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DERIVATION OF EQUATIONS FOR CONVERTING FROM GEODETIC COORDINATES TO GEOCENTRIC COORDINATES

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DERIVATION OF EQUATIONS FOR CONVERTING FROM GEODETIC
COORDINATES TO GEOCENTRIC COORDINATES

F. T. Heuring

In the A.P.L. orbit computation programs, the TRANET Tracking
Stations are specified in a geocentric coordinate system, whereas,
particular positions (such as a TRANET Tracking Site) over the Earth are
expressed initially in a geodetic coordinate system. In order to acquire
geocentric coordinates from a given set of geodetic coordinates a set of
transformation equations were derived.

Section I will define the notation, and Section II will embody
the derivation of the transformation equations.

I. Notation*

Let:

$\phi_i = \text{geodetic latitude of } i\text{-th tracking site in its local datum},$

$\lambda_i = \text{geodetic longitude of } i\text{-th tracking site in its local datum},$

$h_i = \text{elevation of } i\text{-th tracking site above (below) geoid},$

$H_i = \text{geoidal height of } i\text{-th tracking site in its local datum},$

$\xi_i = \text{deflection in meridian at } i\text{-th tracking site},$

$\eta_i = \text{deflection in prime vertical at } i\text{-th tracking site},$

$a_i = \text{equatorial radius of the local datum spheroid of the } i\text{-th}
\text{tracking site, scaled by } R_0,$

$b_i = \text{polar radius of the local datum spheroid on the } i\text{-th tracking}
\text{site, scaled by } R_0.$

*See References 1 and 2 for definition of geodetic, datum, etc.
\[ x_{G_i}', y_{G_i}', z_{G_i}' = \text{cartesian coordinates, scaled by } R_0, \text{ on } i-th \text{ datum spheroid as specified by tracking site } \varphi_G' \text{ and } \lambda_G', (\text{cartesian origin identical to } i-th \text{ datum origin}), \]

\[ x_{H_i}', y_{H_i}', z_{H_i}' = \text{cartesian coordinates, scaled by } R_0, \text{ on geoid as specified by tracking site } H, (\text{cartesian origin identical to } i-th \text{ datum origin}), \]

\[ x_{E_i}', y_{E_i}', z_{E_i}' = \text{cartesian coordinates, scaled by } R_0, \text{ of tracking site on earth's surface, (cartesian origin identical to } i-th \text{ datum origin}), \]

\[ \Delta x_i, \Delta y_i, \Delta z_i = \text{center of spheroid of the } i-th \text{ tracking site datum in the A.P.L. Datum, scaled by } R_0, \]

\[ \zeta_{G_i} = (x_{G_i}'^2 + y_{G_i}'^2)^{\frac{1}{2}} \]

\[ R_0 = \text{equatorial radius of A.P.L. Datum spheroid,} \]

\[ x_{c_i}', y_{c_i}', z_{c_i}' = \text{cartesian coordinates of tracking site in A.P.L. geocentric coordinates, scaled by } R_0, \]

\[ r_{c_i} = \text{radius of } i-th \text{ tracking site in A.P.L. geocentric coordinates, scaled by } R_0, \]

\[ \varphi_{c_i} = \text{latitude of } i-th \text{ tracking site in A.P.L. geocentric coordinates,} \]

\[ \lambda_{c_i} = \text{longitude of } i-th \text{ tracking site in A.P.L. geocentric coordinates,} \]

\[ \zeta_{c_i} = (x_{c_i}'^2 + y_{c_i}'^2)^{\frac{1}{2}}. \]

II. Derivation

A. Given \( \varphi_{G_i}, \lambda_{G_i}, a_i \) and \( b_i \), conversion to \( x_{G_i}', y_{G_i}', z_{G_i}' \) and \( \zeta_{G_i} \) is as follows. Using the equation for an ellipse

\[ \frac{\zeta_{G_i}^2}{a_i^2} + \frac{z_{G_i}^2}{b_i^2} = 1, \]
in particular the ellipse is a meridional plane of the i-th datum; differentiate $z_{G_1}$ with respect to $\zeta_{G_1}$

$$\frac{\partial z_{G_1}}{\partial \zeta_{G_1}} = -\frac{b_1^2}{a_1^2} \frac{\zeta_{G_1}}{z_{G_1}}.$$ 

But (see Figure 1A),

$$\frac{\partial z_{G_1}}{\partial \zeta_{G_1}} = -\frac{1}{\tan \varphi_{G_1}}$$

from which by algebraic manipulation (Figure 1B),

$$\zeta_{G_1} = \frac{a_1}{(1 + (\frac{b_1}{a_1})^2 \tan^2 \varphi_{G_1})^{\frac{1}{2}}},$$

after which,

$$\begin{align*}
x_{G_1} &= \zeta_{G_1} \cos \lambda_{G_1} \\
y_{G_1} &= \zeta_{G_1} \sin \lambda_{G_1} \\
z_{G_1} &= \zeta_{G_1} \frac{b_1^2}{a_1^2} \tan \varphi_{G_1}.
\end{align*}$$
Figure 1A Meridian Plane in i-th Datum

\[
\frac{1}{\tan \phi_{G_i}} = \cot \phi_{G_i} = \tan \epsilon = \frac{\partial z_{G_i}}{\partial \zeta_{G_i}}
\]
Figure 1B
Pictorial view of geodetic \((\phi_{G_i}, \lambda_{G_i})\), cartesian "geodetic" \((x_{G_i}, y_{G_i}, z_{G_i})\) and cartesian "geoidal" \((x_{H_i}, y_{H_i}, z_{H_i})\) coordinates.
B. Compute $x_{H_i}$, $y_{H_i}$, $z_{H_i}$ (Figure 1B). $H_i$ is an extension of the normal to the spheroid, consequently,

$$
\begin{align*}
  x_{H_i} &= x_{G_i} + H_i \cos \varphi_{G_i} \cos \lambda_{G_i} \\
  y_{H_i} &= y_{G_i} + H_i \cos \varphi_{G_i} \sin \lambda_{G_i} \\
  z_{H_i} &= z_{G_i} + H_i \sin \varphi_{G_i}.
\end{align*}
$$

C. Compute $x_{E_i}$, $y_{E_i}$, $z_{E_i}$ by considering $h_i$, $\xi_i$ and $\eta_i$ (see Figure 2).

$$
\begin{align*}
  x_{E_i} &= x_{H_i} + h_i \cos (\varphi_{G_i} + \xi_i) \cos (\lambda_{G_i} + \Delta\lambda_i) \\
  y_{E_i} &= y_{H_i} + h_i \cos (\varphi_{G_i} + \xi_i) \sin (\lambda_{G_i} + \Delta\lambda_i) \\
  z_{E_i} &= z_{H_i} + h_i \sin (\varphi_{G_i} + \xi_i).
\end{align*}
$$

From law of cosines for spherical triangles (Figure 2), $\Delta\lambda_i$ can be approximated.

$$
\cos \eta_i = \sin^2 (\varphi_{G_i} + \xi_i) \cos \Delta\lambda_i = \frac{\cos \eta_i - \sin^2 (\varphi_{G_i} + \xi_i)}{\cos^2 (\varphi_{G_i} + \xi_i)} \quad (4)
$$

(Restrict $\Delta\lambda_i$ to have the same sign as $\eta_i$.)
Figure 2
Diagram of the deflections of the vertical ($\xi_i$ and $\eta_i$) and the associated quantities necessary to acquire the cartesian coordinates on the geoid from earth surface cartesian coordinates.
D. Let us simplify by expanding small quantities. Assume:

\[ \xi_1, \eta_1 \leq 30'' \text{ (of arc)}; \]

and

\[ 1^\circ < |\varphi_G| < 89^\circ; \]

and only take quantities of magnitude \( \xi_1, \eta_1 \) and \( \Delta \lambda \) to second order.

\[ \cos \xi_1 = 1 - \frac{\xi_1^2}{2}, \sin \xi_1 = \xi_1 \]

\[ \cos \eta_1 = 1 - \frac{\eta_1^2}{2}, \sin \eta_1 = \eta_1 \]

\[ \cos \Delta \lambda = 1 - \frac{\Delta \lambda}{2} \]

thus,

\[ \cos^2 (\varphi_G + \xi_1) = \left[ \cos \varphi_G (1 - \frac{\xi_1^2}{2}) - \xi_1 \sin \varphi_G \right]^2 \]

\[ = (1 - \frac{\xi_1^2}{2})^2 \cos^2 \varphi_G + \xi_1^2 \sin^2 \varphi_G \]

\[ - 2 \xi_1 (1 - \frac{\xi_1^2}{2}) \sin \varphi_G \cos \varphi_G \]

\[ = \cos^2 \varphi_G - \xi_1 \sin 2\varphi_G - \xi_1^2 \cos 2\varphi_G + \text{3rd order} \quad (5) \]

*From a personal communication with Mr. L. Simmons, U.S.C. and G.S., deflection of 30'' exist but are in general uncommon.*
\[
\sin^2 (\varphi_G + \xi_1) = \left[ \sin \varphi_G \left( 1 - \frac{\xi_1^2}{2} \right) + \xi_1 \cos \varphi_G \right]^2
\]

\[
= \left( 1 - \frac{\xi_1^2}{2} \right) \sin^2 \varphi_G + \xi_1^2 \cos^2 \varphi_G + 2 \xi_1 \sin \varphi_G \cos \varphi_G
\]

\[
= \sin^2 \varphi_G + \xi_1 \sin 2 \varphi_G + \xi_1^2 \cos 2 \varphi_G + \text{3rd order}
\]

\[\tag{6}\]

\[
\cos (\varphi_G + \xi_1) = \cos \varphi_G \left( 1 - \frac{\xi_1^2}{2} \right) - \xi_1 \sin \varphi_G = \cos \varphi_G - \xi_1 \sin \varphi_G - \frac{\xi_1^2}{2} \cos \varphi_G
\]

\[\tag{7}\]

\[
\sin (\varphi_G + \xi_1) = \sin \varphi_G \left( 1 - \frac{\xi_1^2}{2} \right) + \xi_1 \cos \varphi_G = \sin \varphi_G + \xi_1 \cos \varphi_G - \frac{\xi_1^2}{2} \sin \varphi_G
\]

\[\tag{8}\]

from equation (4):

\[
1 - \frac{\eta_1^2}{2} = \sin^2 (\varphi_G + \xi_1) + \cos^2 (\varphi_G + \xi_1) \left[ 1 - \frac{\Delta \lambda_1^2}{2} \right],
\]

\[= 1 - \frac{\Delta \lambda_1^2}{2} \cos^2 (\varphi_G + \xi_1),\]

\[
\Delta \lambda_1 = \frac{\eta_1}{\cos (\varphi_G + \xi_1)}.
\]
and using equation (7),

$$
\Delta \lambda_i = \frac{\eta_i}{\cos \varphi_{G_i}} \left[ \frac{1}{1 - \xi_i \tan \varphi_{G_i} - \frac{\xi_i^2}{2}} \right] = \frac{\eta_i}{\cos \varphi_{G_i}} \left[ 1 + \xi_i \tan \varphi_{G_i} + \frac{\xi_i^2}{2} + \frac{\xi_i^2}{2} \tan^2 \varphi_{G_i} \right]
$$

$$
= \eta_i \sec \varphi_{G_i} \left[ 1 + \xi_i \tan \varphi_{G_i} \right] + 3\text{rd order.}
$$

Further,

$$
\sin (\lambda_{G_i} + \Delta \lambda_i) = \sin \lambda_{G_i} \left( 1 - \frac{\eta_i^2 \sec^2 \varphi_{G_i}}{2} \right) + \cos \varphi_{G_i} \eta_i \sec \varphi_{G_i} \left( 1 + \xi_i \tan \varphi_{G_i} \right)
$$

$$
= \sin \lambda_{G_i} + \eta_i \frac{\cos \lambda_{G_i}}{\cos \varphi_{G_i}} + \frac{\eta_i}{\cos^2 \varphi_{G_i}} \left[ \xi_i \cos \lambda_{G_i} \sin \varphi_{G_i} - \frac{\eta_i}{2} \sin \lambda_{G_i} \right] + 3\text{rd order}
$$

$$
\cos (\lambda_{G_i} + \Delta \lambda_i) = \cos \lambda_{G_i} - \sin \lambda_{G_i} \sec \varphi_{G_i} \left( 1 + \xi_i \tan \varphi_{G_i} \right) \eta_i - \frac{\eta_i^2}{2} \sec^2 \varphi_{G_i} \cos \lambda_{G_i}
$$

$$
= \cos \lambda_{G_i} - \eta_i \frac{\sin \lambda_{G_i}}{\cos \varphi_{G_i}} - \frac{\eta_i}{\cos^2 \varphi_{G_i}} \left[ \xi_i \sin \lambda_{G_i} \sin \varphi_{G_i} + \frac{\eta_i}{2} \cos \lambda_{G_i} \right] + 3\text{rd order.}
$$
E. Using equations (1), (2), (3), (7), (8), (9), (10), and (11), \( x_{E_1} \), \( y_{E_1} \) and \( z_{E_1} \) can be expressed as functions of the geodetic inputs \( (\phi_{G_1}, \lambda_{G_1}, h_1, H_1, \eta_1, \xi_1, a_1, \) and \( b_1) \).

\[
\begin{align*}
x_{E_1} &= \zeta_{G_1} \cos \lambda_{G_1} + H_1 \cos \phi_{G_1} \cos \lambda_{G_1} + h_1 \left( \cos \phi_{G_1} - \xi_1 \sin \phi_{G_1} \right) \left( \cos \lambda_{G_1} - \eta_1 \frac{\sin \lambda_{G_1}}{\cos \phi_{G_1}} \right) \\
&= \zeta_{G_1} \cos \lambda_{G_1} + (H_1 + h_1) \cos \phi_{G_1} \cos \lambda_{G_1} + h_1 \left( \xi_1 \sin \phi_{G_1} \cos \lambda_{G_1} + \eta_1 \sin \lambda_{G_1} \right) + 3\text{rd order.}
\end{align*}
\]

\[
\begin{align*}
y_{E_1} &= \zeta_{G_1} \sin \lambda_{G_1} + H_1 \cos \phi_{G_1} \sin \lambda_{G_1} + h_1 \left( \cos \phi_{G_1} - \xi_1 \sin \phi_{G_1} \right) \left( \sin \lambda_{G_1} + \eta_1 \frac{\cos \lambda_{G_1}}{\cos \phi_{G_1}} \right) \\
&= \zeta_{G_1} \sin \lambda_{G_1} + (H_1 + h_1) \cos \phi_{G_1} \sin \lambda_{G_1} - h_1 \left( \xi_1 \sin \phi_{G_1} \sin \lambda_{G_1} - \eta_1 \cos \lambda_{G_1} \right) + 3\text{rd order.}
\end{align*}
\]

\[
\begin{align*}
z_{E_1} &= \zeta_{G_1} \frac{b_1^2}{a_1^2} \tan \phi_{G_1} + H_1 \sin \phi_{G_1} + h_1 \left( \sin \phi_{G_1} + \xi_1 \cos \phi_{G_1} \right) \\
&= \zeta_{G_1} \frac{b_1^2}{a_1^2} \tan \phi_{G_1} + (H_1 + h_1) \sin \phi_{G_1} + h_1 \xi_1 \cos \phi_{G_1} + 3\text{rd order.}
\end{align*}
\]
F. The cartesian coordinates in the A.P.L. geocentric system are:

\[ x_{ci} = x_{E1} + \Delta x_i \]
\[ y_{ci} = y_{E1} + \Delta y_i \]
\[ z_{ci} = z_{E1} + \Delta z_i \]

where \( \Delta x_i, \Delta y_i, \) and \( \Delta z_i \) are of second order, at best.

H. The cylindrical coordinates \((\zeta_{ci}, \lambda_{ci}, \zeta_{ci})\) in the A.P.L. geocentric system are:

\[ z_{ci} = \zeta_{G1} \frac{b_1^2}{2} \tan \varphi_{G1} + (H_i + h_i) \sin \varphi_{G1} + h_i \xi_1 \cos \varphi_{G1} + \Delta z_i + 3\text{rd order} \quad (13) \]
\[ \zeta_{ci}^2 = x_{ci}^2 + y_{ci}^2 \]

After some algebraic manipulation and using the binominal expansion

\[ \zeta_{ci} = \zeta_{G1} + (H_i + h_i) \cos \varphi_{G1} + \Delta x_i \cos \lambda_{G1} + \Delta y_i \sin \lambda_{G1} - h_i \xi_1 \sin \varphi_{G1} \]
\[ + (H_i + h_i) \cos \varphi_{G1} \cdot \frac{1}{\zeta_{G1}} \left( \Delta x_i \cos \lambda_{G1} + \Delta y_i \sin \lambda_{G1} \right) + 3\text{rd order} \quad (14) \]

+ 3rd order.
In the derivation of $\lambda_{G_1}$, no previously derived quantities were used as was the case with $\zeta_{c_1}$. From Figure 3A, $h_i$ is considered to be zero, thus the angle $\sigma$ can be approximated as follows:

$$\varepsilon_1 + \varepsilon_2 = \Delta x_i \sin \lambda_{G_1}$$

$$\varepsilon_2 = \Delta y_i \cos \lambda_{G_1}$$

where $\varepsilon_1$ and $\varepsilon_2$ are normal to $\zeta_{G_1}$, and

$$\varepsilon_1 = \Delta x_i \sin \lambda_{G_1} - \Delta y_i \cos \lambda_{G_1}.$$  

Since $\varepsilon_1$ considered, at best, second order,

$$\sigma = \frac{\varepsilon_1}{\zeta_{G_1}}$$

and from the geometry,

$$\lambda_{c_1} = \lambda_{G_1} - \sigma = \lambda_{G_1} - \frac{1}{\zeta_{G_1}} (\Delta x_i \sin \lambda_{G_1} - \Delta y_i \cos \lambda_{G_1}) \quad (15)$$

Upon including the station elevation ($h_i$) and deflection in the prime vertical ($\eta_i$) (see Figure 3B)

$$\tau = h_i \sin \eta_i = h_i \eta_i \quad (\eta_i \text{ is of magnitude } \pm 30'' \text{ of arc})$$
Figure 3A  Diagram showing means of determining $\lambda_c$ when $h_i = 0$. 
Diagram showing determination of $\lambda_c$ when $h_1 \neq 0$. 

Figure 3B
and it follows similarly

$$\rho = \frac{\tau}{\zeta G_1} = \frac{h_1 \eta_1}{\zeta G_1}$$

From equation (15) and Figure 3B,

$$\lambda_{c_i} = \lambda_{G_1} + \rho - \sigma$$

$$= \lambda_{G_1} + \frac{h_1 \eta_1}{\zeta G_1} - \frac{1}{\zeta G_1} [\Delta x_1 \sin (\lambda_{G_1} + \rho) - \Delta y_1 \cos (\lambda_{G_1} + \rho)]$$

Assuming $\cos \rho = 1 - \frac{\rho^2}{2}$, $\sin \rho = \rho$,

$$\lambda_{c_i} = \lambda_{G_1} + \frac{1}{\zeta G_1} [h_1 \eta_1 - (\Delta x_1 \sin \lambda_{G_1} - \Delta y_1 \cos \lambda_{G_1})] + 3rd \, order. \quad (16)$$

Equations (13), (14), and (16) are the cylindrical coordinates $z_{c_i}$, $\zeta_{c_i}$, $\lambda_{c_i}$ in the A.P.L. Earth fixed coordinate system expressed as a function of the geodetic coordinates of a tracking station.
References


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