
<td>Force Method</td>
</tr>
<tr>
<td></td>
<td>B.</td>
<td>Stiffness Method</td>
</tr>
<tr>
<td>123</td>
<td>REFERENCES</td>
<td></td>
</tr>
</tbody>
</table>

vi
# List of Illustrations

<table>
<thead>
<tr>
<th>Figure</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Truss Structure</td>
<td>34</td>
</tr>
<tr>
<td>2</td>
<td>Constant Stress Method</td>
<td>34</td>
</tr>
<tr>
<td>3</td>
<td>Constant Strain Method</td>
<td>34</td>
</tr>
<tr>
<td>4</td>
<td>Results of Constant Stress Method for Bar No. 3 of Truss</td>
<td>35</td>
</tr>
<tr>
<td>5</td>
<td>Results of Constant Strain Method for Bar No. 3 of Truss</td>
<td>35</td>
</tr>
<tr>
<td>6</td>
<td>Shear-Lag Specimen Designated SLS1 - From Air Force Report No. RTD-TDR-63-4032; 2024-T4 Aluminum Alloy (1100-F Specimen Similar)</td>
<td>36</td>
</tr>
<tr>
<td>7</td>
<td>Stiffness Method Idealization</td>
<td>36</td>
</tr>
<tr>
<td>8</td>
<td>Force Method Idealization</td>
<td>36</td>
</tr>
<tr>
<td>9</td>
<td>Typical Elements of Force Idealization</td>
<td>37</td>
</tr>
<tr>
<td>10</td>
<td>Typical Elements of Stiffness Idealization</td>
<td>37</td>
</tr>
<tr>
<td>11</td>
<td>2024-T4 Aluminum Alloy Stress-Strain Data and Curve RO2M</td>
<td>38</td>
</tr>
<tr>
<td>12</td>
<td>Comparison of Predicted Effective Stress-Strain Relationship with RO2M at (0, 0) for Various Load Increments</td>
<td>38</td>
</tr>
<tr>
<td>13</td>
<td>Instrumentation of 2024-T4 Aluminum Alloy Specimen</td>
<td>39</td>
</tr>
<tr>
<td>14</td>
<td>Elastic Strains Along x-Axis for P = 1 lb</td>
<td>40</td>
</tr>
<tr>
<td>15</td>
<td>Elastic Strains Along x = 0.8125 in. for P = 1 lb</td>
<td>40</td>
</tr>
<tr>
<td>16</td>
<td>Strains Along x-Axis for P = 11,600 lb and ΔP = 5 lb</td>
<td>41</td>
</tr>
<tr>
<td>17</td>
<td>Strains Along x = 0.8125 in. for P = 11,600 lb and ΔP = 5 lb</td>
<td>41</td>
</tr>
<tr>
<td>18</td>
<td>Strains Along x-Axis for P = 14,600 lb and ΔP = 5 lb</td>
<td>42</td>
</tr>
<tr>
<td>19</td>
<td>Strains Along x = 0.8125 in. for P = 14,600 lb and ΔP = 5 lb</td>
<td>42</td>
</tr>
<tr>
<td>20</td>
<td>Strains Along x-Axis for P = 16,760 lb, and ΔP = 5 lb</td>
<td>43</td>
</tr>
<tr>
<td>21</td>
<td>Strains Along x = 0.8125 in. for P = 16,760 lb and ΔP = 5 lb</td>
<td>43</td>
</tr>
</tbody>
</table>
LIST OF ILLUSTRATIONS (continued)

<table>
<thead>
<tr>
<th>FIGURE</th>
<th>TITLE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>22</td>
<td>Elastic-Plastic Strains at ((0, 0)) for Several Assumed Stress-Strain Curves</td>
<td>44</td>
</tr>
<tr>
<td>23</td>
<td>The Strain Hardening Rule</td>
<td>45</td>
</tr>
<tr>
<td>24</td>
<td>1100-F Aluminum Time-Dependent Behavior and Fitted Curves (Reference 7)</td>
<td>46</td>
</tr>
<tr>
<td>25</td>
<td>1100-F Aluminum Stress-Strain Data at Room Temperature and at 206°C, for Time (t = 0.00)</td>
<td>47</td>
</tr>
<tr>
<td>26</td>
<td>Location of Strain Gages on 1100-F Specimen (Reference 7)</td>
<td>47</td>
</tr>
<tr>
<td>27</td>
<td>Strains Along (x)-Axis for (P = 1600) lb and (t = 0.06) hr</td>
<td>48</td>
</tr>
<tr>
<td>28</td>
<td>Strains Along (x = 1) in. for (P = 1600) lb and (t = 0.06) hr</td>
<td>48</td>
</tr>
<tr>
<td>29</td>
<td>Strains Along (x)-Axis for (P = 2020) lb and (t = 1.10) hr</td>
<td>49</td>
</tr>
<tr>
<td>30</td>
<td>Strains Along (x = 1) in. for (P = 2020) lb and (t = 1.10) hr</td>
<td>49</td>
</tr>
<tr>
<td>31</td>
<td>Strains Along (x)-Axis for (P = 2020) lb and (t = 3.0) hr</td>
<td>50</td>
</tr>
<tr>
<td>32</td>
<td>Strains Along (x = 1) in. for (P = 2020) lb and (t = 3.0) hr</td>
<td>50</td>
</tr>
<tr>
<td>33</td>
<td>Effective Stress vs Strain at Center Node</td>
<td>51</td>
</tr>
<tr>
<td>34</td>
<td>Total (Elastic, Plastic and Creep) Strains at Center of Specimen (Gage 1) for (P = 1600) lb to Time 1 hr, then (P = 2020) lb to Time 3 hr</td>
<td>52</td>
</tr>
<tr>
<td>35</td>
<td>Total (Elastic, Plastic and Creep) Strains at ((1.0) in., 3.0 in.) (Gage 8) for (P = 1600) lb to Time 1 hr, then (P = 2020) lb to Time 3 hr</td>
<td>53</td>
</tr>
<tr>
<td>36</td>
<td>Typical Uniaxial Stress vs Plastic Work, (w), Plot</td>
<td>54</td>
</tr>
<tr>
<td>37</td>
<td>Assumed Effective Stress ((y-y)) Curve, Calculated (x-x) Curves and Test Data</td>
<td>55</td>
</tr>
<tr>
<td>38</td>
<td>Node Strains at ((0, 0)) - Isotropic and Anisotropic Analysis</td>
<td>56</td>
</tr>
</tbody>
</table>
LIST OF ILLUSTRATIONS (continued)

<table>
<thead>
<tr>
<th>FIGURE</th>
<th>TITLE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>39</td>
<td>Strains Along x-Axis for P = 16,760 lb - Isotropic and Anisotropic Analyses</td>
<td>57</td>
</tr>
<tr>
<td>40</td>
<td>Strains Along x = 0.8125 in. for P = 16,760 lb - Isotropic and Anisotropic Analyses</td>
<td>57</td>
</tr>
<tr>
<td>41</td>
<td>Strains Along x-Axis for P = 2020 lb and t = 3.0 hr - Isotropic and Anisotropic Analyses</td>
<td>58</td>
</tr>
<tr>
<td>42</td>
<td>Strains Along x = 1 in. for P = 2020 lb and t = 3.0 hr - Isotropic and Anisotropic Analyses</td>
<td>58</td>
</tr>
<tr>
<td>43</td>
<td>Elastic Stress Distribution Along x-Axis</td>
<td>59</td>
</tr>
<tr>
<td>44</td>
<td>Elastic Stress Distribution Along x = 0.8125 in.</td>
<td>60</td>
</tr>
</tbody>
</table>
SYMBOLS

[ ] Rectangular matrix

( ) Column matrix

$q_i$ Member load, any load acting directly on a member

$\sigma_u$ $u^{th}$ ordinary stress component (normal or shear) at a node point

$\sigma_{3N-2}, \sigma_{3N-1}, \sigma_{3N}$ The normal stress components and shear stress component at node $N$ - another designation for $\sigma_u$

$\sigma_N$ Effective stress at node $N$

$\sigma_N^*$ Relaxed effective stress at node $N$

$\sigma_0$ Reference stress in Ramberg-Osgood Equation

$\Gamma_{um}$ $u^{th}$ stress component in linear redundant structure due to the $m^{th}$ unit applied load

$\Gamma_{uv}$ $u^{th}$ stress component in linear redundant structure due to the $v^{th}$ unit initial strain

$\Gamma_{im}$ $i^{th}$ member load in the redundant structure due to $m^{th}$ applied load

$\Gamma_{ij}$ $i^{th}$ member load in the redundant structure caused by a unit initial strain at the $j^{th}$ member load

$P_m$ $m^{th}$ applied load

$\varepsilon_v$ $v^{th}$ component of initial strain

$\varepsilon(p)_v$ $v^{th}$ component of plastic strain

$\varepsilon(c)_v$ $v^{th}$ component of creep strain

$\tilde{\varepsilon}(p)_N$ Effective plastic strain at node $N$

$\tilde{\varepsilon}(c)_N$ Effective creep strain at node $N$

$\varepsilon$ Total (elastic, plastic, and creep) strain component

$\varepsilon_{xy}$ Engineering shear strain

$\xi$ Error

$E$ Young's modulus
Temperature

t*  Equivalent time at start of creep cycle calculation

\( \Delta t \)  Cycle duration (elapsed time)

\( \alpha, \beta, \gamma \)  Material constants in creep strain equation

\( a_{ij} \)  Subscripted anisotropic parameters

\( \Delta \)  Increment (prefix)

\( n, \theta \)  Nonlinear parameters in Ramberg-Osgood equation

k  Cycle designation, superscript

N  Nodal index

u, v  Nodal stress component or strain component index, related to the nodal index as indicated in equation

\( Y_{ij} \)  Simple directional characteristic stress

\( \sigma_{xx}, \sigma_{yy} \)  Uniaxial stress in the x-x, y-y direction

\( \varepsilon_{xx}, \varepsilon_{yy} \)  Plastic strain in the x-x, y-y direction due to uniaxial x-x, y-y stress

\( \sigma_{xy} \)  Shear stress in x-y plane

\( \varepsilon_{xy} \)  Shear strain in x-y plane
SECTION I
INTRODUCTION

With the introduction of high-speed, large-capacity, digital computers, a number of investigators (References 1, 4, 7) have adapted essentially linear matrix analysis methods to the solution of redundant structures where nonlinear material plasticity and creep properties are considered. These methods have been either iterative or non-iterative step-by-step numerical procedures.

The present work is an effort to explore, revise, and extend the matrix method of analysis in order to apply it to a range of practical aerospace structural problems exhibiting inelastic isotropic or inelastic anisotropic material behavior. Thus, the intent is to concentrate upon methods which are able to predict inelastic strain distributions in irregular idealized structures in a biaxial stress state where materials exhibit strain hardening and creep properties representative of those employed in aerospace construction.

A discussion of complete load reversal, although desirable for plastic fatigue studies, is not included because the theoretical foundation for such a procedure is apparently not yet fully developed. The present formulation of elastic, plastic and creep loading, followed by elastic unloading, while restrictive, is nevertheless of practical interest.

Of the proposed analytical methods, one by Denke (Reference 1) has developed naturally from the matrix force method of analysis and consists of including nonlinear plastic and creep terms in the equations for the gaps at those cuts which are required to make the structure statically determinate. The redundant forces, required to close the gaps and make the structure continuous, are obtained by solving these nonlinear equations by a Newton-Raphson procedure.

A second method reported by Kobayashi and Weikel (Reference 2) has been developed from the direct stiffness method. Here, forces occurring as a consequence of the inelastic effects are included in the nodal force equations. The nodal displacements (or displacement rates, where creep is considered) obtained by solving these equations impose equilibrium at the nodes under the action of internal loads, surface tractions, and prescribed displacements.

When the inelastic effects are accounted for by a flow (or incremental) theory, the deformation is an accumulation of increments each governed by the prevailing stress. Thus, when either of the methods just described is used, one must obtain (for the governing equations) a series of solutions, with one solution corresponding to each load increment. This requires a considerable amount of computer time even for medium-sized analyses.
The approach recommended here, in addition to being conceptually simple, does not require repeated matrix inversions. It was developed from that proposed by R. Gallagher, J. Padlog and others at Bell Aero-system (References 3,4). Input data, as in the case with these other nonlinear analyses, are generated by an elastic analysis; however, for this approach, either the matrix force method or direct stiffness method can be used. The problem, formulated in terms of standard influence coefficients for applied load and initial strain, is reduced from a nonlinear to a linear one by using those strains obtained at the previous load level to approximate the current inelastic strains.

Development of the anisotropic analysis is based on an extension of the proposed anisotropic theory of Hu, Reference 11. The constant-anisotropic-coefficient assumption of Hu is replaced by one in which the coefficients are allowed to vary with the level of stress. The formulation is then a simple modification of the isotropic procedure. It is also shown that anisotropic creep can be included in a manner similar to the isotropic creep.
A. Formulation

An inelastic structural analysis can be carried out in two steps. The first is the standard elastic solution where internal stresses and accompanying strains are related through Hooke's law. The second step, the modification of this elastic system to include inelastic strains, is analogous to a procedure for including superimposed thermal strains. The inelastic strains are defined as the differences between the total stresses and the elastic strains and are generally functions of the final stresses, not those of the linear, elastic state.

The simple, pin-jointed, single-redundant truss, pictured in Figure (1), illustrates the basic notions more clearly. All bars are considered to be elastic, except the vertical diagonal which can become plastic and has a stress-strain relation represented by the curve shown in Figure (2). The applied load $P$ is large enough to cause member 3 to become plastic.

A solution might be obtained by first simply ignoring plastic strain in member 3 and assuming all members elastic. The resulting stress would then be of magnitude $\sigma$. The actual stress for member 3, of lower magnitude due to plastic yielding, is designated $\sigma^{(k)}$ and the associated inelastic strain $\epsilon^{(k)}$ in Figure (2). These stresses and strains are related in the following equation.

$$\sigma^{(k)} = \sigma + \Gamma \epsilon^{(k)}$$

where:

- $\sigma^{(k)}$ is the actual final stress
- $\sigma$ is the elastic stress
- $\epsilon^{(k)}$ the inelastic strain
- $\Gamma$ the redundant elastic stress for a unit value of initial strain

To be more specific $\Gamma$ is equal to the redundant stress in member 3 corresponding to a unit initial strain in member 3. Note that $\Gamma$ must be negative to cause a reduction of stress in the diagonal member.

An important feature to be observed about Equation 1 is that, since the inelastic strain $\epsilon^{(k)}$ is a nonlinear function of the final stress $\sigma^{(k)}$, this equation is really a nonlinear relation to be solved.
for $\sigma^{(k)}$. This characteristic will always be present in the analyses to be discussed in this report.

Equation 1 can be generalized to provide stresses in all members of the structure and provide for inelastic strains in all members for a variety of applied loads. This basic influence coefficient equation* is as follows:

$$ [\sigma_u] = [\Gamma_{um}][P_m] + [\Gamma_{uv}][\varepsilon_v] \quad (2) $$

The $\sigma_u$'s are the ordinary stress components at the various node points of the structure. An element of $[\Gamma_{um}]$ gives the $u^{th}$ stress in the linear redundant structure due to a unit $m^{th}$ applied load, and $[P_m]$ represents the actual applied loads. Also, an element of $[\Gamma_{uv}]$ gives the $u^{th}$ stress component in the linear redundant structure due to a unit initial strain at the $v^{th}$ stress location in the unloaded, statically determinate structure. Finally, an element of $[\varepsilon_v]$ represents the actual initial strain at the $v^{th}$ stress location.

The problem is now reduced in essence to the determination of the inelastic strains to use as the initial strains $[\varepsilon_v]$ in Equation 2. For a structure in a load-temperature-time environment, this task can be rather formidable, because $[\varepsilon_v]$ is a function of local temperature and time, as well as the local stress history.

B. Example Problem

Continuing our study of the simple truss example of Figure (1), we now allow all members to go plastic. The exact results for the deformation and stresses in the truss, for nonlinear properties, are easily obtained by direct numerical solution of the equations, and hence will be used without development.

The step-by-step finite element method for determining the stresses $[\sigma_u]$ and strains $[\varepsilon_v]$ involves the use of Equation 2 and a nonlinear stress-strain relation. This relation will be assumed to be a piecewise linear approximation of the Ramberg-Osgood stress-strain relation.

$$ \varepsilon' = \frac{\sigma}{E} + \varepsilon $$

*The derivation of matrices $[\Gamma_{um}]$ and $[\Gamma_{uv}]$ is given in Appendix VI.
where \( e \) denotes the inelastic (or plastic) strains and is given by

\[
e = \frac{3}{2} \left( \frac{\sigma}{\sigma_0} \right)^{\theta - 1}
\]

and where

- \( \varepsilon \) = total strain
- \( \sigma \) = member stress
- \( E \) = Young's modulus
- \( \sigma_0 \) = reference stress (stress at secant modulus of 0.75)
- \( \theta \) = nonlinear parameter

The first step, in applying the finite element method, is to obtain the influence coefficient matrices \([\Gamma_{um}]\) and \([\Gamma_{uv}]\) for the linear, redundant structure. This requires specification of the geometry of the structure and the (linear) material properties of the individual structural elements. The geometry of the example truss problem is given in Figure (1). The material is assumed to be an aluminum alloy with the following constants.

\[
E = 10^7 \text{ psi}, \quad \sigma_0 = 10^5 \text{ psi}, \quad \theta = 10
\]

The approximate stress-strain curve used matches the Ramberg-Osgood values at 2000 psi stress intervals.

In this case, we find

\[
[\Gamma_{um}] = \begin{bmatrix}
-0.207 \\
-0.293 \\
0.707
\end{bmatrix}, \quad [\Gamma_{uv}] = 10^7 \times \begin{bmatrix}
-0.414 & 0.207 & 0.207 \\
0.586 & 0.293 & 0.293 \\
0.586 & 0.293 & 0.293
\end{bmatrix}
\]

for the case of the single applied load \( P \).
C. Step-by-Step Methods

References 3 and 4 present what appears to be the simplest possible approach to this problem from a computational standpoint. A non-iterative step-by-step calculation is performed in which all quantities including the initial strains $\varepsilon_v$ are incremented and then assumed to remain constant in the ensuing load interval. The inherent difficulty in this approach is to establish the connection between successive steps. Two methods, both of which involve the initial strains from the prior step to predict quantities in the current step are suggested. It is anticipated that by controlling the size of the interval one may achieve any degree of accuracy.

The development herein is discussed only in the detail necessary to analyze the redundant truss of Figure (1) being loaded for the first time into the plastic range. Generalization to biaxial plasticity and creep phenomena are discussed in the succeeding sections.

The step-by-step procedure for solving the problem is introduced by rewriting Equation 2 in the form

$$\{a_u(k)\} = [\Gamma_{um}]\{p_m(k)\} + [\Gamma_{uv}]\{\varepsilon_v(k-1)\}$$

where

$k$ is the cycle designation

This can be regarded as the fundamental equation for the non-iterative, step-by-step methods. The idea in formulating this equation, as indicated by the cycle designating superscript is that the initial strains of the previous cycle can be used to approximate the initial strains of the current cycle. The strains of the previous cycle may be incorporated in several ways, two of which constitute the constant stress and the constant strain methods of analysis.

D. Constant Stress Method

As indicated previously, in the step-by-step procedure considered here, one enters the $k$th cycle with applied loads $\{p_m(k)\}$ and initial strains $\{\varepsilon_v(k-1)\}$, the latter evaluated during the preceding cycle.

*These methods make use of devices previously used by others to solve inelastic problems; for example, S. S. Manson at the Lewis Research Laboratory, NASA, Cleveland, Ohio, has previously carried out inelastic analyses of turbine discs involving somewhat similar techniques (Reference 5).
The first operation of the current cycle is to determine \( \{u(k)\} \) from Equation 3 by direct substitution. The second operation is a determination of \( \{v(k)\} \) for use in the next cycle. The constant stress method does this in the most obvious way, by reading from the given stress-strain curve the plastic strains \( \{v(k)\} \) corresponding to the \( u(k)'s \) (the reason for the name "constant stress" is thus apparent). The operation is indicated schematically in Figure (2).

The results of the application of this method to the example truss problem are shown in Figure (4), where the stress in the vertical member (Bar #3) has been plotted versus load. These results display a striking defect of the method due to the development of a sudden and catastrophic divergence, whose onset depends upon step size. This dependence is such that any attempt to improve accuracy by reducing step size only hastens the occurrence of divergence. An explanation of this behavior is given in Appendix III. Because of this defect, the constant stress method in this form must be eliminated from consideration as an acceptable method for general use.

E. Constant Strain Method

The first operation of the constant strain method is exactly the same as the first operation of the constant stress method; \( u(k) \) is evaluated by direct substitution in Equation 3. Thereafter, one determines \( v(k) \) for use in the next cycle as follows. Referring to Figure (3), for each member, point A is determined with stress-strain coordinates \( u(k) \) and \( v(k)/E + v(k-1) \). A relaxed stress \( u(k) \) is now calculated with the same total strain, corresponding to point B on the given stress-strain curve. Note that here the total strain, rather than stress, remains unchanged—hence the name "constant strain" method. The required initial strain \( v(k) \) is the inelastic strain \( s(k) \) corresponding to the relaxed stress, as indicated on Figure (3).

The results of applying the constant strain method to the truss problem, for the three step sizes 5000, 500 and 50 lb., are shown in Figure (5). The accuracy, for a given step size, is not as good as that of the constant stress method, but the analysis is now free of any instability. The constant strain method is therefore selected for further use herein. The discussion of the step size and of a method of monitoring it is left for a later section.
SECTION III
ISOTROPIC ELASTIC-PLASTIC ANALYSIS

A. Biaxial Theory

Having presented the simple truss example of the step-by-step procedure, we proceed now to the case of more practical interest -- a biaxially stressed structure. The new procedure is identical to the one already discussed for the simple truss with one exception. Because of the biaxial stress we can no longer work directly from the stress-strain curve to obtain the plastic strains for use in Equation 3; instead, we must employ the well-known concept of an "effective" stress-strain relationship in conjunction with a von Mises type yield condition and the associated incremental flow relations.

The biaxial theory is described by a summary of the steps to be used as a guide for a detailed description which follows. The constant strain method used here is a step-by-step procedure which, after incrementing the applied load, can be applied in four parts:

1. Obtain the stress components at each node using the basic Equation 3 by assuming the initial strains from the previous load level.

2. Using these stresses, calculate an effective stress at each node.

3. Assume that the effective stress-strain relation for the material, modified by including the elastic strain, corresponds to data measured in a simple uniaxial tension test. Using this, calculate the effective strain corresponding to the effective stress.

4. Using the incremental flow relations, determine the inelastic strain increments. The proportionality constant in these equations is the ratio of the effective strain increment to the effective stress.

At this point in the calculation, the applied load can be incremented again and the cycle repeated.

When calculating the ordinary stresses \( \sigma_{uj}^{(k)} \) for the \( k \text{th} \) load level using Equation 3 (Step 1), it is convenient to re-identify these stresses by means of a new subscript \( N \), as follows:
The stresses are thus arranged in groups of three components (two normal and one shear) at each node point \( N \).

We now calculate the corresponding effective stresses \( \sigma_N^{(k)} \) for each of the nodes from the von Mises type formula (Step 2)

\[
\begin{pmatrix}
\sigma^{(k)}_1 \\
\sigma^{(k)}_2 \\
\vdots \\
\sigma^{(k)}_{3N-2} \\
\sigma^{(k)}_{3N-1} \\
\sigma^{(k)}_{3N}
\end{pmatrix} = \begin{pmatrix}
\sigma_{3N-2} \\
\sigma_{3N-1} \\
\vdots \\
\sigma_{3N}
\end{pmatrix}
\]

The stresses are thus arranged in groups of three components (two normal and one shear) at each node point \( N \).

We now calculate the corresponding effective stresses \( \sigma_N^{(k)} \) for each of the nodes from the von Mises type formula (Step 2)

\[
\sigma_N^{(k)} = \left[ \left( \sigma_{3N-2}^{(k)} \right)^2 - \left( \frac{\sigma_{3N-2}^{(k)}}{\sigma_{3N-1}^{(k)}} \right) + \left( \frac{\sigma_{3N-1}^{(k)}}{\sigma_{3N}^{(k)}} \right)^2 \right]^{1/2}
\]

Note that by this definition \( \sigma_N^{(k)} \) must be positive and is proportional to the octahedral shear stress. This formula together with the stress-strain data constitutes the strain hardening criterion.

We now go to the tensile stress-strain curve (which is also the stress vs strain curve) for the material of interest and, using the constant strain method, read from it the corresponding effective plastic strain \( \tilde{\varepsilon}_N^{(p)} \). This operation (Step 3) is identical to that previously described for the uniaxial case on page 7.

In accordance with the flow theory of plasticity, the increment in the effective strain \( \Delta\tilde{\varepsilon}_N^{(p)} \) over that of the preceding interval must be calculated (Step 4). The increment will be either positive or zero, depending upon whether plastic loading or elastic unloading (or reloading) is taking place. Thus,

\[
\Delta\tilde{\varepsilon}_N^{(p)} = \tilde{\varepsilon}_N^{(p)} - \tilde{\varepsilon}_N^{(p-1)}
\]

when \( \tilde{\varepsilon}_N^{(p)} \) is greater than any previous \( \tilde{\varepsilon}_N \) (inelastic strain increasing)
when $\bar{\alpha}_N^{(k)}$ is smaller than a previous $\bar{\alpha}_N$ (elastic unloading or re-loading, inelastic strain constant)

The increments in the ordinary plastic strain components may now be obtained using a Prandtl-Reuss incremental relationship.

$$\Delta \varepsilon(p)^{(k)}_{N} = 0$$  \hspace{1cm} (6b)

$$\Delta \varepsilon(p)^{(k)}_{3N-2} = \frac{\Delta \varepsilon(p)^{(k)}_{N}}{\bar{\alpha}_N^{(k)}} \left[ \sigma^{(k)}_{3N-2} - \frac{1}{2} \sigma^{(k)}_{3N-1} \right]$$

$$\Delta \varepsilon(p)^{(k)}_{3N-1} = \frac{\Delta \varepsilon(p)^{(k)}_{N}}{\bar{\alpha}_N^{(k)}} \left[ \sigma^{(k)}_{3N-1} - \frac{1}{2} \sigma^{(k)}_{3N-2} \right]$$  \hspace{1cm} (7)

$$\Delta \varepsilon(p)^{(k)}_{3N} = \frac{\Delta \varepsilon(p)^{(k)}_{N}}{\bar{\alpha}_N^{(k)}} \left[ 3\sigma^{(k)}_{3N} \right]$$

The total, ordinary plastic strain components are obtained by addition,

$$\begin{bmatrix}
\vdots \\
\varepsilon(p)^{(k)}_{3N-2} \\
\varepsilon(p)^{(k)}_{3N-1} \\
\varepsilon(p)^{(k)}_{3N} \\
\vdots
\end{bmatrix} =
\begin{bmatrix}
\vdots \\
\varepsilon(p)^{(k-1)}_{3N-2} \\
\varepsilon(p)^{(k-1)}_{3N-1} \\
\varepsilon(p)^{(k-1)}_{3N} \\
\vdots
\end{bmatrix} +
\begin{bmatrix}
\vdots \\
\Delta \varepsilon(p)^{(k)}_{3N-2} \\
\Delta \varepsilon(p)^{(k)}_{3N-1} \\
\Delta \varepsilon(p)^{(k)}_{3N} \\
\vdots
\end{bmatrix}$$  \hspace{1cm} (8)

These components together with the new applied loads $p_{m}^{(k+1)}$ may be substituted in Equation 3 to obtain $\sigma_{u}^{(k+1)}$ in the next load cycle.
B. Determination of Calculation Step Size

It should be noted that, according to the constant strain method, every predicted value of effective strain \( \varepsilon(p)^{(k)}_N \) together with its accompanying value of effective stress \( \sigma^{(k)}_N \) constitutes an approximation to a point on the actual effective stress-strain curve. The excellence of the approximation is directly related to the loading increment, as is shown in the truss results Figure (5). Thus it is only necessary to monitor this agreement for one or more of the critically loaded nodes to determine whether the step size is satisfactory. This is illustrated below in connection with the shear lag structure investigation.

C. Description of Shear Lag Structure

Several very useful tests have been performed for the Air Force upon shear lag structures (Reference 7). The structure, loaded as shown in Figure (6), is an integrally machined part of 2024-T4 aluminum alloy stiffened along the loading (y) axis. The stiffener is tapered in thickness from each end towards the center of the structure.

This structure was chosen originally because it is simple to work with and well adapted to analysis by both matrix methods when appropriate idealizations are employed. When tension forces are applied to the ends of the stiffener, high stress gradients are induced in a manner analogous to those encountered in aircraft structures.

The material properties essential to this analysis were obtained from tension tests reported in Reference 7. These tests were performed on coupons, machined from the parent plate, in the longitudinal or x-direction and the transverse or y-direction of Figure (6). The data resulting from these tests, Figure (11), indicate the presence of a considerable degree of anisotropy. In the present study, three piecewise linear representations of stress-strain curves were fitted to these points; two, RO1 and RO2, in Figure (11), are equivalent to Ramberg-Osgood curves used in Reference 7; the third, RO3M, is a Grumman modification. The modulus of elasticity of all the curves is taken as 10.3 x 10^6 psi. Note also that the maximum strains recorded are of the order of 0.010 in/in, whereas the maximum strains reached in the shear lag tests are around 0.020 in/in. Thus there is some doubt as to whether the idealized curve correctly represents the test material in this high strain region.

The locations of the strain gages for the test of the stiffened plate are shown in Figure (13). The plate was loaded by applying tension to the stiffener in steps of 1000 pounds to 6000 pounds, gage readings being taken at each step. It was then unloaded in steps of 1500 pounds to zero, and finally progressively loaded to failure. Buckling occurred at a load of 23,000 pounds and fracture at 25,800 pounds. Data from this test are plotted on Figures (14) through (22).
D. Elastic Shear Lag Structure Analysis

The idealizations of the upper right quadrant of the shear lag structure for a direct stiffness and a force method analysis are shown in Figures (7) and (8) respectively. A typical element of the force method analysis, Figure (9), consists of conventional bars and rectangular shear panels. Many previous idealizations have omitted the Poisson's Ratio effect. The present idealization, however, incorporates this effect in the manner described in Appendix II.

The basic element of the stiffness approach consists of a cluster of four "Turner triangles" (References 8,9) to form a rectangle as shown in Figure (10). The manner of obtaining the stresses is discussed in Appendix I.

An elastic analysis under a unit applied load was performed by both the force and the stiffness methods. These results are compared in Appendix I.

The inelastic analysis can be made using either of the two approaches (stiffness or force method). For the present investigation, only the force method is used.

E. Flow Theory Shear Lag Structure Analysis

The inelastic analysis was carried out using the Fortran 2 program listed in Appendix V. This program is capable of carrying out isotropic or anisotropic, plastic or creep flow theory analyses. The flow charts and instructions for preparation and submission of data are also included in Appendix V.

Before comparing the analytical and test results, let us look at a plot of the tensile stress-strain curve, R02N, and compare it with the predicted effective stress-strain relationship for various step sizes. Such a comparison is found in Figure (12) for the node corresponding to the center of the specimen which is the point of highest strain in the structure. It can be seen there that for a step size of $\Delta P = 500 \text{ lb.}$, the agreement is rather poor. For $\Delta P = 50 \text{ lb.}$ the agreement is much better, while for $\Delta P = 5 \text{ lb.}$, the predicted value lies directly on the stress-strain curve. The IBM 7094 computer time for this best result and a maximum load of $P = 16,760 \text{ lb.}$ is approximately 20 minutes.

The predicted strain distributions are shown in Figures (14) through (21), together with the corresponding test values, along the two strain gage lines. In these plots, the calculated results are linearly interpolated values between node points. Figures (14) and (15) give elastic results; the agreement with test data is seen to be rather good, giving the necessary confidence in the accuracy of the basic influence coefficient matrices. It is observed also that the specimen achieves its basic purpose of displaying a pronounced shear lag effect with the highest strain occurring at the central node, as expected.
As the applied load increases through 7070 lbs., the strains at the central node become plastic. During the tests, the strain gages continued to function through an applied load level of 14,600 lb.; beyond this point the x-gage failed to record. The y-gage failed also above a load level of 16,760 lb., which, consequently, is the highest level considered in the comparison of test and analysis even though the plate did not buckle until \( P = 23,000 \) lb.

The proportional limit for the test specimen material occurs at a strain of approximately .004 in/in, as shown in Figure (11). Bearing this in mind during an examination of Figures (16) through (21), it is seen that plastic behavior is primarily confined to a fairly small region around the central node, and we thus have a case of contained plasticity. The analysis predicts a very pronounced strain redistribution extending somewhat beyond this region. This is indicated by a comparison of the plastic results with extrapolated elastic results shown as dotted curves on Figures (20) and (21).

As for agreement with test values, the analysis substantially underpredicts the strain gage readings where plasticity is most pronounced. Since the elastic results agree so much better, one must assume that the difficulty lies somewhere in the plasticity part of the correlation. It was mentioned previously that the idealization of the stress-strain curve \( RO2M \) of Figure (11) was open to question at the high strain end because of the absence of test points. Accordingly, an additional run was made for a revised curve extending horizontally beyond the last indicated test point. The plastic strains at the critical central node are increased approximately 10% by so doing. This represents an appreciable closing of the gap, but the gap nevertheless remains.

Calculations were also made based upon the previously mentioned stress-strain curves \( RO1 \) and \( RO2 \) of Figure (11). The results for the central node are shown on Figure (22). As might be expected, they depart appreciably from the \( RO2M \) predictions.

F. Deformation Theory Analysis

Solutions by a deformation theory have traditionally been considered to be more easily obtainable than flow theory solutions. This is, of course, because only the stresses at the final applied load level need be considered, rather than the stress histories developed during loading. It is therefore of interest to determine whether similar benefits are attainable in the case of the finite element analyses currently being considered.

Once again, a solution of Equation 3 is required, this time such that the initial strains \( \varepsilon_i \) satisfy the deformation theory of plasticity. This can be accomplished as follows. Equation 3, the "\( k^{th} \) cycle" stress equation of the preceding section, can be used intact if it is understood that \( P_m^{(k)} \) is the peak load at which the results are required and does not change from one cycle to another as before. We must iterate to a solution in order to obtain a satisfactory approximation to the plastic strains.
The intra-cycle procedure employed for the determination of the equivalent strain for the \(k\)th cycle is the same as before, namely the constant strain method. At this point, however, the equivalent strain itself, not its increment, is resolved into the node plastic strains by utilizing an engineering adaptation of the incremental relations, Equations 7, thus:

\[
\varepsilon(p)_{3N-2} = \frac{\tilde{\varepsilon}(p)_{N}}{s(k)} \left[ \sigma^{(k)}_{3N-2} - 1/2 \sigma^{(k)}_{3N-1} \right]
\]

\[
\varepsilon(p)_{3N-1} = \frac{\tilde{\varepsilon}(p)_{N}}{s(k)} \left[ \sigma^{(k)}_{3N-1} - 1/2 \sigma^{(k)}_{3N-2} \right] \tag{9}
\]

\[
\varepsilon(p)_{3N} = \frac{\tilde{\varepsilon}(p)_{N}}{s(k)} \left[ 3\sigma^{(k)}_{3N} \right]
\]

These are now available for the stress equation of the next cycle.

Three analyses, one at \(P = 11,600\) lbs., one at \(P = 14,600\) lbs., and one at \(P = 16,760\) lbs., were performed on the shear lag specimen by this deformation theory procedure. The results were practically identical with those shown in Figures (16) to (21). The convergence to each of these results was obtained after less than thirty cycles of iteration. The machine time for each calculation was approximately four minutes.

In the case of solutions like this, where the two analyses give practically identical results, the deformation approach is naturally very attractive because of the greatly reduced machine time. However, the question remains of determining when to expect the results to agree in this manner.

The IBM program presented in Appendix V cannot be used for deformation theory analyses. However, minor changes can be made in the program to permit calculations of this type to be carried out.
SECTION IV
ISOTROPIC ELASTIC-PLASTIC-CREEP ANALYSIS

A. Introduction to Creep Theory

Strains due to creep constitute an additional form of inelastic strain, and can be handled in a way analogous to that already discussed for biaxial plasticity by the flow theory. It is only necessary to select a method for evaluating these time-dependent strains based upon the material properties and to add them to the plastic strains prior to insertion in the basic Equation 3.

A familiar relationship used to match the creep behavior in a tensile creep test, performed at constant stress and constant temperature, is (Reference 4):

\[ e(c) = \alpha t \gamma (e^{\beta \sigma} - 1) \]  

(10)
in which

- \( e(c) \) is the tensile creep strain
- \( t \) is the elapsed time
- \( \sigma \) is the constant tensile stress
- \( \alpha, \beta, \gamma \) are empirical constants for the particular test temperature

For this analysis the assumption is made that there exists an effective creep strain \( \varepsilon(c) \) in a biaxial situation which can be calculated using Equation 10. In doing this the stress \( \sigma \) is taken to be the von Mises effective stress obtained from Equation 5. The further assumption is made that this effective creep strain can be resolved into nodal creep strains by use of a Prandtl-Reuss type of flow law.

The creep strain calculation must be generalized to situations in which the stresses vary with time. One well-known procedure for doing this, the strain-hardening rule (Reference 4), has been determined to be most appropriate for the present purposes. Its use will be described presently in connection with the \( k^{th} \) calculation cycle.

B. Creep Theory Details

The calculation cycle follows a sequence similar to that described previously for the isotropic elastic-plastic analysis. An additional step is necessary, just after the plastic strains are obtained, to determine the creep strains. The intra-cycle order of calculation is as follows.
Entering the $k^{th}$ cycle with applied loads $\{p_m^{(k)}\}$, and with the initial strains, plastic and creep, as calculated during the preceding cycle, the stresses $\{\sigma_u^{(k)}\}$ are calculated from Equation 3.

The effective stresses $\{\sigma_e^{(k)}\}$ obtained from Equation 5 are used to obtain the effective plastic strains.

The creep strain increments at each node, for a specified time step, are now determined by the strain hardening rule which relates the strain at a node to the corresponding stress and strain for the previous cycle by the introduction of an assumed elapsed time.

Referring to Figure (23), one goes to the constant effective stress-temperature curve ($\sigma^k, T^k$) relevant to the node and the cycle, and locates upon it the point with ordinate $\tau^k(k-1)$. The corresponding abscissa, designated $t^*(k)$, is called the reference time and is generally different from the actual elapsed time at the start of the cycle. The required effective creep strain increment $\Delta \tau^k(k)$ is that corresponding to the increase in time from $t^k$ to $(t^k + \Delta t^k)$ as shown on Figure (23), $\Delta t^k$ being the selected calculation time increment.

The increment $\Delta \tau^k(k)$ is substituted into the Prandtl-Reuss type incremental relations, Equations 7, together with the stresses indicated there. The creep initial strains are then obtained as in Equation 8.

In summary, the steps in the $k^{th}$ calculation take the following order:

1. Evaluation of Equation 3 to obtaining the stress components, $\sigma_u^{(k)}$
2. The calculation of effective stress according to Equation 5
3. The determination of the node plastic strains
4. The determination of the node creep strains
5. The addition of the nodal plastic and creep strains to give the initial strains for the next cycle.

C. Description of Structure and Tests

The description of the shear lag structure to be analyzed in this section and tests for the material properties may be found in Reference 7. The shear lag structure was manufactured from 1100-F aluminum. It was of the same physical dimensions as the structure of Figure (6). The idealization of the upper right quadrant remains unchanged.
Material properties for the creep analysis were obtained from uni-
axial strain-time tests for constant tensile stress. The temperature
at which these tests were conducted was 206°C, the temperature identical
to that of the structural test. The curves for these tests are presented
in Figure (24) together with the fitted curves from Equation 10. The
constants of the equation were obtained from Reference 7 and are as
follows:

\[ \begin{align*}
\alpha &= 0.650 \times 10^{-4} \\
\gamma &= 0.500 \\
\beta &= 0.700 \times 10^{-3} \text{ in}^2/\text{lb}
\end{align*} \]

Ordinary tensile stress-strain tests were performed at room tempera-
ture on coupons cut from the x and y orientations of the plate material.
The data and faired curve are presented in Figure (25). Tensile stress-
strain data for 206°C are also plotted here. These data were obtained
from the intersections of the test curves on the zero time axis in Fig-
ure (24). A piecewise linear representation TCI was fitted to this
latter data.

The location of the strain gages on the structure is given in Fig-
ure (26). The shear lag specimen was tested for a total of three hours
at 206°C. An initial applied load of 1600 lb. was increased to 2020 lb.
at the end of the first hour. It was held constant thereafter to the
end of the test.

D. Results of Creep Shear Lag Analysis

The predicted strain distributions along the x-axis and along the
section \( x = 1 \text{ in.} \) at \( t = 0.06 \text{ hr.}, \ t = 1.10 \text{ hr.}, \) and \( t = 3.00 \text{ hr.} \)
elapsed times, are given in Figures (27) through (32), together with
the experimental data of Reference 7. Test data are not available for
the y-node strains at the center node, and so this correlation point of
critical significance does not exist.

The curve of Figure (33), effective stress versus strain at the
central node, exhibits the shapes characteristic to the various regions
of the load-time sequence. The initial linear segment, representing
elastic loading is followed by the region of the negative curvature
representing loading into the plastic range, all at assumed zero time.
Thereafter, the applied load remains constant for one hour, during which
time there is a stress redistribution in the structure due to creep.
This particular node unloads, as evidenced by the reduction in effective
stress, although the total strain is growing continuously. The applied
load is now increased to 2020 lb. Because of the previous elastic strain
recovery, the effective stress at first goes up elastically, and then
becomes plastic once more. Once the applied load reaches its final value,
redistribution due to creep effects again takes place.

In the initial stages of creep the curve is very sensitive to time
increment size and it is necessary to choose exceptionally small time
increments in this region if good accuracy is to be achieved. The time increments employed are as shown in Figure (33).

Considering the simplicity of the expressions employed to describe as complex a phenomenon as creep and the liberal assumptions made in the process, the correlation between analysis and experiment, as evidenced by the preceding graphs and also by Figures (34) and (35), is surprisingly good.
SECTION V
ANISOTROPIC ELASTIC-PLASTIC ANALYSIS

A. Anisotropy in Structures

Several "expanding yield surface" theories for extending the isotropic plastic theory to provide for anisotropy have become available within the last two decades. Each is based on experimentally determined parameters and, therefore, each is biased in favor of specific test data. The complexity and amount of testing required to obtain these parameters differ considerably. Under these circumstances, no single theory can be completely acceptable, but it is thought that a suitable theory must, at least, be capable of evaluating the type of anisotropy associated with biaxially stressed structures used in flight vehicle design without being unduly complex in application. It would be desirable to have the procedure based on a well-known, accepted theory.

A particular type of anisotropy, the so-called "orthotropic symmetric" type, develops during a cold rolling process where the material is lengthened and thinned with no appreciable change in width. Since cold rolled sheet and plate are frequently used in aerospace structures, this type of anisotropy may be anticipated and is considered here.

A theory proposed by Hill, Reference 10, has been widely accepted as the most straightforward extension of the isotropic theory. The formulation, however, is not very convenient for numerical step-by-step computation. A modification of Hill's theory proposed by Hu, Reference 11, however, is very tractable to formulation into the matrix inelastic program discussed previously in this report. The Hu procedure has two distinct advantages:

1. It employs a von Mises type hardening surface, associated flow law and effective stress-strain relationships in appropriate form.

2. It requires a minimum of material data: simple uniaxial and shear stress-strain tests on coupons cut in the directions of the orthotropy.

B. Hu's Strain Hardening Theory

A summary of Hu's theory is presented to establish its limitations and provide background for the necessary modifications to obtain a more general theory.

The isotropic expressions for effective stress, Equation 5, and associated incremental relations, Equations 7, are modified by means of anisotropic parameters \( \alpha_{ij} \). These are constants in Hu's theory. Here it is more convenient to introduce the 1-1, 2-2, etc., directions instead
of x-x, y-y, etc. The modified expression for the effective stress is

\[
\bar{\sigma} = \left[ \alpha_{12}(\sigma_{11} - \sigma_{22})^2 + \alpha_{23}(\sigma_{22} - \sigma_{33})^2 + \alpha_{31}(\sigma_{33} - \sigma_{11})^2 \\
+ 3\alpha_{44}\sigma_{12}^2 + 3\alpha_{55}\sigma_{23}^2 + 3\alpha_{66}\sigma_{31}^2 \right]^{\frac{1}{2}}
\]

The incremental flow equations are

\[
\begin{align*}
\dot{\varepsilon}_{11} &= \frac{\partial}{\partial \bar{\sigma}} \left[ \alpha_{11} - \alpha_{12}(\sigma_{22} - \sigma_{33}) - \alpha_{31} \right] \\
\dot{\varepsilon}_{22} &= \frac{\partial}{\partial \bar{\sigma}} \left[ - \alpha_{12}^{\sigma_{11}} + \alpha_{22}(\sigma_{22} - \sigma_{33}) - \alpha_{23} \right] \\
\dot{\varepsilon}_{33} &= \frac{\partial}{\partial \bar{\sigma}} \left[ \alpha_{31}^{\sigma_{11}} - \alpha_{23}^{\sigma_{22}} + \alpha_{33}^{\sigma_{33}} \right] \\
\dot{\varepsilon}_{12} &= \frac{\partial}{\partial \bar{\sigma}} \left[ 3\alpha_{44}\sigma_{12} \right] \\
\dot{\varepsilon}_{23} &= \frac{\partial}{\partial \bar{\sigma}} \left[ 3\alpha_{55}\sigma_{23} \right] \\
\dot{\varepsilon}_{31} &= \frac{\partial}{\partial \bar{\sigma}} \left[ 3\alpha_{66}\sigma_{31} \right]
\end{align*}
\]

Equations 11 and 12 are written for the case where the reference axes are the principal axes of anisotropy.

The anisotropic parameters are determined by means of a total of six, simple, directional, stress-strain tests (i.e., uniaxial and shear tests), where, alternately, all stress components are equal to zero except one. From each of the six tests a characteristic stress, such as an approximate yield stress, is read off. Then substituting each of these results into Equation 11 in succession, we may write

\[
\begin{align*}
\alpha_{11} &= \alpha_{12} + \alpha_{31} = \left( \frac{K}{Y_{11}} \right)^2 \\
\alpha_{22} &= \alpha_{23} + \alpha_{12} = \left( \frac{K}{Y_{22}} \right)^2 \\
\alpha_{33} &= \alpha_{31} + \alpha_{23} = \left( \frac{K}{Y_{33}} \right)^2 \\
\alpha_{44} &= \frac{1}{3} \left( \frac{K}{Y_{12}} \right)^2 \\
\alpha_{55} &= \frac{1}{3} \left( \frac{K}{Y_{23}} \right)^2 \\
\alpha_{66} &= \frac{1}{3} \left( \frac{K}{Y_{31}} \right)^2
\end{align*}
\]

where

- \( K \) = the effective characteristic stress
- \( Y_{ij} \) = the simple directional characteristic stress
It remains to assume an effective stress-effective strain relationship. Hu shows that it is acceptable to assume the stress-strain data associated with one particular simple tension test as the effective relationship. It can be seen from Equations 13 that this implies setting one $\alpha_{44}$ equal to unity.

Thus, in particular for $\alpha_{22} = 1$, from Equations 11 and 12:

$$\sigma = \sigma_{22}$$
$$d\varepsilon = d\varepsilon_{22}$$

Now consider Equation 12 for simple stress-strain tests in the other directions. Then

$$d\varepsilon_{11} = \alpha_{11} \sigma_{11} d\varepsilon$$
$$d\varepsilon_{33} = \alpha_{33} \sigma_{33} d\varepsilon$$
$$d\varepsilon_{12} = 3\alpha_{14} \sigma_{12} d\varepsilon$$
$$d\varepsilon_{23} = 3\alpha_{55} \sigma_{23} d\varepsilon$$
$$d\varepsilon_{31} = 3\alpha_{66} \sigma_{31} d\varepsilon$$

or

$$\frac{d\varepsilon}{\sigma} = \frac{1}{\alpha_{11}} \frac{d\varepsilon_{11}}{\sigma_{11}} = \frac{1}{\alpha_{33}} \frac{d\varepsilon_{33}}{\sigma_{33}} = \frac{1}{3\alpha_{14}} \frac{d\varepsilon_{12}}{\sigma_{12}} = \frac{1}{3\alpha_{55}} \frac{d\varepsilon_{23}}{\sigma_{23}} = \frac{1}{3\alpha_{66}} \frac{d\varepsilon_{31}}{\sigma_{31}}$$

These equations say that, with the anisotropic parameters constant for strain hardening, the simple, stress-incremental strain relations must be proportional to the effective stress-incremental strain relationship. The implication is that the integrated forms of Equations 15, that is, the simple, directional stress strain curves, are thus prescribed. These may or may not be a reasonable fit to the test data for the material of interest. Obviously, only when the fit is good can one hope for Hu's theory to give acceptable results for all types of loading.

Based upon the Hu theory, it becomes relatively easy to obtain anisotropic solutions using the previously developed isotropic inelastic procedure and corresponding digital computer program. It is only necessary to substitute the appropriate anisotropic constants for their
isotropic counterparts, for which $\alpha_{11} = \alpha_{66} = 1$, $\alpha_{12} = \alpha_{23} = \alpha_{31} = \frac{1}{2}$.

C. Extension of Hu’s Theory

It is in the determination of the parameters that an extension to Hu’s theory is made. Except for the special case pointed out in the preceding section, these parameters should not be constant for a strain-hardening material, but should be variables dependent upon stress level. The objective, obviously, is to determine the variation in a manner that allows for all of the simple, directional stress-strain curves to be correctly reproduced. This can be accomplished by a consideration of plastic work.

In the current approach, one continues with the assumption of the existence of an effective stress-strain relationship. Then the basic notion is that the anisotropic parameters are determined such that, for equal amounts of plastic work done during simple directional stress-strain tests in all directions, the effective stress level reached will be identical.

Accordingly, for a tensile specimen in the 1-1 direction, one calculates the plastic work $w$ performed during a uniaxial test by the formula

$$w = \int \alpha_{11} \, d\varepsilon_{11} = \int \delta \, d\varepsilon$$

(16)

Let the corresponding maximum stress reached be identified by the superscript (I), i.e. $\sigma_{11}^{(I)}$. For a similar test in the 2-2 direction, and for which the amount of plastic work performed is identical, the corresponding maximum stress reached is $\sigma_{22}^{(I)}$. Since the amounts of work done in the two cases are the same, $\sigma_{11}^{(I)}$ and $\sigma_{22}^{(I)}$ correspond to the same effective stress $\delta^{(I)}$. By means of Equation 11 we have

$$(\delta^{(I)})^2 = \alpha_{11} \left(\frac{\sigma_{11}^{(I)}}{\delta}ight)^2 = \alpha_{22} \left(\frac{\sigma_{22}^{(I)}}{\delta}ight)^2$$

(17)

This expression constitutes a relationship defining $\alpha_{11}$ and $\alpha_{22}$ as functions of $\delta$. Similar relationships can clearly be found for the other $\alpha_{11}$'s. Thereafter, the $\alpha_{ij}$'s can be determined as functions of $\delta$ by recourse to the $\alpha$ definitions of Equations 13.

It is convenient to again select the 2-2 direction stress-strain curve as the one defining the effective stress-strain relationship. This results in $\alpha_{22}$ being equal to unity once more.
The actual evaluation of the other \( a' \)'s as functions of \( \phi \) follows easily. Figure (36) indicates schematically how the plastic work done in each simple directional stress-strain test can be plotted as a function of stress. Then for a given amount of plastic work, and reading off the corresponding stresses \( \sigma_{11}^I \) and \( \sigma_{22}^I \) for example, by Equation 17 one finds that

\[
a_{11}^I = \left( \frac{\sigma_{22}^I}{\sigma_{11}^I} \right)^2
\]

(18)

Using this and similar relationships in the other directions, curves representing all of the \( a' \)'s as functions of \( \phi \) may be constructed. The incorporation of this information in the step-by-step calculation procedure is discussed in the next section.

D. Anisotropic Theory Details

The detailed step-by-step calculation procedure to be followed in the case of anisotropic material in a biaxially stressed structure is very similar to that previously discussed for isotropic materials. Accordingly, only the differences will be stressed.

As before, one starts the \( k \)th calculation cycle by evaluating \( [\sigma_{\eta}^{(k)}] \) by means of Equation 3. This operation employs the initial strains of the preceding cycle \( [\varepsilon_{\nu}^{(p)}]^{(k-1)} \).

The next operation is to evaluate the effective stresses at each of the nodes, \( \bar{a}_{N}^{(k)} \), by Equation 11, modified for the biaxial case to

\[
\bar{a}_{N}^{(k)} = \left[ \alpha_{11}^{(k-1)} \left( \sigma_{22}^{(k)} \right)^2 - 2 \alpha_{12}^{(k-1)} \left( \sigma_{33}^{(k)} \right) \left( \sigma_{33}^{(k)} \right)^2 \right]^{1/2}
\]

(19)

We continue here our assumption that the 2-2 direction has been selected as that in which the effective stress-strain relationship is defined; hence \( \sigma_{22} = 1 \).
Note that if the Hu theory is being used, the $a_{ij}$'s are all known constants. If the modified theory is being employed, the variable $a_{ij}$'s are those that have been evaluated during the preceding cycle, $(k-1)$. This is in keeping with the overall nature of the analysis as a step-by-step procedure.

Having determined the $a(k)_s$, one can now go to the curves representing the $a_{ij}$'s as functions of $a$ to evaluate $a_{11}(k)$, $a_{12}(k)$ and $a_{44}(k)$. After having also determined $e_N(p)(k)$ and $\Delta e_N(p)(k)$ as before by the constant strain method, these quantities may be substituted into a finite equivalent of Equations 12 specialized for the biaxial case, to yield

$$\Delta e_N(p)(k) = \frac{\Delta e_N(p)(k)}{\Delta e_N(k)} \left[ a_{11} \sigma_{3N-2} - a_{12} \sigma_{3N-1} \right]$$

$$\Delta e_N(p)(k) = \frac{\Delta e_N(p)(k)}{\Delta e_N(k)} \left[ \sigma_{3N-1} - a_{12} \sigma_{3N-2} \right]$$ (20)

$$\Delta e_N(p)(k) = \frac{\Delta e_N(p)(k)}{\Delta e_N(k)} \left[ 3a_{44} \sigma_{3N} \right]$$

From these, the strain components $e_p(k)$, $e_3(k)$ and $e_4(k)$ are obtained by addition, as before, using Equation 8.

After incrementing the applied load, the sequence can now be repeated for the next load cycle.

E. Rotation of Axes of Anisotropy

Anisotropic symmetry may occur in a structure for which it is convenient to choose coordinate axes that are rotated from the orthogonal axes of anisotropy. The corresponding expressions for theincremental flow equations and effective stress equations are derived in Appendix IV. The derivation is limited to the case of biaxial stress where the 3-3 and z-z axes coincide. A method of obtaining the shear anisotropic coefficients is also indicated.
F. Anisotropic Analysis of Shear Lag Structure

It has been pointed out earlier that the 2024-T4 aluminum alloy of the shear lag structure tested for the Air Force (Reference 7) displayed considerable anisotropy as shown in Figure (11). The structure had been analyzed for isotropic strain hardening based upon the curve RO2M in this figure, and the results discussed in Section III. A corresponding anisotropic strain hardening analysis has also been carried out, employing first the Hu theory and then the proposed extension.

Material stress-strain data is available along two of the axes of anisotropy, the rolling or x-x (1-1) direction, and the long transverse or y-y (2-2) direction. This data has been plotted and discussed in connection with Figure (11); it is replotted for convenience in Figure (37).

The additional required, but missing, test data is (a) tensile stress-strain data in the short transverse or z-z (3-3) direction and (b) shear stress-strain data in the x-y plane. A reasonable assumption to make for engineering purposes for (a) is that the long and short transverse properties are identical; this is made in the analyses to follow. In the case of (b), the missing shear data, the following is done. First, a shear stress-strain curve is obtained based upon the tensile curve in the rolling direction, together with the assumption that the material is isotropic and governed by the incremental theory of Section III -- specifically, Equations 5 and 7. Next, a similar shear stress-strain curve based upon the tensile curve in the long transverse direction is obtained. Finally, a faired-in average of these two curves is taken to represent the missing shear stress-strain relation.

In order to apply the Hu theory, one must first select four characteristic stresses to represent the directional stress-strain curves, as discussed in Section V-B. These are the quantities $Y_{11}$, $Y_{22}$, $Y_{33}$ and $Y_{12}$ of Equations 13. We arbitrarily choose the proportional limits from the four curves just discussed for these values; they are

$$
Y_{11} = 51 \text{ ksi} \\
Y_{22} = 32 \text{ ksi} \\
Y_{33} = 51 \text{ ksi} \\
Y_{12} = 22 \text{ ksi}
$$

Since the simple directional stress-strain curve in the 2-2 direction, i.e. RO2M, is selected for the effective stress relationship, the effective stress characteristic value $K$ is also 32 ksi. Substituting these values in Equations 13, one can solve for the necessary anisotropic parameters. This information is all that is required to carry out an analysis of the shear lag structure by the Hu theory.

As pointed out previously, once the anisotropic parameters $\alpha_{11}$, $\alpha_{22}$, $\alpha_{12}$ and $\alpha_{14}$ have been specified and one of the stress-strain curves chosen as the effective stress-strain curve, the remaining simple direc-
tional stress-strain curves are prescribed. Let us now examine the consequences of our current parameter selection. Since only test data in the 1-1 and 2-2 directions are available, we shall concentrate on these. For this special case, Equations 11 and 12 can be easily manipulated to yield

\[
\bar{\sigma} = \sqrt{\sigma_{11} \sigma_{11}} \quad (21)
\]

\[
\frac{d\varepsilon_{11}}{d\sigma_{11}} = \frac{\sigma_{11}}{\bar{\sigma}}
\]

Using these expressions, one can construct a \( \sigma_{11} \) stress-strain curve.

Such a curve is shown on Figure (37). As can be seen, the fit with the 1-1 test data is very poor. Apparently, one could do better by arbitrarily choosing for the 1-1 characteristic value \( Y_{11} \) a much lower value than 51 kpsi -- perhaps in the neighborhood of 35 kpsi. Nevertheless, it is clear that in this case it would still be impossible to get a really good fit because of the fundamentally different shapes of the two stress-strain curves, especially in the region of their knees. It will turn out that, in the case of the shear lag structure, this selection is not critical, because of the fact that stresses in the 1-1 direction are very low, compared to those in the 2-2 direction.

In order to carry out an analysis of the shear lag structure based upon the extension to the Hu theory, one must first evaluate the plastic work done in each of the simple directional stress-strain tests as a function of the applicable stress. Using this information in the manner discussed previously, one can then obtain the anisotropic parameters as functions of effective stress. This has been done, assuming that it is sufficiently accurate to represent the curves by a small number of connected straight line segments. The key values of the resulting \( \alpha' \)'s are given in the accompanying table. Corresponding total strains \( \varepsilon = \varepsilon / E + \varepsilon \) are also listed.

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>( \varepsilon )</th>
<th>( \sigma_{11} = 2\alpha_{12} )</th>
<th>( \alpha_{22} )</th>
<th>( \alpha_{11} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>32000</td>
<td>.00310</td>
<td>.394</td>
<td>1.000</td>
<td>.697</td>
</tr>
<tr>
<td>37000</td>
<td>.00378</td>
<td>.510</td>
<td>1.000</td>
<td>.735</td>
</tr>
<tr>
<td>40000</td>
<td>.00435</td>
<td>.590</td>
<td>1.000</td>
<td>.795</td>
</tr>
<tr>
<td>43000</td>
<td>.00509</td>
<td>.660</td>
<td>1.000</td>
<td>.833</td>
</tr>
<tr>
<td>46000</td>
<td>.00592</td>
<td>.742</td>
<td>1.000</td>
<td>.877</td>
</tr>
<tr>
<td>48000</td>
<td>.00734</td>
<td>.809</td>
<td>1.000</td>
<td>.905</td>
</tr>
<tr>
<td>49000</td>
<td>.00800</td>
<td>.835</td>
<td>1.000</td>
<td>.918</td>
</tr>
<tr>
<td>51000</td>
<td>.00960</td>
<td>.892</td>
<td>1.000</td>
<td>.945</td>
</tr>
<tr>
<td>54000</td>
<td>.03800</td>
<td>.930</td>
<td>1.000</td>
<td>.965</td>
</tr>
</tbody>
</table>
In the preceding table, the reduction in number of independent anisotropic parameters is a consequence of the assumptions which had to be made for the missing stress-strain curves.

The value of unity for \( a_{22} \) is in accordance with our continued selection of the tensile stress-strain curve RO2M in the y-y or 2-2 direction as the basis for our effective stress-strain relationship.

It might be reiterated that, because of the manner in which the \( a's \) are derived in this case, all of the simple directional stress-strain curves are matched as closely as desired.

Using first the anisotropic parameters selected as described for the Hu theory, and then for its extension, the shear lag structure has been analyzed. The results are presented in the next section.

G. Discussion of Results

Some of the highlights of the two anisotropic analyses are presented in Figures (38) through (40). Corresponding results obtained previously and based upon isotropic theory are also included for comparison.

Figure (38) refers to the central node of the shear lag structure and shows the variation of the total strains in the x and y directions with increasing applied load. The isotropic result is replotted from Figure (22) -- specifically those curves based upon the effective stress-strain curve RO2M.

It can be seen from Figure (38) that the most flexible analysis predictions, that is, those for which the total strains are the largest, are obtained by the isotropic analysis. The analysis based upon the extension to the Hu theory is somewhat less flexible, while the Hu theory results are the stiffest. Also, the differences between the three analyses are less in the y direction than in the x direction.

Comparing the analyses with the test data, all three analyses substantially underpredict the test points in the high applied load regime, and more so in the y than in the x direction. It is interesting to note however, that while the anisotropic analyses make these differences even greater, they do have the virtue of making the comparison more consistent as between the x and y directions. Thus it would appear that the...
anisotropic nature of the 2024-T4 material does influence the strain distribution in the shear lag structure, and that the anisotropic analyses can detect this tendency.

Figures (39) and (40) show the strain distributions along the two strain gage lines of the test structure. In addition to the isotropic predicted results, replotted from Figures (20) and (21), anisotropic results based upon the extension to the Hu theory are presented. They indicate that while the differences are not dramatic, they do in fact exist.
A. Anisotropic Creep Theory

The application of matrix analysis procedures to problems involving anisotropic creep is at present academic. Neither meaningful analytical research nor appropriate test data has been found. Therefore the procedure presented here for anisotropic creep is a simple extension of that already described in previous sections. Accordingly, only the difference and additional assumptions are discussed.

Strains due to anisotropic creep can be handled in a way similar to those due to isotropic creep. The step-by-step procedure has only to be modified for the anisotropic behavior by substituting Equations 19 and 20 for Equations 5 and 7 respectively in the manner described for time independent plastic anisotropy in Section V-D. The additional assumptions implied by this simple extension are the following:

1. The anisotropic parameters calculated from zero time simple directional tests (by either Hu theory or the proposed extension) are valid for anisotropic creep.

2. The effective creep strain equation, Equation 10, remains valid and the empirical constants \( \alpha, \beta, \gamma \) are determined for the tensile creep test in the assumed effective stress-strain direction.

Because testing of structures exhibiting anisotropic creep has apparently not been done, the anisotropic creep procedure cannot be checked out against tests at this time. However, the results of a sample calculation are presented for a hypothetical material having this characteristic.

B. Sample Problem

The 1100-F aluminum shear lag specimen, already analyzed for isotropic creep, is used for an anisotropic creep analysis making the following assumptions:

1. All the uniaxial data (Figure 25) employed in the isotropic creep analysis is assumed to refer to the \( \gamma-\gamma \) (2-2) effective direction of the anisotropic creep analysis.

2. The plastic anisotropic parameters (extension of Hu's theory) for the effective stress-strain curve of the 2024-T4 aluminum alloy material for successive levels of stress, 32 ksi, 37 ksi, 40 ksi, etc., are arbitrarily chosen for the stress levels, 2 ksi, 3 ksi, 4 ksi, etc., of the 1100-F aluminum effective stress-strain curve T01 (Figure 25). Thus:
This means that the uniaxial x-x (1-1) predicted stress-strain curve and the shear x-y stress-strain curve are arbitrarily stiffer than the corresponding isotropic curves.

The node strains obtained, using the anisotropic assumptions for load \( P = 2020 \) lbs at time \( t = 3 \) hrs, are presented for the x-axis and along \( x = 1 \) in. in Figures (41) and (42) respectively.

In general, the anisotropic results appear to be stiffer than those for the isotropic case, and this is especially true for the vertical gage line, where the anisotropic shear strains are of the order of \( 3/4 \) of the isotropic values. This should be expected, due to the fact that the anisotropic parameters used are all less than, or at the most equal to, their isotropic equivalents.
SECTION VII
CONCLUSION

The linear, matrix structural analysis methods currently in general use throughout the aerospace industry have recently been extended to include structures loaded into the inelastic material behavior regime. However, very little published information is available correlating predicted results with the test data.

The current report recommends a simple analytical approach to such problems. It is based upon the concept of initial strains in combination with a suitable matrix of influence coefficients, obtained by standard linear matrix structural analysis methods. The initial strains are those associated with plasticity and creep.

Some particularly useful tests of aluminum and aluminum alloy shear lag structures have been performed previously for the Air Force. These structures have been analyzed by the recommended method, and the resulting agreement (for both the plasticity and the creep tests) is considered to be very encouraging. On the other hand, additional testing must be carried out and correlations with analysis made before the method can be considered as fully evaluated.

In the meantime, the writers believe that sufficient confidence in the method has been established that it may now be used in practical engineering applications. For example, it should be immediately useful in such problems as predicting inelastic strain distributions around stress raisers in simple structural components.

As for the influence of anisotropy, it has been shown that this material property can be readily accommodated in the recommended procedure. Calculations made for the 2024-T4 shear lag structure indicate that, in this case at least, anisotropy plays a minor but discernable role in determining the strain distributions. This evidence is inconclusive, because in this particular test, while the material itself is decidedly anisotropic, the stresses normal to the direction of the applied loads are quite small.

It is recommended that in cases where doubt exists as to the importance of anisotropy, it be included in the analysis. Certainly in such examples as the shear lag structure, the additional complexity in the use of the computer program is very slight.

There is one very important restriction implicit in the method proposed in this report. Essentially, this method applies only to a structure in which the material is initially in the virgin state, and thereafter experiences only continually-increasing applied loads until failure occurs. This limitation can actually be relaxed to the extent that elastic unloading followed by reloading can also be accommodated, but
completely reversed loading is specifically excluded. For the latter case, a different plasticity theory is needed. It is important that a technique for handling such problems be developed because of the need for such applications in fatigue work. Efforts toward removing this restriction are currently under way under government contract.

Other than the preceding, the largest remaining obstacle to complete inelastic analysis of practical aerospace structures is believed to be the dearth of appropriate material property data, and constitutive laws to describe them.
SECTION VIII

FIGURES

33
Fig. 1 Truss Structure

Fig. 2 Constant Stress Method

Fig. 3 Constant Strain Method
Fig. 4 Results of Constant Stress Method for Bar No. 3 of Truss

Fig. 5 Results of Constant Strain Method for Bar No. 3 of Truss
Fig. 6 Shear-Lag Specimen Designated S181 - From Air Force Report No. RTD-TDR-63-4032; 2024-T4 Aluminum Alloy (1100-F Specimen Similar)

Fig. 7 Stiffness Method
Idealization

Fig. 8 Force Method
Idealization

36
Fig. 9 Typical Elements of Force Idealization

Fig. 10 Typical Elements of Stiffness Idealization
Fig. 11 2024-T4 Aluminum Alloy Stress-Strain Data and Curve R02M

Fig. 12 Comparison of Predicted Effective Stress-Strain Relationship with R02M at (0, 0) for Various Load Increments
<table>
<thead>
<tr>
<th>Gage Nos.</th>
<th>Co-ordinates</th>
<th>Gage Nos.</th>
<th>Co-ordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>Front</td>
<td>Back</td>
<td>x in.</td>
<td>y in.</td>
</tr>
<tr>
<td>1</td>
<td>11</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>0.5</td>
<td>0.0</td>
</tr>
<tr>
<td>3</td>
<td>13</td>
<td>1.125</td>
<td>0.0</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
<td>1.750</td>
<td>0.0</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>3.000</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Fig. 13 Instrumentation of 2024-T4 Aluminum Alloy Specimen
Fig. 14 Elastic Strains Along x-Axis for P = 1 lb

Fig. 15 Elastic Strains Along x = 0.8125 in. for P = 1 lb

40
Fig. 16 Strains Along x-Axis for P = 11,600 lb and ΔP = 5 lb

Fig. 17 Strains Along x = 0.8125 in. for P = 11,600 lb and ΔP = 5 lb
Fig. 18 Strains Along x-Axis for P = 14,600 lb and \( \Delta P = 5 \) lb

Fig. 19 Strains Along \( x = 0.8125 \) in. for P = 14,600 lb and \( \Delta P = 5 \) lb
Fig. 20 Strains Along x-Axis for $P = 16,760 \text{ lb}$, and $\Delta P = 5 \text{ lb}$

Fig. 21 Strains Along $x = 0.8125 \text{ in.}$ for $P = 16,760 \text{ lb}$ and $\Delta P = 5 \text{ lb}$
Fig. 22 Elastic-Plastic Strains at (0, 0) for Several Assumed Stress-Strain Curves
Fig. 23 The Strain Hardening Rule
Fig. 24 1100-F Aluminum Time-Dependent Behavior and Fitted Curves (Ref. 7)
Fig. 25 1100-F Aluminum Stress-Strain Data at Room Temperature and at 206° C, for Time t = 0.00

Fig. 26 Location of Strain Gages on 1100-F Specimen (Ref. 7)
Fig. 27 Strains Along x-Axis for P = 1600 lb and t = 0.06 hr

Fig. 28 Strains Along x = 1 in. for P = 1600 lb and t = 0.06 hr
Fig. 29 Strains Along x-Axis for $P = 2020$ lb and $t = 1.10$ hr

Fig. 30 Strains Along $x = 1$ in. for $P = 2020$ lb and $t = 1.10$ hr
Fig. 31 Strains Along x-Axis for $P = 2020$ lb and $t = 3.0$ hr

Fig. 32 Strains Along $x = 1$ in. for $P = 2020$ lb and $t = 3.0$ hr
Fig. 31 Strains Along x-Axis for P = 2020 lb and t = 3.0 hr

Fig. 32 Strains Along x = 1 in. for P = 2020 lb and t = 3.0 hr
Fig. 33 Effective Stress vs Strain at Center Node
Fig. 34 Total (Elastic, Plastic and Creep) Strains at Center of Specimen (Gage 1) for P=1600 lb to Time 1 hr, then P=2020 lb to Time 3 hr
Fig. 35 Total (Elastic, Plastic and Creep) Strains at (1.0 in., 3.0 in.) (gage 8) for P = 1600 lb to time 1 hr, then P = 2020 lb to time 3 hr
Figure 36 Typical Uniaxial Stress vs Plastic Work, w, Plot.
Fig. 37 Assumed Effective Stress (y-y) Curve, Calculated x-x Curves and Test Data
Isotropic Analysis Based on $\epsilon_y$ Stress-Strain Curve (RO2X)

Anisotropic Analysis

$\Delta = \epsilon_y$, Experimental Data
$
\Theta = \epsilon_x$, Based on Extension to Hu's Theory

Based on Hu's Theory

Based on Extension to Hu's Theory

Fig. 38 Node Strains at (0, 0) - Isotropic and Anisotropic Analysis
Fig. 39 Strains Along x - Axis for $P = 16,760$ lb.
Isotropic and Anisotropic Analyses

Fig. 40 Strains Along $x = 0.8125$ in. for $P = 16,760$ lb.
Isotropic and Anisotropic Analyses
Fig. 41 Strains Along x - Axis for P = 2020 lb and t = 3.0 hr.
Isotropic and Anisotropic Analyses

Fig. 42 Strains Along x = 1 in. for P = 2020 lb and t = 3.0 hr.
Isotropic and Anisotropic Analyses
Fig. 43 Elastic Stress Distribution Along x-Axis
Fig. 44 Elastic Stress Distribution Along \( x = 0.8125 \) in.

Note: Force and Stiffness Method Results identical except as noted.
APPENDIX I
ELASTIC ANALYSIS OF SHEAR LAG STRUCTURE

A. Idealization of Shear Lag Structure

As stated in the introduction, when a problem is formulated by means of a standard influence coefficient approach, the necessary linear analysis may be carried out using either the force or displacement method. Since published correlations between results of the matrix force and direct stiffness methods of linear elastic analyses for redundant structures have, in the past, left room for doubt as to the equivalence of results, this Appendix presents a comparison of the stresses from two idealizations of the simple shear-lag stiffened-plate structure, Figures (7) and (8). In the past, discrepancies have been due in part to a marked difference in the arrangement of node points for corresponding idealizations, and also to the fact that techniques for obtaining node stresses in finite element analyses are still being improved. An attempt was made to keep the idealizations as comparable as possible with respect to location of nodes and the determination of stresses.

B. Force Method

The idealization for the force method may be seen in Figure (8). It comprises conventional bars and shear panels located in the manner shown. The analyses of some previous idealizations of this type have omitted the Poisson's ratio effect. This effect can be incorporated in the manner described in Appendix II.

C. Stiffness Method

The idealization for the stiffness method consists of "Turner triangles," which are located as shown in Figure (7). The basic theory of the triangle is to be found in References 8 and 9.

While the conventional procedure was used to obtain node stresses for the force method, comparable stiffness-method stresses can be calculated in several ways. A recent paper, Reference 9, suggests two means of obtaining node stress, one of which was employed in the analysis. This method will be reviewed briefly.

Figure (10) represents a cluster of triangles. It is required to find the stresses at the node common to triangles P to W. The node forces for each triangle at this apex are obtained as described in Turner's former papers. Summing the forces on a vertical section through l in both directions gives

\[ X_V = X_P + X_Q + X_R + X_S \]
\[ Y_V = Y_P + Y_Q + Y_R + Y_S \]
Analogous results $X_n$ and $Y_n$ are obtained for the corresponding horizontal cut.

The node stresses at the node are

$$
\sigma_x = \frac{X_v}{\frac{1}{2} (ct_p + dt_s)}
$$

$$
\sigma_y = \frac{Y_h}{\frac{1}{2} (at_r + bt_u)}
$$

$$
\sigma_{xy} = \frac{1}{2} \left[ \frac{X_h}{\frac{1}{2} (at_r + bt_u)} + \frac{Y_v}{\frac{1}{2} (ct_p + dt_s)} \right]
$$

where $a$, $b$, $c$, and $d$ are as indicated and $t$ is the thickness.

D. Comparison of Elastic Results and Perspective

The results are correlated by means of the curves appearing in Figures (43) and (44) with calculated values from the experimental data of Reference 7.

The correlation between the stresses derived from the force method and from the stiffness method of analysis is excellent and may be regarded as exact for engineering applications. The largest discrepancy is in the direct stresses in the $x$ direction at the middle of the plate as shown in Figure (43). Even in this region the difference is quite small. It is believed that an even closer agreement could be obtained by modifying the idealized structure to provide square shear panels adjacent to the reinforcement and a finer grid at the plate center.

The largest discrepancy between analysis and test results is located in the region of the plate center. Reference 7 indicated that considerable bending was exhibited by the structure as the ends of the stiffener were loaded. The extent to which this affects the gage readings was not determined; however, it may be anticipated that the effect be greatest near the middle of the plate. The curves on Figure (43) reinforce this impression.

On the basis of the excellent agreement noted here, it can be concluded that the incorporation of plastic and creep effects into the present method of structural analysis will not be restricted in any way by the particular linear analysis method employed.
APPENDIX II

POISSON'S RATIO EFFECT

The strain energy relationship for an elastic plate in terms of in plane stresses is given by the volume integral:

\[ U = \frac{1}{2E} \iint_V \left[ \sigma_{xx}^2 + \sigma_{yy}^2 - 2\nu \sigma_{xx} \sigma_{yy} + 2(1 + \nu) \sigma_{xy}^2 \right] dV \]

The idealized structure corresponding to a rectangular plate for a finite element matrix force analysis has orthogonal bars taking only normal stresses and panels in pure shear. Figure (8), the shear-lag specimen, represents a typical idealization of this type. Defining \( \tilde{\sigma}_{xx} \) and \( \tilde{\sigma}_{yy} \) as the axial stress in the bars and \( \tilde{\sigma}_{xy} \) as the shear stress in the panels, the strain energy \( U' \) of the idealized structure is sometimes taken as:

\[ U' = \frac{1}{2E} \iint_V \left[ \tilde{\sigma}_{xx}^2 + \tilde{\sigma}_{yy}^2 + 2(1 + \nu) \tilde{\sigma}_{xy}^2 \right] dV \]

Comparing the plate strain energy \( U \) with the idealized structure strain energy \( U' \), it is obvious that the finite element expression neglects to account for the interaction term of the plate \( -2\nu \sigma_{xx} \sigma_{yy} \) which is due to Poisson's ratio. This uncoupling of the normal stresses has the effect of making the idealized plate less rigid than the actual plate. The finite element idealization is refined by including the term \( -2\nu \sigma_{xx} \sigma_{yy} \) in \( U' \), making the model more consistent with the elastic plate.

The sketch, Figure (9), shows a shear panel with adjacent axial load carrying bars. Assume the structure represents a portion of a plate structure \( l \) inches long, \( b \) inches wide and \( t \) inches thick. The normal stresses at one corner of the idealized structure are designated \( \sigma_1 \) and \( \sigma_2 \) in the \( x \) and \( y \) directions respectively. It is sufficiently accurate in accounting for the interaction term to assume that the normal stresses are constant over the plate corresponding to the shaded quadrant and also to assume that these stresses are equal to \( \sigma_1 \) and \( \sigma_2 \), the values at the corner. The strain energy term to be included is represented by

\[ \frac{1}{2} \sigma_1 \sigma_2 \sigma_{12} \]
where $a_{12}$ represents the interaction flexibility influence coefficient. Carrying out the integration over the quadrant of the plate, the influence coefficient is evaluated.

$$a_{12} = -\frac{1}{E} \nu \int \frac{dV}{V} = -\frac{\nu b^4 t}{E}$$

This term together with the reciprocal term $a_{21}$ and similar terms for other biaxially stressed areas, when included in the flexibility matrix, account for the Poisson's ratio effect.
APPENDIX III
INSTABILITY ANALYSIS OF THE CONSTANT STRESS AND CONSTANT STRAIN METHODS

The stability of the two methods is easily tested by determining whether perturbations introduced into the analysis grow or decay with succeeding steps. It is instructive, however, to take the following approach. In place of the computed quantities, introduce exact quantities, signified by a caret and associated error terms, in the manner

\[
\{q_1^{(k)}\} = \{\bar{q}_1^{(k)}\} + \{\hat{e}_1^{(k)}\}.
\]

The exact relation is taken to be

\[
\{q_1^{(k)}\} = \left[\Gamma_{lm}\right]\{p_1^{(k)}\} + \left[\Gamma_{ij}\right]\{q_j^{(k)}\}.
\]

Equation 3 can be written in terms of member loads

\[
\{q_1^{(k)}\} = \left[\Gamma_{lm}\right]\{p_1^{(k)}\} + \left[\Gamma_{ij}\right]\{q_j^{(k-1)}\}
\]

Substituting Equation III-1 into Equation III-3 gives

\[
\{q_1^{(k)}\} + \{\hat{e}_1^{(k)}\} = \left[\Gamma_{lm}\right]\{p_1^{(k)}\} + \left[\Gamma_{ij}\right]\{e_j^{(k-1)}\}
\]

from which Equation III-2 may be subtracted to yield the following expression for the errors

\[
\{\hat{e}_1^{(k)}\} = \left[\Gamma_{ij}\right]\{e_j^{(k-1)}\} - \{q_1^{(k)}\}
\]

The constant stress and constant strain methods are now distinguished by the manner in which the \(e_j^{(k-1)}\) are specified. The Ramberg-Osgood stress-strain relation, which we may write in the form

\[
e_1^{(k)} = \frac{q_1^{(k)}}{E_1A_1} + \frac{3}{7} \frac{\sigma_{ol}}{E_1} \left| \frac{q_1^{(k)}}{\sigma_{ol}A_1} \right|^n \frac{q_1^{(k)}}{\left| q_1^{(k)} \right|}
\]

will be used in examining both methods.
The nonlinear strains, in the constant stress method, are given by

\[
\varepsilon_j^{(k-1)} = \frac{3}{7} \frac{\sigma_{0j}}{E_j} \left[ \frac{q_j^{(k-1)}}{\sigma_{0j} A_j} \right] n \frac{n-1}{n-1} \frac{\sigma_{0j}}{\sigma_{0j} A_j} + \ldots
\]

which, on using Equation III-1, may be written in the expanded form

\[
\varepsilon_j^{(k-1)} = \frac{3}{7E_j} \frac{\sigma_{0j}}{q_j^{(k-1)}} \left[ \frac{q_j^{(k-1)}}{\sigma_{0j} A_j} \right] n \frac{n-1}{n-1} \frac{\sigma_{0j}}{\sigma_{0j} A_j} + \ldots
\]

A similar expansion can be constructed for the \( \varepsilon_j^{(k)} \) on introducing

\[
\{ \varepsilon_j^{(k)} \} = \{ \varepsilon_j^{(k-1)} \} + \{ \Delta \varepsilon_j^{(k)} \}
\]

which gives

\[
\varepsilon_j^{(k)} = \frac{3}{7E_j} \frac{\sigma_{0j}}{q_j^{(k-1)}} \left[ \frac{q_j^{(k-1)}}{\sigma_{0j} A_j} \right] n \frac{n-1}{n-1} \frac{\sigma_{0j}}{\sigma_{0j} A_j} + \ldots
\]

Substituting the foregoing expression into Equation III-4, and assuming small errors in the sense that \( \varepsilon_1 \ll q_1 \) and small steps in the loading such that \( \Delta q_1 \ll q_1 \), so that terms containing the products of these quantities may be dropped, we obtain

\[
\{ \xi_1^{(k)} \} = \left[ 1 + \frac{3n}{E_j} \frac{q_j^{(k-1)}}{\sigma_{0j} A_j} \right] \left[ \frac{q_j^{(k-1)}}{\sigma_{0j} A_j} \right] \left[ \frac{\Delta \xi_1^{(k)}}{\xi_j^{(k-1)}} \right] \left\{ \varepsilon_j^{(k-1)} - \Delta \xi_1^{(k)} \right\}
\]

which may be re-written

\[
\{ \xi_1^{(k)} \} = \left[ 1 + \frac{3n}{E_j} \frac{q_j^{(k-1)}}{\sigma_{0j} A_j} \right] \left[ 1 - \frac{\Delta \xi_1^{(k)}}{\xi_j^{(k-1)}} \right] \left\{ \varepsilon_j^{(k-1)} \right\}
\]

III-5

where \( \bar{E}_j \) give the slope of the stress versus inelastic-strain curve at the individual element stress levels.
Instability develops when the errors in the $k^{th}$ step increase over those in the $k-1^{st}$ step. Clearly, the method would be expected to be unstable, if any of the inequalities $e_i(k) > e_i(k-1)$ were satisfied directly. A more critical check, however, is to consider the entire set of $\{e_i\}$ as a vector (in an $n$-dimensional space where $n$ is equal to the number of elements) and apply the condition that the Euclidean length of this vector does not increase. Mathematically, this means that for stability

\[ \{e_i(k)\}^T \{e_i(k)\} \leq \{e_i(k-1)\}^T \{e_i(k-1)\} \]

where the critical condition is defined by using the equality sign.

For simplicity, we consider the case of infinitesimally small steps in the loading, so that the $\Delta q_i$ terms may be neglected, and further, denote

\[ \left[ \begin{array}{c} \Gamma_{ij} \end{array} \right] \left[ \begin{array}{c} 1 / E_j^{(k-1)} \end{array} \right] \left[ \begin{array}{c} A_j \end{array} \right] = \left[ \begin{array}{c} \delta_i^{(k-1)} \end{array} \right] \]

The critical condition for instability then becomes

\[ \{e_i(k)\}^T \{e_i(k)\} = \{e_i(k-1)\}^T \left[ \begin{array}{c} B_i^{(k-1)} \end{array} \right] \left[ \begin{array}{c} B_i^{(k-1)} \end{array} \right]^T \{e_i(k-1)\} \]

\[ = \{e_i(k-1)\}^T \{e_i(k-1)\} \]

It is now observed that the eigenvalues of

\[ \left[ \begin{array}{c} B_i^{(k-1)} \end{array} \right] \left[ \begin{array}{c} B_i^{(k-1)} \end{array} \right]^T \]

will all be positive, hence the condition that none of these eigenvalues be greater than one (which becomes the stability requirement), can be replaced by the more severe condition that the sum of the eigenvalues not be greater than one. This latter condition can be assured by requiring that the sum of the squares of all elements of $B_{ij}$ not be greater than one, and in addition, that the absolute sum of any row or
column in $B_{ij}$ not be greater than one. The critical stress is then
found from the largest of these sums.

In the case of the example truss problem, where $E_i = E = 10^7$ psi,
$n = 10$, $A_i = 1.0$ sq. in., and $\sigma_{i1} = \sigma_o = 10^5$ psi, the sum of the squares
approach, viz.,

$$\left\{ \frac{3n}{\pi^2} \left( \frac{\sigma_{cr}}{\sigma_o} \right)^{n-1} \right\} \times \sum_{ij=1}^{3} \Gamma_{ij}^2 = 1$$

yields $\sigma_{cr} = 83,800$ psi., while the rows and columns approach gives
$\sigma_{cr} = 82,900$ psi.

Recall that the foregoing development has determined the minimum
value at which instability might occur. It is of interest to compute
a critical value of stress at which instability is strongly assured to
occur. This can be done by returning to the initial notion that in-
stability will occur if $\xi_{i1}^{(k)} > \xi_{i1}^{(k-1)}$. This amounts to restricting
attention to the diagonal elements in the $B_{ij}$ matrix. The corresponding
critical stress (lowest value) will be given by the largest (in absolute
value) element on the diagonal in the $B_{ij}$ matrix. This corresponds to
the $-0.414 \times 10^7$ term in the $\Gamma_{ij}$ matrix, so that the critical stress
is given by

$$\frac{3n}{\pi^2} \left( \frac{\sigma_{cr}}{\sigma_o} \right)^{n-1} \times (0.414 \times 10^7) = 1$$

which gives $\sigma_{cr} = 93,800$ psi. The lowest stress at which instability
would develop, in the case of the example truss problem, would therefore
be expected to occur between 82,900 psi and 93,800 psi.

Note that if only one structural element were inelastic, then the
diagonal term in the $\Gamma_{ij}$ matrix corresponding to this element would
give the correct critical stress by the latter procedure. Both of the
foregoing values of critical stress have been indicated in Figure 4,
where they are seen to correlate with the experimental (computer) re-
sults. The simple approach of considering only the diagonal elements
appears to be advantageous in the present problem. With this approach, it is easy to see that finite values of $\Delta q_1$, which make the elements in the second diagonal matrix in Equation III-5 less than one, must raise the stress for instability, which explains the progression of critical stresses (with increasing load step size) appearing in Figure 4.

Finally, it is noted that certain of the foregoing results, viz., the value of $\sigma_{cr}$ based on one diagonal elements alone, can be obtained simply by introducing $q_1^{(k)} + \psi_1^{(k)}$ in place of $q_1^{(k)}$ directly into Equation III-3, and regarding the $q_1^{(k)}$ as perturbations on the $q_1^{(k)}$. The additional results, such as the demonstration of the effects of finite $\Delta q_1$, however, are not obtained by this procedure.

In the case of the constant strain method, in addition to the error quantities in the member loads

$$\{q_1^{(k)}\} = \{q_1^{(k)}\} + \{e^{(k)}_1\}$$

we also introduce error quantities for the relaxed loads

$$\{q_1^{(k)}\} = \{q_1^{(k)}\} + \{e^{(k)}_1\}$$

The equation defining the relaxed loads, written for the generic $i^{th}$ member, is

$$\frac{q_1^{(k)}}{E_1 A_1} + e_1^{(k-1)} = \frac{q_1^{(k)}}{E_1 A_1} + \frac{3q_1^{(k)}}{7E_1 A_1} \frac{\Delta q_1^{(k)}}{\sigma_{cr} A_1}$$

which may be written in the following form

$$\begin{vmatrix} q_1^{(k)} + e_1^{(k)} \\ E_1 A_1 \end{vmatrix} = \begin{vmatrix} q_1^{(k)} \\ E_1 A_1 \end{vmatrix} + \begin{vmatrix} 3q_1^{(k)} + \Delta q_1^{(k)} \\ \sigma_{cr} A_1 \end{vmatrix} \begin{vmatrix} q_1^{(k)} \\ \Delta q_1^{(k)} \end{vmatrix}$$

or

$$\begin{vmatrix} q_1^{(k)} + e_1^{(k)} \\ E_1 A_1 \end{vmatrix} = \begin{vmatrix} q_1^{(k)} \\ E_1 A_1 \end{vmatrix} + \begin{vmatrix} 3q_1^{(k)} + \Delta q_1^{(k)} + e_1^{(k)} \\ \sigma_{cr} A_1 \end{vmatrix} \begin{vmatrix} q_1^{(k)} \\ \Delta q_1^{(k)} \end{vmatrix}$$
If we now apply the condition of no load reversals, then the terms \( q_1^{(k)}/q_1^{(k)} \) etc., will all produce the same sign (for a given element) and hence may be cancelled out. Applying the condition that all error terms \( r_1^* \) and load increments \( Aq_1 \) are much smaller than the load magnitude \( |q_1| \), leads to the result

\[
\xi_1^{(k)} = \xi_1^{(k)} - \frac{E_1}{E_{*1}^{(k-1)}} \left( \xi_1^{(k-1)} - \xi_1^{(k)} - Aq_1^{(k)} \right)
\]

where

\[
\frac{1}{E_{*1}^{(k-1)}} = \frac{3n}{(E_1)} \frac{\sigma_{*1}^{(k-1)}}{|\sigma_{*1}^{(k-1)}|^{n-1}}
\]

The corresponding form of Equation III-4 may now be written by introducing the exact load-reduction-increments \( \Delta q_1^{(k)} \), where

\[
\Delta q_1^{(k)} = q_1^{(k)} - q_1^{(k)}
\]

which leads to

\[
\begin{bmatrix}
\xi_1^{(k)} \\
\end{bmatrix} = \begin{bmatrix}
\frac{3\sigma_{*1}^{(k-1)}}{E_{*1}^{(k-1)}} \frac{\sigma_{*1}^{(k-1)} + \xi_1^{(k-1)}}{|\sigma_{*1}^{(k-1)}|^{n-1}} A_1^{(k-1)} \\
\frac{3\sigma_{*1}^{(k-1)}}{E_{*1}^{(k-1)}} \frac{\sigma_{*1}^{(k-1)} + \Delta q_1^{(k-1)} + Aq_1^{(k-1)}}{|\sigma_{*1}^{(k-1)}|^{n-1}} q_1^{(k)} \\
\end{bmatrix}
\]

Applying, once again, the smallness requirement on the \( r_1^* \) and \( Aq_1 \), yields

\[
\begin{bmatrix}
\frac{E_1}{E_{*1}^{(k-1)}} \xi_1^{(k)} \\
\end{bmatrix}
= 
\begin{bmatrix}
\begin{bmatrix}
\frac{3\sigma_{*1}^{(k-1)}}{E_{*1}^{(k-1)}} \frac{\sigma_{*1}^{(k-1)} + \xi_1^{(k-1)}}{|\sigma_{*1}^{(k-1)}|^{n-1}} A_1^{(k-1)} \\
\frac{3\sigma_{*1}^{(k-1)}}{E_{*1}^{(k-1)}} \frac{\sigma_{*1}^{(k-1)} + \Delta q_1^{(k-1)} + Aq_1^{(k-1)}}{|\sigma_{*1}^{(k-1)}|^{n-1}} q_1^{(k)} \\
\end{bmatrix}
\end{bmatrix}
\]

\[
III-7
\]

70
where the absolute value signs have been omitted for simplicity.

The check for the occurrence of instability may now be carried out in the same manner as for the constant stress method. Thus, considering only the case of infinitesimal load increments, the corresponding form of the $B_{ij}$ matrix, as defined by Equation 111-6, is found to be

$$
B_{ij} \approx \frac{E_{ij}(\sigma)_{i}}{E_{ij}(\sigma)_{i} + E_{ij}} \left[ \begin{array}{c}
\left[ \Gamma_{ij} \right] \left[ \frac{1}{E_{ij}(\sigma)_{j}} A_{j} \right] + \left[ \frac{E_{ij}}{E_{ij}(\sigma)_{i}} \right]
\end{array} \right]
$$

A numerical check for the special case of the truss problem shows that the critical condition of the eigenvalues summing to unity calls for physically inadmissible values of $E_{ij}(\sigma)$. A simpler demonstration of this property is provided by the "direct approach" (i.e., setting $\xi_{ij}^{(k)} = \xi_{ij}^{(k-1)}$). In this case, the critical stress is given by the lowest value corresponding to the "n" equations obtained by equating the diagonal elements of $F_{ij}$. In the example truss problem, where $E_{ij} = E$, $\sigma_{ij} = \sigma_{ij}$, and $A_{ij} = 1.0$, and where the three diagonal elements in the $\Gamma_{ij}$ matrix can be denoted by $-\xi_{ij}^{E}$, where in turn $0 < \xi_{ij}^{E} < 1$, the foregoing matrix equation reduces to the following simple algebraic equation

$$
\frac{-\xi_{ij}^{E} + E}{E_{ij}(\sigma)_{i} + E} = \pm 1
$$

The indicated critical values are easily seen to be $-\xi_{ij}^{E}$ and $(2 - \xi_{ij}^{E})$, both of which are physically inadmissible for the Ramberg-Osgood stress-strain relation. Thus, the constant strain method is indicated to be free of instability in this case. The problem of accuracy, of course, is another matter, due to the necessity of working with finite (and preferably large) load steps. These, apparently, are responsible for the slow divergence of the computer results shown in Figure 5.
APPENDIX IV

ROTATION OF AXES OF ANISOTROPY

Let the x and y axes be rotated from the axes of anisotropy in a positive sense so that from the strain transformation equations we get:

\[
\begin{align*}
\delta e_x &= \lambda^2 \delta e_{11} + m^2 \delta e_{22} + 2\mu \delta e_{12} \\
\delta e_y &= m^2 \delta e_{11} + \lambda^2 \delta e_{22} - 2\mu \delta e_{12} \\
\delta e_{xy} &= -\lambda m \delta e_{11} + \mu m \delta e_{22} + 2(\lambda^2 - m^2) \delta e_{12}
\end{align*}
\]

when \(\lambda\) and \(m\) are the usual direction cosines of the x-axis with respect to the orthogonal axes.

Similarly from the stress transformation equations:

\[
\begin{align*}
\sigma_{11} &= \lambda^2 \sigma_x + m^2 \sigma_y - 2\mu \sigma_{xy} \\
\sigma_{22} &= m^2 \sigma_x + \lambda^2 \sigma_y + 2\mu \sigma_{xy} \\
\sigma_{12} &= \mu \sigma_y - \lambda \sigma_x + (\lambda^2 - m^2) \sigma_{xy}
\end{align*}
\]

Substituting Equation IV-2 into the appropriate expressions in Equation 12 gives a set of equations as follows:

\[
\begin{align*}
\delta e_{11} &= \frac{d\lambda}{\partial x} \left[ (\lambda^2 \sigma_{11} - m^2 \sigma_{12}) \sigma_x + (m^2 \sigma_{11} - \lambda^2 \sigma_{12}) \sigma_y ight] \\
&\quad - 2\lambda m (\sigma_{11} + \sigma_{12}) \sigma_{xy} \\
\delta e_{22} &= \frac{d\lambda}{\partial x} \left[ (m^2 \sigma_{22} - \lambda^2 \sigma_{12}) \sigma_x + (\lambda^2 \sigma_{22} - m^2 \sigma_{12}) \sigma_y ight] \\
&\quad + 2\lambda m (\sigma_{22} + \sigma_{12}) \sigma_{xy} \\
\delta e_{12} &= \frac{d\lambda}{\partial x} \left[ 3\sigma_{14} [m(\sigma_x - \sigma_y) + (\lambda^2 - m^2) \sigma_{xy}] \right]
\end{align*}
\]

72
Finally substituting Equation IV-3 into Equation IV-1, simplifying and defining three new coefficients $\delta_1$, $\delta_2$, and $\delta_3$ such that $\delta = \alpha_{12} - \frac{3}{2}m_4$, $\delta_1 = (\alpha_{11} + \delta)$ and $\delta_2 = (\alpha_{22} + \delta)$ the appropriate incremental flow equations become:

$$
\begin{align*}
\dd \epsilon_x &= \frac{d \epsilon_x}{d \delta} \left[ (L^2 \delta_1 + m^2 \delta_2 - \delta) \sigma_x - (\alpha_{12} - L^2 m^2 \delta_1 - L^2 m^2 \delta_2) \sigma_y \right] \\
&\quad - 2a^2(m^2 \delta_1 - m^2 \delta_2) \sigma_{xy} \\
\dd \epsilon_y &= \frac{d \epsilon_y}{d \delta} \left[ (L^2 \delta_2 + m^2 \delta_1 - \delta) \sigma_y - (\alpha_{12} - L^2 m^2 \delta_2 - L^2 m^2 \delta_1) \sigma_x \right] \\
&\quad - 2a^2(m^2 \delta_1 - L^2 \delta_2) \sigma_{xy} \\
\dd \epsilon_{xy} &= \frac{d \epsilon_{xy}}{d \delta} \left[ (3\alpha_{14} + hm^2 \delta_1 + hm^2 \delta_2) \sigma_{xy} \right] \\
&\quad - 2a^2(m^2 \delta_1 - L^2 \delta_2) \sigma_x - 2a^2(m^2 \delta_1 - L^2 \delta_2) \sigma_y
\end{align*}
$$

Insertion of Equation IV-2 into 11 gives the expression for effective stress:

$$
\sigma^2 = \left[ (L^2 \delta_1 + m^2 \delta_2 - \delta) \sigma_x^2 - 2(\alpha_{12} - L^2 m^2 \delta_1 - L^2 m^2 \delta_2) \sigma_x \sigma_y \right] \\
+ (L^2 \delta_2 + m^2 \delta_1 - \delta) \sigma_y^2 + (3\alpha_{14} + hm^2 \delta_1 + hm^2 \delta_2) \sigma_{xy}^2 \\
- 4a^2(m^2 \delta_1 - L^2 \delta_2) \sigma_x \sigma_{xy} - 4a^2(m^2 \delta_1 - L^2 \delta_2) \sigma_y \sigma_{xy}
$$

Now suppose we make an uniaxial stress-strain test for the x-direction. The valid expressions become:

$$
\begin{align*}
\dd \epsilon_x &= \frac{d \epsilon_x}{d \delta} (L^2 \delta_1 + m^2 \delta_2 - \delta) \sigma_x \\
\sigma^2 &= (L^2 \delta_1 + m^2 \delta_2 - \delta) \sigma_x^2
\end{align*}
$$

From the three Equations IV-4, we obtain equations similar to Equation 15. This equation is augmented under restrictions of equal plastic work thus:

$$
\begin{align*}
a \sqrt{\alpha_{11}} &= b \sqrt{\alpha_{22}} \\
&\quad c \sqrt{\alpha_{33}} = d \sqrt{L^2 \delta_1 + m^2 \delta_2 - \delta}
\end{align*}
$$

73
Assigning the value of unity to one "α" as before we obtain the relative values of the others. In particular we now obtain a value for 

\[(t^4 \delta \ + m^4 \delta_2 - \delta)\]

which is a function of \(\alpha_{11}, \alpha_{22}, \alpha_{33}\) all known and \(\alpha_{44}\) unknown. Therefore \(\alpha_{44}\) may be determined.
APPENDIX V

INELASTIC MATRIX COMPUTER PROGRAM

A. Program Description

This is a brief description of the Grumman inelastic matrix program for carrying out elastic-plastic-creep analysis - deck No. 45128 - Elastic Unloading (follows Hooke's Law when unloading), including anisotropy.

This program for elastic-plastic analysis has proved to be quite adaptable for analytic investigations. It has been modified to include an option for anisotropic analysis. Previously, the program was modified to use time-hardening theory, deformation theory, inelastic unloading, and constant stress theory (none of which have been retained in the final program). It was not necessary to do major program revisions to accommodate these variations. The program housekeeping is arranged so that modifications to the manipulation of data will not affect the housekeeping. Thus the program makes a convenient framework to explore various calculation procedures.

The program is written in Fortran II to run on the Grumman IBM 7094s. These are 2 channel (A and B) machines with 6 drives per channel; 32,768 words of core storage; on-line card reader; on-line printer; and printer clock. The program is set to run under Fortran Monitor control which uses a $JOB card for identification. Input is on logical tape 7 (A-2), print output on logical tape 6 (A-2) and punched output on logical tape 5 (B-4) (not used by this program). Logical tape 8 (B-1) is used for storage of binary output which is converted to BCD print output in link 6 (at the end of job). The program will accept an input data tape on logical tape 9 (A-5) and will write a binary save tape for restart on logical tape 11 (A-6). These 2 auxiliary tapes are optional for each run (see description of the control cards). The Grumman IOU subroutine, as well as the subroutines for rewinding and unloading a tape (RUN) and for moving to the start of a designated file (FILTAP) are included, as required in the program, in column binary form.

The program tape furnished to Wright Patterson Air Force Base contains all the information needed to duplicate our analysis. It is in the following sequence:

File 1 - a 1-card BCD label tested by the program to distinguish BCD data tape from binary save tape.

File 2 - BCD card images for matrices SIM AND SIL. See pages 79 and 80 for a description of the matrices and their format.
File 3 - BCD card images of Fortran program (6 links), each link including binary subroutines previously mentioned.

Multiple (6) end of file marks.

We recommend that all 3 files of this tape be copied for use, then that file 3 be punched onto Fortran cards. The punching program must be able to handle mixed-mode cards to accommodate the short binary subroutines. This deck of cards, with the proper *I.D. or $JOB card, will then be in proper form for a Fortran compile and execute, using the copied tape on drive A-5 (logical 9). The program uses only the data from files 1 and 2; it will not move into file 3. This 2-file format for the data tape is generated by the GISMO matrix system in use at Grumman and elsewhere.

Each link of the program contains the non-IBM subroutines needed for operation. Standard input-output subroutines etc. will be taken from the library tape. As a point of information, the program contains six links numbered consecutively from 1 to 6.

1) Link 1 reads the first control card, and reads all other decimal input supplied.

2) Link 2 is used only on a restart job. It reads the modified step table, if provided, and part of the binary input tape.

3) Link 3 is used only on a restart job, and reads the balance of the binary input tape.

4) Link 4 is the processing link. It does all the calculation, print control, and writing of binary output on tape B-1 to be converted to BCD print output by Link 6.

5) Link 5 writes a binary tape for restart, then transfers to Link 6. If no binary tape is to be written, exit is from Link 6.

6) Link 6 reads the binary output stored on tape B-1 and converts it to BCD output on tape A-3 for printing. When tape B-1 has been completely processed, a message to the operator indicates that it need not be saved. If, due to machine error or operator intervention, tape B-1 is not processed into prints on A-3, but is saved, then Link 6 can be used as a separate program using B-1 as a data tape and will process B-1 into prints on A-3. Link 6 does not use any data from COMMON. All necessary clues are stored on B-1.

Sequencing and details of the data cards follow. The symbols used in the program for various items of input data are listed on page 77 and are shown on the sample key-punching sheet page 78.
The data cards are used in the following sequence, immediately after the "DATA card required by the Monitor:

1. General clue card (FORMAT 1) containing KLU4, NINCLOD, NA, KJUT, KLUISO, ALPHA, BETA, GAMMA, GNU, and a title or caption.

2. Table of load or time steps desired. Up to ten cards defining ten steps may be used. Load steps and time steps may be used in any sequence. The maximum level for each load or time step may be above or below the previous maximum level of load or time. The program verifies the algebraic sign of the increment, and corrects it if necessary. Each card contains four variables TEMP1, TEMP2, TEMP3, TEMP4 in FORMAT 2.

3. Data matrices. These may be provided in any sequence. Each matrix has a header card in FORMAT 3, one or more data cards in FORMAT 4, and a blank card to end it. The last input matrix on the Monitor input tape must be followed by one added blank card (two total) to trip the program into operation.

In the event that a job is running overtime, and it is desired to stop the program in a restartable form, mount a blank tape on logical tape 11 (drive A-6) and set Sense Switch 6 on. This will write the contents of memory on A-6 in proper form to continue the run later. The program distinguishes between the saved binary tape and a decimal input tape at starting time. Either is mounted on logical tape 9 (Drive A-5).

Built-in pauses in the program are as follows:

1. Pause 11111 to mount the data input tape at the start of the program, if all matrices are not on the Monitor input tape A-2. For this KLU4, in the first data card, should be a "1" to "4" indicating the count of decimal matrices on A-5. If A-5 is a binary saved tape from a preceding run, any digit (except zero) acceptable for KLU4.

2. Pause 1 to mount a blank tape on A-6 to receive memory. This is reached either with Sense Switch 6, or with a card in the table of steps punched TAPE in columns 7-10.

B. Symbols and Format of the Data Cards

1. General Clue Card - FORMAT 1

Cols. Field Symbol

| 1 | 11 | KLU4 |

This gives the number of input matrices on the auxiliary input tape (A-5). If all matrices are on the monitor tape, leave this blank. If using binary restart tape, use a digit. Maximum number of decimal input matrices on the auxiliary input tape is 4.
This gives the number of non-increment re-
cycles at each load or time level. The
total cycles at each level is NINCLD + 1.
When printing, the first, tenth, twentieth,
etc., and last cycle of each printable level
will be printed.

This sets the frame size for the problem
to be handled. NA is three times the
number of nodes. Maximum value is 165
(55 nodes).

0, or blank, prints 5 preselected matrices
on cycles indicated by the step table;
1 prints all matrices on cycles indicated
by the step table. 2 prints 5 preselected
matrices on all cycles; 3-9 print all
matrices on all cycles.

0 indicates an isotropic run
1 indicates an anisotropic run

Not used

A variable defined by the creep-strain
equation.

A variable defined by the creep-strain
equation.

A variable defined by the creep-strain
equation.

Poisson's ratio "nu"

Not used

Any 30 characters of alpha-numeric text
to be printed as a heading for identifica-
tion purposes.

LOAD indicates a load step, TIME indicates a
time step, TAPE indicates write memory on a
save-tape on drive A-6 (logical #11) then exit.
Col. Field

FINS or blank indicates end of table (this may be the 11th card in the table).

11-20 E10.6 Upper limit of step in pounds or hours

21-30 E10.1 Interval or increment for calculation

31-40 E10.1 Interval or increment for print output. Prints are generated on tape A-3 for the current cycle when the current load or time level is an integral multiple of the print interval. If the print interval is left blank, no prints are generated for cycles in this step. If print interval is very small compared to the current level, then numeric problems sometimes occur in the print control subroutine (OUTPU), and it may be necessary to re-run with the every-cycle print control "2" or "3" for K1U7 punched in the first control card.

41-80 Ignored

3. Data Matrices - Header Card - FORMAT 3

Cols. Field

1-6 6X Not used

7-10 4X Not used. We use the letters MTRX for compatibility with the GISMO Matrix System, which reads and writes matrices in this format.

11 1X Not used

12-17 A6 This is the identification name for the input matrices and must correspond exactly with one of the following names:

- bbbSIM Matrix of stresses for applied loads maximum size 165x1
- bbbSLJ Matrix of stresses for member strains maximum size 165x165
- bbbEPSN Table of strain values 11x1

These two matrices define the stress-strain curve as a series of chords. The data is entered in this format.

T9
merely to conform to the format of the SDM-SIJ matrices which were generated using the GIXNO Matrix System. Note that the first value in both TSIGN and TEPSN must be zero to avoid upsetting the interpolation procedure.

\[
\begin{align*}
TALF_{12} & \; \alpha_{12} \\
TALF_{23} & \; \alpha_{23} \\
TALF_{31} & \; \alpha_{31} \\
TALF_{44} & \; \alpha_{44}
\end{align*}
\]

Basic anisotropic parameters

\[
\begin{array}{ccc}
18 & 1X & \text{Not used} \\
19-21 & I3 & \text{Number of rows in this matrix} \\
22-24 & I3 & \text{Number of columns in this matrix} \\
25-32 & 8X & \text{Not used} \\
33 & 11 & \text{The digit 3} \\
34-80 & \text{Not used}
\end{array}
\]

4. Data Matrices - Data Cards - FORMAT 4

<table>
<thead>
<tr>
<th>Cols.</th>
<th>Field</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1X</td>
</tr>
<tr>
<td>2-4</td>
<td>I3</td>
</tr>
<tr>
<td>5-7</td>
<td>I3</td>
</tr>
<tr>
<td>8-23</td>
<td>E16.8</td>
</tr>
<tr>
<td>24</td>
<td>1X</td>
</tr>
<tr>
<td>25-27</td>
<td>I3</td>
</tr>
<tr>
<td>28-30</td>
<td>I3</td>
</tr>
<tr>
<td>31-46</td>
<td>E16.8</td>
</tr>
<tr>
<td>47</td>
<td>1X</td>
</tr>
<tr>
<td>48-50</td>
<td>I3</td>
</tr>
<tr>
<td>51-53</td>
<td>I3</td>
</tr>
</tbody>
</table>
The last card of a matrix must be completely blank (tested in Col. 2-4). The last matrix on the Monitor input tape 7 (drive A-2) must be followed by one added blank card (two total) to trip the program into operation.

The input matrices on the Monitor tape may be in any sequence as long as each matrix starts with a header card, has all its data cards next, and ends with a blank card.

C. Anisotropic Parameter Matrices

If a run is indicated as isotropic in the first control card (KUISO in Col. 9 is zero or blank) the program will read in the anisotropic parameters, if provided; then it will replace them with the built-in parameters for the isotropic condition. If a run is indicated as anisotropic in the first control card (KUISO is 1 to 9), then each of the anisotropic parameter matrices must be provided or the program will terminate on an error.

D. Restart Procedure

If the run being set up is a restart, the input deck can be in several forms. The first control card must have a digit in KU4 so that the program will read tape A-5; NINCUL and KU7 are read from this card. The other factors are carried from the previous run. The continuation of an isotropic run will always be isotropic, and conversely, regardless of the clue provided (KUISO).

The table of steps may be read in again (modified) if the previous run is stopped with sense switch 6, or it may be retained and continued from the previous run. However, if the previous run is stored on tape A-6 by using a TAPE card in the step table, then a new step table must be read in.

In either case, the last data card on a restart job must be FINS or blank in columns 7-10. This means a restart data deck will have a minimum of two cards (clue card and a blank), or a maximum of 12 cards (clue card, 10 step cards and a FINS or blank).

E. Time Estimates

For time estimates, allow 3.5 minutes to compile the Fortran, 1.5 minutes to read the input tape, 2.0 seconds per cycle printed, and 100 to 110 cycles of calculation per minute. Each printed cycle writes approximately 3 feet of print tape (55 node problems).
Matrix Analysis of an Inelastic Plate

### GENERAL CONTROL CARD - ONE REQUIRED

```
| 01 | 02 | 03 | 04 | 05 | 06 | 07 | 08 | 09 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
```

**FORMAT 1**

- **EX**
- **A4**
- **E16.6**
- **E10.3**
- **F5.2**

### STEP TABLE (ONE TO TEN CARDS PLUS A FINS OR BLANK CARD)

```
| 01 | 02 | 03 | 04 | 05 | 06 | 07 | 08 | 09 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
```

- **LOAD**
- **UPPER LIMIT OF STEP**
- **INCREMENT FOR CALCULATION**
- **INCREMENT FOR PRINT**

### MATRIX HEADER CARDS (DEFINE THE NAME, SIZE AND START OF A MATRIX)

```
| 01 | 02 | 03 | 04 | 05 | 06 | 07 | 08 | 09 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
```

- **NAME**
- **NRROWS**
- **NCOLOLS**
- **I-NFORM**

### MATRIX DATA CARDS (THREE ELEMENTS PER CARD - ROW SORT - L&S)

```
| 01 | 02 | 03 | 04 | 05 | 06 | 07 | 08 | 09 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
```

- **MR(1)**
- **ND(1)**
- **EL(1)**
- **MR(2)**
- **ND(2)**
- **EL(2)**

### EACH MATRIX IS TERMINATED BY ONE COMPLETELY BLANK CARD

**CODING INSTRUCTIONS:**

1. Alphabetical characters are written as follows:
   - A B C D E F G H I J K L M N O P Q R S T U V W X Y Z
2. Numerical characters are written as follows:
   - 1 2 3 4 5 6 7 8 9 0

**GRUMMAN AIRCRAFT ENGINEERING CORP.**
Anisotropic - Anisotropic

Elastic Unloading (Folllows Hooke's Law)

1. Vary load level at constant time
2. Vary time, hold load level constant
3. Store memory on tape A-8 (logical no. 11) for restart, then exit
   end of table entries (blank card is also accepted)

FORMAT 3

3. TALF31 and TALF44 are similar

LAST CARD MAY HAVE 1 OR 2 ELEMENTS FORMAT 4

<table>
<thead>
<tr>
<th>Type</th>
<th>IX</th>
<th>I3</th>
<th>I3</th>
<th>E16.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2)</td>
<td>NR(2)</td>
<td>ND(2)</td>
<td>RL(2)</td>
<td></td>
</tr>
</tbody>
</table>

RENT | ROW COL | ELEMENT

R0

December, 1965
FLOW CHART FOR
INELASTIC PLATE
ANALYSIS
Enter Link 1

Is This a Continuation Run Or a New One

New

Read In All Decimal Data

Link 4

Set Print Control. Do All Calculations. Write All Output On Tape B1

Is Saved Tape Needed (Sense SW. 6 or Input Card)

Yes

Link 5

Write Memory Tables On Save-Tape-A-6

No

Link 2

Read In Step Table and Saved Binary Data

Link 3

Read In The Save Of Saved Binary Data

Link 6

Convert Binary Output On Tape B-1 To Output On The Print Tape

Exit
Enter Link 2

Read Saved Control Variables. KLU6, KLU8 and NIN/LD Are Used From This Run's Initial Clue Card

Write headings

Read Step Table Proc. Tape 7

Is Control Word Valid

Yes

Is Control Word KINS or Blank

Yes

Has Any Step Control Been Read

Yes

Skip Over Saved Control Tables Arrays PMTM and KIMTM

Read Saved Tables of Control Information Arrays PMTM and KIMTM

Write Out List of Steps And Increments

Write Out Message For Non-Increment Cycles

No

Write Message Exit

Store Control
: Format File:
Arrays: PMTM
A : KIMTM

Is Increment Blank In A Load Or Time Step

Yes

Write Message Exit

No

Read Binary Matrices From Tape 9

Call Chain (3,3)
Enter Link 3

Read The Balance of the Saved Binary Matrices From Tape 9

Rewind and Unload the Auxiliary Input Tape 9

Print Operator Message Relative to Updating the Saved Binary Tape

Write Out Modulus of Elasticity and Shear Modulus

Write Out Stress-Strain Table

Call Chain (4,3)
Enter Link 3

Rewind Tape 11 Drive A-6

Write Coded BCD Label on A-6

Write End-of-file on A-6

Write Various Control Clues and Single-Valued Variables on A-6

Write Matrices To Be Saved For Re-Start On A-6

Write End-of-Files on A-6

Rewind and Unload A-6

Print Message To Operator To Save Tape A-6 For Re-Start At This Point

Call Chain (6, 3)
Enter Link 6

Rewind Tape 8 Drive B-1

Read A Record From Tape 8 B-1

Is First Word Between 1 and 9

Yes

Is Clue 7

Yes

Write Operator Message

Final Write A-3

Rewind 8

Exit

No

Branch to Proper Output Statement

Write Output Tape 6 (A-3)
FORTRAN II LISTING
FOR INELASTIC PLATE
ANALYSIS
MATRIX ANALYSIS FOR ISOTRUPIC OR ANISOTROPIC INELASTIC STRUCTURES
IN THE PLASTIC AND CREEP REGIME

THIS PROGRAM WRITTEN FOR AIR FORCE CONTRACT AF 33(615)-2260

** TABLE OF SYMBOLS USED

* AL1212 - MATRIX OF ANISOTROPIC PARAMETERS FOR EACH NUDE
* AL1223 - MATRIX OF ANISOTROPIC PARAMETERS FOR EACH NUDE
* AL1231 - MATRIX OF ANISOTROPIC PARAMETERS FOR EACH NUDE
* ALFA44 - MATRIX OF ANISOTROPIC PARAMETERS FOR EACH NUDE
* DELEPK - MATRIX OF NUDE STRAIN CHANGES (DELTA EPSILON)
  FOR THE X AND Y DIRECTIONS, AND THE SHEAR STRAIN CHANGE
* DELEPN - MATRIX OF EFFECTIVE INELASTIC STRAIN CHANGES
* EPBARN - MATRIX OF EFFECTIVE INELASTIC STRAINS (EPSILON BAR SUB N)
  FOR K-TH CYCLE
* EPBARP - MATRIX OF EFFECTIVE INELASTIC STRAINS (EPSILON BAR SUB N)
  FOR K-1 CYCLE (THE PRECEDING CYCLE)
* EPCNCP - MATRIX OF EFFECTIVE CREEP STRAINS FOR THIS CYCLE
* EPSK - MATRIX OF NUDE PLASTIC STRAINS (EPSILON SUB U) FOR THE
  K-TH CYCLE
* KPMTM - TABLE OF LOAD INCREMENTS, PRINT CONTROL INCREMENTS AND
  UPPER LOAD LEVEL PER STEP (CONTROL INFORMATION)
* SGBARN - MATRIX OF EFFECTIVE NUDE STRESSES FOR LAST CYCLE THAT
  SHOWED AN INCREASE AT A PARTICULAR NODE
* SIGUK - MATRIX OF NUDE STRESSES (SIGMA SUB U) FOR K-TH CYCLE
* SIJ - MATRIX OF STRESSES FOR MEMBER STRAINS
* SIM - MATRIX OF STRESSES FOR APPLIED LOADS
* TALF12 - MATRIX OF ANISOTROPIC PARAMETERS (INPUT DATA)
* TALF23 - MATRIX OF ANISOTROPIC PARAMETERS (INPUT DATA)
* TALF31 - MATRIX OF ANISOTROPIC PARAMETERS (INPUT DATA)
* TALF44 - MATRIX OF ANISOTROPIC PARAMETERS (INPUT DATA)
* TEFSTN - TABLE OF TOTAL NUDE EFFECTIVE STRAINS
* TEPSN - TABLE OF STRAIN VALUES DEFINING THE STRESS-STRAIN CURVE
* TIMK1 - MATRIX OF REFERENCE CREEP TIMES FOR ALL NODES
* TOTEPS - MATRIX OF TOTAL NUDE STRAINS
* TSIGN - TABLE OF STRESS LEVELS DEFINING THE STRESS-STRAIN CURVE

* VARIABLES AND CLUES
  * ALPHA - PARAMETER USED IN THE CREEP EQUATIONS
  * BETA - PARAMETER USED IN THE CREEP EQUATIONS
  * K - THE CYCLE COUNTER
  * GAMMA - PARAMETER USED IN THE CREEP EQUATIONS
  * GNU - POISSONS RATIO
  * KERAS - CLUE USED FOR TEMPORARY INDICATOR BETWEEN LINKS
  * KLU4 - CLUE INDICATING TOTAL COUNT OF MATRICES ON AUXILIARY TAPE
  * KLU5 - CLUE INDICATING MATRICES STILL NOT READ FROM AUX. TAPE
KLUB - CLUE FOR PRINT CONTROL (WHICH CYCLES) 0060
KLUB - CLUE FOR PRINT CONTROL (WHICH MATRICES) 0061
KLUI50 - CLUE FOR ISOTROPIC OR ANISOTROPIC RUN (INPUT) 0062
KSET - CLUE FOR THE LOAD OR TIME LEVEL CURRENTLY IN USE 0063
NA - 3 TIMES THE NODE COUNT (INPUT) 0064
NC - NUMBER OF NODES 0065
NINCLD - NUMBER OF NON-INCLUSION CYCLES AT EACH LOAD LEVEL (INPUT) 0066
PM - CURRENT LOAD LEVEL 0067
SHRMOD - SHEAR MODULUS 0068
TIME - CURRENT TIME LEVEL 0069
OUTPUT - SUBROUTINE TO CONTROL PRINTING AT VARYING LOAD LEVELS 0070

ELASTIC UNLOADING (FOLLOWS HOOKE'S LAW WHEN UNLOADING)
COMMON TEFSTII, TEPSI 0074
COMMON KLUS, KLUS, KLUS, KLUS, NA, NC, K 0075
COMMON KERRSW, NINCLD, KSET, PM, E, GNU, SHRMOD 0076
COMMON ALPHA, BETA, GAMMA, TIME, KLUI50 0077
COMMON PNTM, KPTM, SIM, SIIJ, TSIGN, TEPSN 0078
COMMON TALF12, TALF23, TALF31, TALF44 0079
COMMON AL12, AL123, AL1231, ALFA44 0080
DIMENSION TALF12(), TALF23(), TALF31(), TALF44() 0081
DIMENSION AL12(), AL123(), AL1231(), ALFA44() 0082
EQUIVALENCE (TLMKII, TEFSTII, TTEPSI, TDELEPSI) 0083

THESE FOUR ARRAYS ARE EQUIVALENT TO SAVE CORE SPACE.
DIMENSION TLMKII(), TEFSTII(), TDELEPSI() 0084
DIMENSION PNTM(), KPTM(), SIM(), SIIJ(), TSIGN() 0085
DIMENSION TEPSN() 0086
DIMENSION BL() 0087
DIMENSION PSTEP() 0088
EQUIVALENCE (ID(), ID(), ID(), ID(), ID(), ID(), ID(), ID()) 0089

AD1 = 606060606060 0090
AD2 = 606060606060 0091
AD3 = 606060606060 0092
AD4 = 606060606060 0093
AD5 = 606060606060 0094
AD6 = 606060606060 0095
AD7 = 606060606060 0096
AD8 = 606060606060 0097
AD9 = 606060606060 0098
BL1() = 606060606060 0099
BL2() = 606060606060 0100
BL3() = 606060606060 0101
BL4() = 606060606060 0102
BL5() = 606060606060 0103
KERRSW = 1 0104
INTAPE = 7 0105
KLUS = 0 0106
KLUS = 0 0107
KLUS = 0 0108
KLUS = 0 0109
KLUS = 0 0110
KLUS = 0 0111
KLUS = 0 0112
PSTEP(1) = 0.0 0113
PSTEP(2) = 0.0 0114
PSTEP(3) = 0.0 0115
DO 51 J22 = 1, 10  
51 KPM(M(J22)) = 0  
LROW = 0  
READ INPUT TAPE 7, I, KLU4, KLUT, KLUSO, ALPHA, BETA, GAMMA,  
IGNU, TA, TB, TL, TO, TE  
WRITE OUTPUT TAPE 6, 21, TA, TB, TL, TO, TE  
C KLU7 (INPUT) = 0 TO PRINT PRE-SELECTED MATRICES ON SELECTED CYCLES  
C KLU7 (INPUT) = 1 TO PRINT ALL MATRICES ON SELECTED CYCLES  
C KLU7 (INPUT) = 2 TO PRINT PRE-SELECTED MATRICES ON ALL CYCLES  
C KLU7 (INPUT) = 3-9 TO PRINT ALL MATRICES ON ALL CYCLES  
IF(KLU7) 384, 386, 381  
381 IF(KLU7-1) 383, 383, 382  
382 CONTINUE  
C KLU6 (OUTPUT) = 1 TO PRINT EACH CYCLE, 0 TO PRINT SELECTED CYCLES  
IF(KLU6-2) 384, 384, 383  
383 KLU8 = 0  
C KLU8 (OUTPUT) = 0 TO PRINT ALL MATRICES  
C KLU8 (OUTPUT) = 10000 TO PRINT SELECTED MATRICES  
384 CONTINUE  
IF(KLU8) 302, 302, 301  
301 PRINT 17  
PAUSE 11111  
C TEST INPUT TAPE TO DISTINGUISH GISMU BCD FROM INELASTIC PHOC. SAVE  
READ INPUT TAPE 9, 31, LTAP  
REWD 9  
C SUBROUTINE FILTAP POSITIONS A TAPE AT THE FIRST RECORD OF ANY FILE  
CALL FILTAP(9, 2)  
IF(LTAP-21) 343, 343, 344  
344 CONTINUE  
C SAVED TAPE FROM PREVIOUS RUN OF INELASTIC PLATE  
C THIS RUN IS A CONTINUATION  
CALL CHAIN (2, 3)  
343 CONTINUE  
C GISMU FORMAT BCD TAPE  
C THIS RUN IS A NEW ONE  
KLUS = KLU4  
INTAPE = 9  
302 CONTINUE  
C KLUSO = 0 FOR ISOTROPIC RUN  
C KLUSO = 1 FOR ANISOTROPIC RUN  
WRITE OUTPUT TAPE 6, 9  
IF(KLUSO) 380, 380, 379  
379 CONTINUE  
WRITE OUTPUT TAPE 6, 10  
380 CONTINUE  
52 READ INPUT TAPE 7, 2, TEMP1, TEMP2, TEMP3, TEMP4  
LROW = LROW + 1  
DO 53 J22 = 1, 5  
53 CONTINUE  
B IF(TEMP1-1) 53, 54, 53  
54 GO TO(53, 55, 55, 60, 60) J22  
55 IF(TEMP2-1) 56, 56, 53  
95
C
KPMTH(N) = 1 FOR A LuaU step
KPMTH(N) = 2 FOR A TIME step
KPMTH(N) = 3 TO OUMP MEMORY INTO A SAVE TAPE
0172
C
PMTM(N+1) = UPPER LIMIT OF step
0173
PMTM(N-1) = INTERVAL (INCREMENT) FOR CALCULATION
0174
PMTM(N-2) = INTERVAL (INCREMENT) FOR PRINT OUTPUT
0175
56 KPMTH(LROW) = J22
0176
PMTM(LROW,1) = TEMP2
0177
TEMP5 = TEMP2-PSTEP(J22)
0178
PMTM(LROW,2) = SIGNF(TEMP5,TEMP2)
0179
PSTEP(J22) = TEMP2
0180
IF(PMTM(LROW+2))57,61,57
0181
03 GO TO(995,9995,57),J22
0182
57 CONTINUE
0183
PMTM(LROW,3) = TEMP4
0184
GO TO 52
0185
50 CONTINUE
0186
DO 361 J23 = 1,10
0187
14(KPMTM(J23))361,361,362
0188
362 IF(KPMTM(J23)-4)363,361,361
0189
363 J24 = KPMTM(J23)
0190
GO TO (364,366,368),J24
0191
364 WRITE OUTPUT TAPE 6,22,PIMM(J23,2),PMTM(J23,1)
0192
365 WRITE OUTPUT TAPE 6,23,PIMM(J23,2)
0193
GO TO 361
0194
366 WRITE OUTPUT TAPE 6,24,PIMM(J23,2),PMTM(J23,1)
0195
367 WRITE OUTPUT TAPE 6,26,PIMM(J23,3)
0196
GO TO 361
0197
368 WRITE OUTPUT TAPE 6,27
0198
361 CONTINUE
0199
WRITE OUTPUT TAPE 6,28,NINC6
0200
WRITE OUTPUT TAPE 6,25,ALPHA,DETA,OMMA
0201
IF(GNU)=87,87
0202
86 GNU = 87
0203
87 CONTINUE
0204
IF(INA)9999,91,92
0205
91 NA = 169
0206
92 IF(NA-165193,93,999
0207
93 CONTINUE
0208
96 IF(INA-NA/3)9999,97,999
0209
97 NC = NA/3
0210
IF(KLUS)304,304,303
0211
303 IF(6KUS-61306,306,994
0212
306 CONTINUE
0213
INTAPE = 9
0214
KLUS = KLUS + 1
0215
108 READ INPUT TAPE INTAPE, J,NAME,NAWS,NCOLS,IFORM
0216
00 110 (1,110,110,110)
0217
110 CONTINUE
0218
BAD INPUT - MATRIX NAME NOT ACCEPTABLE
0219
GO TO 997
0220
311 GO TO (321,322,323,324,325,326,327,328,111,121
0221
321 DO 331 11 =1,165
0222
96
331  SIM(11) = 0.0
     WRITE OUTPUT TAPE 6,20,NOCOLS,INTAPE
     GO TO 111
322  DO 332 11 = 1,165
     DO 332 12 = 1,165
332  SIJ(11,12) = 0.0
     GO TO 111
323  DO 333 11 = 1,11
333  TSIGN(11) = 0.0
     GO TO 111
324  DO 334 11 = 1,11
334  TEP5N(11) = 0.0
     GO TO 111
325  DO 335 11 = 1,11
335  TALF12(11) = 0.0
     KLUALF = KLUALF + 1
     GO TO 111
326  DO 336 11 = 1,11
336  TALF23(11) = 0.0
     KLUALF = KLUALF + 2
     GO TO 111
327  DO 337 11 = 1,11
337  TALF31(11) = 0.0
     KLUALF = KLUALF + 4
     GO TO 111
328  DO 338 11 = 1,11
338  TALF44(11) = 0.0
     KLUALF = KLUALF + 8
     GO TO 111
112  READ INPUT TAPE INTAPE,4, (NR(122),ND(122),EL(122),122=1,3)
     IF(NR(11))996,109,113
109  IF(KLU5)308,308,308
     INTAPE = 7
308  GO TO 108
804  GO TO 308
113  GO TO (121,122,125,126,201,202,203,204,1501,121)
C  READ IN ARRAY SIGMA-IM
121  MROW = NR(1)
     WRITE OUTPUT TAPE 6,4, (NR(122),ND(122),EL(122),122=1,3)
     SIM (MROW) = EL(1)
     IF(EL(1))127,128,127
127  MROW = NR(2)
     SIM (MROW) = EL(2)
128  IF(EL(3))129,112,129
129  MROW = NR(3)
     SIM (MROW) = EL(3)
     GO TO 112
C  READ IN ARRAY SIGMA-IJ
122  MROW = NR(1)
     MCOL = ND(1)
     SIJ (MROW,MCOL) = EL(1)
     IF(EL(2))130,131,130
130  MROW = NR(2)
     MCOL = ND(2)
     SIJ (MROW,MCOL) = EL(2)

97
131 IF(EL(3)) 132, 112, 132
132 MROW = NR(3)
MCOL = ND(3)
SJL (MROW, MCOL) = EL(3)
GO TO 112

C READ IN ARRAY TSIGN (TABLE OF SIGMA BAR N)
125 MROW = NR(1)
TSIGN(MROW) = EL(1)
IF(EL(2)) 139, 140, 139
139 MROW = NR(2)
TSIGN(MROW) = EL(2)
140 IF(EL(3)) 141, 112, 141
141 MROW = NR(3)
TSIGN(MROW) = EL(3)
GO TO 112

C READ IN ARRAY TEPSN (TABLE OF EPSILON BAR N)
126 MROW = NR(1)
TEPSN(MROW) = EL(1)
IF(EL(2)) 114, 114, 114
142 MROW = NR(2)
TEPSN(MROW) = EL(2)
143 IF(EL(3)) 112, 112, 112
144 MROW = NR(3)
TEPSN(MROW) = EL(3)
GO TO 112

C READ IN ALPHA TABLES
201 MROW = NR(1)
TALF12(MROW) = EL(1)
IF(EL(2)) 221, 222, 221
221 MROW = NR(2)
TALF12(MROW) = EL(2)
222 IF(EL(3)) 223, 223, 223
223 MROW = NR(3)
TALF12(MROW) = EL(3)
GO TO 112
202 MROW = NR(1)
TALF23(MROW) = EL(1)
IF(EL(2)) 224, 225, 224
224 MROW = NR(2)
TALF23(MROW) = EL(2)
225 IF(EL(3)) 226, 226, 226
226 MROW = NR(3)
TALF23(MROW) = EL(3)
GO TO 112
203 MROW = NR(1)
TALF31(MROW) = EL(1)
IF(EL(2)) 227, 227, 227
227 MROW = NR(2)
TALF31(MROW) = EL(2)
228 IF(EL(3)) 229, 229, 229
229 MROW = NR(3)
TALF31(MROW) = EL(3)
GO TO 112
204 MROW = NR(1)
TALF44(MROW) = EL(1)
IF(EL(2)) 230, 231, 230
230 MROW = NR(2)
231 IFIEL(3) = 112, 232
232 MROW = NR(3)
TALF44(MROW) = EL(3)
GO TO 112
150 CONTINUE
161 IF(KLUAF-15) = 167, 162
162 WRITE OUTPUT TAPE 6,3,2, KLUAFA
IF(KLUAF) = 164, 163
C SUBROUTINE RUN REWINDS AND UNLOADS THE DESIGNATED TAPE
163 CALL RUN(9)
164 CALL EXIT
165 DO 166 I = 1, 11
TALF23(I) = 0.5
TALF23(I) = 0.5
TALF31(I) = 0.5
166 TALF44(I) = 1.0
167 CONTINUE
E = T5IGN(2)/TEPSN(2)
81 SHRMD = E/(1.0+GNUM)
WRITE OUTPUT TAPE 6,3,E,SHRMU,GNU
WRITE OUTPUT TAPE 6,6
DO 149 II = 1 11
149 WRITE OUTPUT TAPE 6,7,II, TALF23(II), TALF31(II), TALF44(II)
K = 0
PM = 0.0 IF(KLU3) = 152, 152, 209
C SUBROUTINE RUN REWINDS AND UNLOADS THE DESIGNATED TAPE
209 CALL RUN(9)
151 PRINT 19
152 CALL CHAIK(4,3)
994 WRITE OUTPUT TAPE 6,18,KLUA CALL EXIT
995 WRITE OUTPUT TAPE 6,16,KLUA CALL EXIT
996 CONTINUE
WRITE OUTPUT TAPE 6,14,NAME
WRITE OUTPUT TAPE 6,6,1(NR(122),NU(122),EL(122),122=1,3)
997 CONTINUE
WRITE OUTPUT TAPE 6,13, NAME
CALL EXIT
999 WRITE OUTPUT TAPE 6,11,NA
CALL EXIT
1 FORMAT(11X,213,211,1X,2E10,3.2F5,2,1UX,5A6)
2 FORMAT(6X,4,10,6,10,1,E10,11)
3 FORMAT(11X,6,1X,213,8X,11)
4 FORMAT(3(1X,213,11X,8)
5 FORMAT(26H1 MODULUS OF ELASTICITY = ,FI1.0,4H PSI,6X,16H SHEAR MODU
1LUS = ,FI1.0,4H PSI,6X,5HNU = ,Fo.31
6 FORMAT(4H TABLE OF VALUES FOR STRESS-STRAIN CURVE //
15X,29H POINT STRESS LEVEL SINAIN,7X,8H ALPH A 12,7x,8H ALPHA 23,3X
1,8H ALPH A 31,7X,8H ALPH A 44/16X,3HPSI,9X,7MIN,7IN//1

99
7 FORMAT(6X,15,2X,F11.2,5X,F12.4)
  9 FORMAT//79H THIS ISOTROPIC KIN USES ELASTIC UNLOADING (HOUKES L
 1AM WITH STRAIN HARDENING)
10 FORMAT(1H4,5X,2HAN)
11 FORMAT(10H ERROR NA=14)
12 FORMAT(2BH ERROR- INCREMENT CARD NO.; 13,5M N.G.)
13 FORMAT(1H4 ERROR-MATRIX (AO))
14 FORMAT(17H ERROR-NEG. INDEX AO)
16 FORMAT(3H ERROR- INTERVAL-INCREMENT CARD, 13)
17 FORMAT(15HO PAUSE 1111 INPUT DATA TAPE ON DRIVE A-5/)
18 FORMAT(1H4 ERROR- 15,18H INPUT MATRICES NG)
19 FORMAT(11H DEMOUNT AND SAVE TAPE A-5, DATA HAS BEEN READ INTO MEM.)
  IRY, THE PROGRAM CONTINUES ON RUN///
20 FORMAT///5X,7HMA triX (AO,IX;13,6H KUWS X,13,1H LU LUNSN FROM TAPE
  1,12)
 21 FORMAT///L1H,29X,5A6)
 22 FORMAT(5X,16HLOAD INCREMENTS ,PY.2;11H POUNDS TO ,F10.2;7H JUNOS)
 23 FORMAT(1H4,6LX,19HPRINT OUTPUT EVERY ,F9.2;7H POUNDS)
 24 FORMAT///5X,6HTIME INCREMENTS ,RY.4;11H MOURS TO ,F10.4;5H MOURS)
 25 FORMAT///9H ALPHA = ,E10.3;5X,7HUE = ,E10.3;5X,7HUMM = ,E10.3)
 26 FORMAT///1H4,6LX,19HPRINT OUTPUT EVERY ,F9.4;6H MOURS)
 27 FORMAT///5X,5HSTORE MEMORY ON TAPE A-6, THEN EXIT)
 28 FORMAT///5X,13,48H NON-INCREMENT CYCLES AT EACH LOAD OR TIME LEVEL
  1)
 19
31 FORMAT///LX,12)
 32 FAYOUT///5X,12,65H ALPHA TABLES WERE READ IN UP 4 REQUIRED, CHECK YU
 1UR INPUT CARDS.)
        END1110,0,0,0,1,1,0,1,0,0,0,0,0)
*  CHAIN(2,3)
*  LIST8
C451282 MATRIX ANALYSIS OF INELASIC PLATE - LINK 2 - WITH CREEP

COMMON TEFSTN, TDEPS
COMMON KL44, KL55, KL66, KL88, NA, NC, K
COMMON KERRSW, NINC1D, KSET, PM, E, GNU, SHRMD
COMMON ALPHA, BETA, SIGMA, TIME, KLISO
COMMON PSTM, KSTM, S11, SIJ, TIGN, TEPSN
COMMON TALF12, TALF23, TALF31, TALF44
COMMON ALL122, ALL123, ALL124, ALFA44
DIMENSION TALF12(11), TALF23(11), TALF31(11), TALF44(11)
DIMENSION ALL122(155), ALL123(155), ALL124(155), ALFA44(155)
COMMON SIGUK, EPSUK
EQUIVALENCE (TIMK1, TEFSTH, TTEPSN, TULIEPK)

* THESE FOUR ARRAYS ARE EQUIVALENCED TO SAVE CORE SPACE.

DIMENSION TIMK(355), TEFSTN(355), TDEPS(355), ULEPK(355)
DIMENSION PSTM(10,3), KSTM(10,3), S11(165), SIJ(165), TIGN(11)
DIMENSION TEPSN(11), SIGUK(165), EPSUK(165)
DIMENSION BL(15)
DIMENSION PSTM(3)

READ TAPE 9, KL44, KL55, KL66, KL88, NA, NC, K, KERRSW, KSET, PM, E
COMMON SIGUK, EPSUK

* THESE FOUR ARRAYS ARE EQUIVALENCED TO SAVE CORE SPACE.

DIMENSION TIMK(355), TEFSTN(355), TDEPS(355), ULEPK(355)
DIMENSION PSTM(10,3), KSTM(10,3), S11(165), SIJ(165), TIGN(11)
DIMENSION TEPSN(11), SIGUK(165), EPSUK(165)
DIMENSION BL(15)
DIMENSION PSTM(3)

WRITE OUTPUT TAPE 6,9

IF (KLU4 = 1) THEN WRITE OUTPUT TAPE 6,10
IF (KLU4 = 2) THEN WRITE OUTPUT TAPE 6,10
READ TAPE 5, KL44, KL55, LA, LB, NA, NC, K, KERRSW, KSET, PM, E

101
C KPMTM(N) = 2 FOR A TIME STEP
C KPMTM(N) = 3 TO DUMP MEMORY INTO A SAVE TAPE
C PMTH(N,1) = UPPER LIMIT OF STEP
C PMTH(N,2) = INTERVAL (INCREMENT) FOR CALCULATION
C PMTH(N,3) = INTERVAL (INCREMENT) FOR PRINT OUTPUT
56 KPMTMLNKON) = J22
NCLU = 2
KSET = 1
PMTMLROW(1) = TEMP2
TEMP5 = TEMP2-STEP(J22)
PMTMLROW(2) = SIGN(TEMP5,TEMP5)
STEP(J22) = TEMP2
IF(PMTMLROW(2)) = 157,63,57
63 GO TO 1995,995,57,J22
57 CONTINUE
PMTMLROW(3) = TEMP4
GO TO 52
60 CONTINUE
GO TO (61,62),NCLU
62 CONTINUE
READ TAPE 9,JUNK
READ TAPE 9,JUNK
GO TO 64
61 CONTINUE
READ TAPE 9,PMTM
READ TAPE 9,KPMTM
64 CONTINUE
WRITE OUTPUT TAPE 6,1
DO 361 J23 = 1,10
IF(KPMTM(J23)) = 361,361,361
362 IF(KPMTM(J23)) = 4) 363,361,361
363 J24 = KPMTM(J23)
GO TO (364,365,366,367),J24
364 WRITE OUTPUT TAPE 6,22,KPM(J23,2),PMTM(J23,1)
365 WRITE OUTPUT TAPE 6,23,KPM(J23,3)
GO TO 361
366 WRITE OUTPUT TAPE 6,24,KPM(J23,2),PMTM(J23,1)
367 WRITE OUTPUT TAPE 6,25,KPM(J23,3)
GO TO 361
368 WRITE OUTPUT TAPE 6,27
361 CONTINUE
WRITE OUTPUT TAPE 6,26,NINCLU
READ TAPE 9,ALI
READ TAPE 9,TSIGN
READ TAPE 9,TEPSN
READ TAPE 9,TALF12
READ TAPE 9,TALF23
READ TAPE 9,TALF31
READ TAPE 9,TALF44
READ TAPE 9,ALI212
READ TAPE 9,ALI223
READ TAPE 9,ALI231
READ TAPE 9,ALI244

102
READ TAPE 9, SIGUK
READ TAPE 9, EPSUK
CALL CHAIN (3,3)
945 WRITE OUTPUT TAPE 6,16, LNUM
CALL EXIT
1 FORMAT(1H0,29X,37HCONTINUATION RUN - INELASTIC ANALYSIS)
2 FORMAT(6X,A4,E10.6, E10.1,E10.1)
9 FORMAT(79H THIS ISOTROPIC RUN USES ELASTIC UNLOADING (HOOKE'S LAW) WITH STRAIN HARDENING)
10 FORMAT(1H+,5X,2IAN)
12 FORMAT(26H ERROR- INCREMENT CARD NU.-13,5H N.G.)
16 FORMAT(33H ERROR-NU INTERVAL-INCREMENT CARD-0,13)
22 FORMAT(5X,16HLOAD INCREMENTS,F9.2,F11H POUNDS TO ,F10.2,F7H POUNDS)
23 FORMAT(1H+,61X,19HPRINT OUTPUT EVERY ,F9.2,F7H POUNDS)
24 FORMAT(5X,16HTIME INCREMENTS,F9.4,F11H HOURS TO ,F10.4,F6H HOURS)
25 FORMAT(1H+,61X,19HPRINT OUTPUT EVERY ,F9.4,F6H HOURS)
26 FORMAT(5X,13,48H NON-INCREMEN CYCLES AT EACH LOAD OR TIME LEVEL 1)
27 FORMAT(5X,35HSTORE MEMORY ON TAPE A-6, THEN EXIT)
END(1,1,0,0,0,0,1,1,0,1,0,0,0,0,0)
* CHAIN(3,3)
* LIST8
C451283 MATRIX ANALYSIS OF INELASTIC PLATE LINK 3 - WITH CREEP

C THIS IS THE SECOND HALF OF UU LINK 2

C THIS LINK READS IN A SAVED BINARY TAPE - SECOND PART

COMMON TEFSTN, TOTEPS
COMMON KLU4, KLU5, KLU6, KLU7, NA, NC, K
COMMON KERRS, NINCLO, KSET, PM, E, GUN, SHRMUD
COMMON ALPHA, BETA, GAMMA, TIME, KLUSU
COMMON PHM, KPHM, TSHM, TS1J, TSIGN, TEPSN
COMMON TALF12, TALF23, TALF31, TALF44
COMMON ALI12, ALI23, ALI31, ALFA4
DIMENSION TALF12(11), TALF23(11), TALF31(11), TALF44(11)
DIMENSION ALI12(55), ALI23(55), ALI31(55), ALFA4(55)
COMMON SIGUK, EPSUK
COMMON SUBARP, SUMBAR, EPBARP, EPBARN, DELEPN
COMMON EPCNK, EPCNP
EQUIVALENCE (TIMK, TEFSTN), (IUTEPS, DELEPN)

C THESE FOUR ARRAYS ARE EQUIVALENCED TO SAVE CURE SPACE.

DIMENSION TIMK(10,15), TEFSTN(155), IUTEPS(165), DELEPN(165)
DIMENSION PMTM(10,3), PMTPM, PMIM, IUTPS(165), DELEPN(165)
DIMENSION SIGUK(155), EPSUK(155)
DIMENSION SUBARP(155), SUMBAR(155), EPBARP(155), DELEPN(155), EPCNK(155)

READ TAPE 9, SUBARP
READ TAPE 9, SUMBAR
READ TAPE 9, EPBARP
READ TAPE 9, DELEPN
READ TAPE 9, EPCNK
READ TAPE 9, SUBARP
READ TAPE 9, SUMBAR
READ TAPE 9, EPBARP
READ TAPE 9, DELEPN
READ TAPE 9, EPCNK

COMMON SIGUK, EPSUK
COMMON SUBARP, SUMBAR, EPBARP, EPBARN, DELEPN
COMMON EPCNK, EPCNP
EQUIVALENCE (TIMK, TEFSTN), (IUTEPS, DELEPN)

C THESE FOUR ARRAYS ARE EQUIVALENCED TO SAVE CURE SPACE.

DIMENSION TIMK(10,15), TEFSTN(155), IUTEPS(165), DELEPN(165)
DIMENSION PMTM(10,3), PMTPM, PMIM, IUTPS(165), DELEPN(165)
DIMENSION SIGUK(155), EPSUK(155)
DIMENSION SUBARP(155), SUMBAR(155), EPBARP(155), DELEPN(155), EPCNK(155)

READ TAPE 9, SUBARP
READ TAPE 9, SUMBAR
READ TAPE 9, EPBARP
READ TAPE 9, DELEPN
READ TAPE 9, EPCNK

CALL RUN(9) (I
PRINT 31
WRITE OUTPUT TAPE 6,5,5, SHRMUD, 100
WRITE OUTPUT TAPE 6,7, 11, F
DO 149 11 = 1, 11
149 WRITE OUTPUT TAPE 6,7, 11, TSUM(11), TEPSN(11), TALF12(11), TALF23(11)
CALL CHAIN(4, 3)
5 FORMAT(26H MODULUS OF ELASTICITY = , F11.0, 4H PSI, 6X, 16H SHRMUD MOU
11US = , F11.0, 4H PSI, 6X, 5H WJ = , F0.3)
6 FORMAT(41H TABLE OF VALUES FOR STRESS-STRAIN CURVE /
15X, 29 POINT STRESS LEVEL SINAIN, TX, BHALPHA 12, TX, BHALPHA 23, TX
1, BHALPHA 31, TX, BHALPHA 44, 10X, 3HPSI, 9X, THIN, ., 7)
7 FORMAT(1X, 13, 2X, F11.2, 51X, F14.8)
31 IF NECESSARY I STOP THIS RUN BECAUSE IT EXCEEDS THE TIME ESTIMATE, PUT A RING IN THE SAVE TAPE ON A-5 AND CHANGE IT / 256H TO DRIVE A-6 TO UPDATE IT, THEN PUT SENSE SWITCH 6 ON. //
END(1, 1, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 0)

SAVE TAPE B-1 UNLESS ON-LINE PRINT SAYS IT HAS BEEN PROCESSED
PUT SENSE SWITCH 6 ON TO END THE RUN IF IT EXCEEDS THE TIME ESTIM.
IF PAUSE OCCURS (SENSE SWITCH 6 OR INTERNAL CONTROL), MOUNT A
BLANK TAPE ON A-6. THIS TAPE WILL HAVE RESTART DATA WRITTEN ON IT.
AND MUST BE SAVED.
CHAIN(4, 3)
LIST

104
C  THIS LINK USES THE CALCULATION AND WRITES PRINT OUTPUT

COMM TEFSTN, TUTEPS

COMM KLU4, KLU5, KLU6, NA, NC, K

COMM KERRSW, NINCLO, KSET, PM, E, GNU, SHRMD

COMM ALPHA, BETA, GAMMA, TIME, KLUISO

COMM PMTM, KPMTM, SIJM, SIJ, TSIGN, TEPSN

COMMON TALF12, TALF23, TALF31, TALF44

COMMON AL1212, AL1223, AL1231, ALFA44

COMMON SIGUK, EPSUK

COMMON SGBARN, SGBARP, SUGARN, EPSUK, EPBARN, EPCNP

COMMON EPCNK, EPCNP

EQUIVALENCE (TIMK, TEFSTN), (TUTEPS, DELEPK)

C THESE FOUR ARRAYS ARE EQUIVALENCED TO SAVE CORE SPACE.

C THEY ARE NOT THE SAME, BUT ARE NOT NEEDED AT THE SAME TIME

C AND DO NOT CARRY FROM CYCLE TO CYCLE.

DIMENSION TIMK(55), TEFSTN(55), TUTEPS(165), DELEPK(165)

DIMENSION PMTM(10,3), KPMTM(10,3), SIJM(165), SIJ(165,165), TSIGN(11)

DIMENSION TEPSN(11), SIGUK(55), EPSUK(55), SGBARN(55), SGBARP(55)

DIMENSION EPCNK(55), EPCNP(55)

KSET = KSET

KERRSW = KERRSW

GO TO (418, 419), KERRSW

C INITIALIZE WORK AREAS

DO 102 I = 1, 165

EPSUK(I) = 0.0

102 SIGUK(I) = 0.0

DO 103 J = 1, 55

DELEPK(I) = 0.0

103 SGBARN(J) = 0.0

SGBARP(J) = 0.0

EPBARN(J) = 0.0

EPCNP(J) = 0.0

AL1212(I) = 2.*TALF12(I)

AL1223(I) = TALF23(I) + TALF44(I)

AL1231(I) = TALF12(I) + TALF31(I)

ALFA44(I) = TALF44(I)

EPBARN(I) = 0.0

EPCNP(I) = 0.0

KSET = 1

TIME = 0.0

CONTINUE

REWIND 8

KSET = KSET

KERRSW = KERRSW

NINCMP = NINCLO + 2

NINTUT = NINCLO + 1

151 K = K + 1

KLU2 = 1

C IF KLU2 = 0, THE CYCLE OF OPERATIONS WILL BE PRINTED

C KLU2 = 1 WILL PRINT EVERY CYCLE, U WILL PRINT ONLY SELECTED CYCLES
C  KLU2 = 0  WILL PRINT ALL MATRICES, NON-ZERO PRINTS SELECTED ONES
   IF(KLU6)1248,248,249
248 KLU2 = 0
249 CONTINUE
   IF(K-1)270,270,271
270 KLU2 = 0
271 CONTINUE
   GO TO (416,417), KERRSW
417 KLU2 = 0
418 KERRSW = 1
419 CONTINUE
C  KSET IS THE ROW OF KPMTM ON KPMTM BEING USED (CURRENT LOAD OR
C  TIME LEVEL)
300 IF(KSET=11)301,319,319
301 IF(KPMTM(KSET))302,302,303
302 KSET = KSET + 1
   GO TO 300
303 IF(KPMTM(KSET)-4)304,304,302
304 J24 = KPMTM(KSET)
   KSET = KSET
   KERRSW = KERRSW
   IF(KPMTM(KSET,2))342,342,341
341 GO TO (305,315,423), J24
342 GO TO (345,355,423), J24
305 IF(KPMTM(KSET,1))306,306,307
306 IF(KPMTM(KSET,1)>(TIME+PMTM(KSET,2)))308,308,309
307 IF(NINCLD)309,309,308
309 KLU2 = 0
308 DELPM = PMTM(KSET,2)
   GO TO 154
310 IF(NINCPLM-2)316,316,154
316 KLU2 = 0
   GO TO 302
315 IF(KPMTM(KSET,1)>(TIME+PMTM(KSET,2)))308,317,318
317 IF(NINCLD)314,314,318
314 KLU2 = 0
318 DELTM = PMTM(KSET,2)
   GO TO 330
319 IF(KLU2)320,320,320
320 KLU2 = 0
   KSET = KSET - 1
   IF(KSET)321,321,321
321 IF(KPMTM(KSET))322,322,323
322 J24 = KPMTM(KSET)
   GO TO (308,318), J24
323 CONTINUE
1: CONTINUE
324 CONTINUE
1: CONTINUE
   IF(NINCLD)301,501,510
510 IF(KPMTM(KSET))511,511,511
511 NINCPLM = NINCPLM - 1
   IF(NINCPLM-31513,515,515
513 CONTINUE

106
GO TO (501,512),NINCPM 0732
C NINCSW = 1 MEANS THIS IS A K-E CYCLE STEP (NO LOAD OR TIME INCREASE) 0733
C NINCSW = 2 MEANS THIS IS LAST K-E CYCLE AT THIS LOAD OR TIME LEVEL 0734
C NINCSW = 3 MEANS THIS IS A LOAD OR TIME INCREMENT STEP 0735
512 NINCSW = 2 0736
GO TO 514 0737
515 NINCSW = 1 0738
GO TO 514 0739
501 GO TO (502,503),J24 0740
502 PM = PM + DELPM 0741
GO TO 504 0742
503 TIME = TIME + DELTIM 0743
504 NINCPM = NINCLD + 2 0744
NINCSW = 3 0745
514 CONTINUE 0746
IF(NINCLD)1524,524,525 0747
525 NINKCY = NINCLD + 3 - NINCPM 0748
GO TO (523,524,524),NINCSW 0749
523 NINKLU = NINKCY - (NINKCY/LUL)*10 0750
IF(NINKLUI)1328,524,328 0751
C NINKCY IS COUNT OF TOTAL CYCLES AT THIS LOAD OR TIME LEVEL 0752
C NINKLU = 0 WILL PRINT EACH TENTH CYCLE, IF THIS LOAD OR 0753
C TIME LEVEL IS TO BE PRINTED 0754
524 GO TO (324,325),J24 0755
324 CALL OUTPUT(PM,PHTM(KSET,J),LUL) 0756
GU TO 326 0757
325 CALL OUTPUT(TIME,PHTM(KSET,J),LUL) 0758
326 IF(KLU2)327,328,327 0759
327 KLU2 = LUL 0760
328 CONTINUE 0761
IF(TIME)333,333,333 0762
332 GO TO (333,337),J24 0763
C COMPUTE REFERENCE CREEP TIME FOR ALL NODES 0764
337 DU DU 331 13 = 1*NC 0765
EPCNP(13) = EPCNK(13) 0766
FACNUM = ABSF(EPCNP(13)) 0767
EXPON = BETA*ABSF(SGBARN(13))*1. 0768
FACON1 = 2.71828183**EXPON 0769
TIMK11(13) = (FACNUM/(ALPHA*FACON1))**(1./GAMMA) 0770
GO TO (336,336,334),NINCSW 0771
336 CONTINUE 0772
EPCNK(13) = (ALPHA*TIMK11(13))**(GAMMA)*FACON1 0773
GO TO 331 0774
334 CONTINUE 0775
EPCNK(13) = (ALPHA*TIMK11(13)+DELTIM)**GAMMA*FACON1 0776
331 CONTINUE 0777
333 CONTINUE 0778
B 156 FA = 214747433125 0779
KLU9 = 1 0780
IF(KLU2)250,250,251 0781
250 CONTINUE 0782
C LABEL = APPLIED LOAD 0783
B FB = 24604462124 0784
B FC = 606060606060 0785
KLUSIZ = 2 0786
WRITE TAPE 6,KLU9,KLUSIZ,FA,FB,FC,PM,TIME 0787
107
251 CONTINUE
  KLU9 = 2
  IF(KLU2+KLU8)252,252,253
252 CONTINUE
  C LABEL = REF. CREEP TIME
  B FA = 51252636023
  B FB = 512525476063
  B FC = 31425606000
  WRITE TAPE 8,KLU9, NC,FA,FB,FC,(TIMK1(J1),J1=1,NC)
  B FA = 252626335023
  B FB = 512525476062
  B FC = 635121314560
  WRITE TAPE 8,KLU9, NC,FA,FB,FC,(EPCKM(K1),J1=1,NC)
253 CONTINUE
  C SAVE PRECEDING CYCLE VALUES OF EPBARN
  DO 152 I1 = 1,55
152 EPBARN(I1) = EPBARN(I1)
  C CALCULATE NODE STRESSES - MATRIX SIGUK - FRAME SIZE 165 X 1
  DO 661 I5 = 1,NA
    SIGUK(I15) = SIM(I5)*PM
    DO 661 I6 = 1,NA
      SIGUK(I15) = SIGUK(I15) + S1J(I5,I6)*EPSUK(I6)
661 CONTINUE
  IF(KLU2)254,254,255
254 CONTINUE
  C LABEL = NOUE STRESSES
  B FA = 454224256062
  B FR = 63512526225
  B FC = 626060606060
  WRITE TAPE 8,KLU9, NA,FA,FB,FC,(SIGUK(J1),J1=1,NA)
255 CONTINUE
  C CALCULATE MAGNITUDE AND SIGN OF EFFECTIVE STRESS AT EACH NODE
  C CALCULATE EFFECTIVE STRESSES - MATRIX SGBARN - FRAME SIZE 55 X 1
  DO 166 I7 = 1,NC
    SGBARN(I7) = SGBARN(I7)
    M3 = 3*I7
    M32 = M3-2
    M31 = M3-1
    SGBARN(I7) = SQRT(AL123117)*SIGUK(M32)*2-AL121217*SIGUK(M32)*
                 SIGUK(M31)+AL122317*SIGUK(M31)**2+3.*ALFA44(I7)*SIGUK(M3)**2
166 CONTINUE
  IF(KLU2)256,256,257
256 CONTINUE
  C LABEL = EFF. STRESSES
  B FA = 252626336062
  B FR = 63512526225
  B FC = 626060606060
  WRITE TAPE 8,KLU9, NC,FA,FB,FC,(SGBARN(J1),J1=1,NC)
257 CONTINUE
  C CALCULATE EFFECTIVE INELASTIC STRAIN FOR EACH NODE - INTERPOLATE
  C IN TABLE (TSIGN VS. TEPNS)
  DO 181 I8 = 1,NC
258 CONTINUE
  C SGBARN IS EFFECTIVE STRESS OF PREVIOUS CYCLE
  IF(SGBARN(18)=SGBARN(18))411,401,401
  C EFFECTIVE STRESS IS ABOVE PREVIOUS LEVEL
  C SGBARN IS EFFECTIVE STRESS OF LAST CYCLE TO SHOW AN INCREASE
C EFFECTIVE STRESS IS AVOID KNEE OF PREVIOUS DROP-OFF, IF ANY

401 IF(SGBARN(I8) > SGBARM(I8)) 1411, 402, 403
402 IF(SGRAMP(I8) = SGBARM(I8)) 013, 403
403 SGBARM(I8) = SGBARN(I8)
        DU 171 19 = 1,11
        ESUPRK = (SGBARN(I8) / E) + EPBARP(I8) + IF(ESUPRK - TEPSN(I8)) 1/3, 1/2, 1/1
    171 CONTINUE
    GO TO 998

172 BARSIGN = TSIGN(I9)
        IF(SGBARN(I8) = TSIGN(I9)) 177, 178
177 AL1212(I8) = 2. * TALF12(I9)
        AL1223(I8) = TALF12(I9) + TALF34(I9)
        AL1231(I8) = TALF12(I9) + TALF31(I9)
        ALFA44(I8) = TALF44(I9)
    178 CONTINUE
    GO TO 174

174 KKK2 = 19
        KKK1 = 19 - 1
        SINRAT = (ESUPRK - TEPSN(KKK1))/TLEPSN(KKK2) - TEPSN(KKK1)
        BARSIGN = TSIGN(KKK1) + (TSIGN(KKK2) - TSIGN(KKK1)) * SINRAT
        IF(SGBARN(I8) = TSIGN(KKK1)) 175, 176
175 CONTINUE
        ALFA12 = TALF12(KKK1) + (TALF23(KKK2) - TALF12(KKK1)) * SINRAT
        ALFA23 = TALF23(KKK1) + (TALF23(KKK2) - TALF23(KKK1)) * SINRAT
        ALFA31 = TALF31(KKK1) + (TALF31(KKK2) - TALF31(KKK1)) * SINRAT
        ALFA44(I8) = TALF44(KKK1) + (TALF44(KKK2) - TALF44(KKK1)) * SINRAT
        AL1212(I8) = 2. * ALFA12
        AL1223(I8) = ALFA12 + ALFA23
        AL1231(I8) = ALFA12 + ALFA31
    176 CONTINUE
    GO TO 174

174 EPBARN(I8) = ESUPRK - BARSIGN / E
        IF(EPBARN(I8) = EPBARN(I8)) 178, 179
178 CONTINUE
        TEFSN(I8) = ESUPRK
179 CONTINUE
145 TEFSN(I8) = TEFSN(I8) + EPCNK(I8)
        C CALCULATE INCREMENTAL EFFECTIVE INELASTIC STRAIN
        DELEPN(I8) = EPBARN(I8) - EPBARP(I8)
        GO TO (420, 425, 426)
425 DELEPN(I8) = DELEPN(I8) + EPCNK(I8) - EPCNP(I8)
        GO TO 420
        CONTINUE
420 CONTINUE
        GO TO 181

C DROP-OFF OF EFFECTIVE STRESS

C OR STILL BELOW THE KNEE OF PREVIOUS DROP-OFF

411 EPBARN(I8) = EPBARP(I8)
        TEFSN(I8) = EPBARN(I8) + (SGBARN(I8) / E)
        TEFSN(I8) = TEFSN(I8) + EPCNK(I8)
        DELEPN(I8) = 0.0
        GO TO (424, 426, 427)
426 DELEPN(I8) = EPCNK(I8) - EPCNP(I8)
        GO TO 424
424 CONTINUE
181 CONTINUE
        IF(IKLUZ+KLU8) 266, 266, 267
266 CONTINUE
C LABEL = EFF.PLASTIC STRAIN
B FA = 252626334743
B FB = 216263132360
d FC = 626351213145
WRITE TAPE 8,KLU9, NC,FA,FB,FC,(EP3BARN(J1),J1=1,NC)
267 CONTINUE
IF(KLU2>125,125,125)
258 CONTINUE
B LABEL = TOTAL EFF. STRAIN
B FA = 634663214360
B FB = 2342623360062
d FC = 635121314560
WRITE TAPE 8,KLU9, NC,FA,FB,FC,(EPESTN(J1),J1=1,NC)
259 CONTINUE
DO 862 125=1,NC
862 T3FSTN(I25) = 0.0
C CALCULATE NODE STRAIN CHANGE - MATRIX DELEPK - FRAME SIZE 165 X 1
DO 191 111 = 1,NC
TEMPA = DELEPK(111)/SUBARN(111)
M3 = 3*111
M32 = M3-2
M31 = M3-1
O(3)DELEPK(M32) = TEMPAL(A(123)(111)*SIGUK(M32)-.5*AL(123)(111)*SIGUK(M31)
O(1)DELEPK(M31) = TEMPAL(A(123)(111)*SIGUK(M31)-.5*AL(123)(111)*SIGUK(M32)
O(1)DELEPK(M3) = TEMPAL(A(123)(111)*SIGUK(M3))
191 CONTINUE
C CALCULATE NODE STRAIN CHANGE - MATRIX EPSUK - FRAME SIZE 165 X 1
C CALCULATE NODE POINT STRAINS
DO 192 123=1,NA
192 EPSUK(I23) = EPSUK(I23) + DELEPK(I23)
IF(KLU2+KLU3)260,260,261
DO 260 123=1,NA
260 CONTINUE
C LABEL = EFF. STRAIN CHANGES
B FA = 252626336263
B FB = 512131456023
B FC = 302145273262
WRITE TAPE 8,KLU9, NC,FA,FB,FC,(DELEPK(J1),J1=1,NA)
C LABEL = NUDE STRAIN CHANGE
B FA = 4346242560062
B FB = 635121314560
B FC = 233021452725
WRITE TAPE 8,KLU9, NC,FA,FB,FC,(DELEPK(J1),J1=1,NA)
C LABEL = NUDE INELAS. STRAIN
B FA = 4346242560031
B FB = 435243216223
B FC = 626351213145
WRITE TAPE 8,KLU9, NC,FA,FB,FC,(EPSUK(J1),J1=1,NA)
C CALCULATE TOTAL NODE STRAINS - MATRIX TOTEPS - FRAME SIZE 165 X 1
DO 261 114=1,NC
261 CONTINUE
C
TOTEPSI (M3) = EPSUK (M3) + SIGUK (M3) / 2
201 TOTEPSI (M3) = EPSUK (M3) / SIGUK (M3) / 5
IF (KLU2) 262, 262, 263
262 CONTINUE
C
LABEL = TOT. NODE STRAINS
B
FA = 634663336045
B
FB = 46245606263
B
FC = 512131456260
WRITE TAPE 8, KLU9, NA, FA, FB, FC, (TOTEPSI (J1), J1 = 1, NA)
263 CONTINUE
IF (KLU2) 268, 268, 269
268 CONTINUE
KL9 = 3
KLU9Z = 2
WRITE TAPE 8, KLU9, KLU9Z, K, PM, PM, PM, TIME
IF (FININCLO) 269, 269, 273
273 CONTINUE
KL9 = 4
KLU9Z = 2
WRITE TAPE 8, KLU9, KLU9Z, PM, PM, PM, PM, INKCY, NMINT
269 CONTINUE
IF (FININCLO) 370, 370, 371
371 IF (FININCW-2) 370, 422
370 CONTINUE
IF (ISENSE SWITCH) 61421, 422
422 CONTINUE
GO TO 151
421 IF (KLU2) 1423, 423, 208
423 CONTINUE
205 KLU9 = 7
B
FA = 777777777777
DO 206 I25 = 1, 10
206 WRITE TAPE 8, KLU9, KLU9, FA, FA, FA, FA, FA, FA, FA, FA, FA
END FILE 8
END FILE 8
END FILE 8
END FILE 8
END FILE 8
END FILE 8
REWIND 8
IF (ISENSE SWITCH) 61207, 209
209 IF (IFJ24) 31210, 207, 210
207 CONTINUE
PAUSE 1
REWIND 11
CALL CHAIN (5, 3)
210 CONTINUE
CALL CHAIN (6, 3)
208 KERRSW = 2
C
KERRSW SET TO 2 TO MAKE KL2 = 0 AND PRINT A CYCLE
GO TO 151
998 KLU9 = 6
KLU9Z = 3
WRITE TAPE 8, KLU9, KLU9Z, K, 18, PM, ESUPRK, SGRARN (18), TIME
GO TO 205
END FILE 8
END FILE 8
REWIND 8
208 KERRSW = 2
C
KERRSW SET TO 2 TO MAKE KL2 = 0 AND PRINT A CYCLE
GO TO 151
998 KLU9 = 6
KLU9Z = 3
WRITE TAPE 8, KLU9, KLU9Z, K, 18, PM, ESUPRK, SGRARN (18), TIME
GO TO 205
END FILE 8
END FILE 8
REWIND 8
C
LIST
SUBROUTINE OUTPUT(VALUE1, STEP1, KLU1)

C THIS SUBROUTINE SETS KLU1 = 0 IF THE CURRENT CYCLE IS TO BE PRINTED

VALUE = ABSF(VALUE1)
STEP = ABSF(STEP1)
102 IF (VALUE - STEP) LT 131, 100, 100
100 NTEST1 = (VALUE/STEP) * 1.00001
NTEST2 = (VALUE/STEP) * .999
IF (NTEST1 - NTEST2) LT 131, 130, 111
130 KLU1 = 0
GO TO 135
131 KLU1 = 1
135 RETURN
END(1,1,0,0,0,0,1,1,0,0,0,0,0,0)

* CHAIN(5,3)
* LIST8
C651286 MATRIX ANALYSIS OF INELASTIC PLATE - LINK 6 - WITH CREEP
C
C THIS LINK CONVERTS BINARY OUTPUT ON TAPE B TO HCD
C
ON THE MONITOR PRINT TAPE
C
COMMUN IEFSTN, TUTEPS
COMMUN KLU4, KLU5, KLU6, KLU7, NA, NC, K
COMMUN KEKSW, NINCLUD, KSEL, PM, E, SHPMJS
COMMUN ALPHA, BETA, GAMMA, TIME, KLUISO
DIMENSION TEFSN(55),TOLPS(163)
DIMENSION ARRAY(155),LIS(111)
EQUIVALENCE (FA,IFA),(F0,F0A),(FL,IFC),(ARRAY,LIST)
RECORD 8
101 READ TAPE B,KLU9,KLU12,TAPE 8
IF(KLU9)101 101,106
106 IF(KLU9)9111,111,101
111 GO TO 121,122,123,124,125,126,127,128,129,130,131,132
121 CONTINUE
WRITE OUTPUT TAPE 6,22,FA,F0,FL,ARRAY(1),ARRAY(2)
GO TO 101
122 CONTINUE
WRITE OUTPUT TAPE 6,21,FA,F0,FL,ARRAY(11),ARRAY(2)
GO TO 101
123 CONTINUE
WRITE OUTPUT TAPE 6,31,FA,ARRAY(11),ARRAY(2)
GO TO 101
124 CONTINUE
WRITE OUTPUT TAPE 6,32,LIS(11),LIS(2)
GO TO 101
126 CONTINUE
WRITE OUTPUT TAPE 6,12,ARRAY(11),ARRAY(2)
WRITE OUTPUT TAPE 6,13,FA,F0,FL,ARRAY(3)
GO TO 101
127 CONTINUE
PRINT 14
WRITE OUTPUT TAPE 6,14
RECORD 8
IF(KERSSW)51132,131,132
131 PRINT 33
132 CALL EXIT
125 CONTINUE
128 CONTINUE
129 CONTINUE
GO TO 101
12 FORMAT(46H VALUE NOT FOUND IN TABLE FOR EPSILON BAR N = ,E15.8,
119H ( SIGMA BAR N = ,E15.8,2H ) )
13 FORMAT(16H CYCLE NUMBER = ,I5,20H ELEMENT INDEX = ,I4,17H LO
11AD LEVEL = ,F9.2,11H TIME = ,F6.4)
14 FORMAT(76H DO NOT SAVE TAPE B - I - IT HAS BEEN COMPLETELY PROCESSED
111 ONTO THE PRINT TAPE)
21 FORMAT(1X,3A6,5(IPE6.7)/(19X,3E16.7))
22 FORMAT(1X,3A6,2X,F12.2,3X,6HFMAC = ,F12.6)
31 FORMAT(28H,16,27H CYCLES COMPLETEU - LOAD = ,F9.2,
1112H TIME = ,F6.4)
32 FORMAT(1H1,68X,6HCYCLE,13,4H OF,13,27H AT THIS LOAD OR TIME LEVEL
1111L)
33 FORMAT(52H ,** SAVE TAPE A-0 FOR RESTART AT THIS POINT **)
END

DATA 1151
APPENDIX VI

STRESS DISTRIBUTIONS DUE TO UNIT INITIAL STRAINS

The basic matrix utilized in the nonlinear and inelastic analysis described in this report is the initial strain influence coefficient matrix \( [\Gamma_{uv}] \). Elements of this matrix provide the \( u \)th stress component caused by a unit initial strain at the \( v \)th stress location. The first description and derivation of this useful matrix was made by Denke in 1954, Reference 20. The matrix is generated by a structural analysis which may be of the force or stiffness type.

A. Force Method Application

The method utilized here is a simple extension of redundant structure analysis. Essentially, the additional work involved is the calculation of displacements at the applied loads and redundants in the statically determinant structure caused by initial strains. These displacements are combined with those caused by the applied loads and redundants. The final step is to adjust the magnitude of the redundants to eliminate the total displacements at the redundants. From this point the determination of stress distributions for the redundant structure is carried out as before.

The displacement in the statically determinate structure at the \( t \)th applied load or redundant due to initial strain can be expressed by use of the principle of virtual work as follows:

\[
\alpha_{tv} = \sum_a \int (\sigma_{xt} \cdot \epsilon_{xy}) dV
\]

\( (A-1) \)

Volume of \( a \)th member

In the above expression, \( \alpha_{tv} \) is the displacement at the \( t \)th unit applied load or redundant due to the \( v \)th initial strain, \( \sigma_{xt} \) is the direct stress in the \( a \)th member due to the unit load, and \( \epsilon_{xy} \) is the initial strain in the \( a \)th member at the corresponding stress point. The summation \( \sum_a \) indicates that the virtual work in all members affected by the induced strain, \( \epsilon_{xy} \), must be considered.

Equation (A-1) is written for the uniaxial direct stress condition that is commonly assumed to exist in bar members as pictured in Figure (1). The effect of a shear panel has been omitted in the derivation for simplicity. Terms necessary for inclusion of shear may be derived in a similar manner.
Suppose that the \( a^{th} \) member of a structure is a bar, with cross sectional area \( A \), linearly varying axial load \( \sigma_{xt}A \), and linearly varying initial strain \( \varepsilon_{xy} \).

Then

\[
\sigma_{xt} = \frac{1}{A} \left[ \gamma_{1t}(1 - \frac{X}{l}) + \gamma_{2t}\frac{X}{l} \right]
\]

\[
\varepsilon_{xy} = \varepsilon_{1y}(1 - \frac{X}{l}) + \varepsilon_{2y}\frac{X}{l}
\]

and after evaluation

\[
\int \sigma_{xt} \cdot \varepsilon_{xy} \, dv = \left[ \gamma_{1t} \quad \gamma_{2t} \right] \left[ \begin{array}{cc} L_{a_{11}} & L_{a_{12}} \\ L_{a_{21}} & L_{a_{22}} \end{array} \right] \left[ \begin{array}{c} \varepsilon_{1y} \\ \varepsilon_{2y} \end{array} \right]
\]

where

\[
L_{a_{11}} = L_{a_{22}} = \frac{1}{3}
\]

\[
L_{a_{12}} = L_{a_{21}} = \frac{\xi}{12}
\]

The \( L_{a_{ij}} \) matrix can be similarly determined for other types of structural elements. For rectangular shear panels, of dimensions \( b \) and \( h \), it can be shown that the \( L_{a_{ij}} \) factor for shear strain \( \gamma_{xy} \), is \( bh \), the panel flat plate area. For nonorthogonal structure (swept panels) other \( L_{a_{ij}} \) factors may be developed. The sum of such matrices for the entire structure is designated \( [L_{11}] \).
The matrix expression for the displacements at all of the applied loads and redundants due to unit initial strain is

\[
\begin{bmatrix}
\alpha_{mV} \\
\alpha_{rV}
\end{bmatrix} = \begin{bmatrix}
\gamma_{im} \\
\gamma_{ir}
\end{bmatrix}^T I_{iv}
\]  \hspace{1cm} A-2

The \( \epsilon \)'s are the initial strains corresponding to each of the member loads \( q_v \).

Utilizing Equation A-2, the redundants are evaluated by

\[
\begin{bmatrix}
\alpha_{rm} \\
\alpha_{rv} \\
\alpha_{rtr}
\end{bmatrix}\begin{bmatrix}
p_m \\
q_v \\
q_u
\end{bmatrix} = 0
\]

the solution of which is

\[
\begin{bmatrix}
q_u \\
q_v
\end{bmatrix} = \begin{bmatrix}
\Gamma_{sm} & \Gamma_{sv}
\end{bmatrix}\begin{bmatrix}
p_m \\
\epsilon_v
\end{bmatrix}
\]

where

\[
\begin{bmatrix}
\Gamma_{sm} & \Gamma_{sv}
\end{bmatrix} = -\begin{bmatrix}
\alpha_{ru}
\end{bmatrix}^{-1}\begin{bmatrix}
\alpha_{rm} & \alpha_{rv}
\end{bmatrix}
\]

The member loads in the redundant structure become

\[
\begin{bmatrix}
q_u \\
q_v
\end{bmatrix} = \begin{bmatrix}
\Gamma_{im} & \Gamma_{iv}
\end{bmatrix}\begin{bmatrix}
p_m \\
\epsilon_v
\end{bmatrix}
\]

where

\[
\begin{bmatrix}
\Gamma_{im} & \Gamma_{iv}
\end{bmatrix} = \begin{bmatrix}
\gamma_{im} & \gamma_{ir}
\end{bmatrix} \begin{bmatrix}
I_{rm} & 0 \\
I_{rv} & I_{iv}
\end{bmatrix}
\]

Member stresses are obtained by pre-multiplying member loads by \( [\beta_{ui}] \), the reciprocal values of appropriate bar areas and skin gages.

\[
\begin{bmatrix}
\Gamma_{um} & \Gamma_{uv}
\end{bmatrix} = [\beta_{ui}] \begin{bmatrix}
\Gamma_{im} & \Gamma_{iv}
\end{bmatrix}
\]

then

\[
\begin{bmatrix}
q_u \\
q_v
\end{bmatrix} = \begin{bmatrix}
\Gamma_{um} & \Gamma_{uv}
\end{bmatrix}\begin{bmatrix}
p_m \\
\epsilon_v
\end{bmatrix}
\]

\[A-4\]
Digital computer programs which are available for conventional force method analyses may be used to determine the $\Gamma_{um}$, $\Gamma_{uv}$ matrices. This is readily accomplished by redefining several input matrices.

1) replace $[\gamma_{im}]$, the usual unit load distribution in the statically determinate structure due to applied loads by

\[
\begin{bmatrix}
\gamma_{im} & 0 \\
0 & I
\end{bmatrix}
\]

$(2i) \times (m + i)$

the unit diagonal matrix $I$ has as many elements as there are member loads in the structure.

2) replace $[\gamma_{ir}]$, the usual unit load distribution in the statically determinate structure due to redundants by

\[
\begin{bmatrix}
\gamma_{ir} \\
0
\end{bmatrix}
\]

$2i \times r$

3) replace $[\alpha_{ij}]$, the member flexibility matrix by

\[
\begin{bmatrix}
\alpha_{ij} & L_{ij} \\
L_{ij} & 0
\end{bmatrix}
\]

$2i \times 2i$

The $L_{ij}$ values are the geometrical factors of Equation A-2. Straightforward matrix operation will now yield a load distribution matrix for the redundant structure which may be identified as follows:

\[
\begin{bmatrix}
\Gamma_{im} & \Gamma_{iv} \\
0 & I
\end{bmatrix}
\]

The upper portion $[\Gamma_{im} \mid \Gamma_{iv}]$ are the values defined by Equation A-3, and stresses may be obtained by using Equation A-4.
B. Direct Stiffness Method

Consider a truss shown in Figure (2a) where all the nodes are "locked" (prevented from displacing). If a strain $\varepsilon_o$ is induced in a particular member it will produce a stress in only that member equal to $\sigma_o = -E\varepsilon_o$ since all the nodes are locked. This stress requires node forces as shown in Figure (2b). The negatives of these node forces are applied to the "unlocked" structure and the stresses in all the members, (Figure 2c) are computed. The final stress will be the superposition of the "locked in" stresses and the stresses produced by the node forces acting on the unlocked structure. That is

$$\sigma_T = \sigma + \sigma_o$$

**Figure 2**

Derivation of $[\Gamma_{um}]$ and $[\Gamma_{uv}]$

Consider an element of the total structure, let this element be supported in a statically determinate fashion. The total strain $\varepsilon$ is then written as the sum of two strains namely $\varepsilon_\sigma$ - the strains due to stresses, and $\varepsilon_o$ - a set of induced strains. Thus

$$\{\varepsilon\} = \{\varepsilon_\sigma\} + \{\varepsilon_o\} \quad \text{B-1}$$

Rewriting this,

$$\{\varepsilon_\sigma\} = \{\varepsilon\} - \{\varepsilon_o\} \quad \text{B-2}$$
The stresses may be expressed in terms of the strains by using Hooke’s law,

$$\{\sigma\} = \{b\}\{\varepsilon\}$$

B-3

Thus

$$\{\sigma\} = \{b\}\{\varepsilon\} - \{b\}\{\varepsilon_0\}$$

B-4

$$\{\varepsilon\}$$ (the total strains) may be expressed in terms of the displacements of the nodes that affect the particular member in question. That is

$$\{\varepsilon\} = \{a\}\{\delta\}$$

B-5

It should be noted that the expression contains the basic assumption governing the behavior of the element. For example, for a triangle or bar element it will contain the assumption that the strains are constant throughout the element. Substituting this into Equation B-4

$$\{\sigma\} = \{b\}\{a\}\{\delta\} - \{b\}\{\varepsilon_0\}$$

B-6a

$$= \{s_d\}\{\delta\} - \{b\}\{\varepsilon_0\}$$

B-6b

From virtual work it can be shown that the nodal forces \(f\), associated with \(\{\delta\}\), may be expressed in terms of the stresses by:

$$\{f\} = \int_{\text{Vol}} [a]^T \{\sigma\} dV$$

B-7

Substituting B-6a into B-7 yields:

$$\{f\} = \int_{\text{Vol}} [a]^T \{b\}\{a\}\{\delta\} dV - \int_{\text{Vol}} [a]^T \{b\}\{\varepsilon_0\} dV$$

B-8

For simple elements such as bars or triangles where the strains are assumed to be constant over the element none of the matrices within the integrals will be functions of the coordinates \(x, y\) or \(z\) and hence they are independent of the integration. The first portion of (B-8) yields the standard stiffness matrix \([k]\) while the second portion yields the nodal forces due to induced strains \(\{\varepsilon_0\}\).

$$\{f\} = [k]\{\delta\} - \nabla [a]^T \{b\}\{\varepsilon_0\}$$

B-9

120
In this expression, \( V \) is the volume of the element. Noting that
\[
[a]^T[b] = [b][a]^T = [s_d]^T \text{ yields}
\]
\[
[f] = [k][\delta] - V[s_d]^T[e_0]
\]

The total stiffness matrix for the entire structure is obtained by
superposing the stiffnesses for the individual elements to yield:
\[
[F] = [K][\Delta] - [S_d]^T[V][e_{ot}]
\]

Where \([F]\) represents all of the external nodal forces, \([\Delta]\) represents
all of the nodal displacements, \([e_{ot}]\) represents a column of all the
induced strain components and \([V]\) is a diagonal matrix containing the
values of the volumes of the various elements. (Note that for a triangle,
where three strain components may be induced, the volume will appear
three times). \([S_d]\) is the total stress matrix array which is obtained
by proper arrangement of the individual stress matrices \([s_d]\).

Applying boundary conditions (via matrix \([BC]\)) to equation (B-11)
and introducing the applied external loads \([F]\), with nodal distribution
\([L]\), yields:
\[
\]

which may be solved to give the allowed nodal displacements under the
boundary conditions:
\[
[\Delta'] = [K_{11}]^{-1}[L][F] + [K_{11}]^{-1}[BC]^T[S_d]^T[V][e_{ot}]
\]

where
\[
[K_{11}] = [BC]^T[K][BC]
\]

For the total structure, B-6b may be written as:
\[
[a_T] = [S_d][\Delta] - [B][e_{ot}]
\]

where \([B]\) is a diagonal block of the elastic relations \([b]\) for each
member. Substituting (B-13) into (B-14) and using the relation \([\Delta] =
[B][\Delta']\) yields:

121
\[
\begin{align*}
\{\sigma_T\} &= [S_d][B][K_{11}]^{-1}[L][P] \\
&\quad + \left[[S_d][B][K_{11}]^{-1}[B][S_d]^T[V] - [B]\right]\{\epsilon_T\}
\end{align*}
\]

which, to use the previous notation, may be written as:

\[
\{\sigma_u\} = [\Gamma_{um}][P_m] + [\Gamma_{uv}][\epsilon_v]
\]

The first matrix, \([\Gamma_{um}\]), of this expression, is the conventional distribution obtained for unit applied loads as indicated in Section A of this appendix.
REFERENCES


A number of the references were traced by a machine search of NASA and Department of Defense holding under the following headings:

1) Anisotropic plasticity of metals
2) Anisotropic creep of metals
3) Bauschinger effect of metals
4) Cyclic loading of metals
Most aerospace structural materials exhibit some degree of anisotropic strain hardening. A measure of anisotropy is introduced into structural elements by such fabrication processes as drawing, cold rolling and extrusion. During the past few years, several methods have appeared in the literature for introducing inelastic isotropic material behavior effects into existing matrix analysis routines. A review is presented of one of these methods. It is essentially a step-by-step calculation procedure, and corresponds to the flow theory of plasticity. The method has been extended to include the effects of anisotropic material and is formulated as a standard initial strain influence coefficient problem. Several analyses of an aluminum alloy (2024-T4) shear lag structure which has been tested previously for the Air Force are carried out first assuming isotropic material properties and then anisotropic properties. The resulting correlation between test results and those predicted by isotropic theory is reasonably good.

An analysis of a 11005F aluminum shear lag structure, carried out by the incremental method, gave reasonably good agreement. However, the anisotropic creep capability was not checked for want of test data. The approach is a reasonably good phenomenological model of a complex physical problem. The digital computer program submitted is suited for inclusion of other material nonlinearity.
<table>
<thead>
<tr>
<th>KEY WORDS</th>
<th>LINK A</th>
<th>LINK B</th>
<th>LINK C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Anisotropic Elasticity of Metals</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Anisotropic Creep of Metals</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Matrix Structural Analysis</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**INSTRUCTIONS**

1. **ORIGINATING ACTIVITY:** Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization (corporate author) issuing the report.

2. **REPORT SECURITY CLASSIFICATION:** Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

3. **GROUP:** Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

4. **REPORT TITLE:** Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If the title is meaningful, select without classification, show title classification in all capitals in parenthesis immediately following the title.

5. **DESCRIPTIVE NOTES:** If appropriate, enter the type of report, e.g., interim, progress, summary, final. Give the inclusive dates when a specific reporting period is covered.

6. **AUTHORS:** Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initials. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

7. **REPORT DATE:** Enter the date of the report as day, month, year. If more than one date appears on the report, use date of publication.

8. **TOTAL NUMBER OF PAGES:** The page count may be unclassified. If more than one page appears on the report, use date of publication.

9. **NUMBER OF REFERENCES:** Enter the total number of references cited in the report.

10. **CONTRACT OR GRANT NUMBER:** If appropriate, enter the applicable number of the contract or grant under which the report was written.

11. **PROJECT NUMBER:** Enter the appropriate military department identification, such as project number, subproject number, system number, task number, etc.

12. **ORIGINATOR’S REPORT NUMBER(S):** Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

13. **OTHER REPORT NUMBER(S):** If the report has been assigned any other report numbers (either by the originator or by the sponsor), also enter this number(s).

14. **AVAILABILITY/LIMITATION NOTICES:** Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

   1. “Qualified requesters may obtain copies of this report from DDC.”
   2. “Foreign announcement and dissemination of this report by DDC is not authorized.”
   3. “U.S. Government agencies may obtain copies of this report directly from DDC. Other qualified users shall request through DDC.”
   4. “U.S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through DDC.”
   5. “All distribution of this report is controlled. Qualified users shall request through DDC.”

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

15. **SUPPLEMENTARY NOTES:** Use for additional explanatory notes.

16. **SPONSORING MILITARY ACTIVITY:** Enter the name of the departmental project office or laboratory sponsoring (paying for) the research and development. Include address.

17. **ABSTRACT:** Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as (T), (S), (C) or (U).

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

18. **KEY WORDS:** Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designations, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, value, and weights is optional.