## AD NUMBER

| AD447282 |

## LIMITATION CHANGES

### TO:

Approved for public release; distribution is unlimited. Document partially illegible.

### FROM:

Distribution authorized to U.S. Gov't. agencies and their contractors;
Administrative/Operational Use; APR 1963. Other requests shall be referred to Army Engineer Geodesy Intelligence and Mapping Research and Development, Fort Belvoir, VA. Document partially illegible.

## AUTHORITY

USAERD ltr, 16 May 1967
NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.
Best Available Copy
GIMRADA Research Note No. 11

GENERAL NON-ITERATIVE SOLUTION OF THE INVERSE AND DIRECT GEODETIC PROBLEMS

By Emanuel M. Sodano

April 1963

DDC AVAILABILITY NOTICE

Qualified requestors may obtain copies of this report from DDC.

FORT BELVOIR VA
Research Note No. 11

GENERAL NON-ITERATIVE SOLUTION OF THE
INVERSE AND DIRECT GEODETIC PROBLEMS

Task 4A011001B85002

April 1963

Distributed by
The Director
U. S. Army Engineer
Geodesy, Intelligence and Mapping Research and Development Agency

Prepared by
Emanuel M. Sodano
Research and Analysis Division
U. S. Army Engineer
Geodesy, Intelligence and Mapping Research and Development Agency
Fort Belvoir, Virginia
The views contained herein represent only the views of the preparing agency and have not been approved by the Department of the Army.

GIMRADA Research Notes are published to record and disseminate significant results of research tasks or portions of overall programs during the course of the work.
GENERAL NON-ITERATIVE SOLUTION OF THE

INVERSE AND DIRECT GEODETIC PROBLEMS

I. INTRODUCTORY BACKGROUND

Earlier, at the Army Map Service, the writer published a comprehensive study [1] for the rigorous non-iterative inverse solution of long geodesics, following its presentation at the XI General Assembly of the International Union of Geodesy and Geophysics. (At present, a translation by the German Geodetic Commission and an abstract by the U. S. S. R. Academy of Sciences, complete with final formulas, are also available.) The method received favorable commentaries from several authoritative writers on the subject. A comparative evaluation study published in [2] indicated that the method was as accurate as contemporary solutions, yet simpler and shorter to compute. The procedure does not require any special geodetic tables and calls for relatively few trigonometric interpolations. Of greatest interest was the fact that it was the first rigorous non-iterative inverse solution to go not only beyond the first power of spheroidal flattening but through all the cubic terms of flattening. Yet, it was practical.

II. THEORETICAL REFINEMENT OF NON-ITERATIVE INVERSE

Since [1] represented only the formative stage of development, the next phase consisted of making a thorough mathematical analysis of the formulas in order to determine whether there were any concealed intrinsic properties or relationships which could be used to obtain an optimum solution. This analysis resulted in the uncovering of three basic quantities, \( a \), \( m \), and \( \phi \), the exclusive substitution of which promised a concise, orderly pattern for the terms of the two main spheroidal power series, \( x \) and \( S \), that were originally given in [1], at the top of page 18 and the bottom of page 19, respectively.

III. DEVELOPMENT OF CORRESPONDING DIRECT

Since there were indications that \( a, m, \) and \( \phi \) were a rather unique set of quantities (as will be shown later), an attempt was also made to introduce their equivalent into the formulation of a corresponding Direct solution, which so far was lacking. As if by design, two spheroidal power series resulted, again with a simple, orderly pattern of terms. Moreover, the format was identical to that of the Inverse. The three corresponding basic quantities were denoted as \( a_1, m_2, \) and \( \phi \). This Direct solution was derived from the formulas on pages 14 and 15 of [1]. Essentially, the power series of the quantity \( S \) (now appearing also in Appendix E) was used
to solve for $\phi_0$; then the expression of $\phi_0$ was in turn placed into the $\lambda$ power series. A critical factor in the mathematical determination of $a_1$ and $m_1$ was the proper choice of their common smaller order term $5e^{12}\sin^2 \beta_1$, which is probably one in a series of others required for the orderly theoretical extension of the solution to higher degree.

IV. NOTES ON COMPUTATION FORMS

The above efforts led to the Inverse and Direct computation forms now shown in Appendices A and B, respectively. The main Inverse spheroidal expressions are indicated by $(S - b_0)$ and $(\lambda - L) + c$, while the Direct by $\phi_0$ and $(L - \lambda) + \cos \beta_0$. It is to be noted that their outer coefficients are simple product combinations of $a$ and $m$ or $a_1$ and $m_1$, while their bracketed expressions are functions of only the variable $\phi$ or $\phi_S$ and powers of the spheroidal parameter $f$ or $e^{12}$. By tabulation of the functions of $\phi$ and $\phi_0$, so that they may be rapidly obtained by interpolation, it would then be very simple to multiply them by the easily determined outer coefficients. Since conventional trigonometric tables can be used to obtain the first-order term $\phi$, and no tables are required to obtain the first-order term $\phi_0$, only short tables for the second- and third-order terms need be drawn up. The third-order terms, which are small, vary sufficiently slowly for visual interpolation.

In addition to their two pairs of main power series and the basic quantities necessary to numerically evaluate them, these Inverse and Direct computation forms generally contain one simple closed trigonometric expression for each required final quantity except when, for example, the trigonometric cofunction is given as an alternative for occasions when a weak determination or unlinear interpolation may otherwise result. The forms may appear cluttered with rules for choosing signs, trigonometric quadrants, and so forth. Actually, these do not entail any added calculations but simply define the problem without ambiguity. In addition, the choices provide for greater generality and more varied applications. For example, one may calculate either the shorter geodesic between two given positions, or the longer geodesic around the spheroid's back side. Also, the subscripts 1 or 2 may be assigned to either position without fear of ambiguity. For less accuracy, terms in $f_2$ and $e^{12}$ may be omitted.

V. FORMULAS FOR VERY SHORT AND LONG GEODESICS

From its inception, the non-iterative derivation was developed primarily for very long geodesies; therefore, the auxiliary trigonometric functions which were correspondingly designed provided the greatest simplicity at the expense of generality. Recently, however, this Agency as well as the Army Map Service placed a requirement
for a single set of formulas applicable to very short as well as very long lines, since they were to be used also in the adjustment of triangulation and trilateration ground nets. It was felt that the basic long line formulas given in Appendices A and B appeared particularly convenient for the electronic computers which were to make the calculations. However, in order to obtain the same or even greater accuracy for very short geodesics, the alternate formulas presented in Appendix C were provided. The reason for the increased accuracy requirement is easily understood when one considers, for example, that if the length of a line is decreased a thousandfold, the positions of the new endpoints must be known a thousand times more accurately to maintain a constant azimuth accuracy. This means that the latitude and longitude will have a greater number of decimals and, therefore, additional significant digits. In order to avoid the carrying of too many fixed places, which are more apt to be affected by rounding errors if there is no spare digit capacity, Appendix C provides formulas whose terms are generally very small when a geodesic is very short, so the computations can be done conveniently by floating point for greater decimal accuracy. To obtain the full required accuracy, no additional terms need be added to the power series formulas, because (as shown in the second paragraph of Appendix F) they converge to more good decimal places for shorter geodesies. Since many of the small quantities in Appendix C consist of sines of small angles, their evaluation is especially adaptable to electronic computers, which by means of floating point can readily calculate trigonometric series that inherently converge to additional decimals for such small angles. Actually, the formulas in Appendix C are equally applicable to short and long lines, so only one set of equations need be programmed into an electronic computer. A floating point formula for a more accurate cosine of large absolute latitude is also included in Appendix C.

VI. CONCLUDING REMARKS

Appendix D provides the complete numerical calculations for a very short and a very long geodesic--1 mile and 6,000 miles, respectively. In each case, the Direct solution provides a check on the Inverse. The discrepancies between the two types of solutions are given at the end of Appendix D. The better positional accuracy provided for the short line by the formulas of Appendix C is convincingly shown.

Appendix E provides a non-iterative inverse solution of higher order accuracy (that is, through $f^5$ and $e^{16}$ terms) for use as a theoretical check on Direct or other Inverse formulas. In Appendix F, several interesting types of inter-relations of the terms of the power series are discussed and illustrated. These include relationships between numerical coefficients as well as algebraic terms. In Appendix G, meridional arc formulas are derived as special cases of the Inverse and Direct.
In conclusion, it should be noted that the Inverse case of almost antipodal positions, treated on pages 24 through 25 of [1], is omitted here because of its rare practical occurrence. Also, the elimination of $\beta$ by substitution in terms of the given $B$ is not undertaken because simple closed functions herein would become series expansions.

## APPENDICES

<table>
<thead>
<tr>
<th>Appendix</th>
<th>Item</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>INVERSE COMPUTATION FORM</td>
<td>7</td>
</tr>
<tr>
<td>B</td>
<td>DIRECT COMPUTATION FORM</td>
<td>9</td>
</tr>
<tr>
<td>C</td>
<td>ALTERNATE INVERSE AND DIRECT FORMULAS</td>
<td>12</td>
</tr>
<tr>
<td>D</td>
<td>NUMERICAL ILLUSTRATIONS OF INVERSE AND DIRECT FORMULAS</td>
<td>14</td>
</tr>
<tr>
<td>E</td>
<td>THEORETICAL FORMULAS FOR HIGHER ACCURACY</td>
<td>19</td>
</tr>
<tr>
<td>F</td>
<td>INTER-RELATIONS OF THE TERMS OF THE POWER SERIES</td>
<td>21</td>
</tr>
<tr>
<td>G</td>
<td>MERIDIONAL ARC AS SPECIAL CASE OF NON-ITERATIVE INVERSE AND DIRECT</td>
<td>24</td>
</tr>
</tbody>
</table>
APPENDIX A

INVERSE COMPUTATION FORM

Given: \( B_1, L_1 = \) Geodetic latitude and longitude of any point.

\( B_2, L_2 = \) Latitude and longitude of any other point.

(South latitudes and west longitudes considered negative.)

Required: \( \alpha, S = \) Geodetic azimuths clockwise from north and distance.

\( a_0, b_0 = \) Semimajor and semiminor axes of spheroid.

\( f = \) Spheroidal flattening = \( 1 - \frac{b_0}{a_0} \)

\( L = (L_2 - L_1) \) or \((L_2 - L_1) + [\text{sign opposite of } (L_2 - L_1)] (360^\circ)\)

Use whichever \( L \) has an absolute value < or > 180°, according to whether the shorter or the back-side's longer geodesic is intended. However, for meridional arcs (\(|L| = 0^\circ \) or 180° or 360°), use either \( L \) but consider it (+) for the shorter and (-) for the longer.

\[ \tan \beta = (\tan B) (1 - f) \text{ when } |B| \leq 45^\circ \]

\[ \cot \beta = (\cot B) + (1 - f) \text{ when } |B| > 45^\circ \]

\[ a = \sin \beta_1 \sin \beta_2; \ b = \cos \beta_1 \cos \beta_2; \ \cos \phi = a + b \cos L. \]

\[ \sin \phi = \sqrt{(\sin L \cos \beta_2)^2 + (\sin \beta_2 \cos \beta_1 - \sin \beta_1 \cos \beta_2 \cos L)^2} \]

The \( \sin \phi \) is (+) for the shorter arc and (-) for the longer.

Compute the radical entirely by floating decimals to prevent loss of digits, especially for very short geodesics.

\( \phi = \) Positive radians in proper quadrant, reference angle being determined from \( \sin \phi \) or \( \cos \phi \), whichever has the smaller absolute value.

\[ c = (b \sin L) \div \sin \phi; \quad m = 1 - c^2. \]
\[
\frac{S}{b_0} = \left[ (1 + f + f^2) \phi \right] \\
+ a \left[ (f + f^2) \sin \phi - \left( \frac{f^3}{2} \right) \phi^3 \csc \phi \right] \\
+ m \left[ - \left( \frac{f + f^2}{2} \right) \phi - \left( \frac{f^2}{2} \right) \sin \phi \cos \phi + \left( \frac{f^3}{2} \right) \phi^3 \cot \phi \right] \\
+ a^3 \left[ - \left( \frac{f^2}{2} \right) \sin \phi \cos \phi \right] \\
+ m^2 \left[ \left( \frac{f^2}{16} \right) \phi + \left( \frac{f^2}{16} \right) \sin \phi \cos \phi - \left( \frac{f^3}{2} \right) \phi^3 \cot \phi - \left( \frac{f^3}{8} \right) \sin \phi \cos^3 \phi \right] \\
+ am \left[ \left( \frac{f^2}{2} \right) \phi^2 \csc \phi + \left( \frac{f^2}{2} \right) \sin \phi \cos^3 \phi \right]
\]

\[
\lambda - \frac{L}{c} = \left[ (f + f^2) \phi \right] \\
+ a \left[ - \left( \frac{f^2}{2} \right) \sin \phi - \left( f^2 \right) \phi^3 \csc \phi \right] \\
+ m \left[ - \left( \frac{5}{4} f^2 \phi + \left( \frac{f^2}{4} \right) \sin \phi \cos \phi + \left( f^2 \right) \phi^3 \cot \phi \right] \text{ radians} \right]
\]

cot \alpha_{1-2} = (\sin \beta_2 \cos \beta_1 - \cos \lambda \sin \beta_1 \cos \beta_2) + \sin \lambda \cos \beta_2 \\
cot \alpha_{2-1} = (\sin \beta_2 \cos \beta_1 \cos \lambda - \sin \beta_1 \cos \beta_2) + \sin \lambda \cos \beta_1

For meridional arcs, consider \(\alpha\) as having 0\(^\circ\) reference angle, and obtain only the signs of the cotangents by disregarding the denominators. For other geodesics, replace cotangent by tangent when |cot\(\alpha\)| > 1, by taking the reciprocal of the quotient's value.

<table>
<thead>
<tr>
<th>Quadrant of (\alpha_{1-2})</th>
<th>Quadrant of (\alpha_{2-1})</th>
</tr>
</thead>
</table>
| If \(L\) is (+)  
...and cot (tan) of \(\alpha_{1-2}\) is (+) or (-), \(\alpha_{1-2}\) is in quad I or II, respectively. | ...and cot (tan) of \(\alpha_{2-1}\) is (+) or (-), \(\alpha_{2-1}\) is in quad III or IV, respectively. |
| If \(L\) is (-)  
...and cot (tan) of \(\alpha_{1-2}\) is (+) or (-), \(\alpha_{1-2}\) is in quad III or IV, respectively. | ...and cot (tan) of \(\alpha_{2-1}\) is (+) or (-), \(\alpha_{2-1}\) is in quad I or II, respectively. |
Given: \( B_1, L_1 \) = Geodetic latitude and longitude of any point 1. 
\[ \alpha_{1-2}, S = \text{Azimuth clockwise from north and distance to any point 2.} \]

Required: Geodetic \( \alpha_{2-1}, B_2, \) and \( L_2. \)

(South latitudes and west longitudes considered negative.)

---

\( a_0, b_0 = \text{Semimajor and semiminor axes of spheroid.} \)

\( f = \text{Spheroidal flattening} = 1 - \frac{b_0}{a_0} \)

\( e'^2 = \text{Second eccentricity squared} = \frac{(a_0^2 - b_0^2)}{b_0^2} \)

\[ \tan \beta = (\tan B) (1 - f) \text{ when } |B| \leq 45^\circ \]

\[ \text{or } \cot \beta = (\cot B) (1 - f) \text{ when } |B| > 45^\circ \]

\[ \cos \beta_0 = \cos \beta_1 \sin \alpha_{1-2}; g = \cos \beta_1 \cos \alpha_{1-2}; \]

\[ m_1 = (1 + \frac{e'^2}{2} \sin^2 \beta_1) (1 - \cos^2 \beta_0); \phi_S = (S + b_0) \text{ radians;} \]

\[ a_1 = (1 + \frac{e'^2}{2} \sin^2 \beta_1) (\sin^2 \beta_1 \cos \phi_S + g \sin \beta_1 \sin \phi_S). \]

\[ \phi_0 = [\phi_S]^2 \]

\[ + a_1 [\frac{e'^2}{2} \sin \phi_S] \]

\[ + m_1 [-\frac{e'^2}{4} \phi_S + \frac{e'^2}{4} \sin \phi_S \cos \phi_S] \]

\[ + a_1^2 \left[ \frac{5e'^4}{8} \sin \phi_S \cos \phi_S \right] \]

\[ + m_1^2 \left[ \frac{11e'^4}{64} \phi_S - \frac{13e'^4}{64} \sin \phi_S \cos \phi_S - \frac{e'^4}{8} \phi_S \cos^2 \phi_S + \frac{5e'^4}{32} \sin \phi_S \cos^2 \phi_S \right] \]

\[ + a_1 m_1 [\frac{3e'^4}{8} \sin \phi_S + \frac{e'^4}{4} \phi_S \cos \phi_S - \frac{5e'^4}{8} \sin \phi_S \cos^2 \phi_S] \text{ radians} \]
\[
\cot \alpha_{2-1} = (g \cos \phi_0 - \sin \beta_1 \sin \phi_0) \div \cos \beta_0
\]

For meridional arcs, consider \( \alpha_{2-1} \) as having 0° reference angle, and obtain only the sign of the cotangent by disregarding the denominator. For other geodesies, replace cotangent by tangent when \( |\cot \alpha_{2-1}| > 1 \), by taking the reciprocal of the quotient's value.

<table>
<thead>
<tr>
<th>Quadrant of ( \alpha_{2-1} )</th>
</tr>
</thead>
</table>
| If \((0^\circ \leq \alpha_{1-2} \leq 180^\circ)\) | \(\ldots \) and \( \cot (\tan) \) of \( \alpha_{2-1} \) is (+) or (-), \\
\( \alpha_{2-1} \) is in quad III or IV, respectively. 
| If \((180^\circ < \alpha_{1-2} < 360^\circ)\) | \(\ldots \) and \( \cot (\tan) \) of \( \alpha_{2-1} \) is (+) or (-), \\
\( \alpha_{2-1} \) is in quad I or II, respectively.

\[
\cot \lambda = (\cos \beta_1 \cos \phi_0 - \sin \beta_1 \sin \phi_0 \cos \alpha_{1-2}) \div \sin \phi_0 \sin \alpha_{1-2}
\]

For meridional arcs, consider \( \lambda \) as having 0° reference angle, and obtain only the sign of the cotangent by disregarding \( \sin \alpha_{1-2} \). For other geodesies, replace cotangent by tangent when \( |\cot \lambda| > 1 \), by taking the reciprocal of the quotient's value.

<table>
<thead>
<tr>
<th>Quadrant and Sign of ( \lambda )</th>
</tr>
</thead>
</table>
| When \(0^\circ < \phi_0 \leq 180^\circ\) \\
(sin \( \phi_0 \) considered positive) | When \(180^\circ < \phi_0 \leq 360^\circ\) \\
(sin \( \phi_0 \) considered negative) 
| \(0^\circ \leq \alpha_{1-2} \leq 180^\circ\) | \(\ldots \) then if \( \cot (\tan) \) of \( \lambda \) \\
is (+) or (-), \( \lambda \) is in quad I or II, respectively. 
| \(180^\circ < \alpha_{1-2} < 360^\circ\) | \(\ldots \) then if \( \cot (\tan) \) of \( \lambda \) \\
is (+) or (-), the associated angle is in quad III or IV, respectively, and \\
\( \lambda \) is obtained by subtracting 360°. 
|
\[
\frac{L - \lambda}{\cos \beta_0} = [- \ell \phi_S] \\
+ a_1 \left[ \frac{3f_2}{2} \sin \phi_S \right] \\
+ m_4 \left[ \frac{3f_2}{4} \phi_S - \frac{3f_2}{4} \sin \phi_S \cos \phi_S \right] \text{ radians}
\]

\[L_2 = L_1 + L\]

[If \(|L_3| > 180^\circ\), modify \(L_3\) by adding or subtracting 360\(^\circ\), according to whether it is initially negative or positive.]

\[
\sin \beta_0 = \sin \beta_1 \cos \phi_0 + g \sin \phi_0
\]

\[
\cos \beta_0 = \pm \sqrt{(\cos \beta_0)^2 + (g \cos \phi_0 - \sin \beta_1 \sin \phi_0)^2}
\]

Compute the radical entirely by floating decimals to prevent loss of digits, especially for large absolute latitudes.

\[
\tan \beta_0 = (\sin \beta_2 \div \cos \beta_2)
\]

Use whichever has the smaller absolute value.

Obtain \(\tan\) (or \(\cot\)) of \(\beta_2\) from earlier defined relation of \(\beta\) to \(\beta_0\).

Determine \((-90^\circ \leq \beta \leq 90^\circ\)), applying sign of its \(\tan\) (or \(\cot\)).
APPENDIX C

ALTERNATE INVERSE AND DIRECT FORMULAS
(For very short as well as long geodesics)

The following alternate formulas for corresponding ones in Appendices A and B are designed to maintain or appropriately increase the accuracy of various elements of short geodesics, without decreasing the accuracy of long geodesics. The formulas specifically take advantage of inherently small quantities and of small differences of given large quantities, so as to provide--through the application of floating point calculations--increased decimal place accuracy without requiring additional operational digits. The small angles involved are especially adaptable to electronic computers, which by means of floating point can readily obtain greater decimal accuracy inherent in trigonometric power series of such small angles.

1. FOR INVERSE SOLUTION:

\[
\sin \phi = \pm \sqrt{(\sin L \cos \beta_2)^2 + \left[ \sin(\beta_2 - \beta_1) + 2 \cos \beta_2 \sin \beta_1 \sin^2 \frac{L_2}{2} \right]^2}
\]

\[
\cot \alpha_{1,a} = \frac{[\sin(\beta_2 - \beta_1) + 2 \cos \beta_2 \sin \beta_1 \sin^2 \frac{L_1}{2}] \cos \beta_2 \sin \lambda}{[\sin(\beta_2 - \beta_1) - 2 \cos \beta_1 \sin \beta_2 \sin^2 \frac{L_1}{2}] \cos \beta_1 \sin \lambda}
\]

where

\[
(\beta_2 - \beta_1) = (B_2 - B_1) + 2 \left[ \sin(B_2 - B_1) \right] \left[ (n + n^2 + n^3) a - (n - n^2 + n^3) b \right]
\]

2. FOR DIRECT SOLUTION:

\[
B_2 = B_1 + (\beta_2 - \beta_1) + 2 \left[ \sin(\beta_2 - \beta_1) \right] \left[ (n + n^3) \cos (B_2 + B_1) + n^2 \cos(B_2 - B_1) \right]
\]

where \( \sin(\beta_2 - \beta_1) = \sin \varphi_0 \cos \alpha_{1,a} - 2 \sin^2 \frac{L}{2} \sin \beta_1 \cos \beta_2 \)

and the required approximate \( B_2 \) and \( \cos \beta_2 \) are obtained in Appendix B.
3. FOR INVERSE AND DIRECT AT GIVEN ABSOLUTE LATITUDES > 45°:

\[ \cos \beta = \sin \left( \left( 90 + B \right) \pm 2 \left[ \sin \left( 90 + B \right) \right] \left( n + n^3 + n^5 \right) \sin \beta \right) \]

the upper and lower signs of which are applied for the northern and southern hemispheres, respectively.

In the preceding three sets of formulas, \( n = (a_o - b_o) \pm (a_o + b_o) \). Some smaller coefficients of the almost negligible \( n^3 \) have been removed because they are unsymmetric, and because they become even smaller in Parts 1 and 2 for short geodesics and in Part 3 for large absolute latitudes. It should be noted that terms containing powers of \( n \) are in radians.

The accurate floating point calculations for short geodesics should be applied not only to the formulas of this appendix but, in turn, also to associated formulas in Appendices A and B, as illustrated numerically in Appendix D. The prescribed increase in decimal accuracy in the sine of a small angle, for example, can be obtained not only from the sine series, but also from trigonometric tables by taking the reciprocal of the large interpolated cosecant of the angle. However, in addition to sufficient significant digits, the table should have intervals small enough for accurate linear interpolation. Even better, of course, is a table of high decimal accuracy for the small sines themselves.
APPENDIX D

NUMERICAL ILLUSTRATIONS OF INVERSE AND DIRECT

(Geodesics of approximately 1 and 6,000 miles for each)

The two extreme test distances noted above are chosen to illustrate by calculation not only the basic computation forms of Appendices A and B but also the alternate formulas of Appendix C. The degree of consistency of the answers has been determined below by checking each Inverse solution against the corresponding Direct. The resulting discrepancies, which for each geodesic are summarized at the end of this appendix, therefore represent the combined errors of the Inverse and Direct.

<table>
<thead>
<tr>
<th>Inverse Solution</th>
<th>Long Geodesic</th>
<th>Short Geodesic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_1$</td>
<td>$+20^\circ$</td>
<td>$+45^\circ$</td>
</tr>
<tr>
<td>$L_1$</td>
<td>$0^\circ$</td>
<td>$+12^\circ 11'18&quot;$</td>
</tr>
<tr>
<td>$B_2$</td>
<td>$+45^\circ$</td>
<td>$+45^\circ 00'36&quot;.5$</td>
</tr>
<tr>
<td>$L_2$</td>
<td>$+106^\circ$</td>
<td>$+12^\circ 12'09&quot;.5$</td>
</tr>
<tr>
<td>$a_0$ (meters)</td>
<td>6 378 388.000</td>
<td>6 378 388.000</td>
</tr>
<tr>
<td>$b_0$ (meters)</td>
<td>6 356 911.946</td>
<td>6 356 911.946</td>
</tr>
<tr>
<td>$f$</td>
<td>.00 33670 03367</td>
<td>.00 33670 03367</td>
</tr>
<tr>
<td>$n$</td>
<td>- - -</td>
<td>.00 16863 40641</td>
</tr>
<tr>
<td>$L$</td>
<td>$+106^\circ$</td>
<td>$51&quot;.5$</td>
</tr>
<tr>
<td>$\tan \beta_1$</td>
<td>.36274 47453</td>
<td>.99663 29966</td>
</tr>
<tr>
<td>$\tan \beta_2$</td>
<td>.99663 29966</td>
<td>.99698 57825</td>
</tr>
<tr>
<td>$\cos \beta_1$</td>
<td>.94006 23275</td>
<td>.70820 81969</td>
</tr>
<tr>
<td>$\cos \beta_2$</td>
<td>.70829 81969</td>
<td>.70817 32700</td>
</tr>
<tr>
<td>$\sin \beta_1$</td>
<td>.34100 26695</td>
<td>.70591 33545</td>
</tr>
<tr>
<td>$\sin \beta_2$</td>
<td>.70591 33545</td>
<td>.70603 86817</td>
</tr>
<tr>
<td>Inverse Solution</td>
<td>Long Geodesic</td>
<td>Short Geodesic</td>
</tr>
<tr>
<td>-----------------</td>
<td>---------------</td>
<td>---------------</td>
</tr>
<tr>
<td>a</td>
<td>.24071 83383</td>
<td>.49840 21342</td>
</tr>
<tr>
<td>b</td>
<td>.66584 44515</td>
<td>.50159 78502</td>
</tr>
<tr>
<td>sin L</td>
<td>.96126 16959</td>
<td>.000 24967 90432</td>
</tr>
<tr>
<td>cos L</td>
<td>-.27563 73558</td>
<td>.99999 99688</td>
</tr>
<tr>
<td>cos φ</td>
<td>.05718 67343</td>
<td>.99999 99687</td>
</tr>
<tr>
<td>sin (B₂ - B₁)</td>
<td>- - - -</td>
<td>.000 17695 69927</td>
</tr>
<tr>
<td>(θ₂ - θ₁) radians</td>
<td>- - - -</td>
<td>.000 17695 60928</td>
</tr>
<tr>
<td>sin (θ₂ - θ₁)</td>
<td>- - - -</td>
<td>.000 17695 60919</td>
</tr>
<tr>
<td>sin² L/2</td>
<td>- - - -</td>
<td>.000 00001 55849</td>
</tr>
<tr>
<td>sin φ</td>
<td>.99836 34996</td>
<td>.000 25016 57049</td>
</tr>
<tr>
<td>φ (radians)</td>
<td>1.51357 83766</td>
<td>.000 25016 57075</td>
</tr>
<tr>
<td>c</td>
<td>.64109 99269</td>
<td>.50062 20631</td>
</tr>
<tr>
<td>m</td>
<td>.58899 08837</td>
<td>.74937 75499</td>
</tr>
</tbody>
</table>

(S ÷ b₀) =

\[
\begin{pmatrix}
+1.51869 & 17590 \\
+.00080 & 87665 \\
-.00156 & 22320 \\
-.00000 & 00188 \\
+.00000 & 01279 \\
+.00000 & 18468 \\
\end{pmatrix}
\begin{pmatrix}
+.000 25101 08523 \\
+.000 00042 05152 \\
-.000 00063 22699 \\
-.000 00000 03522 \\
+.000 00000 07962 \\
+.000 00000 10592 \\
\end{pmatrix}
\]

S (meters) = 9 649 412.505 1 594.307 213

(λ - L) ÷ c =

\[
\begin{pmatrix}
+.00511 & 33825 \\
-.00000 & 76243 \\
-.00000 & 16616 \\
\end{pmatrix}
\begin{pmatrix}
+.000 00004 51448 \\
+.000 00000 21203 \\
+.000 00000 00000 \\
\end{pmatrix}
\]

λ (radians) = 1.85331 48325 0.000 25010 10825

sin λ = .96035 63877 0.000 25010 10799

\[
\cos λ = -.27877 51927 \\
\sin² \frac{λ}{2} = - - - - 
\]

sin² λ/2 = .000 0001 56376
<table>
<thead>
<tr>
<th>Inverse Solution</th>
<th>Long Geodesic</th>
<th>Short Geodesic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cot \alpha_1\alpha_2$</td>
<td>1.07455 96453</td>
<td>.99919 16383</td>
</tr>
<tr>
<td>$\cot \alpha_2\alpha_1$</td>
<td>-.47245 22960</td>
<td>.99883 88553</td>
</tr>
<tr>
<td>$\alpha_1\alpha_2$</td>
<td>42°56'30&quot; 03503</td>
<td>45°01'23&quot; 40210</td>
</tr>
<tr>
<td>$\alpha_2\alpha_1$</td>
<td>295°17'18&quot; 59981</td>
<td>225°01'59&quot; 82121</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Direct Check</th>
<th>Long Geodesic</th>
<th>Short Geodesic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_1$</td>
<td>+20°</td>
<td>+45°</td>
</tr>
<tr>
<td>$L_1$</td>
<td>0°</td>
<td>+12°11'18&quot;</td>
</tr>
<tr>
<td>$\alpha_1\alpha_2$</td>
<td>42°56'30&quot; 03503</td>
<td>45°01'23&quot; 40210</td>
</tr>
<tr>
<td>$S$ (meters)</td>
<td>9 649 412.505</td>
<td>1 594.307 213</td>
</tr>
<tr>
<td>$a_0$ (meters)</td>
<td>6 378 388.000</td>
<td>6 378 388.000</td>
</tr>
<tr>
<td>$b_0$ (meters)</td>
<td>6 356 911.946</td>
<td>6 356 911.946</td>
</tr>
<tr>
<td>$f$</td>
<td>.00 33670 03367</td>
<td>.00 33670 03367</td>
</tr>
<tr>
<td>$e^{12}$</td>
<td>.00 67681 70197</td>
<td>.00 67681 70197</td>
</tr>
<tr>
<td>$n$</td>
<td></td>
<td>.00 16863 40641</td>
</tr>
<tr>
<td>$\tan \beta_1$</td>
<td>.36274 47453</td>
<td>.99663 29966</td>
</tr>
<tr>
<td>$\cos \beta_1$</td>
<td>.94006 23275</td>
<td>.70829 81969</td>
</tr>
<tr>
<td>$\sin \beta_1$</td>
<td>.34100 26695</td>
<td>.70591 33545</td>
</tr>
<tr>
<td>$\sin \alpha_1\alpha_2$</td>
<td>.68125 35334</td>
<td>.70739 26381</td>
</tr>
<tr>
<td>$\cos \alpha_1\alpha_2$</td>
<td>.73204 75552</td>
<td>.70682 08083</td>
</tr>
<tr>
<td>$\cos \beta_0$</td>
<td>.64042 07822</td>
<td>.50104 49301</td>
</tr>
<tr>
<td>$g$</td>
<td>.68817 03286</td>
<td>.50063 99041</td>
</tr>
<tr>
<td>$m_1$</td>
<td>.59009 33386</td>
<td>.25146 93699</td>
</tr>
<tr>
<td>$\phi_0$ (radians)</td>
<td>1.51794 02494</td>
<td>.000 25079 90085</td>
</tr>
<tr>
<td>Direct Check</td>
<td>Long Geodesic</td>
<td>Short Geodesic</td>
</tr>
<tr>
<td>--------------</td>
<td>--------------</td>
<td>---------------</td>
</tr>
<tr>
<td>sin $\phi_s$</td>
<td>.99860 34425</td>
<td>.000 25079 90059</td>
</tr>
<tr>
<td>cos $\phi_s$</td>
<td>.05283 14696</td>
<td>.99999 99685</td>
</tr>
<tr>
<td>$a_e$</td>
<td>.24057 82171</td>
<td>.49924 27565</td>
</tr>
<tr>
<td>$\phi_0$ (radians) =</td>
<td>[+1.51794 02494 - .00081 30002 - .00146 29306 + .00000 00374 + .00000 39825 + .00000 25543]</td>
<td>[+ .000 25079 90085 - .000 00042 37199 .000 00000 00000 + .000 00000 17897 .000 00000 00000 + .000 00000 00000]</td>
</tr>
<tr>
<td>sin $\phi_0$</td>
<td>.99848 09807</td>
<td>.000 25037 70783</td>
</tr>
<tr>
<td>cos $\phi_0$</td>
<td>.05509 74693</td>
<td>.99999 99687</td>
</tr>
<tr>
<td>$\cot \alpha_{g-1}$</td>
<td>-.47245 22450</td>
<td>.99883 88542</td>
</tr>
<tr>
<td>$\alpha_{g-1}$</td>
<td>295°17'18&quot;,.59121</td>
<td>225°01'59&quot;,.82132</td>
</tr>
<tr>
<td>$\cot \lambda = \frac{L - \lambda}{\cos \beta_0}$</td>
<td>-.29028 29979</td>
<td>tan $\lambda = .000 25010 10884</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>106°11'13&quot;,.61256</td>
<td>51&quot;.58705 146</td>
</tr>
<tr>
<td>(L - $\lambda$) + $\cos \beta_0$ =</td>
<td>[- .00511 09099 + .00000 40853 + .00000 73513]</td>
<td>[- .000 00084 44411 + .000 00000 21292 .000 00000 00000]</td>
</tr>
<tr>
<td>L (radians)</td>
<td>1.85004 89647</td>
<td>.000 24967 90471</td>
</tr>
<tr>
<td>$L_2$</td>
<td>105°59'59&quot;,.99117</td>
<td>12°12'09&quot;,.50000 027</td>
</tr>
<tr>
<td>sin $\beta_2$</td>
<td>.70591 33687</td>
<td>Preliminary Values</td>
</tr>
<tr>
<td>cos $\beta_2$</td>
<td>.70829 81829</td>
<td>\textit{Values}</td>
</tr>
<tr>
<td>tan $\beta_2$</td>
<td>.99663 30364</td>
<td>.99698 57817</td>
</tr>
<tr>
<td>tan $\beta_2$</td>
<td>1.00000 00399</td>
<td>1.00035 39769</td>
</tr>
<tr>
<td>$B_2$</td>
<td>45°00'00&quot;,.00411</td>
<td>45°00'36&quot;,.49992</td>
</tr>
<tr>
<td>Direct Check</td>
<td>Long Geodesic</td>
<td>Short Geodesic</td>
</tr>
<tr>
<td>---------------------------</td>
<td>------------------------</td>
<td>------------------------</td>
</tr>
<tr>
<td>( \sin^2 \frac{\lambda}{2} )</td>
<td>- - - -</td>
<td>.000 00001 56376</td>
</tr>
<tr>
<td>( \sin (\beta_2 - \beta_1) )</td>
<td>- - - -</td>
<td>.000 17695 60922</td>
</tr>
<tr>
<td>( (\beta_2 - \beta_1) ) radians</td>
<td>- - - -</td>
<td>.000 17695 60931</td>
</tr>
<tr>
<td>( \cos (B_2 + B_1) )</td>
<td>- - - -</td>
<td>-.0017695 69566</td>
</tr>
<tr>
<td>( \cos (B_2 - B_1) )</td>
<td>- - - -</td>
<td>.99999 99841</td>
</tr>
<tr>
<td>( B_3 ) (improved value)</td>
<td>- - - -</td>
<td>45°00'36.5000 005</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Discrepancies between Inverse and Direct</th>
<th>Long Geodesic</th>
<th>Short Geodesic</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta \xi_2 )</td>
<td>0°.00411</td>
<td>0°.00000 005</td>
</tr>
<tr>
<td>( \Delta \xi_3 )</td>
<td>0°.00863</td>
<td>0°.00000 027</td>
</tr>
<tr>
<td>( \Delta \xi_{0-1} )</td>
<td>0°.00860</td>
<td>0°.00011</td>
</tr>
</tbody>
</table>

In addition, the preceding Inverse and Direct illustrative examples contain several common intermediate and secondary components whose values can be compared. Also, since the solutions of the long geodesic are illustrated by the same numerical problem that was used in reference [1] for the earlier form of the Inverse method, opportunities for other comparisons are available. It is apparent that the extremely high positional accuracy for the short geodesic is due to the use of alternate formulas given in Appendix C. The azimuth error is consistent with this positional error, in view of the line’s shortness. Comparable accuracies are also obtainable at large absolute latitudes, but only if interpreted relative to the increasing convergence and closeness of the meridians in polar regions.
APPENDIX E

THEORETICAL FORMULAS FOR HIGHER ACCURACY

The results of the illustrative numerical examples given in Appendix D indicate that the formulas in Appendices A through C provide sufficient practical accuracy. For theoretical purposes, however, the formulas could be extended through $f^3$ and $e^4$ terms or beyond. The outer coefficients of the formula for $(S \div b \omega)$ in Appendix A would then include, for example, the higher order combinations $a^3, m^3, a^5 m, \text{and } a m^4$. Similar orderly extensions should be expected for the $(\lambda - L) \div c$ formula in Appendix A and the $\phi_0$ and $(L - \lambda) \div \cos \beta_0$ in Appendix B, except that in the case of the latter two their outer coefficients will bear the subscript $1$, and their components $a_1$ and $m_1$ would have to be properly defined to higher powers of $e^4$. If necessary, appropriate formulas in Appendix C can also be extended.

In the present appendix, only the $(\lambda - L) \div c$ power series of Appendix A will be given to the next higher order terms, since it provides a non-iterative rigorous solution for the quantity $\lambda$ which is required in most of the classical methods for calculating the Inverse of long geodesics. The unique form of the extended $(\lambda - L) \div c$ power series given below has been derived from the top of page 18 of [1], by substitution in terms of $a, m, \phi, \text{and } f$. The series is followed by accurate Inverse distance and azimuth formulas taken in large part from pages 14 and 15 of reference [1]. The resulting method of solution can be used for precise computation of Inverse problems, especially as a theoretical check on Direct or other Inverse formulas.

\[
\frac{\lambda - L}{c} = \left[(f + f^2 + f^3) \phi\right] + a \left[-\left(\frac{f^2}{2} + f^3\right) \sin \phi - (f^2 + 4f^3) \phi^2 \csc \phi \right] + \left(\frac{3f^3}{2}\right) \phi^3 \csc \phi \cot \phi \right] + m \left[-\left(\frac{5f^2}{4} + 3f^3\right) \phi + \left(\frac{f^2}{4} + \frac{f^3}{2}\right) \sin \psi \cos \psi \right] + (f^2 + 4f^3) \phi^2 \cot \phi - \left(\frac{f^3}{2}\right) \phi^3 \csc^2 \phi - (f^2) \phi^3 \cot^2 \phi \right] + a^2 \left[(f^3) \phi + \left(\frac{f^3}{2}\right) \sin \phi \cos \phi + (f^3) \phi^3 \csc^2 \phi \right]
\]
\[ m^3 \left[ \left( \frac{3f^3}{16} \right) \phi - \left( \frac{g^3}{16} \right) \sin \phi \cos \phi + \left( \frac{f^3}{2} \right) \phi \cos^2 \phi \right. \]

\[ - \left( \frac{g^3}{2} \right) \phi^2 \cot \phi + \left( \frac{f^3}{8} \right) \sin \phi \cos^3 \phi \]

\[ + \left( \frac{f^3}{2} \right) \phi^3 \csc^3 \phi + (2f^3) \phi^2 \cot^2 \phi \]

\[ + \left( \frac{f^3}{2} \right) \phi^3 \csc^3 \phi - \left( \frac{7f^3}{2} \right) \phi^3 \csc \phi \cot \phi \] radians

where the component quantities are again defined in Appendix A, while some alternate definitions are found in Appendix C.

Next, \( \phi_0 \) is obtained in the same manner as \( \phi \), except that the value of \( \lambda \) obtained from above is now to be used in place of \( L \). Then continue as follows:

\[ \cos \beta_0 = (b \sin \lambda) / \sin \phi_0; \cos 2\sigma = (2a + \sin^2 \beta_0) - \cos \phi_0. \]

\[ A_0 = 1 + \frac{e^{12}}{4} \sin^2 \beta_0 - \frac{3e^{14}}{64} \sin^4 \beta_0 + \frac{5e^{16}}{256} \sin^6 \beta_0 \]

\[ B_0 = \frac{e^{12}}{4} \sin^2 \beta_0 - \frac{e^{14}}{16} \sin^4 \beta_0 + \frac{15e^{16}}{512} \sin^6 \beta_0 \]

\[ C_0 = \frac{e^{14}}{128} \sin^4 \beta_0 - \frac{3e^{16}}{512} \sin^6 \beta_0 \]

\[ D_0 = \frac{e^{16}}{1536} \sin^8 \beta_0 \]

\[ S = b_0 \left( A_0 \phi_0 + B_0 \sin \phi_0 \cos 2\sigma - C_0 \sin 2\phi_0 \cos 4\sigma \right. \]

\[ + D_0 \sin 3\phi_0 \cos 6\sigma \]

To complement the above geodetic distance, \( S \), the azimuths \( a_1 - \sigma \) and \( a_2 - \sigma \) are obtained from formulas given in Appendix A or C.
APPENDIX F

INTER-RELATIONS OF THE TERMS OF THE POWER SERIES

As noted earlier, the coefficients \( a \) and \( m \) in the \( (S + b_o) \) and \( (\lambda - L) + c \) Inverse power series in Appendices A and E display a unique set of product combinations. The identical simple pattern is also repeated in the two Direct power series in Appendix B, except that it occurs instead with the subscripted \( a_i \) and \( m_i \). Although not shown in this paper, even the higher degree combinations (such as \( a^3m \), \( m^2a \), \( a^3 \), and \( m^3 \)) appear to enter in orderly fashion in the further extension of the power series. It is of significant importance that the \( a \) and \( m \) (or \( a_i \) and \( m_i \)) combinations are completely factorable from the power series terms, since this permits the latter to be tabulated as a function of only the variable \( \phi \) or \( \phi_0 \), and the parameter \( f \) or \( e'^2 \). Electronic computer programming and calculations also become simpler, whether for producing just the table or for calculating the entire Inverse or Direct.

Another interesting inter-relation of the terms of the series concerns the numerical coefficients of the powers of \( f \) and \( e'^2 \). It should be noted, for example, that in Appendix B the numerical coefficients related to the \( m^2 \) terms of the \( \phi_0 \) power series are:

\[
\begin{align*}
&11 \quad 13 \quad 1 \quad 5 \\
&\frac{64}{64} \quad \frac{64}{64} \quad \frac{8}{8} \quad \frac{32}{32}.
\end{align*}
\]

The total of the above four numbers is found to be exactly zero. Upon closer inspection, it is found from the power series in Appendices A, B, and E that the zero sum occurs with all sets of terms having \( m \) or \( m_i \) as one of the factors, even for the \( (S + b_o) \) series in Appendix A, if it is modified as shown later. When different powers of \( f \) are present, the sum is zero separately for the numerical coefficients of the \( f \) terms, \( f^2 \) terms, and so forth, such as in the \( (\lambda - L) + c \) series in Appendix E. In all instances described, the sum is zero by virtue of the fact that each term—which is a function of \( \phi \) (or \( \phi_0 \))—is first put into a form which satisfies the following condition: The algebraic sum of the exponents of \( \phi \) and \( \sin \phi \) (after all trigonometric functions of \( \phi \) are converted to sines and cosines) is unity. Actually, the above condition can be (and has been) satisfied even for the non-\( m \) and the non-\( m_i \) series terms. For very short geodesies (which of course have a small arc value \( \phi \) and, therefore, \( \sin \phi \) approaches \( \phi \) and \( \cos \phi \) approaches unity), the resulting unity exponent implies that every term is of the small order of \( \phi \), times its numerical coefficient and the proper power of \( f \) or \( e'^2 \). Since even the omitted terms of the series contain that small order of \( \phi \) (or \( \phi_0 \)), the power series converge to a greater number of decimals for short geodesics. This is shown by the much better positional consistency obtained from the numerical example.
for the short geodesic in Appendix D. For terms which have \( m \) or \( m_1 \) as one of the coefficients, the convergency for short geodesics is even greater because (as noted above) the sum of the numerical coefficients is zero separately for each power of \( f \) or \( e^{i\phi} \), and \( \phi \) or \( \phi_S \) is practically a common factor.

As for the \((S + b_0)\) series mentioned in the preceding paragraph, the expression given in Appendix A can be reduced to the following form:

\[
\frac{S}{b_0} = [(1 + f + f^2) \phi]
\]

\[
+ (m \cos \phi - a) \left[ - (f + f^2) \sin \phi + \left( \frac{f^2}{2} \right) \cos \phi \csc \phi \right]
\]

\[
+ m \left[ - (f + f^2) \phi + \left( \frac{f + f^2}{2} \right) \sin \phi \cos \phi \right]
\]

\[
+ (m \cos \phi - a)^2 \left[ - \left( \frac{f^2}{2} \right) \sin \phi \cos \phi \right]
\]

\[
+ m^2 \left[ \left( \frac{f^2}{16} \right) \phi + \left( \frac{f^2}{16} \right) \sin \phi \cos \phi - \left( \frac{f^2}{8} \right) \sin \phi \cos^3 \phi \right]
\]

\[
+ m(m \cos \phi - a) \left[ \left( \frac{f^2}{2} \right) \sin \phi \cos^2 \phi - \left( \frac{f^2}{2} \right) \phi^2 \csc \phi \right]
\]

The compound coefficient \((m \cos \phi - a)\) is an expression which appeared extensively in the course of the original derivation of the Inverse solution. As used above, it causes the numerical coefficients of the terms with the factor \( m \) to add to zero, just like the other power series. It is interesting to note that the next higher order extension of \((S + b_0)\) continues to give the proper zero sum for the numerical coefficients of applicable terms, when the additional prescribed product combinations of the same \((m \cos \phi - a)\) and \( m \) are used.

In conclusion, it is worth noting that, of the four main power series given in Appendices A and B, only \((S + b_0)\) does not lend itself to completely factoring out the ellipsoidal parameter from each series of terms. The capability of factoring for all four power series (at least to the extent of the number of terms given) may be important. It would mean, for example, that the total value of each series of terms could be tabulated independently of any specific spheroid flattening or eccentricity. (Of course, the parameters would then be made a part of the external coefficients instead.) In the \((S + b_0)\) formula given in the present Appendix, only the terms whose coefficient is \((m \cos \phi - a)\) do not lend themselves to factoring out the function of flattening. Those terms, however, can be represented as in the following:
\[
\left[ (m \cos \phi - a) \left( 1 - \frac{f^2}{2 \sin^2 \phi} \right) \right] \left[ - (f + f^3) \sin \phi \right]
\]

where the unwanted portion of flattening has been transferred to the external coefficient. This new compound coefficient may be used in place of the previous \((m \cos \phi - a)\) throughout the \((S \div b_0)\) expression for consistency, since the extraneous \(f^3\) terms which are introduced are negligible.
APPENDIX G

MERIDIONAL ARC AS SPECIAL CASE OF NON-ITERATIVE INVERSE AND DIRECT

An interesting indication of the simplicity and rapid convergence of the non-iterative inverse is to reduce it to the special case of meridional arc distances, for northern latitudes up to 90° from the equator. Since \( \beta_1 \) and \( L \) are then 0°, the following result:

\[
a = 0, \quad m = 1, \quad \varphi = \beta_2 \text{ radians.}
\]

Therefore, such meridional distances, \( S_M \), become:

\[
S_M = b_0 \left[ \left( 1 + \frac{f}{2} + \frac{9f^2}{16} \right) \beta_2 - \left( \frac{f}{2} + \frac{7f^2}{16} \right) \sin \beta_2 \cos \beta_2
\]
\[
- \left( \frac{f^2}{8} \right) \sin \beta_2 \cos^3 \beta_2 \right].
\]

Similarly, \( \beta_2 \) can be derived for the corresponding \( S_M \) by letting \( \beta_1 \) and \( \alpha_1 = \beta_2 \) equal 0° in the Direct solution, whence:

\[
a_1 = 0, \quad m_1 = 1, \quad \beta_2 = \phi_0.
\]

Then by substitution into the \( \phi_0 \) power series, there results:

\[
\beta_2 = \left( 1 - \frac{e^{1/2}}{4} + \frac{11e^{14}}{64} \right) \phi_M + \left( \frac{e^{1/2}}{4} - \frac{13e^{14}}{64} \right) \sin \phi_M \cos \phi_M
\]
\[
+ \left( \frac{5e^{14}}{32} \right) \sin \phi_M \cos^3 \phi_M - \left( \frac{e^{14}}{8} \right) \phi_M \cos^3 \phi_M \text{ radians,}
\]

where \( \phi_M = \left( S_M + b_0 \right) \text{ radians.} \)

As a check (the complete details of which need not be shown), the above Inverse and Direct meridional arc solutions were compared mathematically and found to be fully consistent with each other. Essentially, dividing the Inverse meridional formula by \( b_0 \) produced \( \phi_M \) as a function of \( \beta_2 \), from which \( \sin \phi_M \) and \( \cos \phi_M \) were then obtained by expanding in series around \( \sin \beta_2 \) and \( \cos \beta_2 \), respectively. Substitution into the Direct meridional formula finally made the right side identically equal to the left side's \( \beta_2 \), up through all \( e^{1/2} \) terms.
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>IV</td>
<td>SUMMARY</td>
<td>iv</td>
</tr>
<tr>
<td>I</td>
<td>INTRODUCTORY BACKGROUND</td>
<td>1</td>
</tr>
<tr>
<td>II</td>
<td>THEORETICAL REFINEMENT OF NON-ITERATIVE INVERSE</td>
<td>1</td>
</tr>
<tr>
<td>III</td>
<td>DEVELOPMENT OF CORRESPONDING DIRECT</td>
<td>1</td>
</tr>
<tr>
<td>IV</td>
<td>NOTES ON COMPUTATION FORMS</td>
<td>2</td>
</tr>
<tr>
<td>V</td>
<td>FORMULAS FOR VERY SHORT AND LONG GEODESICS</td>
<td>2</td>
</tr>
<tr>
<td>VI</td>
<td>CONCLUDING REMARKS</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>BIBLIOGRAPHY</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>APPENDICES</td>
<td>6</td>
</tr>
</tbody>
</table>
SUMMARY

Improved practical and theoretical formulas are presented for the calculation of geodetic distances, azimuths, and positions on spheroid. The formulas are designed for use with either electronic computers or desk calculators. For the latter, the formulas lend themselves to the construction of useful interpolation tables.

The report includes convenient computation forms and auxiliary equations which assure a high degree of accuracy for any geodetic line, no matter how short or long (up to half or fully around the earth) and regardless of its orientation or location. Numerical examples illustrate the complete calculation procedure.
<table>
<thead>
<tr>
<th>UNCLASSIFIED</th>
<th>UNCLASSIFIED</th>
<th>UNCLASSIFIED</th>
</tr>
</thead>
</table>

The formulas designed for use with either electronic computers or desk calculators were examined. The formulas and themselves are the construction of useful interpolation tables. The report includes computed computation forms and auxiliary equations which serve a high degree of accuracy for any geodetic line, no matter how short or long (up to half or fully around the earth) and regardless of its direction or location. Examples illustrated the complete calculation procedures.
Improved practical and theoretical formulas are presented for the calculation of geodetic distances, azimuths, and positions on a spheroid. The formulas are designed for use with either electronic computers or hand calculators. In this latter case, the formulas lend themselves to the construction of useful interpolation tables. The report includes convenient computation forms and auxiliary equations which are a high degree of accuracy for any geodetic line, no matter how short or long, or half or fully around the earth and regardless of its orientation or location. Numerical examples illustrate the complete calculation procedure.