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A PROOF OF FOUR-COLORING THE EDGES
OF A REGULAR THREE-DEGREE GRAPH

by

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A graph \( G \) consists of a finite set \( V \) of points called vertices together with a set \( E \) of unordered pairs of vertices called edges. Note that this definition does not permit multiple edges or loops. A subgraph \( G' \) of a graph \( G \) has vertices \( V' \subseteq V \) and edges \( E' \subseteq E \), where \([v_1, v_2] \in E'\) only for \( v_1 \in V' \) and \( v_2 \in V' \). Coloring the edges of a graph means assigning to each edge a color so that no vertex is incident to two edges of the same color. A simple path is a finite sequence of distinct vertices \( v_1, \ldots, v_n \) such that \([v_i, v_{i+1}] \in E\) for \( i = 1, \ldots, n - 1 \). A connected graph has a simple path between every pair of vertices, and a graph not connected consists of connected components. In a connected graph, a bridge is an edge such that its removal from \( E \) causes \( G \) to not be connected. The degree of a vertex is the number of edges incident to that vertex, and a regular graph has the same degree at each vertex.

Coloring the edges of a regular 3-degree graph is of special interest because the 4-color conjecture is equivalent [4] to 3-coloring the edges of a planar, regular 3-degree graph which is connected and without bridges. Edges of a connected, regular 3-degree graph can easily be colored with 5 colors and can be colored with 4 colors provided there is no bridge [1]. Theorem 2 states that the edges of any regular, 3-degree graph can be colored with 4 colors. This result also follows directly from a result on
coloring the nodes of a graph [2], but the method used here is of interest because it gives an algorithm whose length depends linearly on the size of the graph, and it is used to characterize the regular 3-degree graphs in an inductive manner as given in Theorem 1. Theorem 2 does not depend on Theorem 1. The method is similar to "splits" first used by Frink [3].

A regular 3-degree graph having \( n \) vertices must have \( \frac{3}{2} n \) edges, and hence \( n \) must be even. There is one such graph having 4 vertices and two having 6 vertices. They are shown below with a 3-coloring of the edges:

![Graphs with 3-coloring](image)

**FIGURE 1**

An H-tree is defined as a graph with 5 edges and 6 vertices having degrees 3, 3, 1, 1, 1, and 1.

**LEMMA 1.** Any connected, regular graph of degree 3 with \( n \) vertices, \( n \geq 6 \), has an H-tree as a subgraph.

**PROOF:** For any vertex \( v_0 \), there are three distinct edges incident to \( v_0 \), say \( e_1 = [v_1, v_0] \), \( e_2 = [v_0, v_2] \), \( e_3 = [v_0, v_3] \). Consider the edges with both ends in \( \{v_1, v_2, v_3\} \).
There cannot be 3 such edges, because there can be no connected component with 4 vertices. Consider the case of 2 such edges, say $e_4 = [v_1, v_2]$ and $e_5 = [v_2, v_3]$ (Figure 2(a)). Then $v_1$ is incident to another edge $e_6 = [v_1, v_4]$, $v_4$ different from $v_0$, $v_1$, $v_2$, and $v_3$. There are two more edges incident to $v_4$, $e_7 = [v_4, v_5]$ and $e_8 = [v_4, v_6]$. Furthermore $v_5$ and $v_6$ are different from $v_0$, $v_1$, $v_2$, because they already have degree 3. Hence $e_1$, $e_4$, $e_6$, $e_7$, and $e_8$ constitute an H-tree.

Consider the case of exactly one edge, say $e_4 = [v_2, v_3]$, with both ends in $\{v_1, v_2, v_3\}$ (Figure 2(b)). Then $v_1$ is incident to two other edges, $e_5 = [v_1, v_4]$ and $e_6 = [v_1, v_5]$. Furthermore, $v_4$ and $v_5$ are different from $v_0$, $v_1$, $v_2$, and $v_3$ because there is no edge from $v_1$ to $v_2$ or $v_3$. Hence $e_1$, $e_2$, $e_3$, $e_5$, $e_6$ form an H-tree. In case there is no edge with both ends in $\{v_1, v_2, v_3\}$, $v_1$ is incident to two other edges, $e_4 = [v_1, v_4]$ and $e_5 = [v_1, v_5]$, and then $e_1$, $e_2$, $e_3$, $e_4$, $e_5$ form an H-tree (Figure 2(c)). Thus Lemma 1 is proven.

![Figure 2](image_url)
Consider an H-tree in $G$ with vertices $v_0, v_1, v_2, v_3, v_4, v_5$ and edges $e_1 = [v_0, v_1], e_2 = [v_0, v_2], e_3 = [v_0, v_3], e_4 = [v_1, v_4], e_5 = [v_1, v_5]$ (Figure 2(c)). The H-tree is called acceptable if no two edges of $G$ with one end in $\{v_2, v_3\}$ and the other end in $\{v_4, v_5\}$ are incident to the same vertex.

**Lemma 2.** Any connected regular graph $G$ of degree 3 on $n$ vertices, $n \geq 8$, has an acceptable H-tree as a subgraph.

**Proof:** From Lemma 1 there is an H-tree in $G$. Denote the vertices and edges as in Figure 2(c) and consider the edges with one end in $\{v_2, v_3\}$ and the other in $\{v_4, v_5\}$. There cannot be 4 such edges because then there would be a connected component on 6 vertices and $n \geq 8$ for $G$.

Consider first the case of 3 such edges, say $[v_2, v_4]$ is not an edge of $G$ (Figure 3(a)). Let $e_6 = [v_2, v_4]$ and let $e_7 = [v_2, v_5]$ be the other edge incident to $v_2$. Let $e_8 = [v_6, v_7]$ and $e_9 = [v_5, v_6]$ be the other two edges incident to $v_6$. Then $v_7$ and $v_8$ are different from $v_0, v_1, v_2, v_3, v_5$, because they already have degree 3. Then $e_2, e_6, e_7, e_8, e_9$ form an acceptable H-tree, because all the edges with one end in $\{v_0, v_2\}$ have the other end in $\{v_1, v_2, v_3\}$.

Consider now the case (Figure 3(b)) of 2 such edges and suppose they are incident to the same vertex, say $[v_3, v_4], [v_3, v_5]$. If they are not incident to the same vertex, or if there is only one such edge, then the H-tree is already acceptable. Now $v_2$ is incident to two other edges $e_6 = [v_2, v_6], e_7 = [v_2, v_7]$, where $v_6$ and $v_7$ are different from $v_0, v_1, v_2, v_3, v_4$. 

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and \( v_5 \). Then \( e_1, e_2, e_3, e_6, e_7 \) form an acceptable H-tree because all the edges with one end in \( \{v_1, v_3\} \) have the other in \( \{v_0, v_4, v_5\} \).

Let an acceptable H-tree in \( G \) be denoted as in the definition and Figure 2(c). A new graph \( G' \), called the H-reduced graph, can be formed by deleting \( v_0 \) and \( v_1 \) from \( V \) and \( e_1, e_2, e_3, e_4, e_5 \) from \( E \), and adjoining either \( e_1' = [v_2, v_4] \) and \( e_2' = [v_3, v_5] \) or \( e_1'' = [v_2, v_4] \) and \( e_2'' = [v_3, v_5] \) to \( E \) depending on which pair is not in \( E \). In the proofs of theorems 1 and 2, an acceptable H-tree in \( G \) will be assumed to be denoted so that \( e_1' \) and \( e_2' \) are adjoined to form \( G' \).

For any graph \( G \) having two edges \( e_1 = [v_2, v_4] \), \( e_2 = [v_3, v_5] \) with \( v_2, v_3, v_4, v_5 \) distinct vertices, a new graph \( G' \), called the H-enlarged graph can be found by deleting \( e_1 \) and \( e_2 \) from \( E \), and adjoining \( v_0' \) and \( v_1' \) to \( V \) and either the five edges \( [v_0', v_1'], [v_0', v_2], [v_0', v_3], [v_1', v_4] \) and \( [v_1', v_5] \) or the five edges \( [v_0', v_1'], [v_0', v_2], [v_0', v_3], [v_0', v_5], [v_1', v_3], [v_1', v_4] \). Either enlargement can always be done provided \( v_2, v_3, v_4, v_5 \).
$v_4$, $v_5$ are distinct vertices.

**Theorem 1.** For $n \geq 6$, every connected regular 3-degree graph $G$ on $n + 2$ vertices is an $H$-enlargement of a connected, regular 3-degree graph $G'$ on $n$ vertices.

**Proof:** From lemma 2, there is an acceptable $H$-tree in $G$. Clearly the reduced graph $G'$ is also a regular 3-degree graph. Suppose $G'$ is not connected. Then there are two vertices $v$ and $v'$ of $G'$ with no simple path between them. By $G$ connected, there is such a simple path in $G$. The path must include $v_0$ or $v_1$ because otherwise it is a path in $G'$.

**Case 1.** Suppose it includes only $v_0$ (Figure 4(a)). Then there is no simple path from $v_2$ to $v_3$ in $G$ that does not include $v_0$ or $v_1$ because if there were, we could use it in place of $v_2$, $v_0$, $v_3$ and find a simple path from $v$ to $v'$ in $G'$. Hence, there is no simple path in $G$ from $v_2$ to $v_5$, or from $v_3$ to $v_4$, or from $v_4$ to $v_5$ that does not include $v_0$ or $v_1$.

**Case 2.** Suppose now the path includes $v_0$ and $v_1$, say $v_0$ appears first (Figure 4(b)). If $v_0$ and $v_1$ are not adjacent in the path, we can omit the vertices in between to obtain another simple path from $v$ to $v'$ in $G$ with $v_0$ and $v_1$ adjacent. Let $v_2$ be the vertex before $v_0$. Then if the path has $v_2$, $v_0$, $v_1$, $v_4$, we can form a path in $G'$ by using $v_2$, $v_4$ in place of $v_2$, $v_0$, $v_1$, $v_4$. Hence the path has $v_2$, $v_0$, $v_1$, $v_5$. Now, just as in case 1, there is no simple path...
path in $G$ from $v_2$ to $v_5$ that does not include $v_0$ or $v_1$, and hence there is none from $v_2$ to $v_3$, or from $v_3$ to $v_4$, or from $v_4$ to $v_5$ that does not include $v_0$ or $v_1$.

Hence, in either case $[v_2, v_5]$ and $[v_3, v_4]$ are not edges of $G$. Consider the $H$-reduction in $G$ using the same $H$-tree and adjoining the edges $[v_2, v_5]$ and $[v_3, v_4]$ instead of $[v_2, v_4]$ and $[v_3, v_5]$. This $H$-reduction is the same as the preceding one with $v_4$ and $v_5$ interchanged. Hence, if this $H$-reduced graph is also not connected, then there is no simple path in $G$ from $v_2$ to $v_4$ or from $v_3$ to $v_5$ not including $v_0$ or $v_1$.

Therefore, if neither $H$-reduced graph is connected, then there is no simple path in $G$ from $v_2$ to $v_3$, $v_4$, or $v_5$ not including $v_0$ and $v_1$.

![Diagram](image)

Consider, then, the other two edges at $v_2$ in $G$, say $e_6 = [v_2, v_6]$ and $e_7 = [v_2, v_7]$. By the above, $v_6$ and $v_7$ are
different from \( v_0, v_1, v_2, v_3, v_4 \), and \( v_5 \), and there are no edges from \( v_6 \) or \( v_7 \) to \( v_0, v_1, v_3, v_4 \), or \( v_5 \). Hence, \( e_1, e_2, e_3, e_6, e_7 \) form an acceptable H-tree (Figure 4(c)). If neither of its reduced graphs is connected, we can continue along the other two edges of \( v_6 \). In this way we can continue as long as no connected reduced graph is found, and at each step two new vertices will be found because at every step there is no simple path from the new vertices to any of the old ones except using edges already encountered. But the vertices of \( G \) are finite in number, so eventually a connected reduced graph \( G' \) is found.

**THEOREM 2.** The edges of any regular 3-degree graph \( G \) can be colored with 4 colors.

**PROOF:** The theorem is true for \( G \) having 4 or 6 vertices (Figure 1). Suppose the theorem is false. Then there is a smallest graph \( G \) for which it fails. \( G \) is connected because if not, one of its components, which has fewer vertices than \( G \), cannot be 4-colored. \( G \) has 8 or more vertices, so by lemma 3 has an acceptable H-tree. Note that \( G' \) need not be connected, so only methods of lemmas 1 and 2 need be used to find an acceptable H-tree. The edges of the reduced graph \( G' \) can be 4-colored. Denote the colors by \( \alpha, \beta, \gamma, \delta \).

Suppose \( e_1' = [v_2, v_4] \) and \( e_2' = [v_3, v_5] \) are different colors, say \( \alpha \) and \( \beta \) (Figure 5(a)), in the coloring of \( G' \). Then in \( G \) color all the edges except \( e_1, e_2, e_3, e_4, e_5 \) the same color as in \( G' \). Color \( e_2 \) and \( e_4 \) \( \alpha \), and \( e_3 \) and \( e_5 \) \( \beta \). Color \( e_1 \) \( \gamma \).

Suppose \( e_1' \) and \( e_2' \) are colored the same color, say \( \alpha \).
Then color $e_2$ and $e_4$ $\alpha$. Now color $e_3$, the color not incident to $v_3$, and color $e_5$ the color not incident to $v_5$. Then $e_1$ can be colored the fourth color not used to color $e_2$, $e_3$, $e_4$, $e_5$ (Figure 5(b)).

![Diagram](a) and (b)

**FIGURE 5**

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