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SIX DEGREE OF FREEDOM DIGITAL SIMULATION
MODEL FOR UNGUIDED FIN-STABILIZED ROCKETS

BY

LOUIS D. DUNCAN AND RONALD J. ENSEY

ERDA-196 NOVEMBER 1964 964

WHITE SANDS MISSILE RANGE
NEW MEXICO

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SIX DEGREE OF FREEDOM DIGITAL SIMULATION
MODEL FOR UNGUIDED FIN-STABILIZED ROCKETS

By

Louis D. Duncan and Ronald J. Ensey

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U. S. ARMY ELECTRONICS RESEARCH AND DEVELOPMENT ACTIVITY
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ABSTRACT

A six degree of freedom model for digital simulation of the trajectory of an unguided, fin-stabilized rocket is developed. A derivation of the equations and an explanation of the coordinate systems are presented. The development assumes that the trajectory will be over a rotating planet with a variable atmosphere. A space-variable, three-dimensional wind vector is assumed.

The equations of motion are derived from Newton's Laws of Motion. The aerodynamic forces and moments are based on the theory of stability derivatives and the assumption of linear aerodynamics. The body axes are assumed to be principal axes of inertia.

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INTRODUCTION

Various unguided rocket systems have been used at White Sands Missile Range (WSMR) for several years for high-altitude research. Present planning indicates that an increased number and variety of such rockets will be fired at WSMR for novel experimentations and for re-entry studies. The use of the unguided rocket is desirable since it is usually less complex, and, therefore, more reliable; and is also less expensive than one with guidance. The principal disadvantage is the inherent dispersion, since there is no guidance. The largest contribution to the dispersion of unguided rockets is due to the wind encountered during flight. This dispersion can be made less significant by applying a procedure for determining the wind effect and then adjusting the launcher so as to compensate for the wind effect, thereby achieving the desired trajectory.

A real-time prelaunch impact prediction system has been developed for use at WSMR to compensate for the effect of the wind [1,2]. An integral part of this system is the trajectory simulation equations. The equations presently used by the system were developed by Dr. Everett L. Walter [3]. This report presents the development of a new set of equations which will be evaluated for possible replacement of the Walter equations.

The development of a trajectory simulation model can follow numerous approaches and can vary from quite simple to extremely detailed, depending primarily upon the purpose for which the model is to be used. Since this model is to be used solely for flight simulation in a real-time system to determine wind effect on rockets having range and altitude of less than 600 miles, the following assumptions are made:

a. The rocket is assumed to be a rigid body with six degrees of freedom.

b. The body axes are principal axes of inertia.

c. Linear aerodynamics are adequate for determining the aerodynamic forces and moments.

d. The earth is a sphere.

e. Gravity follows the inverse-square law.

f. The thrust vector acts parallel to the longitudinal axis of the rocket.

LIST OF SYMBOLS

X', Y', Z'	Launcher coordinate system
X, Y, Z	Inertial coordinate system
x, y, z	Body coordinate system
T_{X2x}	Linear transformation
k_X, k_Y, k_Z	
$k_{X'}, k_{Y'}, k_{Z'}$	Unit vectors along the axis denoted by the subscript
k_x, k_y, k_z	
R	Position vector of missile center of gravity in inertial system
ω_M	Rotation of x,y,z system in X,Y,Z system
p, q, r	x,y,z components of ω_M
ω	Earth's rotational velocity
γ	Angle determined by the projection of R in the X,Y plane and the X-axis
ϕ_L	Latitude after launch
ϕ_{L_o}	Initial Latitude
l_1, l_2, l_3	
m_1, m_2, m_3	Direction cosines of the x,y,z axes respectively with respect to the X,Y,Z coordinate system
n_1, n_2, n_3	
C_x	
C_{N_α}	Aerodynamic stability derivatives
$C_{N_{p_\alpha}}$	e.g., $C_{N_\alpha} = \frac{\partial C_N}{\partial \alpha}$
C_l	where
C_{l_p}	C_N is the normal force coefficient;

C_{m_α}	
C_{m_q}	$C_{N_{p_\alpha}} = \frac{\partial}{\partial \alpha}\left(\frac{\partial C_N}{\partial P}\right)$: etc.
$C_{m_{p_\alpha}}$	
m	Mass of rocket
P_a	Atmospheric pressure
A_e	Area of exit nozzle of rocket motor
$T_{s.t.}$	Thrust as measured by a static test
$P_{s.t.}$	Atmospheric pressure at static test site
g	Acceleration due to gravity
g_s	Value of g at sea level
R_o	Mean sea level radius of earth
h	Height above mean sea level
θ_o	Elevation angle at launch
α_o	Azimuth angle at launch
F	Total force acting on the missile
M_t	Sum of the external moments acting on the missile
L, M, N	X,Y,Z components of M_t
$\vec{V}$	Missile velocity vector
$\vec{H}$	Angular momentum vector
$\vec{V}^{I}$	$\vec{V}$ referred to the inertial system
$\vec{V}^{b}$	$\vec{V}$ referred to the body system
u, v, w	x,y,z components of $\vec{V}$
$\Sigma F_x, \Sigma F_y, \Sigma F_z$	x,y,z components of F

H_x, H_y, H_z	x,y,z components of $\vec{H}$
$\begin{matrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{matrix}$	Tensor of inertia
$W_{X'}, W_{Y'}, W_{Z'}$	Wind components in the launcher system
W_x, W_y, W_z	Wind components in the body system
V_a	Speed of missile with respect to the wind vector
u', v', w'	x,y,z components of $\vec{V}_a$
α	Angle of attack
β	Angle of sideslip
δ	Absolute angle of attack
α^*	Auxiliary angle of attack
β^*	Auxiliary angle of sideslip
V_s	Speed of sound
ρ	Atmospheric density
q'	Dynamic pressure
s	Reference area for aerodynamics
d	Reference length for aerodynamics

COORDINATE SYSTEMS AND TRANSFORMATIONS

Three right-hand orthogonal coordinate systems are used to describe the missile's position in space. They are the launcher coordinate system with X',Y',Z' axes, the missile coordinate system with x,y,z axes and the inertial coordinate system with X,Y,Z axes.

The X',Y',Z' system has its origin at the launcher and rotates with the earth. The positive X' axis points east, the positive Y' axis points north, and the positive Z' axis points outward along the radius vector from the center of the earth.

The x,y,z system is a moving system with its origin at the center of gravity, C_g, of the missile. The x axis coincides with the longitudinal axis of the rocket and is positive toward the nose. Let θ be the angle between the x axis and the Z' axis measured from the Z' axis. The y axis lies in the X',Y' plane and is positive in the direction of positive $\dot{\theta}$. The positive z axis is chosen so that we have an orthogonal right-hand system.

The X,Y,Z system has its origin at the center of the earth. The system is oriented so that the X and Y axes lie in the equatorial plane with the Y axis initially passing through the longitude of the launcher. The Z axis is coincident with the earth's axis and positive toward the North Pole. This system <u>does not</u> rotate with the earth.

The linear transformations between these systems will be denoted by e.g., T_{X2x} where the left hand subscript denotes the domain of the mapping. To derive these transformations, let $k_X, k_Y, k_Z, k_{X'}, k_{Y'}, k_{Z'}$, k_x, k_y, k_z be unit vectors along the axis denoted by the subscript. Let R be position vector of the C_g and let ω_M define the rotation of the x,y,z system in the X',Y',Z' system with x,y,z components denoted by p,q, and r respectively. Then

$$R = Xk_X + Yk_Y + Zk_Z, \tag{1}$$

and

$$\omega_M = pk_x + qk_y + rk_z. \tag{2}$$

It can be seen from Figure 1 that the direction cosines of R in the X,Y,Z system are ($\cos\phi_L \sin\gamma$, $\cos\phi_L \cos\gamma$, $\sin\phi_L$) where

$$\sin\gamma = X(X^2 + Y^2)^{-1/2},$$

$$\cos\gamma = Y(X^2 + Y^2)^{-1/2},$$

$$\sin\phi_L = Z(X^2 + Y^2 + Z^2)^{-1/2}, \text{ and}$$

$$\cos\phi_L = (X^2 + Y^2)^{1/2} (X^2 + Y^2 + Z^2)^{-1/2}. \tag{3}$$

Also, from Figure 1, the transformation $T_{X2X'}$ is given by the following equations:

$$k_{X'} = -\cos(\omega\Delta t)k_X - \sin(\omega\Delta t)k_Y$$

$$k_{Y'} = \sin\phi_{L_o} \sin(\omega\Delta t)k_X - \sin\phi_{L_o} \cos(\omega\Delta t)k_Y + \cos\phi_{L_o} k_Z$$

$$k_{Z'} = -\cos\phi_{L_o} \sin(\omega\Delta t)k_X + \cos\phi_{L_o} \cos(\omega\Delta t)k_Y + \sin\phi_{L_o} k_Z. \tag{4}$$

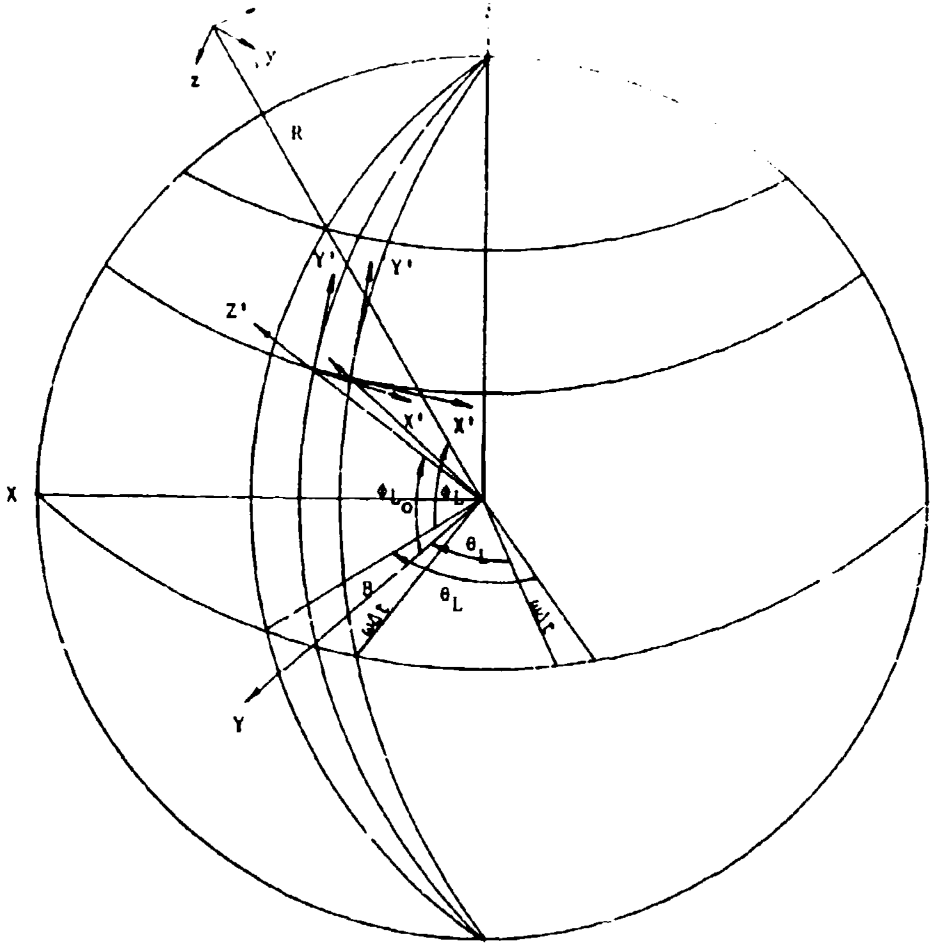

FIGURE 1

The inverse $T_{X'2X}$ of the transformation $T_{X2X'}$ is given by:

$$k_X = -\cos(\omega\Delta t)k_{X'} + \sin\phi_{L_o} \sin(\omega\Delta t)k_{Y'} - \cos\phi_{L_o} \sin(\omega\Delta t)k_{Z'}$$

$$k_Y = -\sin(\omega\Delta t)k_{X'} - \sin\phi_{L_o} \cos(\omega\Delta t)k_{Y'} + \cos\phi_{L_o} \cos(\omega\Delta t)k_{Z'}$$

$$k_Z = \cos\phi_{L_o} k_{Y'} + \sin\phi_{L_o} k_{Z'} . \tag{5}$$

(Observe that since each of the coordinate systems is orthogonal the inverse of a transformation is just the transpose.)

Let (l_1,l_2,l_3), (m_1,m_2,m_3) and (n_1,n_2,n_3) be the respective direction cosines of the x, y, and z axes in the X,Y,Z system. Then the transformation T_{X2x} and its inverse T_{x2X} are given by the sets of equations:

$$\begin{aligned} k_x &= l_1k_X + l_2k_Y + l_3k_Z \\ k_y &= m_1k_X + m_2k_Y + m_3k_Z \\ k_z &= n_1k_X + n_2k_Y + n_3k_Z , \end{aligned} \tag{6}$$

and

$$\begin{aligned} k_X &= l_1k_x + m_1k_y + n_1k_z \\ k_Y &= l_2k_x + m_2k_y + n_2k_z \\ k_Z &= l_3k_x + m_3k_y + n_3k_z , \end{aligned} \tag{7}$$

respectively.

Differential equations for l_i, m_i, n_i, $i = 1, 2, 3$ are obtained as follows: let the linear transformation T_{x2X} be expressed by the matrix

$$C = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} .$$

let the matrix of the linear transformation $T_{X'2X}$ be expressed by (see Equation (4)):

$$D = \begin{pmatrix} -\cos\omega\Delta t & -\sin\omega\Delta t & 0 \\ \sin\phi_{L_o}\sin\omega\Delta t & -\sin\phi_{L_o}\cos\omega\Delta t & \cos\phi_{L_o} \\ -\cos\phi_{L_o}\sin\omega\Delta t & \cos\phi_{L_o}\cos\omega\Delta t & \sin\phi_{L_o} \end{pmatrix};$$

the transformation T_{x2X} is expressed by the matrix

$$A = \begin{pmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{pmatrix}.$$

Hence $A = DC$. Thus,

$$\dot{A} = \dot{D}C + D\dot{C} = \dot{D}D^{-1}A + D\dot{C}\,. \tag{8}$$

Now to determine the elements of $\dot{A}$ it suffices to determine the elements of $\dot{C}$ and express these in terms of A. Consider a rotation, $d\psi$, of the x,y,z system in the X',Y',Z' system. The unit vector k_x changes by an amount of dk_x. The projection of dk_x in the x,y plane is $k_y \cdot dk_x$ (see Figure 2).

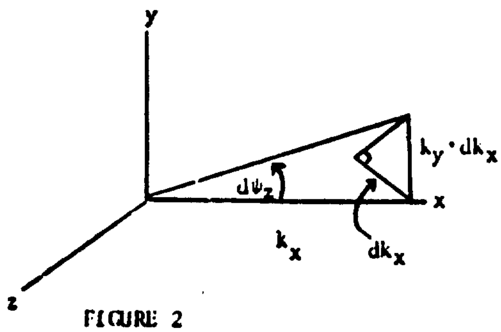

FIGURE 2

For small rotations, i.e., for small changes in time, the components of rotation about the z axis are given by

$$d\psi_z = \sin d\psi_z = k_y \cdot dk_x.$$

Thus

$$r = d\psi_z/dt = k_y \cdot dk_x/dt = k_y \cdot \dot{k}_x. \tag{9}$$

Similarly

$$p = k_z \cdot \dot{k}_y \tag{10}$$

and

$$q = k_x \cdot \dot{k}_z \ . \tag{11}$$

Now $\dot{k}_x$ lies in the y-z plane, $\dot{k}_y$ in the x-z plane, and $\dot{k}_z$ in the x-y plane. Thus

$$\begin{aligned} \dot{k}_x &= k_y A_3 - k_z B_2, \\ \dot{k}_y &= k_z A_1 - k_x B_3, \end{aligned} \tag{12}$$

and

$$k_z = k_x B_2 - k_y B_1 .$$

Since the system is right handed

$$\begin{aligned} k_x &= k_y \times k_z, \\ k_y &= k_z \times k_x, \\ k_z &= k_x \times k_y . \end{aligned} \tag{13}$$

Hence,

$$\dot{k}_x = \dot{k}_y \times k_z + k_y \times \dot{k}_z \tag{14}$$

or

$$\begin{aligned} k_y A_3 - k_z A_2 &= (k_z A_1 - k_x B_3) \times k_z + k_y \times (k_x B_2 - k_y B_1) \\ &= (-k_x \times k_z) B_3 + (k_y \times k_x) B_2 = k_y B_3 - k_z B_2 . \end{aligned} \tag{15}$$

So $A_3 = B_3$ and $A_2 = B_2$. Similarly $A_1 = B_1$. From Equations (9), (10), (12), and (13), we have

$$A_1 = p, \quad A_2 = q, \quad A_3 = r. \tag{16}$$

Now

$$\begin{aligned} \dot{k}_x &= \dot{a}_1 k_{X'} + \dot{a}_2 k_{Y'} + \dot{a}_3 k_{Z'}, \\ \dot{k}_y &= \dot{b}_1 k_{X'} + \dot{b}_2 k_{Y'} + \dot{b}_3 k_{Z'}, \\ \dot{k}_z &= \dot{c}_1 k_{X'} + \dot{c}_2 k_{Y'} + \dot{c}_3 k_{Z'} \ . \end{aligned} \tag{17}$$

Thus from Equations (6), (12), (15), (16), and (17) we have

$$\dot{a}_1 k_{X'} + \dot{a}_2 k_{Y'} + \dot{a}_3 k_{Z'} = \dot{k}_z = (rb_1 - qc_1) k_{X'} + (rb_2 - qc_2) k_{Y'} + (rb_3 - qc_3) k_{Z'} \; .$$

So $$\dot{a} = ra_i - qb_i \qquad i = 1, 2, 3. \tag{18}$$

Similarly

$$\dot{b}_i = pc_i - ra_i \qquad i = 1, 2, 3, \tag{19}$$

and $$\dot{c}_i = qa_i - pb_i \qquad i - 1, 2, 3.$$

Now from the matrix equation A = DC we can determine $\dot{a}_i$, $\dot{b}_i$ and $\dot{c}_i$ $i = 1, 2, 3$ in terms of l_i, m_i, and n_i $i = 1, 2, 3$ Performing this substitution and the matrix algebra and differentiation operations in Equation (8) we obtain, after simplification,

$$\begin{aligned}
\dot{l}_1 &= rm_1 - qn_1 - \omega l_2 \\
\dot{l}_2 &= rm_2 - qn_2 + \omega l_1 \\
\dot{l}_3 &= rm_3 - qn_3 \\
\dot{m}_1 &= pn_1 - rl_1 - \omega m_2 \\
\dot{m}_2 &= pn_2 - rl_2 + \omega m_1 \\
\dot{m}_3 &= pn_3 - rl_3 \\
\dot{n}_1 &= ql_1 - pm_1 - \omega n_2 \\
\dot{n}_2 &= ql_2 - pm_2 + \omega n_1 \\
\dot{n}_3 &= ql_3 - pm_3 \; .
\end{aligned} \tag{20}$$

INITIAL CONDITIONS FOR THE x,y,z SYSTEM

Initially the x axis and the Z' axis form an angle θ_o (this is the elevation of the x axis angle and is measured from Z'); and the projection of the x axis in the X'Y' plane forms an angle α_o with the Y' axis (this is the azimuth angle of the x axis and is measured clockwise from Y'). Since the y axis lies initially in the X',Y' plane, it forms an angle $\alpha_o + 90$ with the Y' axis (see Figure 3). Thus, initially,

$$\begin{aligned}
k_x &= k_{X'} \sin\theta_o \sin\alpha_o + k_{Y'} \sin\theta_o \cos\alpha_o + k_{Z'} \cos\theta_o , \\
k_y &= k_{X'} \cos\alpha_o + k_{Y'} \sin\alpha_o , \\
k_z &= k_{X'} \sin\alpha_o \cos\theta_o + k_{Y'} \cos\alpha_o \cos\theta_o - k_{Z'} \sin\theta_o .
\end{aligned} \tag{21}$$

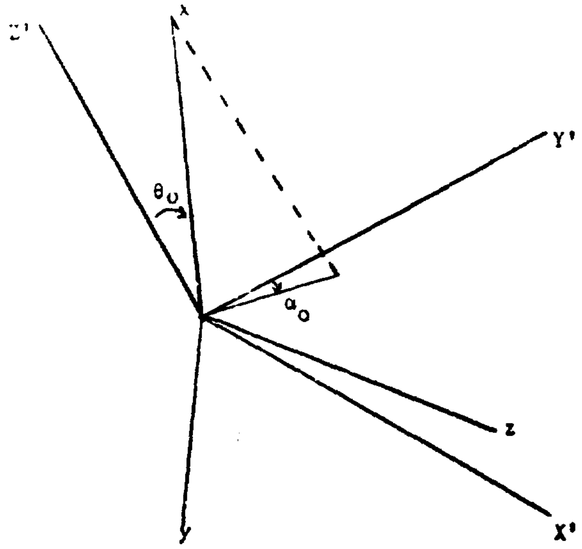

FIGURE 3

From Equation (5) we have, initially,

$$
\begin{aligned}
k_X &= -k_{X'} \,, \\
k_Y &= -\sin\phi_{L_o}\, k_{Y'} + \cos\phi_{L_o}\, k_{Z'} \\
k_Z &= \cos\phi_{L_o}\, k_{Y'} + \sin\phi_{L_o}\, k_{Z'} \;.
\end{aligned} \tag{22}
$$

Thus the initial conditions for l_i, m_i, n_i, $i = 1,2,3$, are given by

$$
\begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} = \begin{bmatrix} \sin\theta_o \sin\alpha_o & \sin\theta_o \cos\alpha_o & \cos\theta_o \\ \cos\alpha_o & -\sin\alpha_o & 0 \\ \sin\alpha_o \cos\theta_o & \cos\alpha_o \cos\theta_o & -\sin\theta_o \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\sin\phi_{L_o} & \cos\phi_{L_o} \\ 0 & \cos\phi_{L_o} & \sin\phi_{L_o} \end{bmatrix} \tag{23}
$$

DERIVATION OF THE EQUATIONS OF MOTION

This section presents the derivation of the equations of motion of a rigid body in inertial space. This derivation can be found in several references [4,5,6,7] and is presented here for completeness only. The two basic equations which define the motion are, by Newton's laws,

$$F = \frac{d}{dt}(m\vec{V}) \qquad (24)$$

$$M_t = \frac{d}{dt}(\vec{H}) \qquad (25)$$

These equations state that the sum of the applied forces, F, is equal to the time rate of change of the linear momentum, $m\vec{V}$, and the sum of the external moment M_t, is equal to the time rate of change of the angular momentum, $\vec{H}$.

THE TRANSLATIONAL ACCELERATIONS

The differential equations describing the motion of a missile system (Equations (24) and (25)) must always be referred to the inertial system (the X,Y,Z system) for solution. This is not meant to imply that every computation is done in the inertial system--to the contrary most computations are performed elsewhere. It does mean that the basic equations of motion must be referred to the inertial system and computations in other frames related by appropriate transformations.

Let $\vec{V}^I$ be the missile velocity in the inertial system and $\vec{V}^b$ the velocity in the body system (throughout this section the superscripts I and b will be used as above). It is clear that $\vec{V}^I = \vec{V}^b$. Since Newton's laws of motion are valid only in an inertial system, the derivatives of (22) and (23) must be in the inertial system. Now

$$\frac{d^I}{dt}(\vec{V}^I) = \frac{d^I}{dt}(\vec{V}^b) \ . \qquad (26)$$

The right-hand side of (26) is mixed and cannot be worked with as it stands. However, the derivative with respect to time in the inertial system can be written as the operator

$$\frac{d^I}{dt} = \frac{d^b}{dt} + \omega_M \times \ . \qquad (27)$$

Hence,

$$\frac{d^I}{dt}(\vec{V}^b) = \left(\frac{d^b}{dt} + \omega_M \times\right)\vec{V}^b = \frac{d^b}{dt}(\vec{V}^b) + \omega_M \times \vec{V}^b. \qquad (28)$$

Expanding the above differential equation gives

$$\dot{\vec{V}}^b = \frac{d^b}{dt}(uk_x + vk_y + wk_z) + \begin{bmatrix} k_x & k_y & k_z \\ p & q & r \\ u & v & w \end{bmatrix} \ , \qquad (29)$$

or

$$\dot{\vec{V}}^b = (\dot{u} + qw - rv)k_x + (\dot{v} + ru - pw)k_y + (\dot{w} + pv - qu)k_z. \qquad (30)$$

Now from the force equation,

$$\Sigma F^b = m\dot{\vec{V}}^b \,, \qquad (31)$$

we have translational equations of motion:

$$\Sigma F_x = m(\dot{u} + qw - rv),$$

$$\Sigma F_y = m(\dot{v} + ru - pw),$$

$$\Sigma F_z = m(\dot{w} + pv - qu), \qquad (32)$$

where ΣF_x, ΣF_y, and ΣF_z are the applied forces in the body coordinate system.

THE ROTATIONAL ACCELERATIONS

The relationships expressing the rotational motion are obtained in a straightforward manner. The components considered come from two sources, the time rate of change of the moment of momentum and the externally applied moments. The moment of a rigid body about its center of mass is given by:

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} . \qquad (33)$$

Since the body coordinate axes are assumed to be principal axes of inertia, the off diagonal terms of the 3 by 3 matrix are equal to zero. Thus we can rewrite Equation (33) as

$$\vec{H} = I_{xx}pk_x + I_{yy}qk_y + I_{zz}rk_z \,. \qquad (34)$$

Thus we obtain

$$\dot{\vec{H}} = k_x[\dot{I}_{xx}p + I_{xx}\dot{p} + (I_{zz} - I_{yy})qr] + k_y[\dot{I}_{yy}q + I_{yy}\dot{q} + (I_{xx} - I_{yy})pr] + k_z[\dot{I}_{zz}r + I_{zz}\dot{r} + (I_{yy} - I_{xx})pq], \quad (35)$$

where the differentiation of moment of inertia refers to that change at constant mass only. The change in angular momentum due to the escaping gases (i.e., due to change in mass) is considered in the external moments as the jet damping moment.

Now, if we denote the sums of the x,y,z components of the external moments by L, M, and N (to adopt standard notation) we have

$$
\begin{aligned}
L &= I_{xx}\dot{p} + \dot{I}_{xx}p + (I_{zz} - I_{yy})qr,\\
M &= I_{yy}\dot{q} + \dot{I}_{yy}q + (I_{xx} - I_{zz})pr,\\
N &= I_{zz}\dot{r} + \dot{I}_{zz}r + (I_{yy} - I_{xx})pq.
\end{aligned}
\qquad (36)
$$

These are the equations which describe the rotations.

THE FORCES AND MOMENTS ACTING ON THE ROCKET

THE AERODYNAMIC FORCES AND MOMENTS

Since the aerodynamic forces and moments depend upon the missile's velocity with respect to the surrounding air, preliminary expressions must be obtained before these forces and moments can be computed.

Let $W_{X'}$, $W_{Y'}$, and $W_{Z'}$ be the components of the wind in the X',Y',Z' system. The wind components in the x,y,z system, W_x, W_y, and W_z, are obtained by applying the transformation $T_{X'2x}$. This can be conveniently written in matrix form as

$$
\begin{bmatrix} W_x \\ W_y \\ W_z \end{bmatrix} = \begin{bmatrix} l_1 l_2 l_3 \\ m_1 m_2 m_3 \\ n_1 n_2 n_3 \end{bmatrix} \begin{bmatrix} -\cos\omega t & \sin\phi_{L_o}\ \sin\omega t & -\cos\phi_{L_o}\ \sin\omega t \\ -\sin\omega t & -\sin\phi_{L_o}\ \cos\omega t & \cos\phi_{L_o}\ \cos\omega t \\ 0 & \cos\phi_{L_o} & \sin\phi_{L_o} \end{bmatrix} \begin{bmatrix} W_{X'} \\ W_{Y'} \\ W_{Z'} \end{bmatrix}. \qquad (37)
$$

The velocity components of the missile relative to the wind are $u' = u - W_x$, $v' = v - W_y$, $w' = w - W_z$, and the relative speed is

$$V_a = [(u')^2 + (v')^2 + (w')^2]^{1/2}.$$

There are several angles which are used to calculate the aerodynamic forces and moments. These angles are shown in Figure 4. They can be expressed in terms of velocity components as follows: the angle of attack, α, the angle of sideslip, β, and the absolute angle of attack, δ, given by

$$\alpha = \tan^{-1} \frac{w'}{u'} \qquad (38)$$

$$\beta = \tan^{-1}\left[\frac{v'}{\sqrt{(u')^2 + (w')^2}}\right] \tag{39}$$

$$\delta = \tan^{-1}\left[\frac{\sqrt{(v')^2 + (w')^2}}{u'}\right]. \tag{40}$$

The auxiliary angle of attack α^* and the auxiliary angle of sideslip β^* are given by

$$\alpha^* = \tan^{-1}\left[\frac{w'}{\sqrt{(u')^2 + (w')^2}}\right] \tag{41}$$

$$\beta^* = \tan^{-1}\left[\frac{v'}{w'}\right] \tag{42}$$

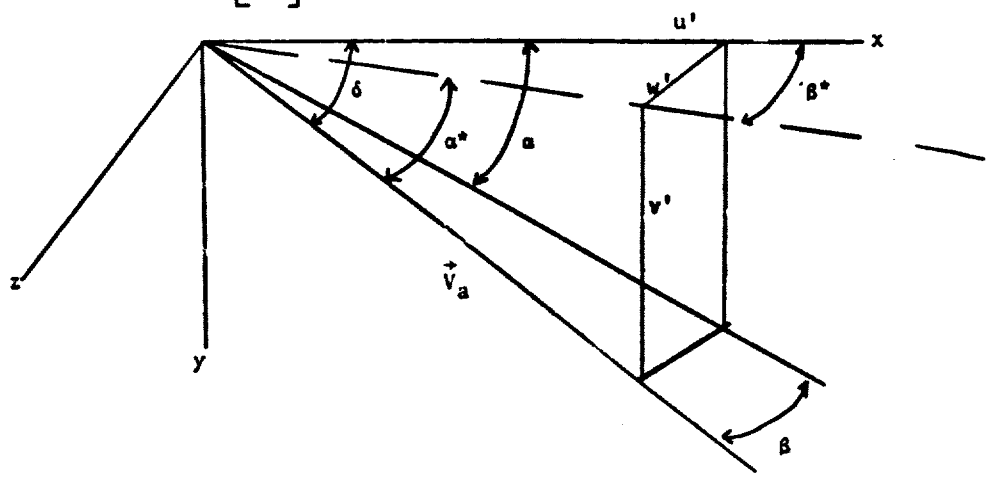

FIGURE 4

It is assumed that speed of sound, V_s, and atmospheric density, ρ, are known as functions of altitude. Now the Mach number and dynamic pressure can be computed from

$$\text{M.N.} = \frac{V_a}{V_s}, \tag{43}$$

and

$$q' = \frac{1}{2}\rho V_a^2 . \tag{44}$$

Since it has been assumed that the aerodynamic forces and moments can be obtained by the use of linear aerodynamics, the forces and moments are [5,6]

$$F_x = (C_x)q'S$$

$$F_y = [-C_{N_\alpha}\sin\beta - C_{N_{p_\alpha}}\left(\frac{pd}{2V_a}\right)\sin\alpha^*]q'S$$

$$F_z = [-C_{N_\alpha}\sin\alpha^* + C_{N_{p_\alpha}}\left(\frac{pd}{2V_a}\right)\sin\beta]q'S$$

$$L = [C_l + C_{l_p}\left(\frac{pd}{2V_a}\right)\sin\beta]q'Sd$$

$$M = [C_{m_\alpha}\sin\alpha^* + C_{m_q}\left(\frac{qd}{2V_a}\right) + C_{m_{p_\alpha}}\left(\frac{pd}{2V_a}\right)\sin\beta]\, q'Sd\ ,$$

$$N = [-C_{m_\alpha}\sin\beta + C_{m_q}\left(\frac{rd}{2V_a}\right) + C_{m_{p_\alpha}}\left(\frac{pd}{2V_a}\right)\sin\alpha^*]q'Sd, \tag{45}$$

where S is the reference area and d is the reference diameter.

THE THRUST FORCE AND THE JET DAMPING MOMENT

Let m be the mass of the rocket including the unspent fuel and let Δm be the change in mass (due to burning of fuel) during a small time interval Δt. By the law of conservation of total momentum we can equate the momentum at a time t to that at a time $t + \Delta t$. This gives us

$$mv = (m + \Delta m)\ (V + \Delta V) + \Delta m(V_e - V) \tag{46}$$

where V_e is the velocity, relative to the rocket, of the exit gases.

By dividing both sides of the above equation by Δt and then taking the limit as Δt approaches zero we get

$$\dot{m}V = -\dot{m}V_e. \tag{47}$$

This is the force on the rocket due to the changing momentum.

Besides $-\dot{m}V_e$ there is an additional force due to the differential in the pressure at the exit nozzle and the atmospheric pressure. If the pressure at the exit nozzle is P_e, the atmospheric pressure P_a, and the area of the exit nozzle A_e, then this additional force is $A_e(P_e - P_a)$, giving a total thrust of

$$T = \dot{m}V_e + (P_e - P_a)\ A_e. \tag{48}$$

If the thrust is measured at a test stand (where $V = 0$) in an atmosphere $P_{s.t.}$ it would be

$$T_{s.t.} = \dot{m}V_e + A_e\,(P_e - P_{s.t.}) . \tag{49}$$

From (48) and (49) we get

$$T = T_{s.t.} + A_e\,(P_{s.t.} - P_a) . \tag{50}$$

We have assumed that the thrust acts parallel to the longitudinal axis of the rocket, hence the only component due to the thrust is along the x-axis and it produces no moments.

Since a rocket rotates about a transverse axis during burning, the gases must be accelerated laterally as they flow down the motor tube. This lateral acceleration produces the so-called jet damping moment. The following expression for the moment is derived in reference [7]:

$$M_p = \dot{m}(l_j^2 - l_p^2)W_p ,$$

where $\dot{m}$ is the mass flow rate, W_p the instantaneous pitching velocity, l_j is the distance between the vehicle's Cg and the exit nozzle and l_p is the distance between the Cg of the vehicle and the propellant Cg.

The components of the moment are, for a symmetric rocket,

$$M_M = \dot{m}(l_j^2 - l_p^2)q \tag{51}$$

$$M_N = \dot{m}(l_j^2 - l_p^2)r . \tag{52}$$

THE FORCE DUE TO GRAVITY

Let g_s be the average value of g at sea level along the trajectory. Then the value of g at an altitude h is, by the inverse-square law,

$$g = g_s \left(\frac{R_o}{R_o+h}\right)^2 . \tag{53}$$

where R_o is the radius of the earth. Since gravity is a central force it acts along the radius vector $\vec{R}$ from the center of gravity of the missile to the origin of the X,Y,Z system and hence has direction cosines, in the X,Y,Z system

$$\left(\frac{X}{\sqrt{X^2 + Y^2 + Z^2}}\right), \left(\frac{Y}{\sqrt{X^2 + Y^2 + Z^2}}\right), \left(\frac{Z}{\sqrt{X^2 + Y^2 + Z^2}}\right).$$

By applying the transformation T_{X2x} we have

$$g_x = \frac{g}{\sqrt{X^2 + Y^2 + Z^2}} (l_1 X + l_2 Y + l_3 Z)$$

$$g_y = \frac{g}{\sqrt{X^2 + Y^2 + Z^2}} (m_1 X + m_2 Y + m_3 Z)$$

$$g_z = \frac{g}{\sqrt{X^2 + Y^2 + Z^2}} (n_1 X + n_2 Y + n_3 Z) \ . \tag{54}$$

THE EQUATIONS OF MOTION

The six equations of motion were developed in the preceding section, and the forces and moments which were assumed to affect the motion were discussed. At this point we collect the previous developments so that the system of equations may be presented together in final form. The translational equations of motion are, by (31), (45), (49), and (54),

$$m\dot{u} = mrv - mqw + C_X q S + T - mg_x \ , \tag{55}$$

$$m\dot{v} = mpw - mru + \{ -C_{N_\alpha} \sin\beta - C_{N_{p_\alpha}} (\frac{pd}{2V_a}) \sin\alpha^* \} q'S - mg_y, \tag{56}$$

$$m\dot{w} = mqu - mpv + \{ -C_{N_\alpha} \sin\alpha^* + C_{N_{p_\alpha}} (\frac{pd}{2V_a}) \sin\beta \} q'S - mg_z. \tag{57}$$

The rotational equations of motion are, by (35), (45), (51), and (52),

$$I_{xx}\dot{p} = (I_{yy} - I_{zz}) qr - \dot{I}_{xx}p + \{ C_{l} + C_{l_p} (\frac{pd}{2V_a}) \sin\beta \} q'dS, \tag{58}$$

$$I_{yy}\dot{q} = (I_{zz} - I_{xx}) pr - \dot{I}_{yy}q + \{ C_{m_\alpha} \sin\alpha^* + C_{m_q} (\frac{qd}{2V_a}) + C_{n_{p_\alpha}} (\frac{pd}{2V_a})$$

$$\sin\beta \} q'dS + M_{\dot{q}} q, \tag{59}$$

$$I_{zz}\dot{r} = (I_{xx} - I_{yy}) pq - \dot{I}_{zz}r + \{ C_{n_\alpha} \sin\beta + C_{n_q} (\frac{qd}{2V_a}) + C_{m_{p_\alpha}} (\frac{pd}{2V_a})$$

$$\sin\alpha^* \} q'dS + M_{\dot{r}} r. \tag{60}$$

These equations are numerically integrated to obtain the motion of the rocket in the x,y,z system. At the same time Equations (18), (19), and (20) are numerically integrated to obtain the transformation matrix required to express the results in the other systems.

SUMMARY

A six-degree-of-freedom digital simulation model has been developed for determining the trajectory and wind effect on a multistage unguided rocket. Although it was not specifically mentioned in the development of the equations of motion, it should be intuitively obvious that there is a discontinuity in these equations at the separation and expulsion of any booster.

Several assumptions were made in the beginning of the development. Several of the assumptions could be easily discarded if such is desirable. The possibility of dropping these assumptions will be discussed below.

<u>Assumption No. 1.</u> The rocket is assumed to be a rigid body with six degrees of freedom. It appears that the problem would be unnecessarily complicated if the rocket were not assumed to be a rigid body.

<u>Assumption No. 2.</u> The body axes are principal axes of inertia. Inspection of Equations (33) and (34) shows that this assumption is clearly not necessary. It was made for two reasons (1) Most rockets are nearly symmetric for stability reasons, and (2) the products of inertia required in lieu of this assumption are not usually available.

<u>Assumption No. 3.</u> Linear aerodynamics are adequate for determining the aerodynamic forces and moments. Inspection of Equation (45) shows that the development is easily adaptable to a change in the type of aerodynamics required.

<u>Assumption No. 4.</u> The earth is assumed to be a sphere. This assumption is not really used in the development of the equations. It is used only in the analysis of the results of the computations. It would be simple enough to consider the earth as an oblate spheroid.

<u>Assumption No. 5.</u> Gravity follows an inverse square law. This assumption is standard for the types of missiles considered.

<u>Assumption No. 6.</u> The thrust vector acts parallel to the longitudinal axis of the rocket. Most rockets are designed for such a thrust orientation. If it were desirable to consider thrust misalignments for dispersion analysis it would be a simple matter to compute the x,y,z component of the thrust and the resulting moments. However, for applications indicated in this report such considerations are not appropriate.

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U. S. ARMY ELECTRONICS RESEARCH AND DEVELOPMENT ACTIVITY
WHITE SANDS MISSILE RANGE
NEW MEXICO

FRANCISCO MEJIA-FLORES
LT COLONEL, SIGNAL CORPS
COMMANDING

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WILLIS L. WEBB
Chief Scientist
Environmental Sciences Directorate

BERNARD F. GUDENKAUF
Major, Signal Corps
Director
Environmental Sciences Directorate

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FOR THE COMMANDER:

L. W. ALBRO
Major, AGC
Adjutant