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ANOMALY SELECTION FOR DEFLECTION INTERPOLATION
PART I: THEORETICAL INVESTIGATION

463072

By
W. E. STRANGE and G. P. WOOLLARD

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ABSTRACT

This report covers work performed under Contract AF23(601)-4009 with the Aeronautical Chart and Information Center, St. Louis, Missouri by the Hawaii Institute of Geophysics of the University of Hawaii. It presents the results of theoretical studies ~~carried out~~ Per Formed ~~in order~~ to determine the combined geodetic and geophysical significance of current gravity reduction techniques. Special emphasis was given to the problem of interpolation of deflections of the vertical by gravimetric means.

An examination of the mathematical theory of physical geodesy revealed that the accuracy of the mathematical equations relating deflections of the vertical and gravity anomalies could not be improved by choice of gravity anomaly. The choice of a gravity anomaly for use in the deflection interpolation equations was found to depend upon the accuracy with which the anomaly could be interpolated between observation points and the relative amount of labor required to carry out numerical computations.

The complete Bouguer anomaly with geologic corrections was found to be the best anomaly to use for interpolation of gravity with a reversion to the normal complete Bouguer anomaly without geologic corrections for use in the mathematical equations.

A

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SECTION 1

INTRODUCTION

This report covers the results of the Investigation and Selection Phases of the Alpine Gravity Reduction Project. As stated in the technical specifications, the work to be carried out in the Investigation and Selection Phases was to: 1) "---evaluate current gravity reduction techniques for their combined geodetic and geophysical significance" and 2) "---select that gravity reduction method which will most accurately reproduce the external gravity field." The over-all aim of this project was to find the best method of utilizing geologic and geophysical information in determining gravity anomalies to be used in the interpolation of deflections of the vertical in areas of rugged topography.

In order to determine what type of gravity anomaly would be the best to use for interpolation of deflections of the vertical in practical problems, two distinct investigations were necessary. First, the mathematical theory leading to the equations relating gravity anomalies and deflections of the vertical was examined. In this way it was possible to determine if the mathematical theory itself led to any special preference for a particular type of gravity anomaly and if controls were placed upon the type of anomaly which could be used by the nature of the mathematical formulation. Second, since gravity is actually known only at a limited number of points, the various types of gravity anomalies were studied to determine their relative value in terms of accuracy of interpolation between points of observation.

Since the equations relating gravity anomalies and deflections of the vertical are obtained by solving a boundary value problem, a study of the mathematical theory is a study of this boundary value problem. The boundary value

problem can be stated in words as follows: "Given the gravitational potential and the magnitude of the force of gravity everywhere on the bounding surface of a gravitating body, determine the shape of the bounding surface and the direction of the force of gravity on it." For a body such as the earth, whose bounding surface is highly irregular, some approximations must be made in order to make the problem tractable. There have been two modes of approach to simplification of the problem. In the sections which follow, these have been labeled "The Classical Geodetic Theory" and "The New Geodetic Theory". These labels were not chosen to have any connotation as to accuracy but were chosen simply because the approach labeled "The Classical Geodetic Theory" was the original approach used in solving the problem while the approach labeled "The New Geodetic Theory" has been developed only recently.

The "Classical Theory" is developed in Section 3. Section 4 gives the development of the "New Theory". Section 2 contains background material which gives the development of certain relations used in Sections 3 and 4 and is primarily to be used for reference when reading these sections.

In the "classical" approach, as developed in Section 3, a solution is obtained for the boundary value problem under the simplifying assumption that the bounding surface is an equipotential surface. In this case, the type of gravity anomaly used is controlled by the necessity for computing the effect of the theoretical transfer of all masses of the actual earth which lie outside the geoid to some point within the geoid. In Section 5, the various types of anomalies which will accomplish such a mass transfer are examined.

In the "new" approach, as developed in Section 4, the free-air anomaly arises naturally in the development. However, as shown in Section 5, this natural appearance of the free-air anomaly is simply a result of the theoretical model chosen. With other choices of theoretical models, other types of gravity anomalies

can be used. The conclusions reached in Sections 3, 4 and 5 are: 1) the "New Theory" is capable of extensions to greater accuracy than is the "Classical Theory". The increased accuracy is of limited importance for computation of height differences but becomes significant for deflection computations in rugged terrain. 2) In using the "Classical Theory" it is difficult to determine which type of anomaly gives the most accurate result since the answer appears to depend in a complex way upon the difference between the actual density distributions above sea level and the actual gravity gradients and the values assumed for these quantities when carrying out the calculations. 3) In using the "New Theory", it is found that gravity anomalies computed using any number of density models will give equally accurate results so long as all of the mass of each model is assumed to lie within the earth's surface.

Since, in the "New Theory" the accuracy of the results is theoretically independent of the type of gravity anomaly used, the choice of anomaly depends, in the case of practical computations, upon: 1) the relative accuracy with which the various types of anomalies can be interpolated, and 2) the amount of work required to compute a particular type of anomaly and carry out computations using it. In Section 6 the various types of anomalies are examined to determine which can best satisfy these two criterias. It is here in connection with the accuracy of interpolation that the question of the degree to which the model densities represent the actual densities of the earth becomes important when using the "New Theory". From the point of view of interpolation accuracy, one would like to use a density model which approaches as nearly as possible to the actual density distribution of the earth above some fixed depth, this depth being dependent upon the distance between points at which gravity has been observed. In the case of deflection interpolation computations which would not normally be attempted without a reasonably close-spaced net of observations, this leads to some type of Bouguer anomaly

with geologic corrections applied down to whatever depth is necessary to cause the resultant anomaly to vary smoothly between observation points.

Using such geologically corrected gravity anomalies directly in the deflection formulae was found to lead to complicated computation procedures. The compromise finally adopted was, therefore, to use the geologic control for gravity interpolation but to revert to normal complete Bouguer anomalies for computation.

The method of solution chosen was one developed by Pellinen (1962), which utilizes terrain corrected free-air anomalies. The theory leading to the final deflection formula using this method is given in Section 7. The theory is presented in a form which allows two contour maps--one of complete Bouguer anomaly and one of elevation--to be used rather than a single map of terrain corrected free-air anomalies. This method of procedure has the advantage that the anomaly used, the Complete Bouguer Anomaly, is a smoothly varying function while the terrain corrected free-air anomaly is nearly linearly related to local elevation changes and therefore changes in a complex manner in regions of rugged terrain.

Detailed computational procedures using the theory of Section 7 are given in Part II of the final report on this contract in conjunction with the results obtained from applying this theory to actual problems in the Test and Application Phases of the contract.

SECTION 2

Basic Potential Theory

The purpose of this section is to summarize those developments in potential theory which are utilized in the development of the various theories of physical geodesy. Only the basic outlines will be given here with the aim of providing an easily accessable source of reference for some of the basic formulae. For a more detailed and mathematically rigorous development one may refer to MacMillan (1958). The results here are largely a summary of the results presented there.

2.1 The Boundary Value Problem

We shall begin with the well known Green's equation which states: If φ and ψ are any two functions of the Cartesian coordinates x,y, and z, and if their first derivatives are continuous and the functions possess second derivatives within a volume V and on its surface S, then the following integral relation holds between φ and ψ.

$$\int_V (\psi \Delta^2 \varphi - \varphi \Delta^2 \psi) dV = \int_S (\psi \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial \psi}{\partial n}) dS \qquad (2\text{-}1)$$

where Δ^2 = the Laplacian operator $\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2}$

$\frac{\partial}{\partial n}$ = the derivative in the direction of the outward normal to the surface S, and

$\int_V$ and $\int_S$ = integrals over the volume and surface respectively.

For the development of this theory the reader is referred to MacMillan sections 53, 54, and 55.

For use in potential theory, Green's formula is utilized as follows. If the functions φ and ψ of (2-1) satisfy Laplace's equation $\Delta^2 \varphi = 0$ and $\Delta^2 \psi = 0$ within the volume V bounded by the surface S then (2-1) becomes

$$\int_S (\psi \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial \psi}{\partial n}) dS = 0 \qquad (2\text{-}2)$$

It is noted that if ξ, η, ζ are the coordinates of a variable point in V and if x,y, and z are the coordinates of any point P then the function

$$\frac{1}{\rho} = \left[(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2\right]^{-1/2} \qquad (2\text{-}3)$$

satisfies the conditions necessary for use as one of the functions in (2-2) everywhere in the volume V provided P lies outside the volume. If P lay within the volume V itself problems would arise where the variable point approached the point P. Thus for P outside V we can use $\psi = \frac{1}{\rho}$ in (2-2) and have the following equation

$$\int_s \frac{1}{\rho}\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial}{\partial n}\left(\frac{1}{\rho}\right) dS = 0 \qquad (2\text{-}4)$$

For our purposes we will let the volume V of equation (2-1) be all of the space lying outside the gravitating body which we are studying except for a small sphere around the point P which is assumed to lie outside the gravitating body. Then the surface of the volume V is the surface of the body S' and the surface of the small sphere around P which we shall call S''. We pause here to re-emphasize that the volume V is all the area outside the gravitating body and the small sphere enclosing P. Then since P lies outside V, we can write (2-4) as

$$\int_{s'}\left[\frac{1}{\rho}\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial}{\partial n}\left(\frac{1}{\rho}\right)\right] dS' + \int_{s''}\left[\frac{1}{\rho}\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial}{\partial n}\left(\frac{1}{\rho}\right)\right] dS'' = 0 \qquad (2\text{-}5)$$

However, it can be shown (See MacMillan, 1958) that

$$\int_{s''}\left[\frac{1}{\rho}\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial}{\partial n}\left(\frac{1}{\rho}\right)\right] dS'' = -4\pi\varphi_p \qquad (2\text{-}6)$$

where the subscript P indicates the function φ is to be evaluated at the point P.

Then (2-5) becomes

$$\varphi_p = \frac{1}{4\pi}\int_{s'}\left[\frac{1}{\rho}\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial}{\partial n}\left(\frac{1}{\rho}\right)\right] dS \qquad (2\text{-}7)$$

Stated in words (2-7) says that if a function φ satisfies the conditions:

1. φ and its first derivatives are single valued and continuous within the volume V and on its boundary S.

2. the second derivatives of φ exist

3. φ satisfies the equation of Laplace

within a volume which is bound by a closed surface S' then the value of φ at any interior point of the volume can be computed from the values of φ and its first spatial derivatives on S'.

In geodetic theory, the function φ will be the potential of some mass distribution lying entirely outside the volume V (i.e., within the gravitating body bounded by S') or on the surface S'. Such a potential function may be used since it satisfies $\Delta^2\varphi = 0$ outside the gravitating body. The potential related to centrifugal force does not satisfy the Laplace equation and thus is not a function which will satisfy (2-7). It is for this reason that in the development of the theory of Section 3 the potential T must be considered as purely a gravitation potential. In this case T satisfies (2-7) and we can write

$$T_p = \frac{1}{4\pi} \int_{s'} \left[\frac{1}{\rho} \frac{\partial T}{\partial n} - T \frac{\partial}{\partial n} \left(\frac{1}{\rho}\right)\right] ds' \qquad (2\text{-}8)$$

2.2 Development in Spherical Harmonics

Since development in spherical harmonics is fundamental in physical geodesy, two ways of arriving at a spherical harmonic representation of gravitational potential are outlined below. It is believed that taken together, the two give a clear understanding of what spherical harmonics imply.

Let us begin the first development by considering the gravitational potential, at a point P lying outside a given mass distribution, due to a small incremental volume V, whose mass is dM and which is located at a point Q. Using the symbol U to represent potential we have

$$dU_p = \frac{k\,dM}{r} = \frac{k\sigma\,dV}{r} \qquad (2\text{-}9)$$

where: k = gravitational constant

dV = incremental volume element

σ = density

r = distance from P to Q

Thus the gravitational potential at the point P due to the entire mass distribution is given by

$$U_p = k \int_V \frac{\sigma(r', \theta', \lambda')}{r(r', \theta', \lambda', \rho, \theta, \lambda)} \, dr' \, d\theta' \, d\lambda' \tag{2-10}$$

where the coordinates have the meaning illustrated in Figure (2-1)

From Figure (2-1) we see by the law of cosines that

$$r = (\rho^2 + r'^2 - 2\rho r' \cos \psi)^{1/2} = \rho(1 + (\frac{r'}{\rho})^2 - 2(\frac{r'}{\rho}) \cos \psi)^{1/2} \tag{2-11}$$

Thus

$$\frac{1}{r} = \frac{1}{\rho} (1 + (\frac{r'}{\rho})^2 - 2 (\frac{r'}{\rho}) \cos \psi)^{-1/2} \tag{2-12}$$

The binomial series expansion is given by

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + - - - - \tag{2-13}$$

provided $x<1$. Thus, if in equation (2-12) $\rho > r'$ we can use (2-13) to expand the quantity in parenthesis in (2-12) with $n = -1/2$ and $x = [(\frac{r'}{\rho})^2 - 2(\frac{r'}{\rho}) \cos \psi]$ If we can carry out such an expansion and collect terms with common powers of $(\frac{r'}{\rho})$ we get

$$\frac{1}{r} = \frac{1}{\rho} \left\{ 1 + (\frac{r'}{\rho}) \cos \psi + (\frac{r'}{\rho})^2 (3/2 \cos^2 \psi - 1/2) + (\frac{r'}{\rho})^3 (5/2 \cos \psi - 3/2 \cos \psi) + - - - - - \right\} \tag{2-14}$$

It can be shown that the function of $\cos \psi$ appearing in the nth term of the expansion can be written

$$\frac{1}{2^n n!} \frac{d^n (\cos \psi^2 - 1)^n}{d \cos \psi^n} \tag{2-15}$$

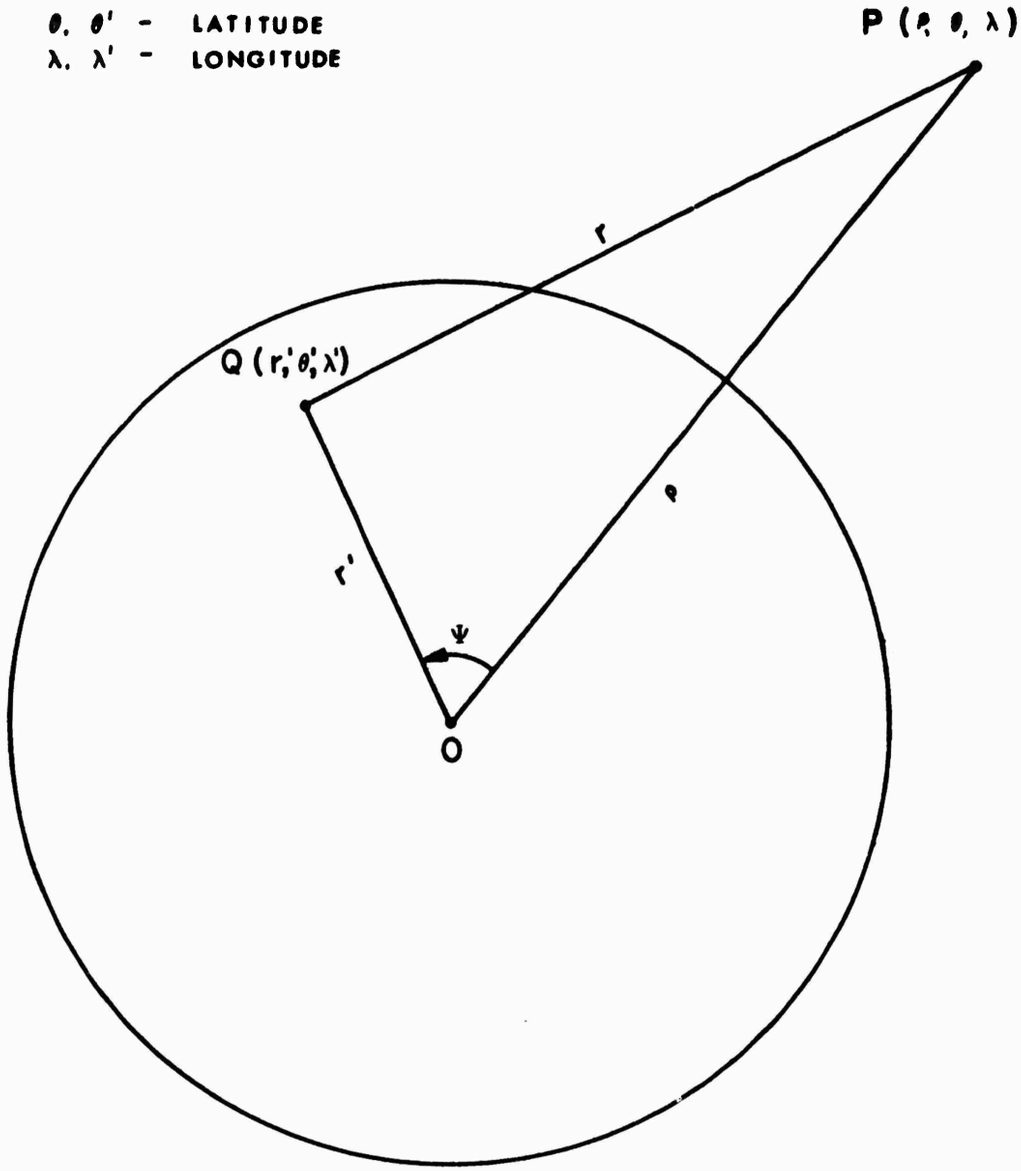


FIGURE 2.1

The functions generated by this formula are called Legendre polynomials or zonal harmonics and are represented by the symbol $P_n(\cos\psi)$. Using the P_n symbology we can therefore write

$$\frac{1}{r} = \sum_{n=0}^{\infty} \frac{r'^n}{\rho^{n+1}} P_n(\cos\psi) \tag{2-16}$$

Thus taking note of (2-14) and (2-16) we can write (2-10) as

$$U_p = k \int_v \sigma \sum_{n=0}^{\infty} \frac{r'^n}{\rho^{n+1}} Pn(\cos\psi)dV = \frac{k}{\rho}\int_v \sigma\, dV + \frac{k}{\rho^2}\int_v \sigma r' \cos\psi dV + \frac{k}{\rho^3}\int_v \sigma\, r'^2(3/2\cos\psi - 1/2)dV + \cdots \tag{2-17}$$

It is instructive to evaluate the first few integrals of the right hand side of equation (2-17).

The first integral is simply

$$\int_v \sigma dV = M_e \quad = \text{total mass of the earth} \tag{2-18}$$

The second integral can be written

$$\int_v \sigma\, r' \cos\psi dV = \cos\theta \int_v \sigma\, z' dV + \sin\theta \cos\lambda \int_v \sigma x' dV + \sin\theta \sin\lambda \int_v \sigma y' dV \tag{2-19}$$

But the coordinates $(\bar{x}, \bar{y}, \bar{z})$ of the center of mass of a body are by definition

$$\bar{x} = \frac{\int_v \sigma x' dV}{M_B}; \quad \bar{y} = \frac{\int_v \sigma y' dV}{M_B}; \quad \bar{z} = \frac{\int_v \sigma z' dV}{M_B} \tag{2-20}$$

where M_B is the mass of the body. Thus if we choose the origin of our coordinate system to be the center of mass of the body, then $\bar{x} = \bar{y} = \bar{z} = 0$ which leads to the result

$$\int_v \sigma x' dV = \int_v \sigma y' dV = \int_v \sigma z' dV = 0 \tag{2-21}$$

The third integral on the right hand side of (2-17) can be written

$$\int_v \sigma r'^2(3/2\cos^2\psi - 1/2) = (3/2\cos^2\theta - 1/2)\int_v \sigma\left[z'^2 - 1/2(x'^2 + y'^2)\right]dV + 3/2\sin 2\theta\cos\lambda\int_v x'z'dV + 3/2\sin 2\theta\sin\lambda \int_v \sigma y'z'dV + 3/2\sin^2\theta\sin 2\lambda\int_v \sigma x'y'dV + 3/2\sin^2\theta\cos 2\lambda \int_v \sigma 1/2(x'^2 - y'^2)dV \tag{2-22}$$

If the z' axis of the coordinate system coincides with one of the principal axes of inertia of the gravitating body

$$\int_V \sigma x'z'dV = \int_V \sigma y'z'dV = 0 \tag{2-23}$$

In the case of the earth we chose the z' axis to coincide with the rotational axis of the earth, which we know is to a very close approximation a principal axis of inertia, so that we can assume (2-23) holds.

Thus, taking all of the above results into account we can write (2-17) as

$$Up = k\left\{\frac{M_e}{\rho} + \frac{1}{\rho^3}\left[(3/2\cos^2\theta - 1/2)\int_V \sigma\left[z'^2 - 1/2(x'^2 + y'^2)\right]dV + 3/2\sin^2\theta\sin 2\lambda\int_V \sigma x'y'dV + 3/2\sin^2\theta\cos 2\lambda\int_V \sigma 1/2(x'^2 - y'^2)dV\right] + \int_V \sigma \sum_{n=3}^{\infty}\frac{r'^n}{\rho^{n+1}}Pn(\cos\psi)dV\right\} \tag{2-24}$$

The above development shows that the constants in spherical harmonic development of the gravitational field are various moments and demonstrates the reason for certain terms being zero. However, the above method does not demonstrate how to get the general expansion in terms of θ and λ. Briefly this is arrived at by seeking a general solution to Laplaces equation in spherical coordinates (ρ, θ, λ) i.e.

$$\frac{1}{\rho^2}\left\{\frac{\partial}{\partial\rho}\left(\frac{\rho^2\partial U}{\partial\rho}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 U}{\partial^2\lambda}\right\} = 0 \tag{2-25}$$

The general solution is found to be

$$U = \sum_{n=0}^{\infty} A_n \rho^n Y_n + \sum_{n=0}^{\infty} B_n \frac{Y_n}{\rho^{n+1}} \tag{2-26}$$

where the A_n and B_n are constants and

$$Y_n = \sum_{i=0}^{n} a_{ni} P_n^i(\cos\theta)\cos i\lambda + \sum_{i=0}^{n} b_{ni} P_n^i(\cos\theta)\sin i\lambda \tag{2-27}$$

where a_{ni} and b_{ni} are also constants and

$$P_n^o(\cos\theta) = \frac{1}{2^n n!}\frac{d^n}{d\cos\theta^n}(\cos^2\theta - 1)^n$$

$$P_n^i(\cos\theta) = (\cos^2\theta - 1)^{i/2}\frac{d^{n+i}}{d\cos\theta^{n+i}}(\cos^2\theta - 1)^n \quad i \neq 0 \tag{2-28}$$

Since the potential function must vanish at infinity, A_n must be zero and it is conventional procedure to set all B_n equal to the gravitational constant k, then (2-26) becomes

$$U = k \sum_{n=0}^{\infty} \frac{Y_n}{\rho^{n+1}} \tag{2-29}$$

Equation (2-29) is a general equation expressing the potential of a mass distribution in spherical harmonics. In the practical case in physical geodesy we wish to express the attraction potentials of the actual earth and of the ellipsoidal normal field in spherical harmonics and then to take the difference to get a spherical harmonic representation of the anomalous potential T. In normal practice the center of gravity of the actual earth is chosen as the center of the coordinate system to be used and the center of gravity of the normal field is chosen to coincide with the c.g. of the actual earth. In this case, as has been shown, the Y_1 term will be zero for both potentials and thus for their difference T. It is also normal to choose the mass of the normal model to equal the mass of the actual earth. In this case, the Y_o terms will be identical and will cancel when computing T. Thus it is permissable to write T as

$$T = k \sum_{n=2}^{\infty} \frac{Y_n}{\rho^{n+1}} \tag{2-30}$$

The symbol Y_n is used to represent the spherical harmonic of degree n when referring to a number of different bodies throughout this report. To avoid confusion, one should point out that the symbol Y_n does not refer to the same quantity for all bodies. For any particular body the constants of (2-27) have particular values characteristic of that body and related to the moments of the body.

SECTION 3

Classical Geodetic Theory

The present section discusses the determination of the shape of the bounding equipotential surface of body B according to the theory usually employed in classical geodesy.

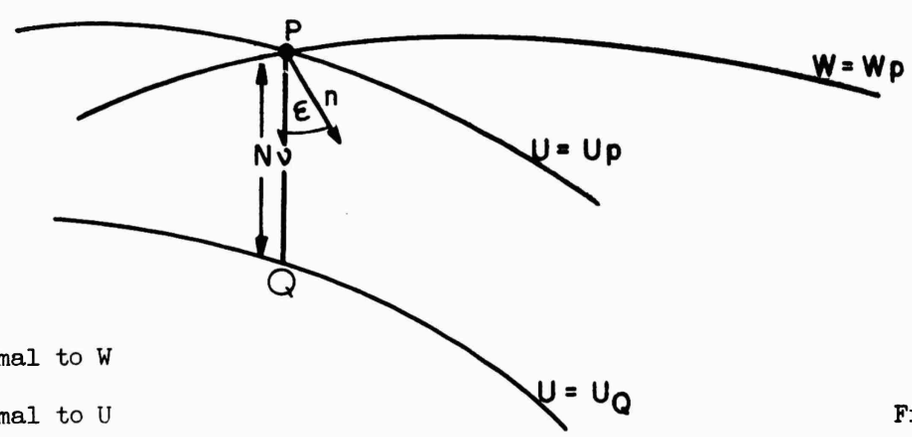
Let γ be a reference gravitational plus rotational field whose corresponding potential field is represented by U. In practice, for computation of gravity anomalies, the reference field is taken to be that given by the International Gravity formula with the gravity gradient taken to be a constant .09406 mgls/ft. A number of papers have been written on more exact expressions for the normal field. These have been examined by Daugherty and Define (1963) who found that for geodetic purposes all are sufficiently accurate.

If the gravitational plus rotational potential of the body B is represented by the symbol W, the potential at any point is given by

$$W = U + T \tag{3-1}$$

where T is by definition the difference between W and U.

Consider the figure below:



n = normal to W

ν = normal to U

Figure 3-1

The equipotential surface W = Wp is taken to be the bounding equipotential surface, S, of the body B. P is a point on this surface S. Up is the equipotential of the normal field passing through the point. U_Q is the equipotential surface of the reference field such that

$$U_Q = Wp \tag{3-2}$$

The point Q is the point on the equipotential surface U_Q intersected by the normal to the potential field U which passes through P. N is the distance $\overline{PQ}$.

Two points should be mentioned here. First, there are other methods of establishing the point Q on U_Q to be related to the point P on Wp. These lead to slightly different forms of development leading to equation (3-5) below, but the final results are essentially the same (See, for example, Jung, 1956). Second, the use of (3-2) implies that the value of Wp is known so that one can establish which equipotential surface of the normal field U_Q is equal to it. In fact, for problems concerned with the earth, Wp is not known. For a discussion of the errors which result, see Molodenski et al (1960). We shall proceed under the assumption that Wp is known.

From definition (3-1) we have

$$Wp = Up + Tp \tag{3-3}$$

Now assume that if the potential function U is expanded about the point Q in a Taylor series, the value of U at P can be accurately enough represented by the first two terms of the expansion. Then we have

$$Up = U_Q - \left(\frac{\partial U}{\partial \nu}\right)_Q N = U_Q - \gamma_Q N \tag{3-4}$$

where the negative sign results from the fact that the direction of γ is toward Q from P. The accuracy of the assumption that the first two terms are sufficient depends upon the magnitude of the distance N. If we consider that geoidal heights

seldom exceed 100 feet, we can get an idea of the magnitude of the error

$N = 100$ feet $\qquad N^2 = 10{,}000$

$\gamma \approx 980{,}000$ mgls. $\qquad \frac{\partial \gamma}{\partial \nu} \approx .1$ mgls./ft.

Then $\gamma_Q N = 98 \times 10^6 \approx 10^8$

$$\frac{\partial^2 U}{\partial \nu^2} N^2 = .1 \times 10{,}000 = 1000 = 10^3$$

Thus for reasonable N values, the neglected terms are negligible (i.e., of the order 10^{-5} of the second term).

Then substituting (3-4) into (3-3) gives

$$Wp = U_Q - \gamma_Q N + Tp$$

or making use of (3-2)

$$N = \frac{Tp}{\gamma_Q} \tag{3-5}$$

In classical geodetic theory the bounding equipotential Wp is the co-geoid differing from the geoid or sea level equipotential of the earth by an amount which depends upon the method used to reduce the surface anomalies of the earth to geoid level anomalies and the equipotential surface U_Q is the international ellipsoid. Then N is the distance between ellipsoid and co-geoid. The implications of the reduction methods used in classical geodesy will be discussed in Section 5. For the present assume gravity is known on Wp.

Since γ_Q is known, only Tp must be determined in order to compute N. To do this we will establish a differential equation relating gravity on the equipotential surface W = Wp to Tp and then solve this equation.

Referring to (3-1) we see that it is possible to write the identity

$$\left(\frac{\partial W}{\partial n}\right)_p = \left(\frac{\partial U}{\partial n}\right)_p + \left(\frac{\partial T}{\partial n}\right)_p \tag{3-6}$$

where the subscripted P indicates the derivatives are to be evaluated at the point P.

Now
$$\left(\frac{\partial W}{\partial n}\right)_p = g_p \tag{3-7}$$

the "observed" gravity at P and

$$\left(\frac{\partial U}{\partial n}\right) = \left(\frac{\partial U}{\partial \nu}\right) \cos \epsilon = \gamma_p \cos \epsilon \tag{3-8}$$

where ϵ is the angle between the normals at P to the equipotentials Wp and Up, i.e., it is the "deflection of the vertical" at P. Since ϵ seldom exceeds 1', the approximation $\cos \epsilon = 1$ is permissable. Using this approximation and substituting (3-7) and (3-8) into (3-6) gives

$$g_p = \gamma_p + \left(\frac{\partial T}{\partial n}\right)_p \tag{3-9}$$

Expanding the function γ about the point Q in a Taylor series and keeping only two terms in the expansion yields

$$\gamma_p = \gamma_Q - \left(\frac{\partial \gamma}{\partial \nu}\right)_Q N \tag{3-10}$$

Again note that the error here is of the same order as discussed after equation (3-4).

Substituting for N from (3-5) gives

$$\gamma_p = \gamma_Q - \left(\frac{\partial \gamma}{\partial \nu}\right)_Q \frac{T_p}{\gamma_Q} \tag{3-11}$$

Substituting this into (3-9) gives

$$g_p = \gamma_Q - \left(\frac{\partial \gamma}{\partial \nu}\right)_Q \frac{T_p}{\gamma_Q} + \left(\frac{\partial T}{\partial n}\right)_p$$

or

$$(g_p - \gamma_Q) = \left(\frac{\partial T}{\partial n}\right)_p - \frac{T_p}{\gamma_Q} \left(\frac{\partial \gamma}{\partial \nu}\right)_Q \tag{3-12}$$

This formula is called the boundary value problem of classical geodesy. It describes the relation between gravity anomalies and T at points on the boundary of the body. For the case of the earth both U and W are potentials of attraction plus a rotational potential. It is usually assumed that the two rotational potentials are equal.

Thus, T = W - U is a pure gravitational potential and is an harmonic function outside the body B. The assumption that the rotational potential of W and U are equal may not be strictly accurate. The problems connected with this possibility are discussed for example by Molodenski, et al (1960). Normally, this problem is ignored.

If the reference gravitational field were spherical rather than the ellipsoidal normal field usually employed we would have $\gamma_Q = \frac{kM}{R_Q^2}$ and

$$\left(\frac{\partial \gamma}{\partial \nu}\right) = - \left(\frac{\partial \gamma}{\partial R}\right)_Q = + \frac{2kM}{R^3_Q} \tag{3-13}$$

where k = universal gravitational constant

M = mass of model usually assumed to be equal to mass of earth

R_Q= distance from center of spherical field to Q

Substituting equations (3-13) into equation (3-12) gives

$$(g_p - \gamma_Q) = \left(\frac{\partial T}{\partial n}\right)_p - \frac{2}{R} T_p \tag{3-14}$$

Now if it is also sufficiently accurate to use the approximation

$$\left(\frac{\partial T}{\partial n}\right)_p \approx - \left(\frac{\partial T}{\partial R}\right)_p \tag{3-15}$$

we get

$$(g_p - \gamma_Q) = - \frac{2}{R} T_p - \left(\frac{\partial T}{\partial R}\right)_p \tag{3-16}$$

In the derivation of Stokes and Vening Meinesz equations the boundary condition (3-16) is used. Note that in deriving (3-16) the approximations (3-13) and (3-15) are applied to the right hand side of equation (3-12) but the values of γ_Q used on the left hand side of equation (3-16) in deriving the gravity anomalies remain the International Formula values, i.e., the ellipsoidal approximation is retained on the left hand side of the equation.

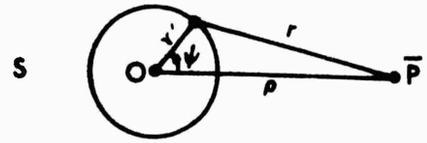
The error involved in using equation (3-16) rather than (3-12) would be expected to be of the order of the flattening of the earth. The result of proceeding from (3-12) retaining the assumption of an ellipsoidal normal field and obtaining a result correct to the order of the square of the flattening has been studied most intensively by Zagrebin (1956) and by Bjerhammar (1962) who corrected some errors in Zagrebin's final formulae. The results for computing N are very complicated and have received little usage. Molodenski et al (1960) outline earlier work of Molodenski and arrive at a somewhat simpler formulation which he extends to deflection of the vertical computations in the classical theory. We shall not at present pursue this refinement further insofar as it should not have any significant effect on deflection interpolation problems.

The next step in the normal procedure is to determine a formula for T which will be correct everywhere outside of and on the surface, S, of the body B. The value of T anywhere on or outside a gravitating body can be expressed by the use of Green's formula (See Section 2). Letting $\overline{P}$ be any point on or outside the surface S, we have

$$T\overline{p} = 1/4\pi \iint_s \left[\frac{1}{r}\frac{\partial T}{\partial n} - T\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] dS \tag{3-17}$$

where r = distance from $\overline{P}$ to variable point on surface S and n is the normal to the surface S directed inward.

Consider the situation below:



r'= distance between 0 and incremental surface element dS

r = distance from $\bar{P}$ to incremental surface element dS

ρ = distance from $\bar{P}$ to 0

ψ = angle between r' and ρ

Assume that the point $\bar{P}$ lies sufficiently far from the origin that, for all elements dS of the surface, $\rho > r'$. Then, as shown in Section 2,

$$\frac{1}{r} = \frac{1}{\rho}\left[1 + \frac{r'^2}{\rho^2} - 2\frac{r'}{\rho}\cos\psi\right]^{-1/2} = \sum_{n=0}^{\infty} \frac{r'^n}{\rho^{n+1}} Pn(\cos\psi) \tag{3-18}$$

For $\rho \leq r'$ as will occur in the case of points lying on the co-geoid of the earth, it is not clear that the development which follows is valid. It has usually been assumed that any error would be of the order of the flattening of the earth, but recently Molodenski et al (1962) have shown the error may be much greater.

In Section 2 it is shown that the perturbing potential at $\bar{P}$ can be written as a series of spherical harmonics to give

$$T\bar{p} = k \sum_{i=2}^{\infty} \frac{Yi}{\rho^{i+1}} \tag{3-19}$$

where k = universal gravitational constant and

Y_i = spherical harmonic of degree i.

Going to the spherical approximation in (3-17) i.e., assuming that we can replace $\frac{\partial}{\partial n}$ by $\frac{-\partial}{\partial r'}$, with sufficient accuracy gives

$$\frac{\partial}{\partial n}\left(\frac{1}{r}\right) = \frac{-\partial}{\partial r'}\left(\frac{1}{r}\right) = \frac{-\partial}{\partial r'}\left[\sum_{n=0}^{\infty} \frac{r'^n}{\rho^{n+1}} Pn(\cos\psi)\right]$$

or

$$\frac{\partial}{\partial n}\left(\frac{1}{r}\right) = -\sum_{n=0}^{\infty} \frac{n r'^{n-1}}{\rho^{n+1}} Pn(\cos\psi) \tag{3-20a}$$

Let the value of T at any point P on S be given by

$$Tp = k \sum_{i=2}^{\infty} \frac{Yi}{r'^{i+1}} \tag{3-20b}$$

Again using $\frac{\partial}{\partial n} \approx \frac{\partial}{\partial r'}$, gives

$$\left(\frac{\partial T}{\partial n}\right) = - \left(\frac{\partial T}{\partial r'}\right) = k \sum_{i=2}^{\infty} \frac{(i+1)Yi}{r'^{i+2}} \tag{3-20c}$$

If we now substitute equations (3-18), (3-19), (3-20a), and (3-20c) into (3-17) we get

$$T\bar{p} = \frac{k}{4\pi} \int_s \left[\sum_{n=0}^{\infty} \frac{r'^n}{\rho^{n+1}} Pn (\cos \psi) \sum_{i=2}^{\infty} \frac{(i+1)Yi}{r'^{i+2}} + \sum_{i=2}^{\infty} \frac{Yi}{r'^{i+1}} \sum_{n=0}^{\infty} \frac{nr'^{n-1}}{\rho^{n+1}} Pn (\cos \psi) \right] dS \tag{3-21}$$

If the surface S were a sphere ($r' = R$), it could be shown that

$$\iint_s Pn\ Yi\ dS = 0 \text{ if } n \neq i \tag{3-22}$$

Thus, again assuming that S may be considered a sphere with sufficient accuracy that (3-22) is satisfied, yields the following result. In (3-21) only those terms where $n = i$ will be different from zero after carrying out the indicated multiplications and integrations. This result leads to the conclusions that:

A. Since there are no Y_o or Y_1 terms the P_o and P_1 terms in the summations with respect to n will contribute nothing to the final answer. Thus we will drop the P_o and P_1 terms and begin the summation at $n = 2$;

B. The following identity may be used

$$\sum_{n=2}^{\infty} Pn \sum_{i=2}^{\infty} (i + 1)\ Yi = \sum_{n=2}^{\infty} (n + 1)\ Pn \sum_{i=2}^{\infty} Yi \tag{3-23}$$

Using the results (A) and (B) above and the approximation $r' = R$ we can write (3-21) in the form

$$T\bar{p} = \frac{1}{4\pi} \iint_s \left[\left(\sum_{n=2}^{\infty} \frac{(n+1)R^{n-1}}{\rho^{n+1}} Pn\right)\left(k \sum_{i=2}^{\infty} \frac{Yi}{R^{i+1}}\right) + \left(\sum_{n=2}^{\infty} \frac{nR^{n-1}}{\rho^{n+1}} Pn\right)\left(k \sum_{i=2}^{\infty} \frac{Yi}{R^{i+1}}\right)\right] dS \tag{3-24}$$

$$T\bar{p} = \frac{1}{4\pi} \iint_s \left[\sum_{n=2}^{\infty} \frac{(2n+1)R^{n-1}Pn}{\rho^{n+1}}\right] \left[k \sum_{i=2}^{\infty} \frac{Yi}{R^{i+1}}\right] dS \tag{3-25}$$

Substituting (3-20b) and (3-20c) into (3-16) after allowing the approximation r' = R gives for every point on S

$$(g_p - \gamma_Q) = \frac{-2k}{R} \sum_{i=2}^{\infty} \frac{Yi}{R^{i+1}} + k \sum_{i=2}^{\infty} \frac{(i+1)Yi}{R^{i+2}}$$

or

$$\Delta g = (g_p - \gamma_Q) = k \sum_{i=2}^{\infty} \frac{(i-1)Yi}{R^{i+2}} \tag{3-26}$$

Now returning to the integral of (3-25) and altering its form slightly, we get

$$T\bar{p} = \frac{1}{4\pi R} \iint_s \left[\sum_{n=2}^{\infty} (2n+1)\, Pn \left(\frac{R}{\rho}\right)^{n+1}\right] \left[k \sum_{i=2}^{\infty} \left(\frac{i-1}{i-1}\right) \frac{Yi}{R^{i+2}}\right] dS \tag{3-27}$$

Or making use of (3-23)

$$T\bar{p} = \frac{1}{4\pi R} \iint_s \left[\sum_{n=2}^{\infty} \left(\frac{2n+1}{n-1}\right) Pn \left(\frac{R}{\rho}\right)^{n+1}\right] \left[k \sum_{i=2}^{\infty} \frac{(i-1)Yi}{R^{i+2}}\right] dS \tag{3-28}$$

Substituting from (3-26) into (3-28) gives

$$T\bar{p} = \frac{1}{4\pi R} \iint_s \left[\sum_{n=2}^{\infty} \left(\frac{2n+1}{n-1}\right) Pn \left(\frac{R}{\rho}\right)^{n+1}\right] \Delta g\, dS = \frac{1}{4\pi R} \sum_{n=2}^{\infty} \frac{2n+1}{n-1} \left(\frac{R}{\rho}\right)^{n+1} \int_s \Delta g\, Pn\, dS \tag{3-29}$$

Substituting (3-29) into (3-5) after letting the point $\bar{P}$ lie on S so that $\rho \approx R$, gives for the undulation of the co-geoid at the point $\bar{P}$.

$$N\bar{p} = \frac{T\bar{p}}{\gamma_{\bar{Q}}} = \frac{1}{4\pi R \gamma_{\bar{Q}}} \sum_{n=2}^{\infty} \frac{(2n+1)}{(n-1)} \iint_s \Delta g\; Pn\, dS \tag{3-30}$$

Equation (3-30) is Stokes' equation relating the co-geoidal height at $\bar{P}$ to the integral of the gravity anomalies and a function of position over the co-geoidal surface in its spherical approximation.

There are of course many ways of arriving at equation (3-30) (See for example Heiskanen and Vening-Meinesz, 1958). The method chosen here is meant

to show as clearly as possible what approximations are made and where they enter the calculations. To allow for numerical calculations it is desirable to have the summations of (3-29) or (3-30) in closed form. In the following development, which follows closely that given by Nagy (1962), the closed form of (3-29) will be obtained. The closed form of (3-30) would immediately follow by letting $\rho = R$ in (3-29) and substituting into (3-5).

To begin let us do the following to (3-29)

a. Factor out $\frac{\rho}{R}$ from beneath the integral sign

b. Use the abbreviated notation $X = \frac{R}{\rho}$ beneath the integral

c. Use the identity $\frac{2n+1}{n-1} = 2 + \frac{3}{n-1}$

Then the equation (3-29) becomes

$$T\bar{p} = \frac{\rho}{4\pi R^2} \iint_s \Delta g \sum_{n=2}^{\infty} \left(2 + \frac{3}{n-1}\right) Pn\, X^{n+2} dS \tag{3-31}$$

But from formula (3-18) using the approximation $r' = R$

$$\frac{1}{r} = \frac{1}{\rho(1 + X^2 - 2X\cos\psi)^{1/2}} = \frac{1}{\rho} \sum_{n=0}^{\infty} X^n Pn \tag{3-32}$$

or

$$\frac{1}{r} = \frac{1}{\rho}\left(P_0 + P_1 X + \sum_{n=2}^{\infty} X^n Pn\right) = \frac{1}{\rho}\left(1 + X\cos\psi + \sum_{n=2}^{\infty} X^n Pn\right) \tag{3-33}$$

Then from (3-32) and (3-33) we get

$$\sum_{n=2}^{\infty} X^n Pn = \frac{1}{(1 + X^2 - 2X\cos\psi)^{1/2}} - 1 - X\cos\psi \tag{3-34}$$

Using (3-34)

$$T\bar{p} = \frac{\rho}{4\pi R^2} \iint_S \Delta g \left[2X^2 \left(\frac{1}{(1 + X - 2X\cos\psi)^{1/2}} - 1 - X\cos\psi\right) + \sum_{n=2}^{\infty} \frac{3}{n-1} PnX^{n+2}\right] dS \tag{3-35}$$

If both sides of (3-34) are divided by X^2 and then integrated with respect to X we get

$$\int \sum_{n=2}^{\infty} X^{n-2} Pn \, dX = \int \frac{1}{X^2(1 + X^2 - 2X \cos \psi)^{1/2}} dX - \int \frac{1}{X^2} dX - \int \frac{\cos \psi}{X} dX \tag{3-36}$$

Carrying out the indicated integrations using standard integral tables yields

$$\int \sum_{n=2}^{\infty} X^{n-2} Pn \, dX = \sum_{n=2}^{\infty} \frac{1}{n-1} X^{n-1} Pn$$

$$\int \frac{dX}{X^2(1 - 2X \cos \psi + X^2)^{1/2}} = - \frac{(1 - 2X \cos \psi + X^2)^{1/2}}{X} - \cos\psi \ln \left(\frac{2(1 - 2X \cos \psi + X^2)^{1/2}}{X} + \frac{2}{X} - 2 \cos \psi\right)$$

$$\int \frac{\cos \psi}{X} = \cos \psi \ln X \qquad \int \frac{dX}{X^2} = - \frac{1}{X} \tag{3-37}$$

Since the above are indefinite integrals, there are undefined constants of integration which have not been indicated. Substituting from equations (3-37) into (3-36) and using the symbol C_1 to represent the combined integration constant yields

$$\sum_{n=2}^{\infty} \frac{1}{n-1} X^{n-1} Pn = - \frac{\nu}{X} - \cos \psi \ln \left(\frac{2\nu}{X} + \frac{2}{X} - 2 \cos \psi\right) + \frac{1}{X} - \cos \psi \ln X + C_1 \tag{3-38}$$

where to shorten notation we have defined $\nu = (1 - 2X \cos \psi + X^2)^{\frac{1}{2}} = \frac{r}{\rho}$

Collecting terms and remembering that log a + log b = log ab gives

$$\sum_{n=2}^{\infty} \frac{1}{n-1} X^{n-1} Pn = \frac{1-\nu}{X} - \cos \psi \ln 2 (\nu + 1 - X \cos \psi) + C_1 \tag{3-39}$$

To evaluate C_1 set X = 0 in (3-39). Then,

$$\left(\sum_{n=2}^{\infty} \frac{1}{n-1} X^{n-1} Pn\right)_{X=0} = 0 \tag{3-40}$$

and

$$[\cos \psi \ln 2 (1 + \nu - X \cos \psi)]_{X=0} = \cos \psi \ln 4 \tag{3-41}$$

To evaluate the function $\frac{1-\nu}{X}$, take the limit as X approaches zero. Making use of the fact that: $\lim_{X\to 0} \frac{f(X)}{g(X)} = \lim_{X\to 0} \frac{df(X)}{dg(X)}$

and noting that

$$\frac{d\nu}{dx} = \frac{d}{dx}(1 - 2X\cos\psi + X^2)^{1/2} = 1/2(1 - 2X\cos\psi + X^2)^{-1/2}(-2\cos\psi + 2X)$$

and that

$$\lim_{X\to 0} \nu = 1$$

we get

$$\lim_{X=0} \frac{1-\nu}{X} = \lim_{X\to 0} - \frac{X - \cos\psi}{\nu} = \cos\psi \tag{3-42}$$

Evaluating (3-29) at X = 0 yields

$$0 = \cos\psi - \cos\psi \ln 4 + C_1$$

or

$$C_1 = \cos\psi \ln 4 - \cos\psi \tag{3-43}$$

Using (3-43) and again remembering the laws for combining log terms, we get for (3-39)

$$\sum_{n=2}^{\infty} \frac{1}{n-1} X^{n-1} Pn = \frac{1-\nu}{X} - \cos\psi \ln 2(1 + \nu - X\cos\psi) - \cos\psi + \cos\psi \ln 4$$

or

$$\sum_{n=2}^{\infty} \frac{X^{n-1}}{n-1} Pn = \frac{1-\nu}{X} - \cos\psi \left(1 + \ln \frac{(1 + \nu - X\cos\psi)}{2}\right) \tag{3-44}$$

Using (3-44) and noting that $\sum_{n=2}^{\infty} \frac{X^{n+2}}{n-1} Pn = X^3 \sum_{n=2}^{\infty} \frac{X^{n-1}}{n-1} Pn$

equation (3-35) becomes

$$T\bar{p} = \frac{\rho}{4\pi R^2} \iint_s \Delta g \left[2X^2 \left(\frac{1}{\nu} - 1 - X\cos\psi\right) + 3X^3 \left(\frac{1-\nu}{X} - \cos\psi\left(1 + \ln\left(\frac{1 + \nu - X\cos\psi}{2}\right)\right)\right)\right] dS \tag{3-45}$$

or collecting terms

$$T\bar{p} = \frac{\rho}{4\pi R^2} \iint_s X^2 \left[\frac{2}{\nu} + 1 - 3\nu - 5X\cos\psi - 3X\cos\psi \ln \frac{(1 + \nu - X\cos\psi)}{2}\right] \Delta g dS \tag{3-46}$$

Again note that

$$\nu = (1 + X^2 - 2X\cos\psi)^{1/2} = \frac{r}{\rho} \tag{3-47}$$

By substituting this into (3-46) and remembering that $X = \frac{R}{\rho}$ we get

$$T\bar{p} = \frac{1}{4\pi R} \iint_s R\left[\frac{2}{r} + \frac{1}{\rho} - \frac{3r}{\rho^2} - \frac{5R}{\rho^2}\cos\psi - \frac{3R}{\rho^2}\cos\psi \ln\frac{(\rho + r - R\cos\psi)}{2\rho}\right] \Delta g dS \tag{3-48}$$

If we now assume that point $\bar{P}$ is also on the surface S which can be treated with sufficient accuracy as a sphere then $\rho = R$. Making the spherical assumption we can relate the distance r which is now simply the chord between two points on the sphere to R and ψ by the well known formula

$$r = 2R \sin \tfrac{1}{2}\psi \tag{3-49}$$

Using the two relations $\rho = R$ and $r = 2R \sin \frac{1}{2}\psi$ equation (3-48) becomes

$$T\bar{p} = \frac{1}{4\pi R} \iint_s \left[\frac{1}{\sin 1/2\psi} + 1 - 6 \sin 1/2\psi - 5\cos\psi - 3\cos\psi \ln (\sin 1/2\psi + \sin^2 1/2\psi)\right] \Delta g dS \tag{3-50}$$

Or in shortened form

$$T\bar{p} = \frac{1}{4\pi R} \iint_s S(\psi)\, \Delta g dS \tag{3-51}$$

where the symbol $S(\psi)$ represents the quantity in brackets in equation (3-50).

Then substituting (3-51) into (3-5) gives

$$N\bar{p} = \frac{T\bar{p}}{\gamma_Q} = \frac{1}{4\pi R\gamma_Q} \iint_s S(\psi)\, \Delta g dS \tag{3-52}$$

This is the normal Stokes' equation for co-geoidal undulations. Writing equation (3-48) in shortened form gives for a point not on S

$$T\bar{p} = \frac{1}{4\pi R\gamma_Q} \iint S(\rho, \psi)\, \Delta g dS \tag{3-53}$$

where $S(\rho, \psi)$ is used to represent the quantity in brackets in equation (3-48).

In classical theory one carries out numerical integration of (3-52) over the surface of the earth treated as a sphere. The solutions are set up to

give the geometrical quantities represented by S (ψ) in predetermined form for multiplication by the Δg values.

Deflections of the Vertical in Classical Theory

The deflection of the vertical in a particular direction at a point P on the bounding equipotential surface is the angle between the normal to the bounding equipotential at the point and the normal to the potential field U at the point. This will, of course, be the angle between the two equipotential surfaces at the point. It is usual to express the deflection of the vertical in two components, an east-west component and a north-south component. It is conventional to call the north-south component ξ, and the east-west component η.

If N is computed from Stokes' formula, then the deflection which is the rate of change of N in a direction say l (angular direction φ) is

$$\text{deflection} = \frac{1}{R}\frac{\partial}{\partial\varphi} N = \frac{1}{R}\frac{\partial}{\partial\varphi}\left(\frac{T\bar{p}}{\gamma_Q}\right) \approx \frac{1}{R\gamma_Q}\frac{\partial}{\partial\varphi}(T\bar{p}) \qquad (3\text{-}54)$$

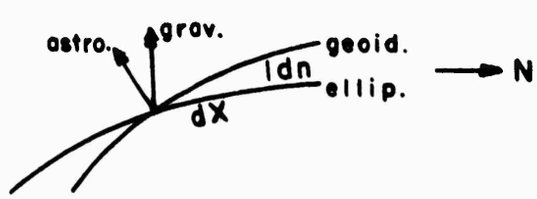
Some confusion can arise as to the sign of the deflections. The sign is, of course, a convention since the deflection is simply a difference. The normal convention is to define the meridional deflection ξ as the astronomic latitude minus the gravimetric latitude and the parallel deflection η as the astronomic longitude minus the gravimetric longitude.

i.e.

$$\xi = \theta \text{ ast} - \theta \text{ grav.}$$

$$\eta = \lambda \text{ ast} - \lambda \text{ grav.}$$

Let us consider the meridional deflection ξ and the picture below



The geoidal height change is positive thus we have $\frac{dN}{dX}$ positive but θ ast < θ grav. so θast - θ grav. is negative and thus

$$\xi = \theta \text{ ast} - \theta \text{ grav.} = -\frac{dN}{dX}$$

or

$$\xi = -\frac{1}{R}\frac{\partial N}{\partial \theta}$$

in a similar manner $\eta = -\frac{1}{R}\frac{\partial N}{\partial \lambda}$

Then the components of deflection in the north and east direction are, using $T\bar{p}$ from (3-52)

$$\xi = \frac{-1}{R}\frac{\partial}{\partial \theta}(N) = \frac{-1}{R\gamma_Q}\frac{\partial T\bar{p}}{\partial \theta} = \frac{-1}{4\pi\gamma_Q R^2}\int_s \frac{\partial S(\psi)}{\partial \theta}\Delta g dS$$

$$\eta = \frac{-1}{R}\frac{\partial}{\partial \lambda}(N) = \frac{-1}{R\gamma_Q}\frac{\partial T\bar{p}}{\partial \lambda} = \frac{-1}{4\pi\gamma_Q R^2}\int_s \frac{\partial S(\psi)}{\partial \lambda}\Delta g dS \qquad (3\text{-}55)$$

but

$$\frac{\partial}{\partial \theta} S(\psi) = \frac{\partial}{\partial \psi} S(\psi)\frac{\partial \psi}{\partial \theta}$$

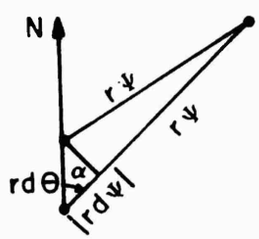
and

$$\frac{\partial}{\partial \lambda} S(\psi) = \frac{\partial}{\partial \psi} S(\psi)\frac{\partial \psi}{\partial \lambda} \qquad (3\text{-}56)$$

If we let α be the angle between the north direction and the ψ direction then

$$\frac{\partial \psi}{\partial \theta} = -\cos\alpha \text{ and } \frac{\partial \psi}{\psi \lambda} = -\sin\alpha \qquad (3\text{-}57)$$

To see the reason for the negative signs in equation (3-57), consider the figure below.



$$\cos\alpha = \frac{rd\psi}{rd\theta} = \frac{d\psi}{d\theta}$$

If $d\theta$ is positive, i.e., it is in the direction of positive change of θ, then ψ is decreasing for a point with $\cos\alpha$ positive and $d\psi$ is negative. Thus to get the correct sign we must have $\frac{d\psi}{d\theta} = -\cos\alpha$.

Thus we get

$$\xi = \frac{1}{4\pi\gamma_Q R^2}\int_s \frac{\partial}{\partial \psi} S(\psi)\,\Delta g \quad \cos\alpha dS$$

$$\eta = \frac{1}{4\pi\gamma_Q R^2}\int_s \frac{\partial}{\partial \psi} S(\psi)\,\Delta g \quad \sin\alpha dS \qquad (3\text{-}58)$$

but $dS = R^2 \sin\psi d\psi dA$

$$\xi = \frac{1}{4\pi\gamma_Q} \int_0^{\pi} \int_0^{2\pi} \left[\frac{\partial}{\partial\psi} S(\psi)\right] \Delta g \cos\alpha \sin\psi \, dA d\psi \tag{3-59}$$

$$\eta = \frac{1}{4\pi\gamma_Q} \int_0^{\pi} \int_0^{2\pi} \left[\frac{\partial}{\partial\psi} S(\psi)\right] \Delta g \sin\alpha \sin\psi \, dA d\psi \tag{3-60}$$

where dA is an incremental angle corresponding to a longitude angle using the computation point as a pole.

Let us evaluate $\frac{\partial}{\partial\psi} S(\psi)$

$$\frac{\partial}{\partial\psi} S(\psi) = \frac{-\cos 1/2\psi}{\sin^2 1/2\psi} - 3\cos 1/2\psi + 5\sin\psi + 3\sin\psi \ln(\sin 1/2\psi + \sin^2 1/2\psi) - \frac{3\cos\psi\cos 1/2\psi\,(1/2 + \sin 1/2\psi)}{\sin 1/2\psi\,(1 + \sin 1/2\psi)}$$

If we now multiply through by $\sin\psi$ and use the identities $\sin\psi = 2\sin\frac{1}{2}\psi \cos\frac{1}{2}\psi$ and $\cos\psi = 1 - 2\sin^2\psi/2$ we get

$$\sin\psi \frac{\partial}{\partial\psi} S(\psi) = \frac{-(1 - \sin^2 1/2\psi)}{\sin 1/2\psi} - 6\sin\frac{\psi}{2}(1 - \sin^2\frac{\psi}{2}) + 20\sin^2\frac{\psi}{2}(1 - \sin^2\frac{\psi}{2}) + 3\sin^2\psi \ln(\sin 1/2\psi + \sin^2 1/2\psi) - 6(1 - 2\sin^2\frac{\psi}{2})(1 - \sin^2\frac{\psi}{2})(1/2 + \sin 1/2\psi)$$

$$\sin\psi \frac{\partial}{\partial\psi} S(\psi) = -\csc\frac{\psi}{2} - 3 + 3\sin^2\psi \ln(\sin 1/2\psi + \sin^2 1/2\psi) - 8\sin\frac{\psi}{2} + 32\sin^2\frac{\psi}{2} + 12\sin^3\frac{\psi}{2} - 32\sin^4\frac{\psi}{2} \tag{3-61}$$

Then if we let $F'(\psi) = \frac{1}{2}\sin\psi S(\psi)$ we get

$$\xi = \frac{1}{2\pi\gamma_Q} \int_0^{\pi} \int_0^{2\pi} F'(\psi) \Delta g \cos\alpha \, dA d\psi$$

$$\eta = \frac{1}{2\pi\gamma_Q} \int_0^{\pi} \int_0^{2\pi} F'(\psi) \Delta g \sin\alpha \, dA d\psi \tag{3-62}$$

We might note here that for small values of ψ $\cos\psi \approx 1$, $\sin\psi \approx 0$ and thus we can use the approximation $F'(\psi) \approx -1/2(\csc\frac{\psi}{2} + 3)$

$$\sin\psi \frac{\partial}{\partial\psi} S(\psi) = -\csc\frac{\psi}{2} - 3 \tag{3-63}$$

Numerical methods of carrying out the integrations of equations (3-62) have been developed by a number of workers. These numerical methods can be, to some extent, carried over to the method of solution adopted for use in the Test

and Application Phases of this contract. The overall numerical method finally adopted is discussed in Section 7.

SECTION 4

The New Geodetic Theory

From the classical viewpoint described in the previous section, the necessary steps in surface level deflection computations are (1) correct transfer of material from above to below sea level, (2) "reduction" of surface gravity anomalies to geoid and then co-geoid level anomalies, (3) computation of deflections at co-geoid level, and (4) computation of surface level deflections from the co-geoid level deflections.

The above procedure appears to result in considerable inaccuracy because of lack of knowledge concerning internal densities and vertical gravity gradients. Because of these inaccuracies and the difficulty involved in assessing the errors introduced from approximations made in the mathematical development of the theory, considerable argument exists as to the "correct" reduction to be used and the type of anomaly which is theoretically more accurate. It has not been possible until recently, because of the way in which the theory was developed, to judge accurately which reduction method is best.

The so-called new theory of gravimetric geodesy developed below is therefore important for two reasons.

1.) It makes possible an evaluation of the accuracy of the results obtained using different types of gravity anomalies with the classical theory.

2.) It provides a mathematical theory capable of extension to any order of accuracy desired.

The following development of the new theory follows very closely that of Molodenski et al (1960). First we will develop an equation analogous to Brun's equation for the co-geoidal height in classical theory.

Let $W(\theta, \lambda, r)$ = potential field of the actual earth including the the rotation potential

$U(\theta, \lambda, r)$ = reference potential field having the same rotational potential as the earth's field

W_o = value of sea level equipotential of the actual earth.

Now assume we pair the equipotential surfaces of the actual earth and the reference field in some way so that

U_o = the equipotential surface of the reference potential field which is paired with equipotential W_o of the actual earth.

The most advantageous way to pair the surfaces, is to pair the equipotentials which have the same value of potential. If this is done we have

$$U_o = W_o \tag{4-1}$$

We cannot do this though, without knowing W_o. As is the case with the classical theory, the assumption is made that W_o is known although it is not in fact known exactly.

Consider some point $P(\theta^*, \lambda^*, H)$ on the earth's surface where H is the height of the point P above the surface $U = U_o$ whose shape is known. Let us define the perturbing potential, T, at any point to be the difference between the actual earth's potential at the point and the reference potential at the point. Thus

$$T_p(\theta^*, \lambda^*, H) = W_p(\theta^*, \lambda^*, H) - U_p(\theta^*, H) \tag{4-2}$$

Note that this equation assumes that U has been chosen so as not to be a function of the coordinate λ. This is the case for the international formula potential. Then

$$W_p(\theta^*, \lambda^*, H) = W_o - \int g\,dh \tag{4-3}$$

where g is the actual value of gravity along the path of integration, and integration takes place along the earth's surface.

Let us now define the equipotential surface U_Q (θ,h) which is paired to W_p by the requirement that

$$W_o - U_o = W_p - U_Q \tag{4-4}$$

Then we have

$$U_Q = U_\bullet - \int \gamma dh \tag{4-5}$$

where $\int \gamma dh$ is defined by

$$\int g dh = \int \gamma dh = h\bar{\gamma} \tag{4-6}$$

$\bar{\gamma}$ = mean value of γ along line of integration.

In principal h can be rigorously determined by gravity and leveling on the earth's surface. In practice h is not so rigorously determined due to the lack of gravity information along the survey lines. However, the difference between normal map elevations and the precise quantities is so negligible as to be unlikely to cause any significant error. From this point on the symbol h will be understood to represent normal orthometric heights as read from maps.

Now by substituting for W_p from (4-4) we can write (4-2) in the form

$$T_p(\theta^*,\lambda^*,H) = W_\bullet - U_\bullet + U_Q(\theta,h) - U_p(\theta^*,H) \tag{4-7}$$

As the differences

$$\begin{aligned} \theta^* - \theta &= \Delta\theta \\ \lambda^* - \lambda &= \Delta\lambda \\ H - h &= N \end{aligned} \tag{4-8}$$

are small quantities, their squares and derivatives can be ignored. If we

expand U about the point Q in a Taylor series keeping only the first terms, we obtain

$$U_p(\theta^*,H) = U_Q(\theta,h) + \left(\frac{\partial U}{\partial h}\right)_Q N + \left(\frac{\partial U}{\partial \theta}\right)_Q \Delta\theta \tag{4-9}$$

Now the error in $\Delta\theta$ is normally less than 1' so that the last term in (4-9) may be ignored. Substituting (4-9) into (4-7) and ignoring the last term in (4-9) we get

$$T_p = W_o - U_o + U_Q - U_Q - \left(\frac{\partial U}{\partial h}\right)_Q N$$

or

$$T_p = W_o - U_o - \left(\frac{\partial U}{\partial h}\right)_Q N \tag{4-10}$$

But to a very close approximation $\left(\frac{\partial U}{\partial h}\right)_Q = -\gamma_Q$ = gravity due to normal potential field at Q. This gives, after rearranging terms and substituting into (4-10),

$$N = \frac{Tp}{\gamma_Q} - \frac{W_o - U_o}{\gamma_Q} \tag{4-11}$$

If we have $W_o = U_o$, we obtain

$$N = \frac{Tp}{\gamma_Q} \tag{4-12}$$

which is completely analgous to Brun's equation. Here, however, T_p is the deviation potential at the earth's surface, γ_Q is normal gravity at a point distance N above or below the earth's surface. N is thus a correction term to be added (with correct sign) to h to give true location above the equipotential $U = U_o$ of the surface point P. N is usually called the height anomaly in this case.

The assumption that $W_o = U_o$ made in deriving (4-12) from (4-11) has been examined in detail by Molodenski et al (1960).

The solution of equation (4-12) requires knowledge of $T_p = W_p - U_p$. To obtain T_p let us first establish the boundary condition which T must satisfy at the earth's surface. Since W = U + T we can write

$$\left(\frac{\partial W}{\partial \nu}\right)_p = \left(\frac{\partial U}{\partial \nu}\right)_p + \left(\frac{\partial T}{\partial \nu}\right)_p$$

ν = direction normal to reference equipotential

Now with an error of the order of the deflection of the vertical, we have $\left(\frac{\partial W}{\partial \nu}\right)_p \approx g_p$ = observed gravity at point P. Expanding $\frac{\partial U}{\partial \nu}$ about Q in a Taylor series and keeping only the first term we get

$$\left(\frac{\partial U}{\partial \nu}\right)_p \approx \left(\frac{\partial U}{\partial \nu}\right)_Q - N\left(\frac{\partial \gamma}{\partial \nu}\right)_Q = \gamma_Q - N\left(\frac{\partial \gamma}{\partial \nu}\right)_Q$$

Substituting this gives

$$g_p = \gamma_Q - N\left(\frac{\partial \gamma}{\partial \nu}\right)_Q + \left(\frac{\partial T}{\partial \nu}\right)_p$$

or, using Brun's formula, we get

$$(g_p - \gamma_Q) = -\frac{T_p}{\gamma_Q}\left(\frac{\partial \gamma}{\partial \nu}\right)_Q + \left(\frac{\partial T}{\partial \nu}\right)_p \qquad (4\text{-}13)$$

Thus the determination of T has been reduced to the solution of the third boundary value problem of potential theory at the earth's surface. This problem can be stated:

Find the function T which:

1.) At every point on the physical surface of the earth S satisfies the condition (4-13)

2.) Is a harmonic function outside S

3.) At infinity is regular or satisfies the stronger condition

$$\lim_{\rho \to \infty} (T_\rho^2) = 0$$

To find the function T it is possible to proceed in a number of ways. Some authors, e.g. Hirvonen (1960), have proceeded much along classical lines by using spherical harmonics and arriving at the generalized Stokes and Vening Meinesz theorems. Molodenski et al (1962) have examined these methods and point out that

the error is greater than the order of the flattening of the earth as has often been supposed. They also stress that the error in deflection computation is much greater than in geoid undulation computation.

Another method of procedure and the one favored by Molodenski involves the introduction of a surface density coating. The use of density coatings dates back to Helmert and is based on that part of potential theory which permits the potential of a volume mass to be represented external to it by the potential of a surface mass distribution on the surface of the volume mass.

Thus to solve equation (4-13) we introduce the auxiliary density function φ representing this surface density and defined by

$$T_p = \int_s \frac{\varphi}{r_p}\, ds \tag{4-14}$$

where r_p is the distance from the point P where T is being computed to the variable point on the surface S.

To substitute into (4-13) we also need $(\frac{\partial T}{\partial \nu})_p$. As the point P lies on the surface S, the derivative contains an extra term due to the density layer causing a discontinuity of the derivative at the surface. Thus we arrive at

$$+(\frac{\partial T}{\partial \nu})_p = +\ 2\pi\varphi_p \cos\alpha + [\frac{\partial}{\partial \nu}\int_s \frac{\varphi}{r}\, ds]_p \tag{4-15}$$

where: $+2\pi\varphi_p \cos\alpha$ is the term appearing due to the discontinuity of the derivative at the density layer S

and α is the angle between the normal to the direction in which the derivative is taken ($\perp$ to the reference field) and the normal to the surface S.

If we can derive a φ and satisfy equation (4-13) by the T determined from it using (4-14) then this T will obviously satisfy boundary conditions 2.) and 3.) i.e., T will be harmonic outside S and regular at infinity. This results from the fact that a T so obtained will be the potential of an actual density distribution and must therefore behave as desired outside S.

To do this we can substitute (4-14) and (4-15) into equation (4-13) which allows us to write this equation in terms of φ . Substituting we get

$$2\pi\varphi_p \cos\alpha + [\frac{\partial}{\partial\nu}\int_s \frac{\varphi}{r_p}\, ds]_p - \frac{1}{\gamma_Q}(\frac{\partial\gamma}{\partial\nu})_Q\int_s \frac{\varphi}{r_p}\, ds = (g_p - \gamma_Q) \tag{4-16}$$

where r_p is the distance from the point P to the variable point on S.

Since the surface S is unknown the equation (4-16) is unsolvable in the form given. However, a sufficiently reliable approximate solution can be obtained by carrying out the operations indicated on the l.h.s. of (4-16) on the surface $\bar{S}$ which is the first approximation to the earth obtained by adding the measured heights h to the international ellipsoid. Then we get

$$2\pi\bar{\varphi}_Q \cos\alpha + \frac{\partial}{\partial\nu}\int_{\bar{s}} \frac{\bar{\varphi}}{r_Q}\, d\bar{s} - \frac{1}{\gamma_Q}(\frac{\partial\gamma}{\partial\nu})_Q\int_{\bar{s}} \frac{\bar{\varphi}}{r_Q}\, d\bar{s} = (g_p - \gamma_Q) \tag{4-17}$$

and the approximation of T_p is given by

$$Tp = \int_{\bar{s}} \frac{\bar{\varphi}}{r_Q}\, d\bar{s} \tag{4-18}$$

If we designate the radius vector of the point Q by ρ_Q , the radius vector of the element $d\bar{s}$ by ρ, and the angle between the two radius vectors by ψ, we have

$$r_Q^2 = \rho_Q^2 + \rho^2 - 2\rho\rho_Q^2 \cos\psi \tag{4-19}$$

If we assume that the reference potential is spherical we have

$$(\frac{1}{\gamma}\frac{\partial\gamma}{\partial\nu})_Q = -(\frac{1}{\gamma}\frac{\partial\gamma}{\partial\rho})_Q = +\frac{2}{\rho_Q}$$

and

$$[\frac{\partial}{\partial\nu}(\frac{1}{r_Q})]_Q = -[\frac{\partial}{\partial\rho}(\frac{1}{r_Q})]_Q = -\frac{\partial}{\partial\rho_Q}(\frac{1}{r_Q}) \tag{4-20}$$

or

$$\frac{\partial}{\partial\rho_Q}[(\rho^2 + \rho_Q^2 - 2\rho\rho_Q\cos\psi)^{-1/2}] = -1/2\,(\rho^2 + \rho^2_Q - 2\rho\rho_Q\cos\psi)^{-3/2}$$

$$(2\rho_Q - 2\rho\cos\psi) = -1/2\,[\frac{2\rho^2_Q - 2\rho_Q\,\rho\cos\psi}{\rho_Q r^3_Q}] = \frac{\rho^2 - \rho^2_Q}{2\rho_Q r^3_Q} - \frac{1}{2\rho_Q r_Q} \tag{4-21}$$

Then substituting (4-20) and (4-21) into (4-17) we get

$$2\pi\ \bar{\varphi}_Q \cos\alpha = (g_p - \gamma_Q) + \frac{3}{2\rho_Q}\int \frac{\bar{\varphi}}{r_Q}\, d\bar{s} + \frac{1}{2\rho_Q}\int_{\bar{S}} \frac{\rho^2 - \rho^2_Q \bar{\varphi} d\bar{S}}{r_Q^3} \qquad (4\text{-}22)$$

What we have assumed here has nothing to do with the surface of integration. We have merely assumed that the equipotentials of U are sufficiently near spherical that equations (4-20) and (4-21) have sufficient accuracy for practical purposes.

Equation (4-22) is a linear integral equation in φ. The method of obtaining a solution to the problem is as follows.

1.) Solve (4-22) for φ.

2.) Use φ to obtain T from (4-18)

3.) Use T to obtain height differences (N) from (4-12)

4.) To obtain deflections of the vertical we would need the formula

$$\frac{\partial T}{\partial m} = \frac{\partial}{\partial m}\int \frac{\varphi}{r}\, dS - 2\pi\varphi \cos(m,n)$$

where m is a direction perpendicular to the radius vector of a point.

As we have made the spherical assumption for the reference potential, let us denote the spherical reference surface as having a radius R then we have

$$\rho_Q = R + h_Q \qquad\qquad \rho = R + h \qquad (4\text{-}23)$$

with an error of the order of the flattening of the earth.

Then with only an error of the order $\frac{h}{R}$ which is permissible since we are retaining only accuracies of the order of the flattening we have

$$\rho^2 - \rho^2_Q = 2hR - 2h_QR + h^2 - h^2_Q$$

$$\rho^2 - \rho^2_Q \approx 2R\ (h - h_Q) \qquad (4\text{-}24)$$

or

and

$$\rho_Q \approx R \qquad (4\text{-}25)$$

This gives

$$2\pi\bar{\varphi}_Q \cos\alpha = (g_p - \gamma_Q) + \frac{3}{2R}\int_s \frac{\bar{\varphi}}{r_Q}\, dS + \int_s \frac{h-h_Q}{r_Q^3}\,\bar{\varphi}\, dS \tag{4-26}$$

From this point one could proceed directly to the iterative solution of Molodenski, but it is perhaps instructive to see how the first approximation which is the one most often used arises. The following development is modeled after Arnold (1960).

If we examine the method of solution of the differential equation (4-13) to arrive at Stokes' equation, we see that what is assumed is that the integration is carried out over a sphere and that both the computation point and the variable points lie on this sphere ($h = h_Q = 0$). Thus the Stokes equation approximation amounts to solving (4-26) in the simplified form

$$2\pi\, \varphi_Q = (g-\gamma) + \frac{3}{2R}\int_x \frac{\varphi}{r_Q}\, dX \tag{4-27}$$

where X denotes integration over the spherical surface and

$$dX = R^2 \sin\psi\, d\psi\, dA \ .$$

We have already shown that starting with the differential equation form of (4-27) we arrive at the solution

$$T = \frac{1}{4\pi R}\int_x (g_p - \gamma_Q)\, S\,(\cos\psi)\, dX \tag{4-28}$$

Thus we get from (4-14)

$$T = \int_s \frac{\varphi}{r_p}\, dS = \frac{1}{4\pi R}\int_x (g_p - \gamma_Q)\, S\,(\cos\psi)\, dX \tag{4-29}$$

where $S\,(\cos\psi)$ is the Stokes function.

Then from (4-27) we have as a first approximation to φ

$$\varphi_1 = \frac{(g_p - \gamma_Q)}{2\pi} + \frac{3}{(4\pi)^2 R^2}\int_x (g_p - \gamma_Q)\, S\,(\cos\psi)\, dX \tag{4-30}$$

Now to arrive at a secondary approximation we go back to equation (4-26). The spherical assumption is retained for integration of the first integral on the r.h.s. of (4-26). In the second integral, the approximation $h = h_Q = 0$ is not made but it is assumed to be accurate enough to replace φ in the integral by φ_1 from (4-30). This gives the equation

$$2\pi\varphi = (g_p - \gamma_Q) + \frac{1}{2\pi}\int_x \frac{h-h_Q}{r_Q^3}\,\varphi_1\,dX + \frac{3}{2R}\int_x \frac{\varphi}{r_Q}\,dX \tag{4-31a}$$

If we assume that within the integral $\int\frac{h-h_Q}{r_Q^3}\,\varphi_1\,dX$ it is accurate enough to replace φ_1 by $(\frac{g_p-\gamma_Q}{2\pi})$ rather than the more accurate expression from (4-30), we get

$$2\pi\varphi = (g_p - \gamma_Q) + \frac{1}{2\pi}\int_x \frac{h-h_Q}{r_Q^3}\,(g_p - \gamma_Q)\,dX + \frac{3}{2R}\int_x \frac{\varphi}{r_Q}\,dX \tag{4-31b}$$

By analogy with the solution of (4-27) we see that if we consider our "anomaly" to be $(g_p - \gamma_Q) + \frac{1}{2\pi}\int_x \frac{h-h_Q}{r_Q^3}\,(g_p - \gamma_Q)\,dX$ in (4-31) as compared to simply $(g-\gamma)$ in (4-27) we should arrive at the same type of solution as (4-28) i.e.,

$$T = \frac{1}{4\pi R}\int_x [(g_p - \gamma_Q) + \Delta g^*]\,S\,(\cos\psi)\,dX \tag{4-32}$$

where

$$\Delta g^* = \frac{1}{2\pi}\int_x \frac{h-h_Q}{r_Q^3}\,(g_p - \gamma_Q)\,dX \tag{4-33}$$

Equation (4-32) is the equation for T from the new theory most commonly used in arriving at corrected formulae for deflections of the vertical. The method of arriving at equation (4-32) as a second approximation may not seem as rigorous as desirable. However, one can arrive at this same result in a more rigorous manner by proceeding as Molodenski et al (1960) have done. The paragraphs below outline their procedure.

We shall start from equation (4-22) which we repeat here as

$$2\pi\varphi_Q \cos\alpha = (g_p - \gamma_Q) + \frac{3}{2\rho_o}\int_s \frac{\varphi}{r_Q}\, dS + \frac{1}{2\rho_o}\int_s \frac{\rho^2 - \rho_o^2}{r_Q^3}\,\varphi dS \tag{4-34}$$

where we have removed the bars to denote another surface. Therefore, we must remember that in what follows the surface S in (4-34) is the first approximation to the actual earth's surface which is obtained by adding the normal height to the international ellipsoid.

Let S of equation (4-34) be the first approximation to the physical surface of the earth. Suppose we have another surface $\bar{S}$ which is related to S by the following transformation. The angular coordinates of the corresponding point on $\bar{S}$ are the same as the angular coordinates of a point on S. The radius vectors of corresponding points are connected by the equation

$$\bar{\rho} = R + k\,(\rho - R) = R + kH \tag{4-35}$$

where: $\bar{\rho}$ = radius vector to transformed surface $\bar{S}$
k = some constant coefficient
ρ = radius vector to surface S
R = radius of mean sphere of earth
H = normal height

If we let $k = 1$ this would correspond to transforming S into itself. If $k = 0$ the surface $\bar{S}$ is a spherical surface of radius R.

Now let us go back to equation (4-34) which gives us an equation at the earth's surface

$$2\pi\varphi_Q \cos\alpha = (g_p - \gamma_Q) + \frac{3}{2\rho_o}\int_s \frac{\varphi}{r}\, dS' + \frac{1}{2\rho_o}\int_s \frac{\rho^2 - \rho_o^2}{r^3}\,\varphi\, dS' \tag{4-36}$$

Let us define a new surface density function by the relation

$$X = \frac{\rho^2}{R^2}\,\varphi \sec\alpha \text{ or } \varphi = X\,\frac{R^2}{\rho^2}\cos\alpha \tag{4-37}$$

Substituting into (4-34) we get $2\pi \frac{R^2}{\rho_o^2} X \cos^2 \alpha = (g - \gamma) + \frac{3}{2\rho_o} \int_s \frac{R^2}{\rho^2} \frac{X}{r}$

$$\cos \alpha \, dS + \frac{1}{2\rho_o} \int_s \frac{\rho^2 - \rho_o^2}{r^3} X \frac{R^2}{\rho^2} \cos\alpha \, dS$$

If we now revert to the incremental solid angle by the relation $dw = \frac{\cos \alpha \, dS}{\rho^2}$ assuming S is near enough a sphere to eliminate integration with respect to ρ and divide both sides by R^2 we get

$$2\pi \frac{X}{\rho_o^2} \cos^2 \alpha = \frac{1}{R^2} (g - \gamma) + \frac{3}{2\rho_o} \int \frac{X}{r} dw + \frac{1}{2\rho_o} \int \frac{\rho^2 - \rho_o^2}{r^3} X dw$$

If we now multiply both sides by ρ_o^2 we get

$$2\pi X \cos^2 \alpha = \frac{\rho_o^2}{R^2} (g - \gamma) + \frac{3\rho_o}{2} \int \frac{X}{r} dw + \frac{\rho_o}{2} \int \frac{\rho^2 - \rho_o^2}{r^3} X dw \qquad (4\text{-}38)$$

In this expression remember that dw is the element of solid angle subtended at the origin by a surface element dS.

Now let us consider a surface $\bar{S}$ related to the surface S' by equation (4-35) with $1 \geq k \geq 0$. If the anomalies $(g - \gamma)$ had been observed on this surface we could write an equation analogous to (4-38) in terms of "barred" quantities related to this surface. Thus we would have

$$2\pi \bar{X} \cos^2 \bar{\alpha} = \frac{\rho_o^2}{R^2} (g - \gamma) + \frac{3}{2} \bar{\rho}_o \int \frac{\bar{X}}{\bar{r}} dw + \frac{1}{2} \bar{\rho}_o \int \frac{\bar{\rho}^2 - \bar{\rho}_o^2}{\bar{r}^3} \bar{X} dw \qquad (4\text{-}39)$$

Equation (4-38) will have a solution provided $(g - \gamma)$ satisfies the condition $\int (g - \gamma) Y_1 \, dw = 0$. When going from S to $\bar{S}$ the element dw does not change and the anomalies $(g - \gamma)$ are the same. So if (4-38) has a solution so does (4-39). The perturbing potential at the surface $\bar{S}$ for the situation given by (4-39) becomes

$$\bar{T} = R^2 \int \frac{\bar{X}}{\bar{r}} dw \qquad (4\text{-}40)$$

To recapitulate what we have shown is that if the gravity anomalies $(g - \gamma)$ belonged to the surface $\bar{S}$ related to S' by (4-35) rather than to S, the external

perturbing potential would be given by (4-40) where $\bar{X}$ was a solution of (4-39)

We can show by geometric reasoning that

$$\bar{r}^2 = r_o^{\ 2} \left(1 + k \frac{H+H_o}{R} + k^2 \frac{HH_o}{R^2}\right) + k^2 (H-H_o)^2 \tag{4-41}$$

where $\bar{r}$, is the distance between the point under consideration on $\bar{S}$ and the moving point on $\bar{S}$, and

$$r_o = 2R \sin \frac{\psi}{2} \qquad (\psi = \text{angle between } \rho \text{ and } \bar{\rho}_o)$$

We note that by geometric reasoning it can also be shown that

$$\tan^2 \bar{\alpha} = \left(\frac{\partial \bar{\rho}}{\partial x}\right)^2 + \left(\frac{\partial \bar{\rho}}{\partial y}\right)^2 = k^2 \tan^2 \alpha \tag{4-42}$$

Now if we expand $\bar{X}$ in powers of k i.e., replace $\bar{X}$ by a series, then

$$\bar{X} = \sum_0^\infty X_n k^n \tag{4-43}$$

where the functions X_n are independent of k.

Since the dependence on k of the functions $\bar{\rho}$, $\bar{r}$, cos $\bar{\alpha}$ are given by (4-35) and (4-42), we may now substitute into (4-39) and expand both sides of the resultant equation in powers of k. The resulting series appearing on the two sides of the equation must then be identical regardless of the value of k ($1>k>0$); thus the coefficients of each k^n term in the left and right hand sides of the equation must be equal. We thus obtain an infinite system of integral equations for which one obtains the functions X_0, X_1, X_2, etc. successively for as many terms as are desired. The results then hold for any surface $\bar{S}$ where $1>k>0$. In particular the results will hold for $k \approx 1$, i.e., for the first approximation to the physical surface of the earth.

Now we will solve (4-39) after making the expansions. First note that (4-39) has a relative error of the order of the flattening of the earth since we have taken the reference figure to be a sphere rather than an ellipse of revolution

in developing the formula. Thus we will simplify the calculations by allowing errors of equal magnitude, i.e., errors of the order $\frac{h}{R}$. This will allow us to make the approximations $\rho_o \approx R \qquad \frac{H}{R} \approx 0$

Then we get

$$\bar{r}^2 = r_o^2 + k^2 (H-H_o)^2 \tag{4-44}$$

$$\cos^2 \bar{\alpha} = (1 + k^2 \tan^2 \alpha)^{-1} \tag{4-45}$$

and

$$2\pi\bar{X} \cos^2 \bar{\alpha} = (g - \gamma) + \frac{3}{2} R \int \frac{\bar{X}}{\bar{r}} dw + R^2 \int \frac{k(H-H_o)}{\bar{r}^3} \bar{X} dw \tag{4-46}$$

Now expanding in series we get

$$2\pi (X_o + kX_1 + k^2X_2 + ----)(1 - k^2 \tan^2 \alpha + k^4 \tan^4 \alpha ----) = (g - \gamma)$$
$$+ \frac{3}{2} R \int \frac{1}{r_o} (1 - \frac{1}{2} k^2h^2 + \frac{3}{8} k^4h^4 ----)(X_o + kX_1 + k^2X_2 + ----)dw$$
$$+ R^2 \int \frac{k}{r_o^3} (H-H_o)(1 - \frac{3}{2} k^2h^2 + \frac{15}{8} k^4h^4 + ----)(X_o + kX_1 + k^2X_2 + ----)dw$$

where $h = (H-Ho)/r_o$

Equating the coefficients of each power of k we get

$$2\pi X_1 = \frac{3}{2} R \int \frac{X_1}{r_o} dw + R^2 \int \frac{X_o(H-H_o)}{r_o^3} dw$$

$$2\pi X_o = (g - \gamma) + \frac{3}{2} R \int \frac{X_o}{r_o} dw \tag{4-48}$$

etc. with similar equations for higher order terms. We note that all of these equations have a solution exactly analogous to the solution of (4-27) with the "anomaly" having different forms. Thus in an analogous manner we are lead to solutions

$$\int \frac{X_o}{r_o} dw = \frac{1}{4\pi R} \int (g - \gamma) S(\cos \psi) dw$$

$$X_o = \frac{(g-\gamma)}{2\pi} + \frac{3}{(4\pi)^2} \int (g - \gamma)[S(\cos \psi)]dw \tag{4-49}$$

Then using this we get

$$\int \frac{X_1}{r_o} dw = \frac{1}{4\pi R} \int [R^2 \int \frac{H-H_o}{r_o^3} X_o dw][S(\cos \psi)]dw \tag{4-49}$$

$$X_1 = \frac{\left[R^2 \int \frac{H-H_o}{r_o^3} X_o dw\right]}{2\pi} + \frac{3}{(4\pi)^2} \int [R^2 \int \frac{H-H_o}{r_o^3} X_o dw] [S(\cos \psi)]dw \tag{4-50}$$

Molodenski et al (1960) write the general equation as

$$2\pi X_n - \frac{3}{2}R \int \frac{Xn}{r_o}\, dw = G_n \tag{4-51}$$

whose solution is

$$\int \frac{Xn}{r_o}\, dw = \frac{1}{4\pi R} \int G_n \; S(\cos \psi)\; dw$$

$$Xn = \frac{G_n}{2\pi} + \frac{3}{(4\pi)^2} \int G_n \; S(\cos \psi)\; dw \tag{4-52}$$

with

$$G_o = (g - \gamma) \qquad\qquad G_1 = R^2 \int \frac{H-H_o}{r_o^3}\, X_o dw \tag{4-53}$$

Then to get values of T we return to

$$T = R^2 \int \frac{\bar{X}}{\bar{r}}\; dw$$

and carry out the series expansions we arrive at

$$\bar{T} = R^2 \int \frac{1}{r_o} \left(1 - \frac{1}{2} k^2 h^2 + \frac{3}{8} k^4 \; h^4 + ----\right)$$
$$(X_o + kX_1 + k^2 X_2 + ----)dw \tag{4-54}$$

where again remember that $h = \frac{H - H_O}{r_O}$

On collecting powers of k

$$T = R^2 \int \frac{X_o}{r_o}\, dw + kR^2 \int \frac{X_1}{r_o}\, dw + k^2 R^2 \left[\int \frac{X_2 dw}{r_o} - \frac{1}{2} \int \frac{(H-H_o)^2}{r_o^3}\; X_o\; dw\right]$$
$$+ ---- = \sum_o^\infty k^n T_n \tag{4-55}$$

where

$$T_o = \frac{R}{4\pi} \int G_o \; S(\cos \psi)\; dw \qquad\qquad G_o = (g - \gamma)$$

$$T_1 = \frac{R}{4\pi} \int G_1 \; S(\cos \psi)\; dw \qquad\qquad G_1 = R^2 \int \frac{H-H_o}{r_o^3}\, X_o dw$$

etc.

To get the value on the first approximation surface we would of course set $k \approx 1$. If we set $k = 0$ we have the result which would have been obtained by letting all the anomalies be referenced to a sphere.

We have in the development assumed that $U_O = W_O$ and thus do not have the constant term 1/2 in the integrals as Molodenski et al (1960) do.

Molodenski et al (1962) point out that the equations for the potential give only the potential at the earth's surface. Thus we know only the quantity T_s = potential at the earth's surface. In determining the deflections we wish to compute the rate of change of T in a circular direction l. The deflection in this direction can be expressed as

$$\theta = \frac{-1}{\gamma_Q} \frac{\partial T}{\partial l} = \frac{-1}{\gamma_Q R} \frac{\partial T}{\partial \varphi} \tag{4-56}$$

where $dl = Rd\varphi$

To get $\frac{\partial T}{\partial \varphi}$ at a point in terms of T_s we note that

$$\frac{\partial T_s}{\partial \varphi} = \frac{\partial T}{\partial \varphi} + \frac{\partial T}{\partial H} \frac{\partial H}{\partial \varphi} \tag{4-57}$$

Thus

$$\frac{\partial T}{\partial \varphi} = \frac{\partial T_s}{\partial \varphi} - \frac{\partial T}{\partial H} \frac{\partial H}{\partial \varphi} \tag{4-58}$$

To evaluate $\frac{\partial T}{\partial H}$ we use equation (4-13) which we approximate by

$$(g_p - \gamma_Q) = - \frac{2Tp}{R} - \frac{\partial T}{\partial R} \tag{4-59}$$

Using the further approximation $\frac{\partial T}{\partial R} \approx \frac{\partial T}{\partial H}$ this gives

$$- \frac{\partial T}{\partial H} = (g_p - \gamma_Q) + \frac{2Tp}{R} \tag{4-60}$$

Substituting into (4-58) this gives

$$\frac{\partial T}{\partial \varphi} = \frac{\partial T_s}{\partial \varphi} + [(g_p - \gamma_Q) + \frac{2Tp}{R}] \frac{\partial H}{\partial \varphi} \tag{4-61}$$

This gives

$$\theta = \frac{-1}{\gamma_Q} \left\{ \frac{1}{R} \frac{\partial T_s}{\partial \varphi} + [(g_p - \gamma_Q) + \frac{2Tp}{R}] \frac{\partial H}{\partial \varphi} \frac{1}{R} \right\} \tag{4-62}$$

Or substituting for T_s from (4-32)

$$\theta = \frac{-1}{\gamma_Q}\left\{\frac{1}{4\pi R^2}\int_x \left[(g_p - \gamma_Q) + \Delta g^*\right] \frac{\partial S}{\partial \psi}\frac{\partial \psi}{\partial \varphi}\, dX + \left[(g_p - \gamma_Q) + \frac{2Tp}{R}\right]\frac{1}{R}\frac{\partial H}{\partial \varphi}\right\} \tag{4-63}$$

This is the basic deflection formula as developed in the new theory. It consists of the Vening Meinesz formula plus two correction terms. The first correction consists of the Δg^* term, which is a correction for the fact that during integration a sphere is used instead of the correct surface of the earth in determining T. The $\frac{\partial H}{\partial \varphi}$ term takes into account the fact that in deflection computation we need to get derivatives in directions other than along the earth's surface.

SECTION 5

Theoretical Aspects of Anomaly Selection

In this section we shall examine the theoretical aspects of anomaly selection. That is, we shall determine which types of gravity anomalies may theoretically be used in making geoidal height or deflection computations both in the classical and in the new geodetic theory. We shall not go into great detail on the mechanics of computation of anomalies since this has been covered in many places but shall be primarily concerned with the logic related in the use of the different types of anomalies.

CLASSICAL THEORY

From the classical geodetic viewpoint the problem is to use gravity observed on the physical surface of the earth which is not an equipotential surface in conjunction with a theory which relates gravity on a bounding equipotential surface with its shape and the direction of the force of gravity on it. In this case, the four distinct steps mentioned at the beginning of Section 4 are necessary. We shall first consider the types of anomalies which result from carrying out the first two steps.

The most common types of gravity reduction methods are:

1. Simple free-air
2. Free-air anomaly with condensation
3. Rudzki inversion
4. Isostatic based on hypothesis of
    a. Pratt-Hayford
    b. Airy-Heiskanen
    c. Vening Meinesz

We shall begin with the isostatic reduction method since it has been so often used and is a clear example of what is done. Physically any type of isostatic anomaly envisions a number of processes.

From the value of gravity measured at P on the earth's surface, proceed as follows:

1. Compute the effect, on gravity at P, of removing all mass above the geoid. This is the complete correction for topography around the world extending above sea level.

2. Compute the effect, at P, of placing this mass below the geoid at some point say between 30 and 50 km. below sea level. The above corresponds to the reduction for topography and compensation in isostatic theory. However, in the isostatic reduction procedure, mass is also moved from below the ocean so as to fill the ocean with matter having the density of rock. Note that this mass transfer is <u>not</u> required in order to use the theory of Section 3 since the oceans are already bounded by the desired equipotential surface. As long as the proper computations are made as to the effect of such an internal mass transfer, it is of course permissible. The same may be said for geologic corrections below sea level.

If a normal isostatic correction is used, the effect of the mass transfer on gravity at the earth's surface is given by

$$-B.C. + L.I.C. + D.T.A.C.$$

where:

B.C. is the complete Bouguer correction as modified by Bullard to extend only through zone 0 and is considered a positive number.

L.I.C. is the local isostatic reduction out through zone 0.

D.T.A.C. is the combined topographic-isostatic correction for material

beyond zone O as this zone is defined by the U.S. Coast and Geodetic Survey and may be either positive or negative.

Then for a land station if g is the observed gravity at a point P on the earth's surface and $\bar{g}$ is the modified value of gravity at P, we have for an inland station

$$\bar{g} = g - B.C. + L.I.C. + D.T.A.C.$$

where for the station the effect of taking away the above sea level near mass is negative, the effect of putting in a "compensating" mass below sea level is positive and the effect of the distant changes may be either positive or negative depending upon the location of the station with respect to the totality of the changes.

3. Compute the value of gravity at the point P' on the surface of the geoid. This means starting with $\bar{g}$ at the point P and using $\frac{\partial \bar{g}}{\partial h}$ along the normal between P and P' to compute the value of gravity at P'. The actual gradient is not known so the average ellipsoidal gradient is normally used, i.e., we apply the "free-air" correction to yield

$$g' = \bar{g} + F.A.C.$$

4. Note the effect of the mass transfer on the equipotential surface which, prior to the mass transfers, coincided with the geoid. This equipotential surface has been moved slightly and we call its present position the co-geoid.

5. Move all matter lying between the geoid and co-geoid vertically to the surface of the co-geoid. We assume that the effect on gravity at P' of this move is negligible.

6. Use the gravity gradient to compute the gravity at P" on the co-geoid which we shall call g'' from the value of gravity at P' by making the correction h' $(\frac{\partial g}{\partial h})$ where h' is the distance between the geoid and co-geoid and $\frac{\partial g}{\partial h}$

is the gravity gradient. This is the so-called indirect effect so we have

$$g'' = g' + I.E.$$

where I.E. = indirect effect = $h'(\frac{\partial g}{\partial h})$

7. We can now get the gravity anomalies needed for Stokes or Vening-Meinesz Theorem by computing $\Delta g = g'' - \gamma_Q$ where γ_Q is the theoretical gravity according to the international formula.

One should note at this point that Vening Meinesz at step 5 rather than transferring the mass between the co-geoid to the co-geoid surface assumes that it is moved to infinity. In this case the effect of the mass transfer on gravity is not negligible but must be computed. The remainder of the steps are as outlined above in 6 and 7.

The details of carrying out isostatic reductions have been covered in a number of places (See for example Heiskanen and Vening Meinesz, 1958) and will not be repeated here. Following the seven steps given above, the formula for any type of isostatic anomaly can be written

$$\Delta gI = (Ob - B.C. + T.C. + R.C. + L.I.C. + D.T.C. + \frac{\partial g}{\partial h} h + \frac{\partial g}{\partial h} h') - Th$$

where:

B.C. = simple Bouguer slab correction

T.C. = terrain correction through zone 0

R.C. = Bullard correction to remove effect of material beyond zone 0 and curvature correction

L.I.C. = correction for compensation of material through zone 0

D.T.C. = correction for topography plus compensation beyond zone 0

$\frac{\partial g}{\partial h} h$ = free-air correction to geoid level from surface level using measured heights

$\frac{\partial g}{\partial h} h'$ = free-air correction from geoid level to co-geoid level

There are of course a number of types of isostatic anomalies. From the geodetic point of view these simply mean that the mass is transferred to different points inside the geoid in different systems. As long as the effect of these transfers is accurately computed it does not matter much which one is employed except possibly, as will be discussed later, in relation to prediction of the anomalies.

Another type of mass transfer which is permissible and can be used is to transfer all mass above geoid level to the geoid surface itself. It can be considered a surface density on the geoid. This is the condensation reduction. It can be shown that such a mass transfer has a negligible effect on the shape of the geoid. The steps in using this method are then reduced to

1. Calculate the effect, on gravity at P, of moving all external mass on to the surface of the geoid.
2. Use $\frac{\partial g}{\partial h}$ h i.e., a free-air reduction to compute gravity on the geoid.

The development of this theory can also be stated in the following slightly different form: Consider a point on the earth's surface P and a corresponding point on the geoid P' a distance h below P. Assume the earth's masses above sea level are in place. Now we wish to arrive at a formula for $\frac{\partial g}{\partial h}$ at points between P and P'.

We begin with Poissons' equation for a rotating body of density ρ

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} = - 4\pi k\rho + 2\omega^2 \tag{5-1}$$

If we have the z axis in the direction of the outward normal to the equipotential surface

$$\frac{\partial^2 W}{\partial z^2} = \frac{\partial}{\partial z}\frac{(\partial W)}{(\partial z)} = - \frac{\partial g}{\partial z} \tag{5-2}$$

Then equation (5-1) becomes

$$\frac{\partial g}{\partial z} = \left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right) + 4\pi k\rho - 2\omega^2 \tag{5-3}$$

But it can be shown that

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = - \left(\frac{1}{r_x} + \frac{1}{r_y}\right) g \tag{5-4}$$

where r_x and r_y are the radii of curvature of the equipotential surfaces in the xz and yz planes.

Then equation (5-3) becomes

$$\frac{\partial g}{\partial z} = - g \left(\frac{1}{r_x} + \frac{1}{r_y}\right) + 4\pi k\rho - 2\omega^2 \tag{5-5}$$

Thus, if g_p is the value of gravity at the point P on the earth's surface, the value g_p' at the point P' on the geoid with the mass above sea level still in place is

$$g_{p'} = g_p - \int_P^{P_1} \left[- g \left(\frac{1}{r_x} + \frac{1}{r_y}\right) + 4\pi k\rho - 2\omega^2\right] dz \tag{5-6}$$

where the minus in front of the integral comes from fact we are integrating in negative z direction. Thus

$$g_{p'} = g_p - 4\pi kM + \int_P^{P_1} \left[g \left(\frac{1}{r_x} + \frac{1}{r_y}\right) - 2\omega^2\right] dz \tag{5-7}$$

The integral on the R.H.S. of equation (5-7) is a free-air correction in the strict sense. It is usual to assume that the simple free-air correction is sufficiently accurate and write $g_{p'} = g_p - 4\pi kM + .09406h$

or, if we write

$$g_o = g_p + .09406h$$

$$g_{p'} = g_o - 4\pi kM$$

Thus according to Green's equivalent layer theorem the gravity field outside the geoid can be taken as equal to the attraction of a layer of density $\frac{g_o}{4\pi R} - M$ on the geoid plus the attraction of material outside the geoid. We

will now remove the material outside the geoid and the surface density of -M and compute their effects on $g_{p'}$. The effects on the shape of the equipotential surface passing through P' will be negligible. We now have a gravity value, $g_{p'}$ that can be used to obtain an anomaly $\Delta g = g_{p'} - \gamma_Q$ which may be substituted into Stokes' or Vening Meinesz' formulae. If the effect of removing the M layer on the geoid and the material above the geoid is the same and each equal to $+2\pi kM$ then $g_{p'} = g_o = g_p + .09406h$ and the simple free-air anomaly could be used. The assumption that the removal of the -M layer and the material above sea level is equal to $4\pi kM$ is fairly accurate in level areas since it simply amounts to the assumption that both can be treated as infinite horizontal sheets.

It is not very accurate in areas of considerable relief since the effect, at P', of condensing the actual topography onto the geoid is considerably different from $4\pi kM$ and a condensation reduction must be applied. We can return to this point later in connection with use of the free air anomaly in the modern theory and the reduction of deflections of the vertical to sea level or the sea level deflections to surface level.

From the point of view of this section the Model Earth anomaly is an offshoot of the condensation reduction. One first carries out a lateral transfer of material at the earth's surface so as to smooth out the earth's topography. The assumption that the M layer on the geoid and the material above the geoid have an effect $4\pi kM$ when carrying out a condensation for the smoothed earth can then be made everywhere with sufficient accuracy. Since the primary purpose of the Model Earth method is to produce an anomaly which is slowly varying for interpolation and extrapolation purposes we shall defer additional discussion of it until Section 6. For present purposes it simply amounts to another way of transferring material from above to below the geoid.

The other type of gravity anomalies sometimes used are the inversion anomalies commonly known as the Rudzki anomalies after the Polish scientist who proposed the method. The reason for this reduction was the desire to avoid any deformation whatsoever of the sea level equipotential (geoid) while carrying out the mass transfers. The Rudzki method has been little used. The equations for its computation can be found in Tengstrom (1962). Strictly from the point of view of the classical theory, the Rudzki reduction leaves something to be desired as it does not actually produce a model where all matter is within the bounding equipotential but leaves a small amount of matter outside this surface.

There are other types of anomalies which could be used in the classical theory relating to different kinds of mass transfer but in fact none have been used and would in any case differ only in computational detail.

The choice of the anomaly which is best to use in the classical theory is beset by many difficulties. Primarily this results from the fact that knowledge of density distributions above sea level, vertical gradients above sea level, and the geometric relation between points on the earth's surface and on the geoid are assumed known. In fact, they are not exactly known and considerations of minimizing errors due to lack of knowledge must be taken into account.

The important points to be remembered from the above discussion are that the necessary requisites for an anomaly for use in the classical theory (considering only the point that they must be theoretically capable of being inserted in the classical equations) is that they envision transfer of all mass so that it lies inside a bounding equipotential surface and then computation of the force of gravity on this surface, using observed gravity on the earth's surface.

To do this requires knowledge of densities above sea level to insure proper mass transfer and of densities below sea level to obtain the proper gravity gradient. Strictly from this viewpoint one needs to have a "geologic correction" above sea level but not below sea level. Again from the viewpoint of satisfying

the conditions for using the equations of Section 3 it should be pointed out that the fact that isostatic equilibrium does not or does exist is entirely irrelevent. It is important only from the point of view of interpolation and extrapolation of anomalies.

THE NEW THEORY

Whereas the theoretical development of the Classical Geodetic Theory does not imply the use of any particular type of anomaly, the New Theory (see Section 4) naturally leads to the free-air anomaly. The question arises as to whether or not some other type of anomaly might be used. The following development, based on the works of Arnold (1961), shows that all of the types of anomalies used in the classical theory can be used in the new theory. It also shows that, considered from the point of view of the new theory, one kind of anomaly gives as accurate a result as another regardless of the extent of knowledge of internal density distributions. This leads to the result that the choice of anomaly is dependent on ease of practical computation and utility in extrapolation and interpolation from known values.

Perhaps one should point out that in the new theory the equality of all types of anomalies should be expected. The development after all is based on small deviations from a known model. The exact form of the model could not be expected to control the accuracy of the method.

Consider a model of the earth consisting of the international ellipsoid model plus some other assumed mass distribution, the only limitation being placed on this mass distribution being that it must lie completely within the physical surface of the earth. Let the potential due to this mass distribution be designated by A.

Then

W = potential due to actual earth

U = potential due to international ellipsoid model

A = potential due to assumed densities in theoretical model other than those of international ellipsoid model.

$T' = W - (U + A)$ = difference between earth's potential and model potential

N = distance from P to Q

Now we will proceed in an exactly analogous way to the procedure in Section 4. Let P be a point on the earth's surface and Q the corresponding point on the equipotential of U such that $U_Q = W_p$.

Now by definition

$$W_p = U_p + A_p + T'_p \tag{5-8}$$

Again using a Taylor expansion of U about Q and assuming N is small enough to use only the first term we get

$$U_p = U_Q + \left(\frac{\partial U}{\partial \nu}\right)_Q N = U_Q - \gamma_Q N \tag{5-9}$$

Or substituting into (5-8) we get

$$W_p = U_Q - \gamma_Q N + A_p + T'_p \tag{5-10}$$

but by definition $U_Q = W_p$ so we get from (5-10)

$$N = \frac{T'_p}{\gamma_Q} + \frac{A_p}{\gamma_Q} \tag{5-11}$$

This equation is analogous to equation 4-12 of Section 4. The term $\frac{A_p}{\gamma_Q}$ was, in the older theory where P was on the geoid rather than the earth's surface, the separation between the geoid and the co-geoid. In equation (5-11), since we supposedly know the density distribution used in our model, we can compute A_p and thus in (5-11) the only term remaining continues to be the potential T'_p.

Now let us get the relation between gravity anomalies and the potential T'_p. From the definition of W, i.e., $W = U + A + T$ we have

$$\left(\frac{\partial W}{\partial n}\right)_p = \left(\frac{\partial U}{\partial n}\right)_p + \left(\frac{\partial A}{\partial n}\right)_p + \left(\frac{\partial T'}{\partial n}\right)_p \tag{5-12}$$

where the derivative is in the direction of the normal to W and the subscript P indicates that the derivatives are to be evaluated at the point P . Now

$$\left(\frac{\partial W}{\partial n}\right)_p = - g_p \tag{5-13}$$

the negative of the observed gravity. Also $\frac{\partial U}{\partial n}$ differs from $\frac{\partial U}{\partial \nu}$, the derivative in the direction of the normal to equal U surfaces, only by a quantity which is proportional to the deflection of the vertical so we shall assume that it is accurate enough to write

$$\left(\frac{\partial U}{\partial n}\right)_p = \left(\frac{\partial U}{\partial \nu}\right)_p = - \gamma_p \, . \tag{5-14}$$

Then again assuming that we can expand γ around the point Q and that N is small enough to keep only the first term we get

$$\gamma_p = \gamma_Q + \left(\frac{\partial \gamma}{\partial \nu}\right)_Q N \, . \tag{5-15}$$

Taking note of (5-13), (5-14) and (5-15) we get for (5-12)

Or rearranging terms $\quad - g_p = - \gamma_Q - \left(\frac{\partial \gamma}{\partial \nu}\right)_Q N + \left(\frac{\partial A}{\partial n}\right)_p + \left(\frac{\partial T'}{\partial n}\right)_p$

$$(g_p - \gamma_Q) = \left(\frac{\partial \gamma}{\partial \nu}\right)_Q N - \left(\frac{\partial A}{\partial n}\right)_p - \left(\frac{\partial T'}{\partial n}\right)_p \tag{5-16}$$

Substituting (5-11) into (5-16) gives

$$(g_p - \gamma_Q) = \left(\frac{\partial \gamma}{\partial \nu}\right)_Q \left(\frac{T'_p}{\gamma_Q} + \frac{A_p}{\gamma_Q}\right) - \left(\frac{\partial A}{\partial n}\right)_p - \left(\frac{\partial T'}{\partial n}\right)_p \tag{5-17}$$

or rearranging terms

$$\left[(g_p - \gamma_Q) + \left(\frac{\partial A}{\partial n}\right)_p - \left(\frac{\partial \gamma}{\partial \nu}\right)_Q \frac{A_p}{\gamma_Q}\right] = \left(\frac{\partial \gamma}{\partial \nu}\right)_Q \frac{T'_p}{\gamma_Q} - \left(\frac{\partial T'}{\partial n}\right)_p \, . \tag{5-18}$$

The expression on the left hand side of equation (5-18) represents some type of anomaly such as an isostatic anomaly. The quantity $(\frac{\partial A}{\partial n})_p$ would in the case of the isostatic anomaly be the attraction at the earth's surface of topography plus compensation. The quantity $(\frac{\partial \gamma}{\partial \nu})_Q \frac{A_p}{\gamma_Q}$ is analogous to the indirect effect in classical theory. However, the term $(\frac{\partial \gamma}{\partial \nu})_Q \frac{A_p}{\gamma_Q}$ must be evaluated at the earth's surface rather than at geoid level. The difference is probably not very great. The above considerations show that if proper computations are carried out all types of anomalies will yield the same result.

One should note the difference in the viewpoint adopted in arriving at equation (5-18) which holds at the earth's surface and that adopted in arriving at an analogous equation in classical theory which holds at co-geoid level. In the case of the classical theory we have thought in terms of transfer of actual masses of the earth from one point to another, which implied knowledge of these masses, and use of gravity observed at one level to compute gravity on another level which implied knowledge of actual vertical gradients. In the viewpoint used in arriving at (5-18) we think in terms of the density distribution of a model which we may hope approximates the actual density distribution of the earth from the viewpoint of ease of interpolation. However, any density deviations between the actual earth and the model do not introduce errors into the result. Also, in arriving at (5-18) we need only vertical gradients of the gravity field of the models which can be computed to any desired degree of accuracy rather than the unknown vertical gradients of the actual earth. Thus even if we are using, for example, an isostatic anomaly, the new theory provides a much sounder basis for assessing the accuracy to be expected and for determining the direction to be taken to improve accuracy.

In the next section we will consider the questions which arise when interpolation and extrapolation of anomalies become important.

It is in connection with this problem of anomaly interpolation that the application of geology and geophysics becomes important.

SECTION 6

Problems of Interpolation of Anomalies

In Sections 2 through 5 we examined the mathematical derivation of the formulae relating gravity and deflections of the vertical. Through the developments given in Section 5 we see that in the "New Geodetic Theory" the accuracy with which one may obtain deflections is entirely independent of any knowledge of the internal density distributions of the earth provided there is a complete knowledge of the gravity field. Thus one can never hope to improve the accuracy of the theory by shifting from one type of anomaly to another or by using any increased knowledge of internal density distributions.

The choice of an anomaly to use in deflection computations will therefore be controlled by two considerations which arise in applying the theory to actual deflection computations. The first consideration is ease of computation. If it is possible to decrease the computational labor by choosing one type of anomaly over another, the anomaly requiring the least labor should, of course, be chosen provided both yield answers of sufficient accuracy. The second consideration is accuracy of gravity interpolation. In practice, gravity is not known everywhere on the earth's surface but at a finite number of points. At intermediate points gravity must be determined by some form of interpolation. If one type of gravity anomaly can be more accurately interpolated than others it should improve the computed deflections by providing a more accurate estimate of gravity between observation points.

In arriving at a final choice of anomaly an attempt must be made to reconcile the two considerations so that the anomaly chosen for substitution in the deflection formulae yields acceptable accuracy with a minimum of labor. This means that one must determine at what point an increase in accuracy no

longer justifies the increase in labor. Another point to be kept in mind in investigating the practical utility of various types of gravity anomalies is the fact that one type of anomaly can be converted to another. Thus it might be possible to use one type of anomaly for interpolation and convert to another type of anomaly for substitution into the deflection formulae. We shall first investigate the relative accuracy with which the various types of gravity anomalies can be interpolated.

Before discussing the individual types of anomalies it will be best to examine some general considerations connected with interpolation of gravity anomalies. The ultimate aim in-so-far as interpolation is concerned is to obtain a type of gravity anomaly which varies as nearly as possible linearly between points of observation. Using this type of anomaly a contour map prepared in the usual way would have maximum accuracy. An alternative would be to use an anomaly which might deviate from linear variation between observation points but in such a way that the deviations could be predicted. Such a procedure might be possible for example by use of geologic and geophysical knowledge to predict non-linear variations.

In order for a gravity anomaly field to vary nearly linearly between observation points it must contain only components whose wavelengths are much greater than the distances between observation points. Any gravity anomaly is difference between observed gravity and the gravity field of a density model. Thus to produce an anomaly field which contains only long wavelength components, the density model used in anomaly computation must correspond as nearly as possible to the actual earth with respect to those density distributions which produce short wavelength gravity components. Long-wavelength and short-wavelength are of course only relative as used here. A particular wavelength might be considered "long" or "short" depending upon the distance between observation points.

It is clear that two factors control the width of the gravity effect of an anomalous density distribution. These are the horizontal dimensions of the anomalous density distribution and its proximity to the surface. Thus the density model used in computing a gravity anomaly for interpolation must correspond to the density of the actual earth in so far as density variations of less than a given horizontal dimension and given depth are concerned. The magnitudes of the limiting values depend upon the distance between observational points. The remarks of the last few paragraphs may seem to be simply a statement of the obvious but overlooking this obvious can lead to considerable extraneous discussion.

Let us now turn to the problem at hand--the use of gravity in interpolation of deflections of the vertical. The accuracy of the computed deflections is critically dependent upon accurate knowledge of the gravity field in the near vicinity of the deflection station, i.e. within 100 kms. Therefore, one would hardly attempt deflection interpolation without a fairly dense set of observations. As a maximum for the distance between stations 30 km is a reasonable value. Thus the problem to be investigated is: What density distributions of the actual earth must be incorporated into a density model in order to produce an anomaly which varies smoothly over distances of 30 km or less?

By far the most important near surface horizontal density variation producing short wavelength gravity effects is topography. One of the basic requirements for any smoothly varying gravity anomaly field is that the density model used in its computation accurately take into account those topographic masses of the actual earth which are near the computation stations.

The common types of gravity anomalies which result in a smoothly varying gravity field are:

1.) Bouguer anomalies
2.) Isostatic anomalies
3.) Model earth anomaly
4.) Rudzki inversion anomalies

In all of these types of anomalies an attempt is made to account for the actual topography of the earth near the computation point in the density model used. In the following discussion we shall examine the ways in which geologic and geophysical information might be used to alter the density models so as to produce an even more smoothly varying gravity anomaly field.

The Bouguer Anomaly

If the Bouguer Anomaly is to be used for interpolation a clear understanding of what the Bouguer anomaly implies must exist.

The normal manner of writing the theoretical model attraction for the complete Bouguer anomaly model is:

Theoretical Gravity of Complete Bouguer Model = International Gravity Formula + Attraction of Infinite Slab of thickness equal to Height of station - Terrain correction.

In abbreviated form this becomes

$$Th_{CMB} = I.F. + 2\pi k\rho h - T.C.$$

where $2\pi k\rho h$ is the well known infinite slab attratction.

The theoretical model attraction for the normal Bouguer anomaly models seem irrational in terms of the actual earth. A closer examination shows that they are not quite so irrational as they first appear.

First, although the term $2\pi k\rho h$ theoretically applies to an infinite slab, more than 99% of this attraction results from the part of the slab material lying within 100 miles (166 km) of the station. Thus to a close approximation the Complete Bouguer anomaly represents a model in which the actual earth's topography out to 166 km has been superimposed on the International Ellipsoid model. Bullard has suggested a slightly modified complete Bouguer model to correct for the minor differences. It is given by

$$Th_{CMB} = I.F. + 2\pi k\rho h - T.C. + b$$

where the small term b (only 3 to 4 mgls at most) takes care of the difference between the infinte slab and a cylinder of radius 166 km and also corrects for curvature effects. This model then represents exactly the International ellipsoid with the earth's topography from the station to 166 km superimposed.

When stated as above it becomes apparent that the density model used in computing a Bouguer anomaly at one point is not the same as the model used at another point. Perhaps the best way to visualize what is actually indicated by the normal Complete Bouguer Anomaly is as follows. Consider a density model, M, which takes into account the actual topography above sea level around the world and the mass deficiencies of ocean water. We might write the attraction, g_M, for such a model as:

$$g_M = IF + N.C. + D.C.$$

where

IF = attraction of International ellipsoid model
NC = gravitational effect of topography within 166 km of a point
DC = gravitational effect of topography beyond 166 km of a point

Both of the functions NC and DC are functions of position on the earth, i.e. of the coordinates θ, λ. Thus we can think of the complete Bouguer anomaly as being written

$$C.B.A. = g_M - D.C.\ (\theta, \lambda).$$

The function D.C. (θ, λ) can be thought of here as simply a slowly varying function of position on the earth's surface. The Complete Bouguer Anomaly can then be considered as simply an anomaly computed using the density model M plus a slowly varying function of position. For the purposes of interpolation over distances of 30 km or less the function D.C. (θ, λ) will be smoothly varying enough to cause little error in interpolation. Of much greater importance are the differences between the model density and the actual density

of the earth within 166 km of a computation point. The more accurately the model densities approach the actual densities of the earth within 166 km of a point the more smoothly varying will be the resultant anomaly. The alteration of the model to more nearly fit the actual density distribution of the earth as indicated by geologic and geophysical information will be termed a geologic correction.

The question of the correct density assumption often arises in using the Bouguer anomaly. The answer depends upon the reason for computing the Bouguer anomaly.

The question to be answered in the present investigation is: What density should be used for the Bouguer correction down to sea level and for the geologic correction below sea level to produce the most easily interpolated anomaly when observation points are 30 km or less apart?

Consider first the suposition that one computes a "geologically corrected" complete Bouguer anomaly using the exact densities of all material above the depth of the deepest sedimentary basin. The horizontal changes in this anomaly will reflect density changes at depths greater than the depth used in computation of the anomaly plus the changes in the effects of density distributions at horizontal distances greater than 166 km. Now, if the surface elevation is 5,000 ft. and the bottom of the deepest basin is at -25,000 ft., one will be at least 30,000 ft. above any remaining density variations. This distance factor in addition to the fact that any density differences at this depth will normally be small and features broad would be expected to result in a gravity effect that is slowly varying in a horizontal direction. From this reasoning one would expect this type of anomaly to be easily interpolated.

Now consider an anomaly computed using a density of 2.88gm/cc above sea level and contrasting density below sea level with a density of 2.88gm/cc.

First we note that the density 2.88gm/cc arose because it is an average value for the mean density of a crustal column. In the first part of section IV of the final report of contract AF 23(601)-3455 it was shown that, regardless of the density of the part of the crustal column above sea level, if one assumes the mean crustal density remains 2.88gm/cc and mantle density remains constant, one would get the correct crustal thickness only by using a density of 2.88gm/cc in the Bouguer correction. In later parts of section IV the problem of changes in mean crustal density and in mantle density were considered. Let us temporarily assume that we have a region where the mean crustal density remains 2.88gm/cc and the mantle density remains constant. The question is: How could the reduction of the observed stations in this area using a density of 2.88gm/cc in the Bouguer anomaly help in the prediction of gravity at other points.

Let us consider the two ways in which the 2.88gm/cc Bouguer anomaly can claim to be better than the Bouguer anomaly using correct densities for interpolation purposes. These ways are

1.) The 2.88gm/cc Bouguer anomalies themselves could be more easily interpolated.

2.) Seismic information could be used to predict Bouguer anomalies (ρ = 2.88gm/cc) more accurately than other types of Bouguer anomalies.

To explore the first contention we note that since almost all topographic features are lower in density than 2.88gm/cc the Bouguer anomalies using 2.88gm/cc will retain a correlation with local elevation changes. If we rigorously take into account the difference in density between material present and 2.88gm/cc either above or below sea level we have simply added a constant to the result we would have gotten using any other density. This can be seen from the following consideration.

Let V be the volume from sea level to a depth h and out to some distance d from the point. Then using the exact densities present we get for the

attraction

$$A = k\int \frac{\sigma}{r^2} \cos\varphi \; dV$$

When using the density difference $\sigma - 2.88 = \Delta\sigma$ we get for the attraction difference

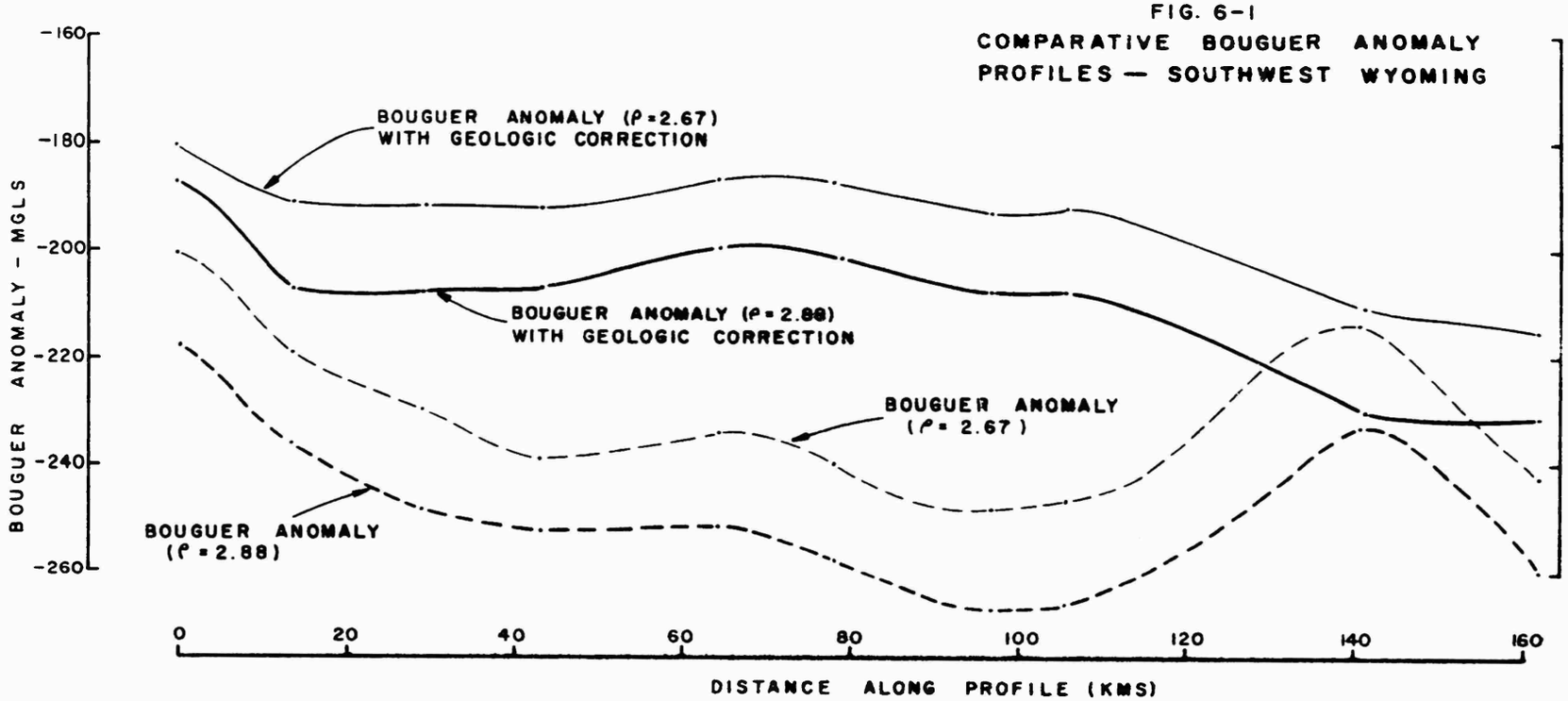
$$A' = k\int \frac{\Delta\sigma}{r^2} \cos\varphi \; dV = k\int \frac{\sigma}{r^2} \cos\varphi \; dV - k\, 2.88 \int \frac{\cos\varphi}{r^2} \; dV$$

Thus if we retain the same area of integration the second integral is a constant at all points. If we use a $\Delta\sigma$ computation only for sediments and not for the basement we will introduce spurious effects correlated with the sediments. It is difficult to visualize what these would be expected to represent.

From the above considerations it is difficult to see how Bouguer anomalies ($\rho = 2.88$) would be more easily interpolated than those using the correct density for the material present. Point anomaly values computed using $\rho = 2.88$ would include, in addition to the effect of deep seated features, correlations with local topography and near surface mass distributions. Examination of the two types of Bouguer anomalies on an empirical basis also failed to show any interpolation advantage to using a density of 2.88gm/cc. Figure 6-1 shows a typical profile along which the two have been compared. Additional large scale profiles not included in this report but provided to ACIC as large scale blackline prints show similar results.

The next question is: Will Bouguer anomalies $\rho = 2.88$ averaged over $1^\circ \times 1^\circ$ square areas by more closely correlated with crustal thickness? If this were the case we could more easily estimate these anomalies than others where there were seismic measurements of the earth's crust. However, this is not a particular reason to compute Bouguer anomalies using 2.88gm/cc in an area where deflection computations are to be undertaken even if it proves to be true. For deflection interpolation purposes the detail needed for the gravity field can

FIG. 6-1
COMPARATIVE BOUGUER ANOMALY PROFILES — SOUTHWEST WYOMING



hardly be expected to be provided by correlations of $1^\circ \times 1^\circ$ average anomalies with crustal thickness.

Perhaps it is well to pause here and again emphasize what we are after. The observed gravity at the earth's surface is a rapidly varying function produced by all of the masses of the earth. Because points where gravity is observed are normally too far apart to adequately sample the short period changes accurate interpolation is not possible. If we can use knowledge of existing density distributions to compute the short period components of the gravity field and remove them we will then be able to interpolate the long period portion of the field between observation points with adequate accuracy. If in the process of removing the short period components of the gravity field we add to the gravity field functions which have only long period components the resultant may also be easily interpolated.

Two factors prevent the use of geologically corrected Bouguer anomalies directly in deflection computation formulae. First as noted previously the Bouguer anomaly as usually computed does not use the same density model at all points. To utilize Bouguer anomalies at all would require using a model taking into account the topography around the world with the increased computations involved. Second, the secondary corrections related to using such a density model would be long and complicated. The is related to computation of the quantities A and $\frac{\partial A}{\partial n}$ of equation (5-18) for this model. Also one often has stations which are sufficiently close together to allow adequate contouring without any geologic correction in one area while in an adjacent area geology must be taken into account. To use geologically corrected gravity values in the formulae would require that the effect of any density change made in the model be computed at all gravity stations. The above factors combined with the fact that Pellinen (1962) has developed a method of using ordinary complete Bouguer

anomalies directly in computations leads to the conclusion that the best method is to use the geologic and geophysical information for interpolation but revert to normal complete Bouguer anomalies to draw up gravity contour maps for use in the actual computations.

Isostatic Anomalies

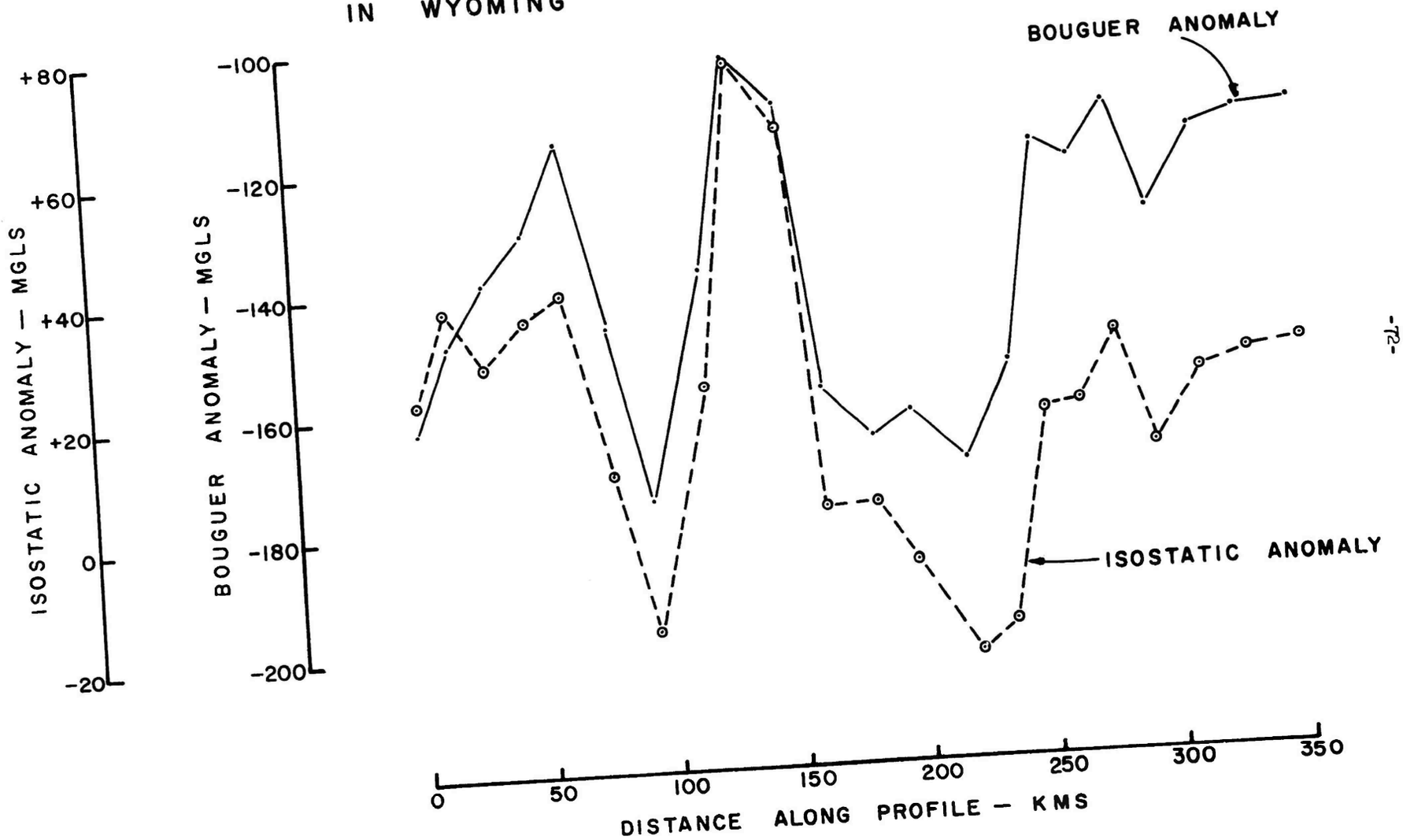
With respect to the problem of interpolation, isostatic anomalies can be thought of as extensions of Bouguer anomalies just as the Bouguer anomalies are themselves an extension of free-air anomalies. In each case one is seeking to improve the density model used in anomaly computation so that it more nearly approximates the actual density of the earth. The free-air anomaly is computed using as a density model an ellipsoid with concentric ellipsoidal shells of constant density. In computing a Bouguer anomaly this density model is altered to take into account the earth's visable topography either locally as is usually done or, if desired, completely around the world using a constant density. In a geologically corrected Bouguer anomaly lateral variations in the density of the upper crust as indicated by geology and geophysics are included in the density model where possible. Both gravity observations and seismic data show that large scale lateral density variations occur at depth; usually in such a way as to balance the topographic and other near surface inequalities. In isostatic density models an attempt is made to include the deep seated density variations as well as those included in the Bouguer density model.

The question to be asked here is whether or not any of the isostatic density models result in an anomaly which is more easily interpolated over distances of 30 km or less than is the geologically corrected Bouguer anomaly. Both from the theoretical point of view and from actual observation the answer would appear to be no. There are a number of reasons for this. First note that we are only interested in the region within 166 km of a point. That the

effect of the densities assumed in the isostatic models beyond 166 km is small is obvious from the publication of Karki, et al (1961) giving the effect of the distant topography plus compensation according to the Airy-Heiskanen model T = 30. Clearly the smoothly varying function shown there could be of no effect on interpolation over distances of 30 km or less. For the area within 166 km of the computation point the amount of improvement in interpolation accuracy to be gained by using an isostatic anomaly depends upon both the accuracy with which the compensating densities of the model actually reflect the conditions within the earth, and the degree to which "local" compensation exists. With respect to the accuracy of the density model assumed it is becoming increasingly clear (Woollard, 1962) that compensation is not achieved in a uniform manner everywhere but is achieved by complex variations in crustal thickness, crustal density and upper mantle density. Thus a density model which assumes that compensation is achieved by variation of a single parameter such as crustal thickness is not likely to give more than a rough approximation to the actual density distribution. The fact that often an isostatic density model will nearly explain the observed gravity even where seismic results indicate it does not correspond to the actual density distribution is in itself an indication that the gravitational effect of the compensating masses is normally a rather smoothly varying function.

The type of compensating masses most likely to have significant short period effects in the gravity field would be local compensation for features of small lateral extent. Every indication is, however, that such local compensation does not exist. The finite strength of the earth's crust would tend to make local compensation unlikely and investigations of gravity traverses across features of small lateral extent (Woollard, 1962) indicate that local compensation does not exist.

FIG. 6-2
COMPARISON OF BOUGUER AND ISOSTATIC ANOMALIES IN WYOMING



In Figure 6-2 an example is given of a typical profile comparing isostatic and complete Bouguer anomaly. It can be seen that although some large scale, long-wavelength trends may be nearly removed by using an isostatic density model, variations over distances of significance for interpolation 30 km or less are virtually unchanged. The same result may be seen on the large scale profiles were also used in the investigations and provided to ACIC as blackline prints. The general conclusion is that for interpolation over short distances of 30 km or less not enough additional interpolation accuracy would be attained to justify the extra work entailed in isostatic anomaly computation and determination of the secondary corrections where the isostatic anomalies were used directly in deflection computations.

Model Earth Anomalies

In so far as interpolation is concerned the utility of the Model Earth Anomaly can be quickly determined from the formula for its computation. The Model Earth Anomaly can be written

$$\Delta g_{me} = g - [\gamma - T - A(H - h)] = g - (\gamma + Ah - T) + AH$$

where Δg_{me} = Model Earth Anomaly
g = observed gravity
γ = International formula value plus free-air correction
T = terrain correction
A = Bouguer correction factor
h = actual height of station
H = height of Model Earth at station

From the meaning of the symbols we see that the Model Earth Anomaly can be written

$$\Delta g_{me} = \text{Complete Bouguer Anomaly} + AH \qquad (6\text{-}2)$$

Since the height H is the weighted mean height of an area of 100 km or more in radius around a point it must be a very slowly varying function of position. Thus the model earth anomaly will be no better than the Complete Bouguer Anomaly for interpolation over short distances of 30 km or less such as

we are interested in here. Since the Model Earth Anomaly will be no better for interpolation than the complete Bouguer anomaly and it requires more computation to attain this anomaly it does not appear to have any particular value for deflection interpolation computations.

Before leaving the Model Earth Anomaly it is perhaps worth while to consider the reason for its developement since it so well illustrates the differences in outlook between the Classical and the New Geodetic Theory.

The Model Earth Anomaly was developed for use with the classical theory. Its aim was to develop an anomaly which could be interpolated and extrapolated as easily as the isostatic anomaly, would be more easily computed than the isostatic anomaly, would avoid assumptions concerning densities down to sea level, and would produce negligable deformation of the geoid.

With the advent of the "New Theory" the questions of incorrect density assumptions and deformations of the geoid are no longer important. Thus the Model Earth Anomaly has lost much of its reason for existing. The Model Earth Anomaly is not appreciably easier to compute than an Isostatic Anomaly in cases where tables such as those of Karki, et al (1961) are available for determining the distant topographic-isostatic effect. In this case the primary computational labor in either method is related to making the terrain correction out to 166 km.

Since, as we have seen, the Model Earth Anomaly is simply the complete Bouguer anomaly with a mean elevation term added it should be smoothly varying. However, DeGraff Hunter's contention that the Model Earth Anomaly is normally somewhat smaller in magnitude and smoother than the isostatic anomaly seems theoretically somewhat unlikely. Since the Model Earth Anomaly is very similar to an average free-air anomaly and average free-air anomalies are positively correlated with average elevation, we would theoretically expect the Model Earth

Anomaly to run consistently positive in mountain areas. Figure 6-3 illustrates this tendency in an area in the Rocky Mountains.

Recently Saunders (1963) has made a study of Model Earth Anomalies in which he computed these anomalies for a large area in Colorado. His results also show the Model Earth Anomaly to be consistently more positive than the isostatic anomaly in mountainous areas.

Rudzki Anomalies

The relative accuracy of Rudzki anomalies for interpolation can be seen from the formula for the attraction of the Rudzki density model. This formula can be written

$$g_R = \text{I.F.} + \text{Top} - \text{T.M.}$$

where

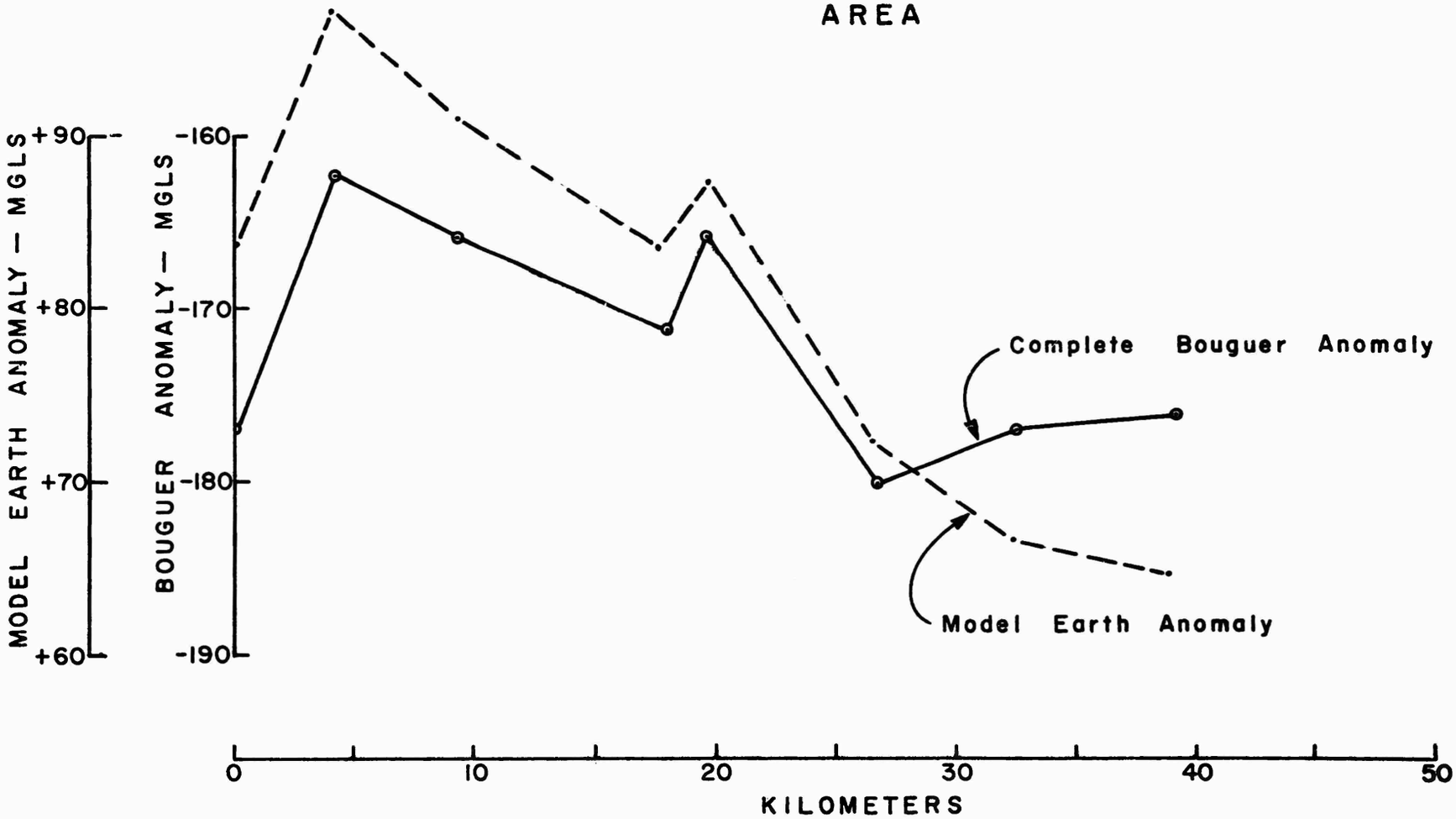
I.F. = International formula attraction

Top = Attraction of topography around the world

T.M. = Attarction of deviation masses below sea level

This attraction formula resembles that of an isostatic anomaly. However, in the case of most isostatic models the deviation mass is chosen so as to try to approximate the actual density distributions below sea level. In the Rudzki density model the deviation mass is chosen in order to avoid secondary corrections when using the classical theory. Clearly an anomaly produced by using a density model in which a portion of the density distribution does not attempt to correspond to that of the actual earth should not be expected to be expecially useful for interpolation in general. The Rudzki anomalies differ from the Bouguer and isostatic anomalies only with respect to the long period components of the anomaly field. Thus it would be possible to apply geologic corrections to a Rudzki anomaly and have a suitable anomaly for interpolation over short distances. Since, however, they would be no better for interpolation

FIG. 6-3
COMPARISON OF ANOMALY TRENDS IN ROCKY MT. AREA



than Bouguer anomalies they have no theoretical advantage in the "new theory" and would require much greater amounts of labor to compute. They have not been considered.

Summary

At the risk of being repetitious, the results of this section will be summarized here since the approach used and the conclusions reached vary somewhat from those anticipated when the investigation was undertaken. In the contract specifications, profiles from "Geoid Correlations: Test Profiles" were designated as profiles to be studied in order to "determine the best geophysical reduction method to transform observed gravity into anomalies of geodetic accuracy and significance". At the time this study was begun it was clear that the use of geologic and geophysical information could improve the accuracy of interpolation and extrapolation of gravity values. However, it was not clear in what way this fact could be incorporated into the theory of physical geodesy in order to achieve the most improvement in the computed geodetic quantities. Primarily this resulted from the confusing claims and counter claims of proponents of the various types of anomalies. Thus to begin this study geodetic theory was carefully examined. As a result of this part of the study it became clear that modern geodetic theory did not indicate any theoretical advantage for one type of anomaly over another. Most of the apparent advantages and the theoretical problems which lead to them no longer existed when using the "New Geodetic Theory". Instead, the accuracy of the results obtained with the "new" theory were dependent primarily on the accuracy with which the gravity field could be interpolated between observation points. The result of this is a reorientation in the way gravity anomalies are viewed. Previously, one considered an anomaly to result from computing the effects on the observed gravity of

transfer of mass, and moving from one point to another in space before comparison with the observed gravity. In the present situation one can think of the gravity anomaly as simply the result of comparing observed gravity at the point of observation with the gravity predicted at that point by some density model. Thus an anomaly is simply the difference between the actual density distribution of the earth and the density distribution of the model. Viewed in this way, one can almost decide even before studying any profiles what type of anomaly will be best for interpolation. The more nearly a density model approximates the density distributions of the actual earth, the more accurately the anomaly computed from it can be interpolated.

For the particular application under consideration in connection with this contract -- interpolation of deflections of the vertical -- a reasonable amount of gravity data is absolutely necessary to achieve an answer of satisfactory accuracy. The question of interpolation over short distances was therefore the question of greatest importance and the one most closely studied. The anomaly profiles prepared as required under sections 3a, 3b and 3c of the technical specifications of the contract, bore out the general conclusions reached theoretically concerning the general problem of interpolation but were too gross in scale to be of great use in the question of interpolation over short distances. These profiles are being provided to ACIC at original scale. They are not included in the report since it was felt that after the reduction necessary to obtain a reasonable size for binding with the report they would provide no useful information.

In order to study empirically the relative accuracy of anomalies for interpolation over short distances, a number of comparative profiles were prepared using closer spaced observational data. These bore out the theoretical expectations.

Of all the types of anomalies computed without using geologic and geophysical control, the complete Bouguer anomaly, using as nearly as possible the actual density of the topography, is as good as any other and requires less computation. Anomalies such as isostatic and Rudzki anomalies differ from the complete Bouguer anomaly only in ways that do not effect interpolation over short distances. No special method of utilizing geologic and geophysical data was discovered under this contract. Actually, none could be expected. Once it was established that the problem was one of simple interpolation it becomes a question of applying all of the results from Woollard (1962) and the results thus far determined under contract AF 23 (601) - 3879 with ACIC to any particular problem at hand. If actual subsurface densities are known, an altered density model to take these into account may be constructed and anomalies computed using this model. Alternately, empirical correlations may be used between geologic and geophysical quantities and various types of anomalies.

The existence of empirical methods which can be applied to improve interpolation even when no exact density model can be determined suggest the use of geologic and geophysical knowledge to improve the interpolations of some type of existing anomaly rather than to compute an anomaly in which the geologic and geophysical information has been inserted directly into the density models. The complete Bouguer anomaly, with geology and geophysics utilized for interpolation between observation points, appears to be the best type of anomaly to use to produce a contour map for use in deflection interpolation computations. The complete Bouguer anomaly also has advantages over other types of anomalies insofar as ease of computation of the anomalies is concerned. In the next section the manner in which the theory of Sections 4 and 5 may be combined to allow the Complete Bouguer Anomalies to be easily utilized for computation is presented.

Section 7

Formulation for Solution

The problem of the formal setup for solution of the deflection computations is important from the point of view of practicality. According to the formulation in the new theory the deflection of the vertical in an arbitrary direction l is

$$\theta = -\frac{R}{4\pi\gamma}\int (\Delta g + G_1)\frac{dS}{d\psi}\frac{d\psi}{dl}\,dw - \left(\frac{\Delta g}{\gamma} + \frac{2N}{R}\right)\frac{dH}{dl} \tag{7-1}$$

where

S = Stokes function
ψ = angle between radius vectors of fixed and variable point
R^2dw = incremental surface element of sphere of radius R
γ = theoretical g (International Formula)
Δg = free air anomaly
$G_1 = \frac{R^2}{2\pi}\int \frac{h\Delta g}{r_o^3}\,dw$
H = normal height of variable point
H_o = normal height of fixed point
$h = H - H_o$
$r_o = 2R\sin\frac{\psi}{2}$ = distance between fixed and variable points
N = height anomaly

Pellinen (1962) seeks an answer by making use of the formulation given in section 5 allowing use of anomalies other than free air in the equations. Pellinen uses the complete Bouguer anomaly in the sense that all mass above sea level is actually considered around the world. This instead of a Bouguer Plate corrected for topography is theoretically used. The following development is taken from Pellinen (1962) with some clarifying remarks added and misprints corrected.

To arrive at the formula for deflection of the vertical we will first consider the formula for height anomaly. Let us call the potential due to the topography $\overline{T}$ and the potential difference between the potential of the actual earth and the model (which is the international formula plus topography above sea level) T'.

Then

$$W_p = U_p + \overline{T}_p + T'_p \tag{7-2}$$

Then according to section 5 we have

$$N = \frac{T'_p}{\gamma_Q} + \frac{\overline{T}_p}{\gamma_Q} = N' + \overline{N} \tag{7-3}$$

and

$$\Delta g' = - \frac{2T'_p}{R} - \left(\frac{\partial T'}{\partial R}\right) \tag{7-4}$$

where

$$\Delta g' = \left[(g_p - \gamma_Q) + \left(\frac{\partial \overline{T}}{\partial R}\right)_p + \frac{2\overline{T}_p}{R} \right] \tag{7-5}$$

Then in a manner completely analgous to the development of section 4 we get

$$T' = \frac{R}{4\pi} \int (\Delta g' + G_1') \; S \; (\cos \psi) \; dw \tag{7-6}$$

where

$$G_1' = \frac{R^2}{2\pi} \int \frac{h \Delta g'}{r_o^3} \; dw \tag{7-7}$$

Then

$$N' = \frac{R}{4\pi\gamma_Q} \int (\Delta g' + G_1') \; S(\psi) \; d\psi \tag{7-8}$$

and

$$\overline{N} = \frac{\overline{T}_p}{\gamma_Q} \tag{7-9}$$

However, there can be no zero or first order harmonics in $\overline{N}$ therefore we must write

$$\overline{N} = \frac{\overline{T}_p - \overline{T}_{p_o} - \overline{T}_{p_1}}{\gamma_Q} \tag{7-10}$$

where $\overline{T}_{p_o}$ and $\overline{T}_{p_1}$ are the zero and first order harmonics in the formal expansion of the potential of the topography.

Now we note that the attraction of the material above sea level at a

point P is

$$\bar{g} = 4\pi k\sigma H_o - k\sigma R \int \left(\frac{1}{r_o} - \frac{1}{r}\right) dw \tag{7-11}$$

where $r = (r_o^2 + h^2)^{1/2}$ = distance from fixed to moving point on surface taking into account topography. We note in passing that the second term comes from the integration of the attraction of a number of line elements with r_o and r being the distances to the two ends of the line. Thus the attraction of material above sea level is broken down into the attraction of a spherical shell of thickness H_o plus deviations of the actual topography around the world from the elevation H_o. Now going back to equation (7-5) we write

$$\Delta g' = \left[\Delta g + \frac{3}{2}\frac{\bar{T}}{R} + \left(\frac{\partial \bar{T}}{\partial R} + \frac{\bar{T}}{2R}\right)\right] \tag{7-12}$$

But according to (7-11) with an accuracy of the order of $\frac{H}{R}$

$$\bar{g} \approx -\frac{\partial \bar{T}}{\partial R} = 4\pi k\sigma H_o - k\sigma R^2 \int \left(\frac{1}{r_o} - \frac{1}{r}\right) dw \tag{7-13}$$

and

$$\frac{\bar{T}}{2R} \approx 2\pi k\sigma H_o + V_2 \tag{7-14}$$

where V_2 = potential of second term on RHS of (7-13)

Then neglecting V_2 which we shall investigate later.

$$\Delta g' = \left[\Delta g + \frac{3}{2}\frac{\bar{T}}{R} - 2\pi k\sigma H + \Delta g_p\right] \tag{7-15}$$

where we are using the symbol Δg_p to represent $k\,\sigma\,R^2 \int \left(\frac{1}{r_o} - \frac{1}{r}\right)dw$. Note that Δg_p is essentially a terrain correction except that it is carried completely around the earth. Pellinen states that it actually differs very little from the simple terrain correction. Whether or not we accept this, the difference should contribute little when used for interpolation of deflections.

Now we shall write $\bar{T}$ as the sum of two parts, $\bar{T}^o$ the potential the masses

above sea level would have if they were condensed on the surface $H = H_o$ which is approximated by the sphere passing through P and $\Delta\overline{T}$ which is the difference between this potential and the actual potential of masses above sea level.

Thus with an accuracy of $\frac{H}{R}$ we have

$$\overline{T}_p{}^o = k\sigma R^2 \int \frac{H}{r_o}\, dw = k\sigma R^2 \int \frac{H_o+h}{r_o}\, dw \tag{7-16}$$

Now to get the potential of the actual topography we go back to (7-13) and divide the potential $\overline{T}$ into two terms analgous to the two terms in $\overline{g}$.

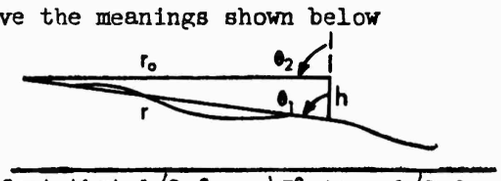
We can write for the potential of a spherical shell

$$k\sigma R^2 \int \frac{H_o}{r_o}\, dw = 4\pi k\sigma H_o R \tag{7-17}$$

Now to get the potential due to the second term. We can see the part of this potential which is most important by examining the equation which holds at distances close enough that earth curvature is unimportant. We get for the potential of a homogeneous line mass (see MacMillan, 1958 p. 42)

$$dV = k\sigma \log \frac{\tan 1/2\ \theta_2}{\tan 1/2\ \theta_1} \tag{7-18}$$

where the symbols have the meanings shown below



Using the fact that $1/2\ \theta_2 = 45°$ $\tan\ 1/2\ \theta_2 = 1$ and substituting for tan. $1/2\ \theta_1$ in terms of r_o, r, h we get

$$dV = k\sigma \ln \frac{r+h}{r_o} \tag{7-19}$$

or

$$V = k\ \sigma\ R^2 \int_o^{r_1} \ln \frac{r+h}{r_o}\, dw \tag{7-20}$$

As stated this formula holds only out to the point where the curvature

of the earth is important. Thus we might write

$$\bar{T}_p = k\sigma R^2 \int \frac{H_o}{r_o}\, dw + k\sigma R^2 \int_o^{r_1} \ln \frac{r+h}{r_o}\, dw + V' \tag{7-21}$$

where V' represents the potential of topography beyond r_1. We shall assume that this can be ignored. If it is not absolutely negligable in magnitude at least it is slowly varying enough to be ignored in deflection interpolation work. Thus we get

$$\Delta\bar{T} = \bar{T}_p - \bar{T}_p^o = - k\sigma R^2 \int \frac{H_o}{r_o}\, dw - k\,\sigma\, R^2 \int \frac{h}{r_o}\, dw + k\sigma R^2 \int \frac{H_o}{r_o}\, dw$$

$$+ k\sigma R^2 \int_o^{r_1} \ln \frac{r+h}{r_o}\, dw \quad \text{OR} \quad \Delta\bar{T} = k\sigma R^2 \int \left(\ln \frac{(r+h)}{r_o} - \frac{h}{r_o}\right) dw \tag{7-22}$$

or expanding $\ln(\frac{r+h}{r_o})$ in series, keeping the first three terms and cancelling the $\frac{h}{r_o}$ terms we get

$$\Delta\bar{T} = k\,\sigma\, R^2 \int \left(\frac{1}{6}\frac{h^3}{r_o^3} + \frac{3}{40}\frac{h^5}{r_o^5}\right) dw \tag{7-23}$$

Now we note that we can think of $k\,\sigma\, H$ in (7-16) as a density layer on the surface of a sphere. Then introduce a function G_p^o associated with R and H by the integral equation

$$G_p^o = 2\pi k\sigma H - \frac{3}{2} k\sigma R \int \frac{H}{r_o}\, dw \tag{7-24}$$

This is an equation exactly analogous to equation (4-51) of section 4 and the solution is

$$k\sigma \int \frac{H}{r_o}\, dw = \frac{1}{4\pi R} \int G_p^o\, S\,(\cos\psi)\, dw \tag{7-25}$$

Or from equation (7-16)

$$\bar{T}_p^o - \bar{T}_{p_o}^o - \bar{T}_{p_1}^o = \frac{R}{4\pi} \int G_p^o\, S(\cos\psi)\, dw \tag{7-26}$$

where the zero and first order harmonics have been left out in the solution.

Substituting (7-16) into (7-24)

$$G_p^o = 2\pi k\sigma H - \frac{3}{2}\frac{\overline{T}_p^o}{R} \tag{7-27}$$

$$\overline{T}_p^o = \frac{R}{4\pi}\int \left(2\pi k\sigma H - \frac{3}{2}\frac{\overline{T}_p^o}{R}\right) dw \tag{7-28}$$

Now going back to equation (7-3)

$$\overline{N} = \frac{\overline{T}_p}{\gamma_Q} = \frac{\overline{T}_p^o}{\gamma_Q} + \frac{\Delta\overline{T}}{\gamma_Q} \tag{7-29}$$

or more properly since there can be no zero or 1st order terms in $\overline{N}$

$$N = \frac{\overline{T}_p^o - \overline{T}_{po}^o - \overline{T}_{p1}^o}{\gamma_Q} + \frac{\Delta\overline{T} - \Delta\overline{T}_o - \Delta\overline{T}_1}{\gamma_Q} \tag{7-30}$$

or substituting from (7-26)

$$\overline{N} = \frac{R}{4\pi\gamma_Q}\int \left(2\pi k\sigma H - \frac{3}{2}\frac{\overline{T}_p^o}{R}\right) S(\cos\psi)\, dw + \frac{\Delta\overline{T} - \Delta\overline{T}_o - \Delta\overline{T}_1}{\gamma_Q} \tag{7-31}$$

Now substituting (7-31) and (7-8) into (7-3) we get

$$N = \frac{R}{4\pi\gamma_Q}\int \left(\Delta g' + G_1' + 2\pi k\sigma H - \frac{3}{2}\frac{\overline{T}_p^o}{R}\right) S(\cos\psi)\, dw + \frac{\Delta\overline{T} - \Delta\overline{T}_o - \Delta\overline{T}_1}{\gamma_Q} \tag{7-32}$$

or substituting for $\Delta g'$ from (7-15) and collecting terms

$$N = \frac{R}{4\pi\gamma_Q}\int \left(\Delta g + \Delta g_p + G_1' + \frac{3}{2}\frac{\Delta\overline{T}_p}{R}\right) S(\cos\psi)\, dw + \frac{\Delta\overline{T} - \Delta\overline{T}_o - \Delta\overline{T}_1}{\gamma_Q} \tag{7-33}$$

Let us now turn our attention to the deflection formulae for this type of development. The formula for deflection is

$$\theta = \frac{-1}{R}\frac{\partial}{\partial\theta}(N) = \frac{-1}{R}\left(\frac{\partial}{\partial\theta}N' + \frac{\partial}{\partial\theta}\overline{N}\right) \tag{7-34}$$

or

$$\theta = \theta' + \overline{\theta} = \frac{-1}{R\gamma_Q}\left\{\frac{\partial}{\partial\theta}T'(\varphi,\lambda,H) + \frac{\partial}{\partial\theta}\overline{T}(\varphi,\lambda,H)\right\} \tag{7-35}$$

but as shown in section 4 for using the potential where it is known we have

$$\frac{\partial}{\partial\theta} T' (\varphi,\lambda,H) = \frac{\partial T'}{\partial\theta}(\varphi,\lambda) + \left\{\Delta g' + \frac{2T'}{R}\right\}\frac{\partial H}{\partial\theta} \tag{7-36}$$

$$\frac{\partial}{\partial\theta} \overline{T} (\varphi,\lambda,H) = \frac{\partial\overline{T}}{\partial\theta} (\varphi,\lambda) + \left\{\overline{\Delta g}+\frac{2\overline{T}}{R}\right\}\frac{\partial H}{\partial\theta} \tag{7-37}$$

but substituting from (7-6)

$$\theta' = \frac{-1}{\gamma_Q R}\left\{\frac{R}{4\pi}\int (\Delta g' + G_1')\frac{\partial S}{\partial\psi}\frac{\partial\psi}{\partial\theta}\,dw + \left[\Delta g' + \frac{2T'}{R}\right]\frac{\partial H}{\partial\theta}\right\} \tag{7-38}$$

and using $\overline{T} = \overline{T}^o + \overline{\Delta T}$ and remembering there can be no first order terms we get

$$\overline{\theta} = \frac{-1}{\gamma_Q R}\left\{\frac{\partial}{\partial\theta}(\overline{T}^o - \overline{T}_1^o) + \frac{\partial}{\partial\theta}(\overline{\Delta T} - \overline{\Delta T}_1) + \left[\overline{\Delta g} + \frac{2\overline{T}}{R}\right]\frac{\partial H}{\partial\theta}\right\} \tag{7-39}$$

but from (7-26)

$$\frac{\partial(\overline{T}^o - \overline{T}_1^o)}{\partial\theta} = \frac{R}{4\pi}\int G_p^o\frac{ds}{d\psi}\frac{d\psi}{d\theta}\,dw \tag{7-40}$$

Using (7-38), (7-39), and (7-40) we get

$$\theta = \theta' + \overline{\theta} = \frac{-1}{4\pi\gamma_Q}\int (\Delta g' + G_1' + G_p^o)\frac{dS}{d\psi}\frac{d\psi}{d\theta}\,dw + \Delta\theta p - \left[\Delta g + \frac{2T}{R}\right]\frac{\partial H}{\partial\theta}\frac{1}{R\gamma_Q} \tag{7-41}$$

$$\text{where } \Delta\theta p = \frac{-1}{\gamma_Q R}\,\frac{\partial(\Delta\overline{T} - \Delta\overline{T}_1)}{\partial\theta}$$

or since $\Delta g' = \Delta g + \frac{3}{2}\frac{\overline{T}}{R} - 2\pi k\sigma H + \Delta g_p$ and $G_p^o = 2\pi k\sigma H - \frac{3}{2}\frac{\overline{T}^o}{R}$

$$\theta = \frac{-1}{4\pi\gamma_Q}\int (\Delta g + \Delta g_p + \frac{3}{2}\frac{\Delta\overline{T}}{R} + G_1')\frac{dS}{d\psi}\frac{d\psi}{d\theta}\,dw + \Delta\theta p - \left[\Delta g + \frac{2T}{R}\right]\frac{\partial H}{\partial\theta}\frac{1}{R\gamma_Q} \tag{7-42}$$

Now we use equation (7-23) to get

$$\Delta\theta p = \frac{-k\sigma R}{\gamma_Q}\int\frac{\partial}{\partial\theta}\left(-\frac{1}{6}\frac{h^3}{r_o^3} + \frac{3}{40}\frac{h^5}{r_o^5}\right)dw \tag{7-43}$$

But one can write

$$\frac{\partial\overline{\Delta T}}{\partial\theta} = \frac{\partial\Delta\overline{T}}{\partial r}\frac{\partial r}{\partial\psi}\frac{\partial\psi}{\partial\theta}$$

Noting that $dr = R \cos \frac{\psi}{2} \, d\psi$

and $d\psi = \cos (r_o, 1) \, d\theta$ (7-44)

where $\cos (r_o, 1)$ is the cosine of the angle between the direction in which the deflection is computed and the direction of the incremental surface element from the station.

Then from equations (7-43) and (7-44) we get

$$\Delta\theta p = \frac{-k\sigma R^2}{\gamma_Q} \int \left(\frac{1}{2} \frac{h^3}{r^4} - \frac{3}{8} \frac{h^5}{r^6}\right) \cos \frac{\psi}{2} \cos (r_o, 1) \, dw \qquad (7\text{-}45)$$

but $dw = \sin \psi d\psi dA = 2 \sin \frac{1}{2}\psi \cos \frac{1}{2}\psi d\psi dA = \frac{1}{R^2} r dr dA$

$$\Delta\theta p = \frac{-k\sigma}{\gamma_Q} \int \left(\frac{1}{2} \frac{h^3}{r^3} - \frac{3}{8} \frac{h^5}{r^5}\right) \cos \frac{\psi}{2} \cos (r_o, 1) \, dr dA \qquad (7\text{-}46)$$

It might be pointed out here that, instead of equation (7-45) one has in the original reference, equation 19 of that reference (Pellinen, 1962).

$$\therefore \Delta\theta p = k\sigma R \int \cos \frac{\psi}{2} \cos (r_o, 1) \left(\frac{1}{2} \frac{h^3}{r^4} - \frac{3}{8} \frac{h^5}{r^6}\right) dw \qquad (7\text{-}47)$$

Clearly the factor γ_Q is incorrectly omitted from equation (7-47). A question also might arise with respect to the correct power of R. This can be examined by use of a dimension check. The units of k are $cm^3/gm\ sec^2$. Remembering that dw is dimensionless we have for (7-45)

$$\frac{k\sigma R^2}{\gamma_Q} \cdot \frac{h^n}{r^{n+1}} = \frac{cm^3}{gm\,sec^2} \cdot \frac{gm}{cm^3} \cdot cm^2 \cdot \frac{sec^2}{cm} \cdot \frac{cm^n}{cm^{n+1}}$$

or the rhs of (7-45) is dimensionless as it should be. If we use equation (7-47) with γ_Q added to the denominator the rhs would be $\frac{1}{cm}$ which would be incorrect as a dimension.

If we integrate (7-46) to obtain the effect of an area lying between $A = A_1$, and $A = A_2$ and $r = r_1$ and $r = r_2$ we get the following, using the assumption that with small angles $\cos \psi/2 \approx 1$,

$$\Delta(\Delta\theta p) = \frac{-k\sigma}{\gamma_Q} \cos (r_o, 1) \, \Delta A \left\{\frac{\bar{h}^3}{4} \left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) - \frac{3\bar{h}^5}{32} \left(\frac{1}{r_1^4} - \frac{1}{r_2^4}\right)\right\} \qquad (7\text{-}48)$$

where $\bar{h}$ is the average value of the height difference and $\Delta A = A_1 - A_2$.

For practical computations the above equations can be simplified in the following manner. The quantities $3/2 \frac{\Delta T}{R}$ and G_1' beneath the integral sign in equation (7-42) can be assumed to be negligible and ignored. Similarly the term $\left[\Delta g + \frac{2T}{R}\right] \frac{\partial H}{\partial \theta} \frac{1}{R\gamma_Q}$ can be assumed to be negligible. If these assumptions are made, equation (7-42) reduces to

$$\theta = \frac{-1}{4\pi\gamma_Q} \int (\Delta g + \Delta g_p) \frac{dS}{d\psi} \frac{d\psi}{d\theta} dw + \Delta\theta p \qquad (7\text{-}49)$$

Similarly we have from equation (7-15)

$$\Delta g' = \Delta g + \Delta g_p - 2\pi k\sigma H \qquad (7\text{-}50)$$

As stated previously we can use for Δg_p the simple terrain correction. In this case $\Delta g'$ is the complete Bouguer anomaly. Thus from (7-50) we get $\Delta g + \Delta g_p = \Delta g' + 2\pi k\sigma H$ and (7-49) becomes

$$\theta = \frac{-1}{4\pi\gamma_Q} \int (\Delta g' + 2\pi k\sigma H) \frac{dS}{d\psi} \frac{d\psi}{d\theta} dw + \Delta\theta p \qquad (7\text{-}51)$$

The integral on the rhs of (7-51) is exactly analgous to the normal integral used for deflection computation in classical theory, only the anomaly used is changed. Thus the normal Rice template may be used to solve the integral expression. In computations it is found best to use the Rice template twice--once for each integral term.

The advantage of carrying out the computations in this way is that the highly variable component requires estimation of average elevations rather than average anomalies of the form $(\Delta g' + 2\pi\ k\ \sigma\ H)$. Since elevation contour maps are already available, whereas, the special anomaly maps required would be fully as complex as the elevation maps and would have to be drafted this is a tremendous advantage.

The complete Bouguer anomaly $\Delta g'$ on the other hand is a very smoothly varying function and, once a realistic representation of the gravity field has been derived, using geological and geophysical data, maps are easily prepared.

An additional advantage of proceeding as in the last paragraph is that the term $\Delta\theta p$ requires averaged elevations. Although the use of the Rice template for integration of the $\Delta\theta p$ integral does not produce a solution in which each section of each ring has the same weight, the average elevations obtained from the Rice template can be used to obtain $\Delta\theta p$ with a minimum of additional calculations. Certainly this method of procedure is much to be preferred over going through the entire process of picking average elevations from another template.

In Part 2 of the Final Report on this investigation the procedure described above will be applied to actual data and the results compared with observed astro-geodetic deflections. A detailed outline of the steps followed in applying the procedure will be included.

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