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Practical Computation of Gravity at High Altitudes

by

R. A. Hirvonen
and
Helmut Moritz

Prepared for
Air Force Cambridge Research Laboratories
Office of Aerospace Research
United States Air Force
Bedford, Massachusetts

Contract No. AF 19(628)-2771
Project No. 7600
Task No. 760002

The Ohio State University
Research Foundation
Columbus, Ohio 43212

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FOREWORD

This report was prepared by Dr. R. A. Hirvonen and Dr. Helmut Moritz, Research Associates, of the Institute of Geodesy, Photogrammetry and Cartography of The Ohio State University, under Air Force Contract No. AF 19(628)-2771, OSURF Project No. 1613, under the supervision of Dr. Weikko A. Heiskanen, Director of the Institute. The contract covering this research is administered by the Air Force Cambridge Research Laboratories, Office of Aerospace Research, Laurence G. Hanscom Field, Bedford, Massachusetts, with Mr. Owen Williams and Mr. Bela Szabo, Project and Task scientists.
This report presents and compares methods for computing the gravity vector outside the earth. The gravity vector is conveniently split up into a normal part, the vector of normal gravity, and an anomalous part, the vector of gravity disturbance.

Part I gives the theoretical foundations and a practical method for the computation of the vectors of normal gravity and gravitation. The method is adapted for electronic computation. It is illustrated by a numerical example.

Then the formulas for the computation of the gravity disturbance vector from free-air gravity anomalies $\Delta g$ at the surface of the earth are developed, (a) for the direct method, which uses $\Delta g$ only, and (b) for the coating method, which requires $\Delta g$ and the geoid heights $N$ but involves simpler formulas.

The components of the gravity vector are first computed in a local coordinate system and then transformed to the geocentric world system by a spatial rotation. If the gravity vector is required along a rocket trajectory, then also the components along and normal to the trajectory can be computed by a spatial rotation.

Part II is concerned with accuracy studies in order to determine the best practical procedure for the computation of the gravity disturbances.

The standard errors of the components of the gravity disturbances, due to the interpolation of the gravity material, are, approximately, inversely proportional to the elevation and very small for high altitudes, provided there is uniform coverage by gravity stations.

The influence of the distant zones is considered in some detail. This influence decreases very slowly beyond a certain radius, so that it is impractical to go farther than about 30° in the direct method and 20° in the coating method (with an error of about ±5 mgal in both cases), unless the integration is extended over the whole earth, which is necessary for higher accuracy.

There is another method, for which the influence of the distant zones is completely negligible and which furthermore is the simplest: the upward continuation of the surface disturbances. If presupposes,
however, the deflections of the vertical $\xi$ and $\eta$ for the horizontal components, and $\Delta g$ and $N$ for the vertical component.

For practical use the following methods are proposed: if only $\Delta g$ is given, the direct method; if $\Delta g$ and $N$ are given, the coating method for the horizontal components and the upward continuation for the vertical component; if $\Delta g$, $\xi$, $\eta$ are given, the upward continuation for all three components.

Finally, a detailed practical computation procedure is described for the practically most important case that $\Delta g$ and $N$ are given.
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III
INTRODUCTION

The purpose of this report is to develop a method of computation of the gravity vector for a great number of points at high altitudes, e.g. for trajectories of moving bodies, when the geocentric coordinates are given. Formulas will be summarized in a form suitable for automatic high speed computers. Numerical examples illustrate the method.

First, the components of the normal gravitation (attraction of a reference ellipsoid) will be computed. Those of the normal gravity can then be obtained by subtraction of the components of the centrifugal force. Finally, the deviations of the actual components from these normal values will be evaluated on the basis of the free air anomalies of the gravity observed on the physical surface of the earth. The accuracy of the method is studied in Part II.

The method is based on the formulas first published in [Hirvonen, 1959], but several practical improvements have been introduced.

1. CONVERSION OF THE COORDINATES

It is supposed that the Cartesian geocentric coordinates \(x, y, z\)
are given for each point \( P \) at which the gravity will be computed.

The \( z \)-axis is the axis of the earth's rotation, the \( x \)-axis has the longitude \( 0^\circ \) (Greenwich) and the \( y \)-axis has the longitude \( 90^\circ \text{E} \). These coordinates must be converted into three other systems.

I. Geocentric coordinates, defined by

\[
\begin{align*}
x &= r \cos \psi \cos \lambda \\
y &= r \cos \psi \sin \lambda \\
z &= r \sin \psi
\end{align*}
\]

and called

\( r \) = radius vector

\( \psi \) = geocentric latitude

\( \lambda \) = geographic longitude

When \( x, y, z \) are given, \( r, \psi, \lambda \) can easily be computed:

\[
\begin{align*}
(1.1) & \quad p^2 = x^2 + y^2 \\
(1.2) & \quad r^2 = p^2 + z^2 \\
(1.3) & \quad \cos \lambda = \frac{x}{p} \\
(1.4) & \quad \sin \lambda = \frac{y}{p} \\
(1.5) & \quad \cos \psi = \frac{p}{r} \\
(1.6) & \quad \sin \psi = \frac{z}{r}
\end{align*}
\]

For the computation of the normal gravitation, we shall use the function
(1.7) \[ F = \cos 2 \Psi = \frac{p^2 - z^2}{r^2} \]

II. Geographic coordinates, defined by

\[ p = (N + H) \cos \varphi \]  
(II)
\[ z = (N + H - e^2 N) \sin \varphi \]

(1.8) \[ N = \frac{a}{w} \]

(1.9) \[ w^2 = 1 - e^2 \sin^2 \varphi \]
\[ e^2 = \frac{a^2 - b^2}{a^2} \]

and called

\[ \varphi = \text{geographic latitude} \]
\[ H = \text{geographic height} \]

The geographic coordinates refer to a reference ellipsoid with equatorial radius \( a \) and polar radius \( b \), and they are used for the computation of triangulations, in spite of the fact that the "orientation" of the ellipsoid (or the "data" of the triangulation system) cannot be determined without world-wide geodetic operations. The names suggested here are, however, not yet in a general use.

The normal gravitation is the attraction of the reference ellipsoid which is supposed to have the same mass as the actual earth. The masses inside the rotating ellipsoid are supposed to be distributed
in such a way that the combined potential of the attraction and of the centrifugal force is constant at the outer surface and very close to the actual potential of the gravity at the mean sea level.

At higher altitudes, the geographic coordinates are less practical for the exact computation of the geodetic quantities. Especially, we cannot use them for the computation of the normal gravity. However, we should use them for the evaluation of the deviations of the actual gravity from the normal gravity on the basis of the anomalies observed on the ground.

The computation of $\phi$ and $H$ from $x, y, z$ is rather complicated. First, we obtain

\[ H = \frac{p}{\cos \phi} - N \]  

\[ H = \frac{z}{\sin \phi} - N + e^2N \]

The difference of these equations gives

\[ \tan \phi = \frac{z}{p} + e^2 \frac{N}{p} \sin \phi \]

This equation will be solved with respect to $\phi$ by successive approximations. The first approximation is usually obtained by $H = 0$ in which case we have

\[ \tan \phi_1 = \frac{1}{1-e^2} \cdot \frac{z}{p} \]
For the higher altitudes, it is slightly better to start from

\[ H = e^a N \approx 43 \text{ km, in which case we obtain} \]

\[ (1.14) \tan \varphi_1 = \frac{(1 + e^a) z}{p} \]

The latter case will be used in our present applications.

Any approximate value \( \tan \varphi_1 \) can be improved as follows. Compute

\[ (1.15) \sin^2 \varphi_1 = \frac{\tan^2 \varphi_1}{1 + \tan^2 \varphi_1} \]

and \( W_1 \) by the aid of the value thus obtained. Then we have a second approximation

\[ (1.16) W_1^2 = 1 - e^a \sin^2 \varphi_1 \]

\[ (1.17) \delta = e^a \frac{\sin \varphi_1}{p W_1} \]

\[ (1.18) \tan \varphi_2 = \tan \varphi + \delta \]

If this new approximation is improved in the same way, the change is already much smaller and it can be estimated roughly:

\[ (1.19) \tan \varphi_3 - \tan \varphi_2 = \delta \cdot \frac{\cos \varphi_3}{\tan \varphi_1} (\tan \varphi_2 - \tan \varphi_1) \]

In the numerical example given in chapter 4 we have \( H = 129 \). Even in this extreme case the error of \( \varphi_3 \) is 0".0002 only. That of \( \varphi_2 \) is 0".035 and that of \( \varphi_1 \) is 9".

In our present problem, we shall use \( \varphi \) only for the computation of the disturbances of gravity. Therefore, we may think that the first approximation \( \varphi_1 \) already is quite sufficient for this partic-
ular purpose. However, we have included the computation of the correct value into our program because it may be useful for other purposes.

Note that $\varphi$ is not the direction of the normal gravity at the elevated point $P$ but the direction of the normal of the reference ellipsoid. The former direction can be computed only when the components of the normal gravity are known:

$$\tan \varphi'' = \frac{\gamma_z}{\gamma_p}$$

We shall call $\varphi''$ the geodetic latitude, bearing in mind the definition:

- component of the deflection of the vertical is the difference of the astronomical latitude $\varphi'$ and geodetic latitude $\varphi''$ at the elevated point $P$.

The difference $\varphi'' - \varphi$ is caused by the curvature of the normal plumb line. In [Hirvonen, 1960], a series is given for the computation of $\varphi'' - \varphi$

$$\varphi'' - \varphi = \sin 2 \varphi (0'' .170293 H$$

$$(1.21) + 0 .001103 H \cos 2 \varphi + 0 .000034 H^2)$$

(H in kilometers). In the present problem, however, we cannot use this formula for the computation of $\varphi$ because $H$ still is unknown.

Therefore, we have to compute $\varphi$ by successive approximations as
described above. When the final approximation has been found, \( H \) can be obtained from (1.10).

For latitudes higher than 45°, our program should be based on an alternative set of formulas. Instead of (1.14) through (1.19), compute

\[
(1.22) \quad \cot \varphi_1 = \frac{p}{z(1+e^2)}
\]

\[
(1.23) \quad \cos^2 \varphi_1 = \frac{\cot^2 \varphi_1}{1+\cot^2 \varphi_1}
\]

\[
(1.24) \quad \cot \varphi_2 = \frac{p}{z} - \frac{e^2 a}{W_1} \frac{\cos \varphi_1}{z} = \cot \gamma - \delta
\]

\[
(1.25) \quad \cot \varphi_3 - \cot \varphi_2 = 8 \cdot \frac{\sin^2 \varphi_1}{\cot \varphi_1} (\cot \varphi_2 - \cot \varphi_1)
\]

With the final approximation of \( \varphi \), compute \( H \) from (1.11).

**III. Elliptic coordinates, defined by**

\[
p = \frac{c}{\sin \varepsilon} \cos \beta
\]

\[
(III) \quad z = \frac{c}{\tan \varepsilon} \sin \beta
\]

\[c = e a\]

and called

\[\varepsilon = \text{angular eccentricity}\]

\[\beta = \text{reduced latitude}\]

At the surface of the reference ellipsoid, \( \varepsilon \) has a constant value
\[ \sin \varepsilon = \varepsilon \]

When \( p \) and \( z \) are given, we could compute \( \varepsilon \) and \( \beta \) by rigorous formulas

\[ k^2 = r^2 + c^2 \]  

(1.27) \[ h^4 = k^4 - 4p^2 c^2 \]  

(1.28) \[ \sin^2 \varepsilon = \frac{k^2 - h^2}{2p^2} \]  

(1.29) \[ \cos^2 \beta = \frac{2p^2}{k^2 + h^2} \]  

For practical computations, however, we shall use power series.

By the aid of

\[ \kappa^2 = \frac{c^2}{r^2} \]  

(1.31) we can replace the formulas above by

\[ k^2 = r^2 (1 + \kappa^2) \]  

(1.32) \[ h^4 = r^4 (1 - 2\kappa^2 \cos 2\varphi + \kappa^4) \]  

(1.33) \[ \sin^2 \varepsilon = \kappa^2 [1 - \kappa^2 \sin^2 \varphi - \kappa^4 \sin^2 \varphi \cos 2\varphi - \kappa^6 \sin^2 \varphi \left( \frac{5}{4} \cos 2\varphi - \frac{1}{4} \right)] \]  

(1.34) \[ \cos^2 \beta = \cos^2 \varphi \left[ 1 - \kappa^2 \sin^2 \varphi - \kappa^4 \sin^2 \varphi \cos 2\varphi \right] \]  

(1.35)
Using the abbreviating symbol

(1.7) \[ F = \cos 2 \varphi \]

we obtain, after lengthy but easy computations,

\[ \varepsilon = \kappa + \frac{1}{12} \kappa^3 (3F - 1) \]

(1.36) \[ + \frac{1}{160} \kappa^5 (35F^2 - 10F - 13) \]

\[ + \frac{1}{128} \kappa^7 (33F^3 - 9F^2 - 21F + \frac{19}{7}) \]

2. THE POTENTIAL OF NORMAL GRAVITATION

The Newtonian attraction of the reference ellipsoid is called the normal gravitation of the earth. The potential of this attraction can be expressed in a closed form:

(2.1) \[ V = \frac{fM}{c} \varepsilon + \frac{w^2a^2}{2q_o} q \left( \frac{2}{3} - \cos^2 \beta \right) \]

where

\[ f : \text{gravitational constant}, \]
\[ M : \text{mass of the earth}, \]
\[ w : \text{rotation speed of the earth}. \]

The auxiliar variable

(2.2) \[ q = \frac{1}{2} [\varepsilon - 3 \cot \varepsilon (1 - \varepsilon \cot \varepsilon)] \]

is a function of \( \varepsilon \) only. At the surface of the reference ellipsoid \( q \) has a constant value \( q_o \).
The closed formula (2.2) of \( q \) is not suitable for practical computations, but must be replaced by power series:

\[
(2.3) \quad q = \frac{2}{15} \tan^3 \epsilon - \frac{4}{35} \tan^5 \epsilon + \frac{2}{21} \tan^7 \epsilon - \ldots
\]

or

\[
(2.4) \quad q = \frac{2}{15} \sin^3 \epsilon + \frac{3}{35} \sin^5 \epsilon + \frac{5}{84} \sin^7 \epsilon + \ldots
\]

Therefore, it is only logical that we use power series throughout for our computations. By the aid of (1.34), (2.4) gives

\[
(2.5) \quad q = \frac{2}{15} \kappa^3 + \frac{1}{10} \kappa^5 (F - \frac{1}{7}) + \frac{1}{560} \kappa^7 (63F^3 - 10F - \frac{59}{3})
\]

Now we have to insert (1.35), (1.36) and (2.5) into (2.1). Using another abbreviating symbol for the constant

\[
(2.6) \quad c = \frac{2}{15} \frac{a^2 a^3 c}{fmq_0}
\]

we can write the result in form

\[
V = \frac{fm}{c} \kappa \left\{ 1 + \frac{1}{4} \kappa^2 (1 - C)(F - \frac{1}{3}) + \frac{1}{224} \kappa^4 (7 - 10C)(7F^2 - 2F - \frac{13}{5}) \right. \\
+ \left. \frac{1}{128} \kappa^6 (3 - 5C)(11F^3 - 3F^2 - 7F + \frac{19}{21}) \right\}
\]

3. THE COMPONENTS OF NORMAL GRAVITY

Differentiation of (2.7) and (1.31) with respect to \( r \) gives

the component of the normal gravitation toward the center of the earth:
\[
\Gamma_r = -\frac{\partial V}{\partial r} = \frac{\tau M}{c^2} \kappa^2 \left\{ 1 + \frac{1}{4} \kappa^2 \left( 1 - \frac{c}{1 - c} \right) (3F - 1) \right. \\
\left. + \frac{1}{224} \kappa^4 \left( 7 - 10C \right) (35F^2 - 10F - 13) \right. \\
\left. + \frac{1}{384} \kappa^6 \left( 3 - 5C \right) (231F^2 - 63F^2 - 147F + 19) \right\}
\]

Differentiation of (2.7) and (1.7) with respect to \( \psi \) and division by \( r \) give the component perpendicular to \( \Gamma_r \):

\[
\Gamma_\psi = -\frac{1}{r} \frac{\partial V}{\partial \psi} = \frac{\tau M}{c^2} \kappa^4 \sin \psi \cos \psi \left\{ 1 - \frac{c}{1 - c} \right. \\
\left. + \frac{1}{28} \kappa^2 \left( 7 - 10C \right) (7F - 1) \right. \\
\left. + \frac{1}{32} \kappa^4 \left( 3 - 5C \right) (333F^2 - 6F - 7) \right\}
\]

In order to facilitate the numerical computations, we replace the increasing powers of \( \kappa^2 \) by those of

\[
G = \frac{\tau M}{r^2}
\]

using convenient units. In other words, we insert:

\[
\kappa^2 = G \frac{c^2}{\tau M}
\]

For the international ellipsoid with the international normal formula of gravity we obtain, when \( r \) is expressed in kilometers and \( \Gamma \) in \( \text{gal} = \text{cm sec}^{-2} \):

\[
G = \frac{3986.290.45}{r^2}
\]
\[ \Gamma_r = 1000 \ G \\
- 0.835 \ 888 \ G^2 \\
+ 2.507 \ 664 \ G^2 \cos 2\ \Psi \\
- 0.005 \ 118 \ G^3 \\
- 0.003 \ 937 \ G^3 \cos 2\ \Psi \\
+ 0.013 \ 778 \ G^3 \cos^2 2\ \Psi \\
+ 0.000 \ 007 \ G^4 \\
- 0.000 \ 054 \ G^4 \cos 2\ \Psi \\
- 0.000 \ 023 \ G^4 \cos^2 2\ \Psi \\
+ 0.000 \ 085 \ G^4 \cos^3 2\ \Psi \]

\[ \Gamma_\Psi = \sin \ \Psi \cos \ \Psi \left[ 3.343 \ 551 \ G^2 \\
- 0.003 \ 149 \ G^3 \\
+ 0.022 \ 045 \ G^3 \cos 2\ \Psi \\
- 0.000 \ 031 \ G^4 \\
- 0.000 \ 026 \ G^4 \cos 2\ \Psi \\
+ 0.000 \ 145 \ G^4 \cos^2 2\ \Psi \right] \]

As \ G \ is always slightly smaller than \ 1, it is easy to see that the accuracy of one milligal can be obtained without the terms with \ G^4.

The components of gravitation in directions of the Cartesian coordinate axes are here chosen to be positive towards the origin. They can be computed by formulas

\[ \Gamma_\Psi = \Gamma_r \cos \ \Psi - \Gamma_\Psi \sin \ \Psi \]

\[ \Gamma_\lambda = \Gamma_r \cos \ \lambda \]

12
\[
(3.10) \quad \Gamma_y = \Gamma_p \sin \lambda
\]

\[
(3.11) \quad \Gamma_z = \Gamma_r \sin \gamma + \Gamma_y \cos \gamma
\]

The components of the normal gravity are obtained by subtraction of those of the centrifugal force:

\[
(3.12) \quad \gamma_p = \Gamma_p - \omega^2 \rho
\]

\[
(3.13) \quad \gamma_x = \Gamma_x - \omega^2 x
\]

\[
(3.14) \quad \gamma_y = \Gamma_y - \omega^2 y
\]

\[
(3.15) \quad \gamma_z = \Gamma_z
\]

The total gravitation is

\[
(3.16) \quad \Gamma = \sqrt{\Gamma_p^2 + \Gamma_y^2}
\]

and the total gravity

\[
(3.17) \quad \gamma = \sqrt{\gamma_p^2 + \gamma_z^2}
\]

The direction of the former is

\[
(3.18) \quad \phi = \arctan \frac{\Gamma_z}{\Gamma_p}
\]

that of the latter was given in (1.20).

4. SUMMARY OF FORMULAS FOR NORMAL GRAVITY

The formulas given above will be summarized here in form of a program for automatic computers. First, the constants are listed which pertain to the international ellipsoid and to the international normal
formula of gravity. Then the computations are described in detail, using a numerical example as illustration.

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<th>Constants</th>
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<table>
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<th>Example</th>
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<tr>
<td>(20)</td>
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</tbody>
</table>
Program of Computations

(21) \( p^3 \) \( (18) \cdot (18) + (19) \cdot (19) \) \( 2484 \ 8065 \)

(22) \( z^2 \) \( (20) \cdot (20) \) \( 1738 \ 2224 \)

(23) \( r^2 \) \( (21) + (22) \) \( 4223.0289 \)

(24) \( \cos 2\psi \) \[ \frac{[ (21) - (22) ]}{(23)} \] \( 0.1767 \ 8877 \)

(25) \( p \) \( \sqrt{21} \) \( 4984.7833 \)

(26) \( r \) \( \sqrt{23} \) \( 6498.4836 \)

(27) \( \cos \psi \) \( (25) \div (26) \) \( 0.7670 \ 6869 \)

(28) \( \sin \psi \) \( (20) \div (26) \) \( 0.6415 \ 6496 \)

(29) \( \tan \psi \) \( (20) \div (25) \) \( 0.8363 \ 8528 \)

(30) \( \cos \lambda \) \( (18) \div (25) \) \( 0.1045 \ 2846 \)

(31) \( \sin \lambda \) \( (19) \div (25) \) \( 0.9945 \ 2191 \)

(32) \( \cot \lambda \) \( (18) \div (19) \) \( 0.1051 \ 0423 \)

(33) \( \cos \psi \cos \lambda \) \( (27) \cdot (30) \) \( 0.0802 \)

(34) \( \sin \psi \cos \lambda \) \( (28) \cdot (30) \) \( 0.0670 \)

(35) \( \cos \psi \sin \lambda \) \( (27) \cdot (31) \) \( 0.7629 \)

(36) \( \sin \psi \sin \lambda \) \( (28) \cdot (31) \) \( 0.6381 \)

(37) \( \lambda \) \( \text{arc cot} (32) \) \( 84^\circ \ 00' \ 0000 \)

(38) \( \tan \varphi_1 \) \( (3) \cdot (29) \) \( 0.8420 \ 0802 \)

(39) \( \tan^2 \varphi_1 \) \( (38) \cdot (38) \) \( 0.7089 \ 7751 \)

(40) \( \sin^2 \varphi_1 \) \( (39) \div [(1) + (39)] \) \( 0.4148 \ 5479 \)

(41) \( W^2 \) \( (1) - (2) \cdot (40) \) \( 0.9972 \ 1107 \)

(42) \( \sin \varphi_1 \) \( \sqrt{(40)} \) \( 0.6440 \ 9222 \)

(43) \( W \) \( \sqrt{(41)} \) \( 0.9986 \ 0456 \)

(44) \( \delta \) \( (5) \cdot (42) \div (25) \) \( 0.0055 \ 4057 \)

(45) \( \delta \) \( (44) \div (43) \) \( 0.0055 \ 4831 \)
<p>| | | |</p>
<table>
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Output

(17), (55), (37), (58), (53), (79), (80), (78) in full
(33), (34), (31), (35), (36), (30), (28), (27) four digit values

17
5. GRAVITY DISTURBANCES

If the free air anomalies $\Delta g$ are known everywhere on the physical surface $S$ of the earth, the potential disturbance $T$ at any point on or above $S$ can be computed. The computation is very complicated, especially if the point is very close to high and steep mountains. In most cases, however, the generalized formula of Stokes gives a sufficient approximation.

We shall use following notations:

$P$: fixed point at which $T$ is wanted;

$M$: "moving" point at $S$;

$r$: distance from the centre of the earth;

$\psi$: angle between radii $r_P$ and $r_M$

$\alpha$: azimuth from $P$ to $M$;

(5.1) $t = \frac{r_M}{r_P}$;

(5.2) $D^2 = 1 - 2t \cos \psi + t^2$;

(5.3) $d\sigma = \sin \psi \, d\psi \, d\alpha = \cos \varphi \, d\varphi \, d\lambda$

is the element of solid angle, situated at point $M$.

The generalized formula of Stokes reads

(5.4) $T_P = \frac{r_P}{4\pi} \int \Delta g_M \, t^2 \left( \frac{2}{D} + 1 - 3D - t \cos \psi \left(5 + 3\ln \frac{D + 1 - t \cos \psi}{2}\right) \right) \, d\sigma$
The integration must be carried out over the entire surface $S$. To a very good approximation,

$$r_M = r_P - H$$

where $H$ is the height of $P$ above $S$. Usually, $r_M$ can even be replaced by a constant $R$, the mean radius of the earth.

Especially, if $H = 0$ or $t = 1$, we have

$$D = 2 \sin \frac{\gamma}{2}$$

and

$$T_S = \frac{r_S}{4\pi} \int \Delta g_M \left\{ \text{cosec} \frac{\gamma}{2} + 1 - 6 \sin \frac{\gamma}{2} - \cos \gamma \left[ 5 + 3 \ln \left( \sin \frac{\gamma}{2} + \sin^2 \frac{\gamma}{2} \right) \right] \right\} d\sigma$$

The quantity

$$\zeta = \frac{T}{\gamma}$$

is called the height anomaly and

$$T_S = \frac{r_S}{\gamma_S}$$

is approximately the elevation of the geoid above the reference ellipsoid.

Because $T_S$ is one of the most important quantities of geodesy, we may often assume that the values of it have already been computed.

Then we can compute the "coating" function of Helmert:

$$\mu = \Delta g + \frac{3}{2} \frac{\zeta_S}{r_S}$$
This function has the advantage that the long formula (5.4) for the computation of \( T \) at higher altitudes can be replaced by a shorter one:

\[
(5.11) \quad T_p = \frac{r^2 M}{2\pi r_p} \int \frac{u}{D} \, d\sigma
\]

In our present problem, we have to compute the components of the gravity disturbance:

\[
(5.12) \quad \delta_n = -\frac{\partial T}{\partial r}
\]

\[
(5.13) \quad \delta_m = -\frac{\cos \alpha}{r} \frac{\partial T}{\partial \psi}
\]

\[
(5.14) \quad \delta_l = -\frac{\sin \alpha}{r} \frac{\partial T}{\partial \psi}
\]

If we take \( T \) from the long formula (5.4), the result can be written in form

\[
(5.15) \quad \delta_n = \frac{1}{2\pi} \int \Delta g \, F_1 \, d\sigma
\]

\[
(5.16) \quad \delta_m = \frac{1}{2\pi} \int \Delta g \, F_2 \cos \alpha \, d\sigma
\]

\[
(5.17) \quad \delta_l = \frac{1}{2\pi} \int \Delta g \, F_2 \sin \alpha \, d\sigma
\]

where

\[
(5.18) \quad F_1 = \frac{1}{2} t^2 \left\{ \frac{1-t^2}{D^3} + \frac{4}{D} + 1 - 6D \ight. \\
\quad \quad \quad \quad - t \cos \psi \left( 13 + 6\pi \ln \frac{D+1-t \cos \psi}{2} \right) \}
\]

\[
(5.19) \quad F_2 = t^3 \sin \psi \left\{ \frac{1}{D^3} + \frac{3}{D} - 4 \ight. \\
\quad \quad \quad \quad + \frac{3}{2} \left( \frac{D+1-t \cos \psi}{D \sin^2 \psi} - \ln \frac{D+1-t \cos \psi}{2} \right) \}
\]
If we use the short formula (5.11), we obtain

\[(5.20)\quad \delta_n = \frac{1}{2\pi} \int \mu f_1 d\sigma\]

\[(5.21)\quad \delta_m = \frac{1}{2\pi} \int \mu f_2 \cos \alpha d\sigma\]

\[(5.22)\quad \delta_\ell = \frac{1}{2\pi} \int \mu f_3 \sin \alpha d\sigma\]

where

\[(5.23)\quad f_1 = \frac{t^3}{D^3} (1 - t \cos \psi)\]

\[(5.24)\quad f_2 = \frac{t^3}{D^3} \sin \psi\]

Table 5.1 shows some values of $F_1$ and $f_1$, Table 5.2 those of $F_2$ and $f_2$. We see that for small values of $\psi$, the functions $F$ and $f$ are almost equal.

In practical computations, we must first find the values of $\psi$ and $\alpha$. If the earth is considered as a sphere, we have

\[(5.25)\quad \cos \psi = \sin \varphi_P \sin \varphi_M + \cos \varphi_P \cos \varphi_M \cos (\lambda_M - \lambda_P)\]

\[(5.26)\quad \sin \psi \cos \alpha = \cos \varphi_P \sin \varphi_M - \sin \varphi_P \cos \varphi_M \cos (\lambda_M - \lambda_P)\]

\[(5.27)\quad \sin \psi \sin \alpha = \cos \varphi_M \sin (\lambda_M - \lambda_P)\]

For small values of $\psi$, we can use the approximate formulas:

\[(5.28)\quad m = \varphi_M - \varphi_P\]
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(5.29) \[ \ell = (\lambda_M - \lambda_P) \cos \varphi_p \]

(5.30) \[ n = \frac{H}{R} + \frac{1}{2} (m^2 + \ell^2) \]

(5.31) \[ \Delta^2 = n^2 + m^2 + \ell^2 \]

(5.32) \[ f_1 = \frac{n}{\Delta^3} \]

(5.33) \[ f_2 \cos \alpha = \frac{m}{\Delta^3} \]

(5.34) \[ f_2 \sin \alpha = \frac{\ell}{\Delta^3} \]

These formulas can be used only when \( \psi < 20^\circ \). On the other hand, the effect of zones \( \psi > 20^\circ \) is often negligible. Otherwise, this effect may be computed for a few points only and interpolated for the other points. When the components \( \delta_n, \delta_m, \delta_l \) have been computed, we must convert them to the system \( x, y, z \) by the matrix multiplication:

\[
\begin{pmatrix}
\delta_x \\
\delta_y \\
\delta_z
\end{pmatrix} =
\begin{pmatrix}
-\cos \varphi \cos \lambda & -\sin \varphi \cos \lambda & -\sin \lambda \\
-\cos \varphi \sin \lambda & -\sin \varphi \sin \lambda & \cos \lambda \\
-\sin \varphi & \cos \varphi & 0
\end{pmatrix}
\begin{pmatrix}
\delta_n \\
\delta_m \\
\delta_l
\end{pmatrix}
\]

(5.35)

The components

\( \delta_A \) along the trajectory

\( \delta_R \) horizontally to the right hand side

\( \delta_D \) perpendicularly (not vertically) down

can be obtained as follows. Take three consecutive points \( P_{i-1}, P_i \) and \( P_{i+1} \) of the trajectory. Compute:
\[
x' = x_{i+1} - x_i
\]
\[
(5.36)\quad y' = y_{i+1} - y_i
\]
\[
z' = z_{i+1} - z_i
\]
\[
x'' = \frac{(x_i x' + y_i y') z_i - z' p_i^2}{r' p_i r_i}
\]
\[
(5.37)\quad y'' = \frac{x'_y y_i - y'_y x_i}{r' p_i}
\]
\[
z'' = \frac{x'_x y' y_i + z'_z z_i}{r' p_i}
\]
\[
(5.38)\quad p'' = \sqrt{x''^2 + y''^2}
\]

\[
\begin{pmatrix}
\delta_A \\
\delta_R \\
\delta_D
\end{pmatrix} =
\begin{pmatrix}
-z'' & x'' & y'' \\
0 & \frac{y''}{p''} & \frac{x''}{p''} \\
p'' & \frac{x'' z''}{p''} & \frac{y'' z''}{p''}
\end{pmatrix}
\begin{pmatrix}
\delta_n \\
\delta_m \\
\delta_2
\end{pmatrix}
\]

**Size of Blocks.** Around the point \( P \) define four regions, each bounded by latitudes \( \phi_1, \phi_2 \) and longitudes \( \lambda_1, \lambda_2 \).

**Region A**

\( \phi_2 - \phi_1 = 3^\circ \)
\( \lambda_2 - \lambda_1 = 4^\circ \)
\( |\phi - \phi_P| < 1^\circ \)
\( |\lambda - \lambda_P| < 1^\circ 30' \)

**Region B**

\( \phi_2 - \phi_1 = 7^\circ \)

outside A

\( \lambda_2 - \lambda_1 = 9^\circ \)
\( |\phi - \phi_P| < 3^\circ \)
\( |\lambda - \lambda_P| < 4^\circ \)

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Region C

\[ \varphi_2 - \varphi_1 = 25^\circ \]
outside B

\[ \lambda_2 - \lambda_1 = 30^\circ \]

\[ |\varphi - \varphi_P| < 10 \]

\[ |\lambda - \lambda_P| < 12^\circ 30' \]

\[ \varphi_1, \varphi_2, \lambda_1, \lambda_2 \] are all divisible by \( 5^\circ \).

Region D outside C

For region A, pick up the mean anomaly cards \( 5' \times 5' \) (\( 36 \times 48 \times 1728 \) cards), for region B, \( 20' \times 20' \) cards \( (21 \times 27 - 9 \times 12 = 459 \) cards), for region C, \( 1^\circ \times 1^\circ \) cards \( (25 \times 30 - 7 \times 9 = 687 \) cards) and for region D, \( 5^\circ \times 5^\circ \) cards. Region D will be used for a few points only.

The influence of zone D on \( \delta_x, \delta_y, \delta_z \) can also be computed directly by means of the Cartesian coordinates \( x, y, z \). If these coordinates \( x, y, z \) are computed for the centers of the \( 5^\circ \times 5^\circ \) squares, then the integration can be carried out by the formulas

\[
\Delta \delta_x = \frac{R^2}{2\pi} \int_D \mu \cdot \frac{x_M - x_P}{\Delta^3} \, d\sigma
\]

\[
\Delta \delta_y = \frac{R^2}{2\pi} \int_D \mu \cdot \frac{y_M - y_P}{\Delta^3} \, d\sigma
\]

\[
\Delta \delta_z = \frac{R^2}{2\pi} \int_D \mu \cdot \frac{z_M - z_P}{\Delta^3} \, d\sigma
\]

where

\[
\Delta^2 = (x_M - x_P)^2 + (y_M - y_P)^2 + (z_M - z_P)^2
\]

and \( \Delta \delta_x, \Delta \delta_y, \Delta \delta_z \) are the contribution of the zone D to \( \delta_x, \delta_y, \delta_z \).
6. INFLUENCE OF AN INACCURATE GRAVITY MATERIAL

For simplicity, the error propagation will be studied for the coating method.

Since, by eqs. (5.20) - (5.24), the components of the gravity disturbance are given by

\[
\delta = \frac{1}{2\pi n} \int_{\alpha=0}^{\pi} \int_{\psi=0}^{\pi} \mu \cdot \frac{t^2}{D^3} (1 - t \cos \psi) \sin \psi \, d\psi \, d\alpha,
\]

we find for their standard errors \( m_n, m_m, m_l \):

\[
m_n = \frac{1}{4\pi^2} \int_{\alpha=0}^{\pi} \int_{\psi=0}^{\pi} \int_{\alpha'=0}^{\pi} \int_{\psi'=0}^{\pi} \sigma(\psi, \alpha, \psi', \alpha') \cdot \frac{t^2}{D^3 D' D'^3} (1 - t \cos \psi) \cdot (1 - t \cos \psi') \sin \psi \sin \psi' \, d\psi \, d\alpha \, d\psi' \, d\alpha',
\]

(6.1)

\[
\left\{ \begin{array}{l}
m_n^2 = \frac{1}{4\pi^2} \int_{\alpha=0}^{\pi} \int_{\psi=0}^{\pi} \int_{\alpha'=0}^{\pi} \int_{\psi'=0}^{\pi} \sigma(\psi, \alpha, \psi', \alpha') \cdot \frac{t^2}{D^3 D' D'^3} \sin \psi \sin \psi' \cdot \\
\end{array} \right.
\]

where \( \sigma(\psi, \alpha, \psi', \alpha') \) is the error covariance function, or error function, of \( \mu \) [Moritz, 1962a, 1963].
We shall assume uniform coverage of the whole earth by gravity stations so that \( \sigma \) approximately depends on the relative position of points \( P(\psi, \alpha) \) and \( P'(\psi', \alpha') \) only. Furthermore we assume that the error function has a sharp maximum at \( P' = P \) and drops off rapidly to zero with increasing distance \( PP' \).

Then we can simplify the integrations considerably by approximately replacing \( \sigma(\psi, \alpha, \psi', \alpha') \) by

\[
(6.2) \quad \frac{S}{\sin \psi'} \delta(\psi' - \psi) \delta(\alpha' - \alpha)
\]

where \( \delta(\psi' - \psi), \delta(\alpha' - \alpha) \) are Dirac's delta functions and \( S \) is a constant given by

\[
(6.3) \quad S = \int_{\alpha'=0}^{\pi} \int_{\psi'=0}^{\pi} \sigma(\psi, \alpha, \psi', \alpha') \sin \psi' \, d\psi' \, d\alpha'.
\]

Since according to a property of the delta function

\[
\int f(x') \, \delta(x' - x) \, dx' = f(x),
\]

the integrations with respect to \( \psi', \alpha' \) can be performed immediately and we get

\[
m^2 = \frac{S}{4\pi^2} \int_{\alpha=0}^{\pi} \int_{\psi=0}^{\pi} \frac{t^4}{D^6} (1 - t \cos \psi)^2 \sin \psi \, d\psi \, d\alpha,
\]

\[
\left\{ \begin{array}{l}
m^2_m = \frac{S}{4\pi^2} \int_{\alpha=0}^{\pi} \int_{\psi=0}^{\pi} \frac{t^6}{D^6} \sin^3 \psi \left\{ \cos^2 \alpha \right\} d\psi \, d\alpha, \\
m^2_L = \frac{S}{4\pi^2} \int_{\alpha=0}^{\pi} \int_{\psi=0}^{\pi} \frac{t^6}{D^6} \sin^3 \psi \left\{ \sin^2 \alpha \right\} d\psi \, d\alpha,
\end{array} \right.
\]

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and finally,

\[ m_n = \frac{S}{2\pi} \left[ \frac{t^4(t^2+1)}{2(t^2-1)^2} + \frac{t^4}{t^2-1} - \frac{t^3}{4} \ln \left| \frac{t+1}{t-1} \right| \right], \]

(6.4)

\[ m_m = m_\ell = \frac{S}{2\pi} \left[ \frac{t^4(t^2+1)}{4(t^2-1)^2} + \frac{t^3}{8} \ln \left| \frac{t+1}{t-1} \right| \right]. \]

Since

\[ t = \frac{R}{R + H} = 1 - \frac{H}{R} + \ldots \]

(R: mean radius of the earth) we can develop in series, and neglecting higher powers of \( \frac{H}{R} \) we get

(6.5) \[ m_n = \frac{SR^2}{8\pi H^2}, \quad m_m = m_\ell = \frac{SR^2}{16\pi H^2}. \]

(For \( H = 500 \text{ km} \) the error of these approximations is smaller than 7%).

Validity of the Simplified Integration

Now we have to investigate in more detail the validity of replacing the error function \( \sigma \) by a product of Delta functions. For simplicity we limit ourselves to the plane approximation. In this case, eqs. (5.20) - (5.22) are simplified to

\[ \delta_n = \frac{H}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mu(x,y)}{(H^2 + x^2 + y^2)^{3/2}} \, dx \, dy, \]

(6.6)

\[ \begin{cases} \delta_m = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mu(x,y)}{(H^2 + x^2 + y^2)^{3/2}} \left\{ x \right\} \, dx \, dy, \\ \delta_\ell = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mu(x,y)}{(H^2 + x^2 + y^2)^{3/2}} \left\{ y \right\} \, dx \, dy, \end{cases} \]

where the xy-plane is horizontal and the z-axis contains point P.
These equations have the general form

\[
(6.7) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(x,y) f(x,y) \, dx \, dy
\]

and the mean square error is given by

\[
(6.8) \quad \sigma^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(x,y,x',y') \, f(x,y) \, f(x',y') \, dx \, dy \, dx' \, dy'.
\]

As before we assume the error function \( \sigma(x,y,x',y') \) to have appreciable magnitude only for \((x',y') = (x,y)\) and to be almost zero elsewhere. Then, only the points \((x',y') = (x,y)\) will contribute significantly to the integral and if we put

\[
x' = x + \xi, \quad y' = y + \eta,
\]

then only small values \(\xi, \eta\) need to be considered. If the usual conditions of continuity and differentiability are satisfied, \(f(x',y')\) can be developed in a Taylor series with respect to \(\xi, \eta\):

\[
f(x',y') = f(x,y) + \xi f_x + \eta f_y +
\]

\[
+ \frac{1}{2} \xi^2 f_{xx} + \xi \eta f_{xy} + \frac{1}{2} \eta^2 f_{yy} + \ldots
\]

where \(f_x, f_{xx}, \ldots\) denote partial derivatives at the point \((x,y)\) and

\[
\xi = x' - x, \quad \eta = y' - y.
\]

Inserting this in (6.8) we can separate the integrations over \(x, y\) and over \(x', y'\), getting
\[ m^2 = \iint \sigma(x,y,x',y') \, dx' \, dy' \cdot \iint [f(x,y)]^2 \, dx \, dy + \]
\[ + \iint (x'-x) \sigma(x,y,x',y') \, dx' \, dy' \cdot \iint f_x \, dx \, dy + \]
\[ + \iint (y'-y) \sigma(x,y,x',y') \, dx' \, dy' \cdot \iint f_y \, dx \, dy + \]
\[ + \frac{1}{2} \iint (x'-x)^2 \sigma(x,y,x',y') \, dx' \, dy' \cdot \iint f_{xx} \, dx \, dy + \]
\[ + \iint (x'-x)(y'-y) \sigma(x,y,x',y') \, dx' \, dy' \cdot \iint f_{xy} \, dx \, dy + \]
\[ + \frac{1}{2} \iint (y'-y)^2 \sigma(x,y,x',y') \, dx' \, dy' \cdot \iint f_{yy} \, dx \, dy \ldots , \]

provided \( \sigma(x,y,x',y') \) depends on \( x-x', y-y' \) only, i.e., we have the same accuracy everywhere. Furthermore, if \( \sigma \) is a symmetrical function of \( x'-x \) and \( y'-y \) which is a very natural assumption, then the integrals containing \( x'-x \) and \( y'-y \) linearly will vanish and there remains

\[ m^2 = S_0 \iint [f(x,y)]^2 \, dx \, dy + \]
\[ (6.9) \]
\[ + \frac{1}{2} S_1 \iint f_{xx} \, dx \, dy + \frac{1}{2} S_2 \iint f_{yy} \, dx \, dy \ldots . \]

where

\[ S_0 = \iint \sigma \, dx' \, dy', \quad S_1 = \iint (x'-x)^2 \sigma \, dx' \, dy', \]
\[ S_2 = \iint (y'-y)^2 \sigma \, dx' \, dy' . \]

(6.10)
If we would have replaced, as before, the error function $\sigma$ by a product of delta functions,

$$S_0 \cdot \delta(x'-x) \delta(y'-y),$$

we would have got

$$m^2 = S_0 \iint_{-\infty}^{\infty} [f(x,y)]^2 \, dx \, dy,$$

which is the first and principal term of (6.9).

Thus the use of delta functions is justified. Transferring these results from the plane to the spherical case offers no difficulties. $S_0$, the integral of the error function over the plane (6.10), is replaced by $S$, the integral over the unit sphere (6.3); we have the relation

$$(6.11) \quad S = \frac{S_0}{R^2}$$

where the scaling factor $R^2$ represents the transition from the terrestrial sphere (or its tangential plane) to the unit sphere.

The error of the delta function method is the effect of $S_1$ and $S_2$ in (6.9).

In order to evaluate the quantities $S$, $S_0$, $S_1$, $S_2$ we have to make some assumption concerning the error function $\sigma$. If this function has the character assumed above - everywhere the same form, sharp maximum at $x' = x$, $y' = y$, vanishing farther away from this
point - then, within the limit of the obtainable accuracy, we can usually take the well-known function

\[ \sigma(x, y, x', y') = \sigma_0 e^{-c^2[(x'-x)^2 + (y'-y)^2]} \]  

(6.12)

(6.10) can then be evaluated immediately to give

\[ S_0 = \frac{\pi \sigma_0}{c^2}, \quad S_1 = S_2 = \frac{\pi \sigma_0}{2c^4} \]

(6.13)

and

\[ S = \frac{\pi \sigma_0}{c^2 R^2} \]

(6.14)

Now we can return to eqs. (6.6). First we consider \( S_n \). Here

\[ f(x, y) = \frac{H}{2\pi} \frac{1}{(H^2 + x^2 + y^2)^{3/2}} \]

Differentiating twice, inserting in (6.9) and integrating yields

\[ m = \frac{3(S_1 + S_2)}{8\pi H^2} - \frac{3}{64\pi H^4} \]

(6.15)

The first term is clearly the same as the first equation in (6.5).

Inserting (6.13) we finally get

\[ m = \frac{\sigma_0}{8c^4 H^2} (1 - \frac{3}{8c^2 H^2} + \ldots) \]

(6.15)

and in a similar way we find

\[ m = \frac{3}{16c^4 H^2} (1 - \frac{3}{8c^2 H^2} + \ldots) \]

(6.16)

The correction term in the parenthesis is practically zero for elevations \( H \gg c^{-1} \), but in view of the practical purpose of these
accuracy formulas, it can already be neglected for $H = c^{-1}$.

Discussion

To sum up the formulas we have for the standard errors of $\delta_n$,

$$\delta_m, \delta_k :$$

$$(6.17) \quad m = \frac{k_1}{H}, \quad m' = \frac{k_2}{H}$$

where

$$(6.18) \quad k_1 = \sqrt{\frac{SR^3}{\delta\pi}}, \quad k_2 = \sqrt{\frac{SR^3}{16\pi}}$$

and

$$S = \frac{\pi c_0}{R^2 \sigma^2}.$$  

These formulas are valid for elevations $H > c^{-1}$. For smaller elevations they cannot be applied (for $H = 0$ they would yield $\infty$, which is obviously wrong).

The error covariance function $\sigma$ is supposed to have the form

$$(6.12) :$$

$$(6.19) \quad \sigma(x,y,x',y') = c_0 e^{-c^2 s^2}$$

where $s$ is the distance of the points $(x,y)$ and $(x',y')$. According to the derivation of the above formulas, $\sigma$ is the error function of $\mu$, the density of the coating. Since $\mu = \Delta g + \frac{3y}{2R} N$, it contains the influence of errors in $\Delta g$ and in $N$. The errors of $N$ have a
small influence but they diminish much less with increasing distance and can hardly be represented in the form (6.19). So, in order to apply our formulas, we must be able to neglect the errors of $N$. That this is possible can be seen in the following way.

We would have avoided the trouble concerning the errors of $N$, had we used the direct formulas (5.15) - (5.19) instead of using the coating method (5.20) - (5.24). These formulas are, however, very difficult to handle for our purpose. On the other hand, for smaller values of $\psi$, where the effect of errors in $\mu$ is largest, $F$ and $f$ are closely equal so that, for this particular purpose, we might replace $\mu$ in (5.20) - (5.24) by $\Delta g$. Then we can take $\sigma$ in (6.1) and in the subsequent developments to be the error covariance function of the gravity anomalies $\Delta g$ only. From this we can conclude that the error of (6.17) due to neglecting the inaccuracy of $N$ must be small.

Thus, we can consider $\sigma$ in (6.19) to be the error function for the interpolation of gravity anomalies. Then,

$$\sigma_0 = m^2$$

is the square of the average standard error of interpolation, $m$.

In Table 6.1 some numerical values of $m$ and $c$ are given, together with the constants $k_1$ and $k_2$ defined above.
Table 6.1

<table>
<thead>
<tr>
<th>s (km)</th>
<th>m (mgal)</th>
<th>( c^{-1} ) (km)</th>
<th>( k_1 ) (mgal.km)</th>
<th>( k_2 ) (mgal.km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1</td>
<td>13</td>
<td>2.5</td>
<td>1.8</td>
</tr>
<tr>
<td>50</td>
<td>9</td>
<td>25</td>
<td>45</td>
<td>32</td>
</tr>
<tr>
<td>100</td>
<td>14</td>
<td>45</td>
<td>126</td>
<td>89</td>
</tr>
<tr>
<td>200</td>
<td>16</td>
<td>90</td>
<td>287</td>
<td>203</td>
</tr>
</tbody>
</table>

To get the standard errors of \( \delta_{m} \) and of \( \delta_{k} \), the above values for \( k_1 \) and \( k_2 \) have to be divided by the elevation \( H \) in kilometers.

For the computation of the dependence of \( m \) and \( c \) on the average station distances, an idealized gravity station net consisting of equilateral triangles with side \( s \) has been assumed. The computation was done in a way described in [Moritz, 1962b, Appendix], using numerical values for the covariance function of gravity anomalies given in [Kaula, 1959] combined with the standard interpolation errors of [Molodenskii et al., 1962, p. 172]. Since the method used is not quite rigorous, the results of Table 2 must be considered somewhat preliminary.
Error Correlation of $\delta_n, \delta_m, \delta_l$

For the error covariances of $\delta_n, \delta_m, \delta_l$ in two points $P, P'$ we have found [Moritz, 1963]:

\[
\sigma_{n,PP'} = \frac{\sigma_0}{8c_n^2H} \left[ 1 - \frac{3}{8} \frac{s^2}{H^3} + \cdots \right],
\]

(6.20) \[
\sigma_{m,PP'} = \frac{\sigma_0}{16c_m^2H} \left[ 1 - \frac{3}{16} \frac{s^2}{H^3} (1 + 2 \cos^2\alpha) + \cdots \right],
\]

\[
\sigma_{l,PP'} = \frac{\sigma_0}{16c_l^2H} \left[ 1 - \frac{3}{16} \frac{s^2}{H^3} (1 + 2 \sin^2\alpha) + \cdots \right].
\]

$P, P'$ are assumed to have the same elevation $H$, $s$ is the distance and $\alpha$ is the azimuth from $P$ to $P'$. These formulas, being the first terms of power series, are valid for small distances $s$ only. For $P = P'$, by (6.5) and (6.14),

(6.21) \[
\sigma_{n,PP} = m_n^2, \sigma_{m,PP} = m_m^2, \sigma_{l,PP} = m_l^2;
\]

the error covariances become variances, i.e., the squares of the standard errors of $\delta_n, \delta_m, \delta_l$.

Dividing the covariances (6.20) by the variances (6.21), we get the correlation coefficients

\[
\rho_{n,PP'} = 1 - \frac{3}{8} \frac{s^2}{H^3} + \cdots,
\]

(6.22) \[
\rho_{m,PP'} = 1 - \frac{3}{16} \frac{s^2}{H^3} (1 + 2 \cos^2\alpha) + \cdots,
\]

\[
\rho_{l,PP'} = 1 - \frac{3}{16} \frac{s^2}{H^3} (1 + 2 \sin^2\alpha) + \cdots.
\]
\( \rho = \pm 1 \) means maximum correlation, i.e., complete functional dependence; \( \rho = 0 \) means independence (more strictly speaking, complete lack of correlation). For a definite \( s \), the correlation coefficients (6.22) will be the closer to 1, the greater the altitude \( H \) is. Thus, with increasing altitude, the standard errors of \( \delta_n \), \( \delta_m \), \( \delta' \), decrease, but the correlation increases. Since the accuracy is characterized by the standard errors and the error correlation, this is significant, too.

7. INFLUENCE OF THE REMOTE ZONES

The purpose of the estimation of the influence of the distant zones is to decide how far the integration must be extended if certain accuracy requirements are prescribed.

Since the effect of the distant zones is almost independent of the elevation \( H \), we may, for simplicity, put \( H=0 \), i.e., we consider \( P \) to be situated directly on the geoid. Then, since

\[
\delta_n = \Delta g + \frac{2\gamma}{R} N, \quad \delta_m = \gamma \xi, \quad \delta' = \gamma \eta,
\]

we can, for the direct method, use one of the evaluations of the effect of distant zones on the components \( \xi, \eta \) of the deflection of the vertical and on \( N \) (e.g., [Kaula, 1957]). For our purpose it is best to use the formulas of [Molodenskii et al., 1962] which can also be
easily adapted to the coating method.

Direct Method (using $\Delta g$). The disturbing potential is

$$(7.1.a) \quad T = \gamma N = \frac{R}{4\pi} \int_0^{2\pi} \int_0^\pi \Delta g \ S(\cos \psi) \sin \psi \, d\psi \, d\alpha$$

where $S(\cos \psi)$ is Stokes' function

$$(7.2.a) \quad S(\cos \psi) = \frac{1}{\sin \frac{\psi}{2}} \left( -3 \cos \psi \ln \left( \frac{\sin \frac{\psi}{2} + \sin \frac{3\psi}{2}}{2} \right) - 6 \sin \frac{\psi}{2} + 1 - 5 \cos \psi \right).$$

If we extend the integration with respect to $\psi$ only up to a spherical radius $\psi_0 < \pi$, we commit an error.

$$(7.3.a) \quad \Delta N = \frac{R}{4\pi} \int_0^{2\pi} \int_0^\pi \Delta g \ S(\cos \psi) \sin \psi \, d\psi \, d\alpha$$

which can be also written

$$(7.4.a) \quad \Delta N = \frac{R}{4\pi} \int_0^{2\pi} \int_0^\pi \Delta g \ \bar{S}(\cos \psi) \sin \psi \, d\psi \, d\alpha$$

where

$$(7.5.a) \quad \bar{S}(\cos \psi) = \begin{cases} 0 & \text{if } \psi < \psi_0 \\ S(\cos \psi) & \text{if } \psi \geq \psi_0 \end{cases}.$$
\( (7.6.a) \quad S(\cos \psi) = \sum_{n=0}^{\infty} \frac{2n+1}{2} Q_n \ P_n (\cos \psi) \)

where the coefficients \( Q_n \) depend on \( \psi \). Then, by multiplying both sides by \( P_n (\cos \psi) \sin \psi \) and integrating from 0 to \( \pi \) we find

\[
Q_n = \int_{0}^{\pi} S(\cos \psi) P_n (\cos \psi) \sin \psi \ d\psi = \int_{0}^{\pi} S(\cos \psi) P_n (\cos \psi) \sin \psi \ d\psi .
\]

By substituting

\( (7.7.a) \quad z = \sin \frac{\psi}{2}, \quad t = \sin \frac{\psi_0}{2} \)

we get

\( (7.8.a) \quad Q_n = -4 \int_{1}^{t} \ P_n (1 - 2z^2) S(1 - 2z^2) \ z \ dz . \)

Performing the integration yields (we need the \( Q_n \) only for \( n \geq 2 \)):

\[
Q_2 = 2 - 4t + 5t^2 + 14t^3 - \frac{53}{2} t^4 - 30t^5 + 18t^6 - \frac{51}{2} t^8 + \\
+ (6t^2 - 24t^4 + 36t^6 - 18t^8) \ln t(1+t),
\]

\( (7.9.a) \quad Q_3 = 1 - 4t + 5t^2 + 22t^3 - 46t^4 - \frac{372}{5} t^5 + 136t^6 + 104t^7 - \\
- 166t^8 - 48t^9 + \frac{352}{5} t^{10} + (6t^2 - 42t^4 + 108t^6 - 120t^8 + \\
+ 48t^{10}) \ln t(1+t),
\]

\[
Q_4 = \frac{2}{3} - 4t + 5t^2 + \frac{98}{3} t^3 - 72t^4 - 156t^5 + 320t^6 + 360t^7 - 645t^8 \\
- \frac{1120}{3} t^9 + 602t^{10} + 140t^{11} - 210t^{12} + (6t^2 - 66t^4 + \\
+ 260t^6 - 480t^8 + 420t^{10} - 140t^{12}) \ln t(1+t).
\]
Higher order $Q_n$ (up to 8th order) may be found in [Molodenskii et al., 1962, p. 149].

By (7.4.a) and (7.6.a) we get

$$\Delta N = \frac{R}{2\gamma} \sum_{n=2}^{\infty} Q_n \Delta g_n$$

if

$$\Delta g = \sum_{n=2}^{\infty} \Delta g_n$$

is the development of $\Delta g$ in Laplace's spherical harmonics ($\Delta g_0$ and $\Delta g_1$ are missing, as usual).

The coating method can be treated in an exactly analogous way.

Coating Method (using $\mu$). The disturbing potential is

$$T = R \gamma N = \frac{R}{4\pi} \int_0^{2\pi} \int_0^{\pi} \mu M(\cos \psi) \sin \psi \, d\psi \, d\alpha$$

where

$$M(\cos \psi) = \frac{1}{\sin \frac{\psi}{2}}$$

If we extend the integration with respect to $\psi$ only up to a spherical radius $\psi_0 < \pi$, we commit an error

$$\Delta N = \frac{R}{4\pi} \int_0^{2\pi} \int_{\psi_0}^{\pi} \mu M(\cos \psi) \sin \psi \, d\psi \, d\alpha$$
which can be also written

\[(7.4,b) \quad \Delta N = \frac{R}{4 \pi^2} \int_0^{2\pi} \int_0^\pi \mu \mathcal{M}(\cos \psi) \sin \psi \, d\psi \, d\alpha \]

where

\[(7.5,b) \quad \mathcal{M}(\cos \psi) = \begin{cases} 0 & \text{if } \psi < \psi_0 \\ \frac{1}{\sin \frac{\psi}{2}} & \text{if } \psi \geq \psi_0 \end{cases} \]

Expand \(\mathcal{M}(\cos \psi)\) in a series of Legendre's polynomials:

\[(7.6,b) \quad \mathcal{M}(\cos \psi) = \sum_{n=0}^{\infty} (n-1) q_n P_n(\cos \psi) \]

where the coefficients \(q_n\) depend on \(\psi_0\). Then, by multiplying both sides by \(P_n(\cos \psi) \sin \psi\) and integrating from 0 to \(\pi\) we find

\[q_n = \frac{2n+1}{2(n-1)} \int_0^\pi \mathcal{M}(\cos \psi) P_n(\cos \psi) \sin \psi \, d\psi = \frac{2n+1}{2(n-1)} \int_{\psi_0}^{\pi} \frac{1}{\sin \frac{\psi}{2}} P_n(\cos \psi) \sin \psi \, d\psi. \]

By substituting

\[(7.7,a) \quad z = \sin \frac{\psi}{2}, \quad t = \sin \frac{\psi_0}{2} \]

we get

\[(7.8,b) \quad q_n = \frac{2(2n+1)}{n-1} \int_0^1 P_n(1-2z^2) \, dz. \]

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Performing the integration yields (we need the $q_n$ only for $n \geq 2$):

\[ q_2 = \frac{2}{1} (1 - 5t + 10t^3 - 6t^5), \]
\[ q_3 = \frac{2}{2} (1 - 7t + 28t^3 - 42t^5 + 20t^7), \]
\[ q_4 = \frac{2}{3} (1 - 9t + 60t^3 - 162t^5 + 180t^7 - 70t^9), \]
\[ q_5 = \frac{2}{4} (1 - 11t + 110t^3 - 462t^5 + 880t^7 - 770t^9 + 252t^{11}), \]

(7.9.b)

\[ q_6 = \frac{2}{5} (1 - 13t + 182t^3 - 1092t^5 + 3120t^7 - 4550t^9 + 3276t^{11} - 924t^{13}), \]
\[ q_7 = \frac{2}{6} (1 - 15t + 280t^3 - 2268t^5 + 9000t^7 - 19250t^9 + 22680t^{11} - 13860t^{13} + 3432t^{15}), \]
\[ q_8 = \frac{2}{7} (1 - 17t + 408t^3 - 4284t^5 + 22440t^7 - 65450t^9 + 111384t^{11} - 109956t^{13} + 58344t^{15} - 12870t^{17}). \]

By (7.4.b) and (7.6.b) we get

\[ \Delta N = \frac{R}{\gamma} \sum_{n=2}^{\infty} \frac{n-1}{2n+1} q_n \mu_n \]

if

\[ \mu = \sum_{n=2}^{\infty} \mu_n \]

if the development of $\mu$ in Laplace's spherical harmonics ($\mu_0$ and $\mu_1$ are missing together with $\Delta g_0$ and $\Delta g_1$).

If

\[ \Delta g = \sum_{n=2}^{\infty} \Delta g_n \]
then

\[ N = \frac{R}{\gamma} \sum_{n=2}^{\infty} \frac{\Delta g_n}{n} \cdot \]

Therefore,

\[ \mu = \Delta g + \frac{3\gamma}{2R} N = \sum_{n=2}^{\infty} \frac{2n+1}{2(n-1)} \Delta g_n \]

so that

\[ \mu_n = \frac{2n+1}{2(n-1)} \Delta g_n \cdot \]

Inserting this in the formula for \( N \) we finally obtain

(7.10.b) \[ \Delta N = \frac{R}{2\gamma} \sum_{n=2}^{\infty} q_n \Delta g_n, \]

which is analogous to (7.10.a).

Mean Square Effect on \( N, \xi, \eta \). Squaring (7.10.a) and averaging over the whole earth we get

\[ \overline{\Delta N^2} = \frac{R^2}{4\gamma^2} \sum_{n=2}^{\infty} \sum_{n'=2}^{\infty} Q_n Q_{n'} \cdot \overline{\Delta g_n \Delta g_{n'}} . \]

Since the integral, over the sphere, of the product of two Laplace harmonics of different degree is zero (Orthogonality !), only the terms where \( n' = n \) remain and we obtain

(7.11.a) \[ \overline{\Delta N^2} = \frac{R^2}{4\gamma^2} \sum_{n=2}^{\infty} Q_n a \overline{\Delta g_n} . \]

where \( \overline{\Delta N} \) is the root mean square influence on \( N \) of the zones beyond a spherical radius \( \Psi_o \), and \( \overline{\Delta g_n} \) is the root mean square average of the Laplace spherical harmonic \( \Delta g_n \).
The derivation of the RMS influence on the components $\xi$, $\eta$ of the deflection of the vertical is rather involved. Since it is omitted in [Molodenskii et. al., 1962], it may be of interest to give it here.

By (7.10.a),

$$\Delta \xi = \frac{1}{R} \frac{\partial N}{\partial \phi} = \frac{1}{2\gamma} \sum_{n=2}^{\infty} Q_n \frac{\partial A_n}{\partial \phi},$$

(7.12)

$$\Delta \eta = \frac{1}{R \cos \phi} \frac{\partial N}{\partial \lambda} = \frac{1}{2\gamma \cos \phi} \sum_{n=2}^{\infty} Q_n \frac{\partial A_n}{\partial \lambda}$$

for the direct method. (For the coating method replace $Q_n$ by $q_n$.)

Hence,

$$\overline{\Delta \xi^2} = \frac{1}{4\gamma^2} \sum_{n=2}^{\infty} \sum_{n'=2}^{\infty} Q_n Q_{n'} \frac{\partial g_n}{\partial \phi} \frac{\partial g_{n'}}{\partial \phi},$$

$$\overline{\Delta \eta^2} = \frac{1}{4\gamma^2} \sum_{n=2}^{\infty} \sum_{n'=2}^{\infty} Q_n Q_{n'} \frac{1}{\cos^2 \phi} \frac{\partial g_n}{\partial \lambda} \frac{\partial g_{n'}}{\partial \lambda},$$

so that

$$\overline{\Delta \xi^2 + \Delta \eta^2} = \frac{1}{4\gamma^2} \sum_{n=2}^{\infty} \sum_{n'=2}^{\infty} Q_n Q_{n'} \frac{\partial g_n}{\partial \phi} \frac{\partial g_{n'}}{\partial \phi} + \frac{1}{\cos^2 \phi} \frac{\partial g_n}{\partial \lambda} \frac{\partial g_{n'}}{\partial \lambda}.$$  (The bar denotes averaging over the sphere.)

$\Delta g_n$ can be written as

$$\Delta g_n = \sum_{m=0}^{\infty} \left( a_{nm} \cos m\lambda + b_{nm} \sin m\lambda \right) P_m^n (\sin \phi)$$

(7.14)

where $a_{nm}$, $b_{nm}$ are constant coefficients and $\phi$, $\lambda$ are geographical coordinates. Now we can differentiate with respect to $\phi$ and $\lambda$:
\[
\frac{\partial \Delta_n}{\partial \phi} = \sum_{m=0}^{n} \left[ a_{nm} \cos m\lambda + b_{nm} \sin m\lambda \right] \frac{dP^m_n}{d\phi},
\]
\[
(7.15)
\frac{\partial \Delta_n}{\partial \lambda} = \sum_{m=0}^{n} \left[ m_{nm} \cos m\lambda - m_{nm} \sin m\lambda \right] P^m_n.
\]

Hence we get, taking \( n' \geq n \),
\[
\frac{\partial \Delta_{n'}^{n}}{\partial \phi} \frac{\partial \Delta_{n'}^{n}}{\partial \phi} = 2\pi \int_{0}^{\pi/2} \int_{-\pi/2}^{\pi/2} \Delta_{n'}^{n} \Delta_{n'}^{n'} \cos \phi d\phi d\lambda =
\]
\[
(7.16)
= 2\pi \int_{0}^{\pi/2} \int_{-\pi/2}^{\pi/2} \frac{dP^0_n}{d\phi} \frac{dP^0_{n'}}{d\phi} \cos \phi d\phi +
\]
\[
+ \pi \sum_{m=1}^{n} \left[ a_{nm} a_{nm} + b_{nm} b_{nm} \right] \int_{-\pi/2}^{\pi/2} \frac{dP^m_n}{d\phi} \frac{dP^m_{n'}}{d\phi} \cos \phi d\phi.
\]

Here we have used the well-known orthogonality relations
\[
2\pi \int_{0}^{\pi} \cos p\lambda \cos q\lambda d\lambda = \begin{cases} 0, & p \neq q \\ \pi, & p = q \neq 0 \\ 2\pi, & p = q = 0 \end{cases},
\]
\[
(7.17)
2\pi \int_{0}^{\pi} \cos p\lambda \sin q\lambda d\lambda = 0 ,
\]
\[
2\pi \int_{0}^{\pi} \sin p\lambda \sin q\lambda d\lambda = \begin{cases} 0, & p \neq q \\ \pi, & p = q \neq 0 \\ 0, & p = q = 0 \end{cases}.
\]
Similarly

\[
\frac{4\pi}{\cos^2 \phi} \frac{\partial \Delta g_n}{\partial \lambda} \frac{\partial \Delta g_{n'}}{\partial \lambda} = \int_{\lambda=0}^{\pi/2} \int_{\phi=-\pi/2}^{\phi=\pi/2} \frac{\partial \Delta g_n}{\partial \lambda} \frac{\partial \Delta g_{n'}}{\partial \lambda} \cos \phi \, d\phi \, d\lambda -
\]

(7.18)

\[
\pi \sum_{m=1}^{n} m^2 (a_{nm} a_{n'm} + b_{nm} b_{n'm}) \int_{-\pi/2}^{\pi/2} \frac{p_m p_{n'}}{\cos^2 \phi} \cos \phi \, d\phi.
\]

Putting

\[
\sin \phi = x, \quad \cos \phi \, d\phi = dx
\]

we find by formulas of [Molodenskii et al., 1962, pp. 165-6]

\[
\frac{\pi}{2} \int_{-\pi/2}^{\pi/2} \frac{dp_m}{d\phi} \frac{dp_{n'}}{d\phi} \cos \phi \, d\phi = \int_{-\pi/2}^{\pi/2} \frac{dp_m}{d\phi} \frac{dp_{n'}}{d\phi} \, dx =
\]

\[
\begin{cases}
\frac{2n(n+1)}{2n+1} \cdot \frac{(n+m)!}{(n-m)!} \cdot m^2 c_m^n, & \text{if } n' = n, \\
- m^2 c_m^n, & \text{if } n' - n \text{ is even, } n' > n, \\
0, & \text{if } n' - n \text{ is odd};
\end{cases}
\]

\[
\frac{\pi}{2} \int_{-\pi/2}^{\pi/2} \frac{p_m p_{n'}}{\cos^2 \phi} \cos \phi \, d\phi = \int_{-\pi/2}^{\pi/2} \frac{p_m p_{n'}}{\cos^2 \phi} \, dx =
\]

\[
\begin{cases}
\binom{m}{n}, & \text{if } n' = n, \\
\binom{m}{n}, & \text{if } n' - n \text{ is even, } n' > n, \\
0, & \text{if } n' - n \text{ is odd};
\end{cases}
\]
where

\[ C^m_n = 2(2n - 1) \frac{(n+m-2)!}{(n-m)!} + 2(2n-5) \frac{(n+m-4)!}{(n-m-2)!} + \ldots, \]

(Congering \( C^m_n \), there is a misprint on p. 165, op. cit.: " + \( C^m_n \) should read " = \( C^m_n \).)

Hence we find, for \( n' \neq n \):

\[
4\pi \frac{\partial \Delta_n}{\partial \phi} \frac{\partial \Delta_{n'}}{\partial \phi} = \begin{cases} 
\pi \sum_{m=1}^{n} \left( a_m a_{n'm} + b_m b_{n'm} \right) \cdot (-m^2 C^m_n), & \text{n' - n even,} \\
0, & \text{n' - n odd,}
\end{cases}
\]

\[
4\pi \frac{1}{\cos^2 \phi} \frac{\partial \Delta_n}{\partial \lambda} \frac{\partial \Delta_{n'}}{\partial \lambda} = \begin{cases} 
\pi \sum_{m=1}^{n} \left( a_m a_{n'm} + b_m b_{n'm} \right) \cdot m^2 C^m_n, & \text{n' - n even,} \\
0, & \text{n' - n odd}
\end{cases}
\]

so that

\[
\frac{\partial \Delta_n}{\partial \phi} \frac{\partial \Delta_{n'}}{\partial \phi} + \frac{1}{\cos^2 \phi} \frac{\partial \Delta_n}{\partial \lambda} \frac{\partial \Delta_{n'}}{\partial \lambda} = 0
\]

for \( n' \neq n \).

For \( n' = n \) we find in the same way

\[
\left( \frac{\partial \Delta_n}{\partial \phi} \right)^2 + \frac{1}{\cos^2 \phi} \left( \frac{\partial \Delta_n}{\partial \lambda} \right)^2 = \frac{n(n+1)}{2n+1} a + \frac{1}{2n+1} \sum_{m=1}^{n} \frac{1}{(n-m)^2} (a_m a_{n'm} + b_m b_{n'm}).
\]

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Forming the average of $\Delta g_n^2$ itself we find easily

$$\overline{\Delta g_n^2} = \frac{1}{4\pi} \int_0^{\pi/2} \int_{\lambda=0}^{\varphi=-\pi/2} \Delta g_n^2 \cos \varphi \, d\varphi \, d\lambda =$$

(7.20)

$$= \frac{1}{2(n+1)} a^2 + \frac{1}{2(2n+1)} \sum_{m=1}^{n} \frac{(n+m)!}{(n-m)!} (a_{nm}^2 + b_{nm}^2)$$

so that

(7.21) $$\left(\frac{\partial \Delta g_n}{\partial \phi}\right)^2 + \frac{1}{\cos^2 \varphi} \left(\frac{\partial \Delta g_n}{\partial \lambda}\right)^2 = n(n+1) \overline{\Delta g_n^2}.$$ 

Taking (7.19) and (7.21) into account, we find by (7.13)

$$\overline{\Delta g^2} + \overline{\Delta \eta^2} = \frac{1}{4\gamma^2} \sum_{n=2}^{\infty} n(n+1) Q_n^2 \overline{\Delta g_n^2}. $$

For reasons of symmetry, $\overline{\Delta g}$ and $\overline{\Delta \eta}$ can be taken to be equal, and we finally get

(7.22,a) $$\overline{\Delta g^2} = \overline{\Delta \eta^2} = \frac{1}{8\gamma^2} \sum_{n=2}^{\infty} n(n+1) Q_n^2 \overline{\Delta g_n^2}$$

for the direct method.

For the coating method we must replace $Q_n$ by $q_n$, obtaining

(7.22,b) $$\overline{\Delta g^2} = \overline{\Delta \eta^2} = \frac{1}{8\gamma^2} \sum_{n=2}^{\infty} n(n+1) q_n^2 \overline{\Delta g_n^2}. $$

49
Relationship between $\overline{\Delta g}_n$ and the Covariance Function. Comparing eq. (7.20) with eq. (33) of [Kaula, 1959] and taking the relationship between conventional and fully normalized spherical harmonics into account, we find that

\[
(7.23) \quad \overline{\Delta g}_n = c_n,
\]

the coefficient of $n$-th degree in the development of the covariance function of gravity anomalies, $C(s)$, in a series of Legendre's polynomials:

\[
(7.24) \quad C(s) = \sum_{n=2}^{\infty} c_n P_n(\cos s).
\]

Mean Square Effect on $\delta_N$, $\delta_M$, $\delta_L$. In order to avoid confusion, we shall change our notation of the components of the gravity disturbance vector, now writing as subscripts, capital letters $N$, $M$, $L$ instead of $n$, $m$, $l$.

Then,

\[
(7.25a) \quad \delta_N = \overline{\Delta g} + \frac{2\gamma}{R} N, \quad \delta_M = \gamma g, \quad \delta_L = \gamma l.
\]

Denoting the RMS effect of the zones beyond $\gamma_0$ on these components by $\overline{\Delta \delta}_N$, $\overline{\Delta \delta}_M$, $\overline{\Delta \delta}_L$, respectively, we find from (7.11.a,b) and (7.22.a,b), taking (7.23) into account, the following formulas:
Direct Method

\[ \Delta \delta \mathbf{a}_N = \sum_{n=2}^{\infty} q_n^a c_n , \]

(7.26.a)

\[ \Delta \delta \mathbf{a}_M = \Delta \delta \mathbf{a}_L = \frac{1}{8} \sum_{n=2}^{\infty} n(n+1) q_n^a c_n . \]

If we use the coating \( \mu \), then

(7.25.b) \( \delta_N = \mu + \frac{\gamma \Delta N}{2R_N} \), \( \delta_M = \gamma \xi \), \( \delta_L = \gamma \eta \),

and we have instead:

Coating Method

\[ \Delta \delta \mathbf{a}_N = \frac{1}{16} \sum_{n=2}^{\infty} q_n^a c_n , \]

(7.26.b)

\[ \Delta \delta \mathbf{a}_M = \Delta \delta \mathbf{a}_L = \frac{1}{8} \sum_{n=2}^{\infty} n(n+1) q_n^a c_n . \]

Numerical Results. First we give tables for the \( q_n \) (Table 7.1.a, taken from [Molodenskii et.al., 1962]), and for \( q_n \) (Table 7.1.b, computed by eqs. (7.9.b)).
Table 7.1.a

Coefficients \( Q_n \) for the direct method as functions of radius \( Y_o \).

<table>
<thead>
<tr>
<th>( Y_o )</th>
<th>( Q_2 )</th>
<th>( Q_3 )</th>
<th>( Q_4 )</th>
<th>( Q_5 )</th>
<th>( Q_6 )</th>
<th>( Q_7 )</th>
<th>( Q_8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>+2.000</td>
<td>+1.000</td>
<td>+0.667</td>
<td>+0.500</td>
<td>+0.400</td>
<td>+0.333</td>
<td>+0.286</td>
</tr>
<tr>
<td>5°</td>
<td>+1.801</td>
<td>+0.802</td>
<td>+0.470</td>
<td>+0.304</td>
<td>+0.206</td>
<td>+0.141</td>
<td>+0.095</td>
</tr>
<tr>
<td>8.99°(1000km)</td>
<td>+1.634</td>
<td>+0.639</td>
<td>+0.312</td>
<td>+0.152</td>
<td>+0.061</td>
<td>+0.004</td>
<td>-0.032</td>
</tr>
<tr>
<td>10°</td>
<td>+1.593</td>
<td>+0.596</td>
<td>+0.274</td>
<td>+0.118</td>
<td>+0.030</td>
<td>-0.024</td>
<td>-0.056</td>
</tr>
<tr>
<td>13.49°(1500km)</td>
<td>+1.457</td>
<td>+0.472</td>
<td>+0.159</td>
<td>+0.015</td>
<td>-0.058</td>
<td>-0.095</td>
<td>-0.110</td>
</tr>
<tr>
<td>17.99°(2000km)</td>
<td>+1.306</td>
<td>+0.339</td>
<td>+0.049</td>
<td>-0.070</td>
<td>-0.116</td>
<td>-0.126</td>
<td>-0.118</td>
</tr>
<tr>
<td>20°</td>
<td>+1.247</td>
<td>+0.290</td>
<td>+0.011</td>
<td>-0.094</td>
<td>-0.127</td>
<td>-0.125</td>
<td>-0.105</td>
</tr>
<tr>
<td>25°</td>
<td>+1.133</td>
<td>+0.204</td>
<td>-0.044</td>
<td>-0.116</td>
<td>-0.120</td>
<td>-0.093</td>
<td>-0.057</td>
</tr>
<tr>
<td>30°</td>
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<td>+0.161</td>
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<td>-0.103</td>
<td>-0.087</td>
<td>-0.050</td>
<td>-0.013</td>
</tr>
<tr>
<td>40°</td>
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<td>+0.144</td>
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<td>-0.080</td>
<td>-0.057</td>
<td>-0.024</td>
<td>+0.002</td>
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<tr>
<td>50°</td>
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<td>-0.116</td>
<td>-0.067</td>
<td>-0.006</td>
<td>+0.033</td>
</tr>
<tr>
<td>60°</td>
<td>+1.040</td>
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<td>-0.188</td>
<td>-0.134</td>
<td>-0.009</td>
<td>+0.069</td>
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</tr>
<tr>
<td>70°</td>
<td>+0.958</td>
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<td>-0.240</td>
<td>-0.053</td>
<td>+0.095</td>
<td>+0.081</td>
<td>-0.015</td>
</tr>
<tr>
<td>80°</td>
<td>+0.807</td>
<td>-0.260</td>
<td>-0.187</td>
<td>+0.073</td>
<td>+0.111</td>
<td>-0.016</td>
<td>-0.075</td>
</tr>
<tr>
<td>90°</td>
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<td>-0.068</td>
<td>+0.128</td>
<td>+0.023</td>
<td>-0.076</td>
<td>-0.011</td>
</tr>
<tr>
<td>100°</td>
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<td>+0.024</td>
<td>+0.090</td>
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<tr>
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<td>+0.049</td>
<td>+0.038</td>
<td>-0.043</td>
<td>+0.008</td>
<td>+0.020</td>
</tr>
<tr>
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<td>+0.047</td>
<td>+0.028</td>
<td>-0.035</td>
<td>+0.010</td>
<td>+0.010</td>
</tr>
<tr>
<td>130°</td>
<td>+0.430</td>
<td>-0.237</td>
<td>+0.078</td>
<td>+0.019</td>
<td>-0.050</td>
<td>+0.034</td>
<td>-0.003</td>
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<td>-0.263</td>
<td>+0.140</td>
<td>-0.039</td>
<td>-0.025</td>
<td>+0.050</td>
<td>-0.045</td>
</tr>
<tr>
<td>150°</td>
<td>+0.291</td>
<td>-0.234</td>
<td>+0.171</td>
<td>-0.107</td>
<td>+0.050</td>
<td>-0.006</td>
<td>-0.023</td>
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<tr>
<td>160°</td>
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<td>+0.127</td>
<td>-0.107</td>
<td>+0.086</td>
<td>-0.064</td>
<td>+0.044</td>
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<td>170°</td>
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<td>-0.044</td>
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<td>-0.041</td>
<td>+0.039</td>
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<td>0.000</td>
<td>0.000</td>
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<td>0.000</td>
<td>0.000</td>
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Table 7.1.b

Coefficients $q_n$ for the coating method as functions of radius $\gamma_o$

<table>
<thead>
<tr>
<th>$\gamma_o$</th>
<th>$q_2$</th>
<th>$q_3$</th>
<th>$q_4$</th>
<th>$q_5$</th>
<th>$q_6$</th>
<th>$q_7$</th>
<th>$q_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>+2.000</td>
<td>+1.000</td>
<td>+0.667</td>
<td>+0.500</td>
<td>+0.400</td>
<td>+0.333</td>
<td>+0.286</td>
</tr>
<tr>
<td>5°</td>
<td>+1.565</td>
<td>+0.697</td>
<td>+0.408</td>
<td>+0.265</td>
<td>+0.179</td>
<td>+0.123</td>
<td>+0.083</td>
</tr>
<tr>
<td>8.99°(1000km)</td>
<td>+1.226</td>
<td>+0.465</td>
<td>+0.215</td>
<td>+0.095</td>
<td>+0.026</td>
<td>-0.016</td>
<td>-0.042</td>
</tr>
<tr>
<td>10°</td>
<td>+1.142</td>
<td>+0.408</td>
<td>+0.170</td>
<td>+0.056</td>
<td>-0.007</td>
<td>-0.044</td>
<td>-0.066</td>
</tr>
<tr>
<td>13.49°(1500km)</td>
<td>+0.858</td>
<td>+0.222</td>
<td>+0.024</td>
<td>-0.062</td>
<td>-0.102</td>
<td>-0.119</td>
<td>-0.121</td>
</tr>
<tr>
<td>17.99°(2000km)</td>
<td>+0.512</td>
<td>+0.009</td>
<td>-0.128</td>
<td>-0.170</td>
<td>-0.173</td>
<td>-0.156</td>
<td>-0.129</td>
</tr>
<tr>
<td>20°</td>
<td>+0.366</td>
<td>-0.075</td>
<td>-0.182</td>
<td>-0.202</td>
<td>-0.185</td>
<td>-0.152</td>
<td>-0.113</td>
</tr>
<tr>
<td>25°</td>
<td>+0.033</td>
<td>-0.251</td>
<td>-0.275</td>
<td>-0.233</td>
<td>-0.169</td>
<td>-0.101</td>
<td>-0.040</td>
</tr>
<tr>
<td>30°</td>
<td>-0.255</td>
<td>-0.374</td>
<td>-0.309</td>
<td>-0.206</td>
<td>-0.103</td>
<td>-0.018</td>
<td>+0.040</td>
</tr>
<tr>
<td>40°</td>
<td>-0.676</td>
<td>-0.460</td>
<td>-0.228</td>
<td>-0.045</td>
<td>+0.066</td>
<td>+0.103</td>
<td>+0.085</td>
</tr>
<tr>
<td>50°</td>
<td>-0.878</td>
<td>-0.363</td>
<td>-0.037</td>
<td>+0.116</td>
<td>+0.127</td>
<td>+0.057</td>
<td>-0.021</td>
</tr>
<tr>
<td>60°</td>
<td>-0.875</td>
<td>-0.156</td>
<td>+0.138</td>
<td>+0.153</td>
<td>+0.040</td>
<td>-0.058</td>
<td>-0.071</td>
</tr>
<tr>
<td>70°</td>
<td>-0.707</td>
<td>+0.070</td>
<td>+0.206</td>
<td>+0.061</td>
<td>-0.074</td>
<td>-0.073</td>
<td>+0.007</td>
</tr>
<tr>
<td>80°</td>
<td>-0.433</td>
<td>+0.235</td>
<td>+0.149</td>
<td>-0.064</td>
<td>-0.090</td>
<td>+0.014</td>
<td>+0.060</td>
</tr>
<tr>
<td>90°</td>
<td>-0.121</td>
<td>+0.293</td>
<td>+0.018</td>
<td>-0.119</td>
<td>-0.007</td>
<td>+0.068</td>
<td>+0.003</td>
</tr>
<tr>
<td>100°</td>
<td>+0.165</td>
<td>+0.241</td>
<td>-0.101</td>
<td>-0.070</td>
<td>+0.070</td>
<td>+0.020</td>
<td>-0.050</td>
</tr>
<tr>
<td>110°</td>
<td>+0.376</td>
<td>+0.115</td>
<td>-0.147</td>
<td>+0.026</td>
<td>+0.060</td>
<td>-0.048</td>
<td>-0.011</td>
</tr>
<tr>
<td>120°</td>
<td>+0.484</td>
<td>-0.028</td>
<td>-0.105</td>
<td>+0.086</td>
<td>-0.012</td>
<td>-0.039</td>
<td>+0.040</td>
</tr>
<tr>
<td>130°</td>
<td>+0.483</td>
<td>-0.137</td>
<td>-0.014</td>
<td>+0.068</td>
<td>-0.058</td>
<td>+0.020</td>
<td>+0.015</td>
</tr>
<tr>
<td>140°</td>
<td>+0.406</td>
<td>-0.178</td>
<td>+0.065</td>
<td>+0.002</td>
<td>-0.035</td>
<td>+0.041</td>
<td>-0.029</td>
</tr>
<tr>
<td>150°</td>
<td>+0.275</td>
<td>-0.153</td>
<td>+0.092</td>
<td>-0.050</td>
<td>+0.018</td>
<td>+0.004</td>
<td>-0.016</td>
</tr>
<tr>
<td>160°</td>
<td>+0.138</td>
<td>-0.088</td>
<td>+0.066</td>
<td>-0.051</td>
<td>+0.038</td>
<td>-0.027</td>
<td>+0.018</td>
</tr>
<tr>
<td>170°</td>
<td>+0.037</td>
<td>-0.025</td>
<td>+0.021</td>
<td>-0.019</td>
<td>+0.017</td>
<td>-0.015</td>
<td>+0.014</td>
</tr>
</tbody>
</table>
| 180°       | 0.000  | 0.000  | 0.000  | 0.000  | 0.000  | 0.000  | 0.000  | 53
Now we are going to evaluate formulas (7.26.a,b) for the direct and the coating method, for different sets of $c^*_n$. First we use the maximum estimates for $c^*_n$ of [Kaula, 1959, p. 2419]:

$$c_2 = 15, \ c_3 = 43, \ c_4 = 30, \ c_5 = c_6 = c_7 = c_8 = 25 \ \text{mgal}^2$$

The results are given in Table 7.2. The summation has first been extended to $n = 8$; higher order $c^*_n$ have been neglected. In order to see the convergence of the series (7.26.a,b) we have also performed the summation up to $n = 5$ only.
Table 7.2.

RMS influence of the zones beyond a radius $\psi_o$ on $\delta_N$, $\delta_M$, $\delta_L$, based on the maximum estimates for the degree variances of [Kaula, 1959]

<table>
<thead>
<tr>
<th>$\psi_o$</th>
<th>Summation up to 8th order</th>
<th>Summation up to 5th order</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Direct $\Delta \delta_N$, $\Delta \delta_M = \Delta \delta_L$</td>
<td>Direct $\Delta \delta_N$, $\Delta \delta_M = \Delta \delta_L$</td>
</tr>
<tr>
<td></td>
<td>mgal</td>
<td>mgal</td>
</tr>
<tr>
<td>$5^\circ$</td>
<td>9.4 10.6</td>
<td>2.0 9.3</td>
</tr>
<tr>
<td>$8.99^\circ (1000\text{km})$</td>
<td>7.8 8.1</td>
<td>1.4 6.0</td>
</tr>
<tr>
<td>$10^\circ$</td>
<td>7.5 7.7</td>
<td>1.3 5.4</td>
</tr>
<tr>
<td>$13.49^\circ (1500\text{km})$</td>
<td>6.6 6.7</td>
<td>1.0 4.3</td>
</tr>
<tr>
<td>$17.99^\circ (2000\text{km})$</td>
<td>5.7 5.9</td>
<td>0.7 4.3</td>
</tr>
<tr>
<td>$20^\circ$</td>
<td>5.4 5.6</td>
<td>0.6 4.4</td>
</tr>
<tr>
<td>$25^\circ$</td>
<td>4.7 4.8</td>
<td>0.7 4.5</td>
</tr>
<tr>
<td>$30^\circ$</td>
<td>4.4 4.1</td>
<td>0.8 4.7</td>
</tr>
<tr>
<td>$40^\circ$</td>
<td>4.2 3.8</td>
<td>1.1 5.2</td>
</tr>
<tr>
<td>$50^\circ$</td>
<td>4.2 4.0</td>
<td>1.1 4.6</td>
</tr>
<tr>
<td>$60^\circ$</td>
<td>4.3 4.3</td>
<td>0.9 4.0</td>
</tr>
<tr>
<td>$70^\circ$</td>
<td>4.1 4.3</td>
<td>0.8 3.3</td>
</tr>
<tr>
<td>$80^\circ$</td>
<td>3.8 4.2</td>
<td>0.6 3.1</td>
</tr>
<tr>
<td>$90^\circ$</td>
<td>3.3 3.7</td>
<td>0.5 2.8</td>
</tr>
<tr>
<td>$100^\circ$</td>
<td>2.7 3.0</td>
<td>0.5 2.6</td>
</tr>
<tr>
<td>$110^\circ$</td>
<td>2.3 2.4</td>
<td>0.5 2.2</td>
</tr>
<tr>
<td>$120^\circ$</td>
<td>2.2 2.3</td>
<td>0.5 2.2</td>
</tr>
<tr>
<td>$130^\circ$</td>
<td>2.4 2.6</td>
<td>0.5 2.2</td>
</tr>
<tr>
<td>$140^\circ$</td>
<td>2.4 3.0</td>
<td>0.5 2.2</td>
</tr>
<tr>
<td>$150^\circ$</td>
<td>2.2 2.8</td>
<td>0.4 1.8</td>
</tr>
<tr>
<td>$160^\circ$</td>
<td>1.6 2.4</td>
<td>0.2 1.3</td>
</tr>
<tr>
<td>$170^\circ$</td>
<td>0.6 1.1</td>
<td>0.1 0.5</td>
</tr>
<tr>
<td>$180^\circ$</td>
<td>0.0 0.0</td>
<td>0.0 0.0</td>
</tr>
</tbody>
</table>

Fig. 7.1 shows a graphical representation of the first part of Table 7.2.
Figure 7.1

Graphical representation of the values of Table 7.2 for summation up to 8th order.
One would expect a rather monotonous decrease of the curves from 0° to 180°. The non-monotonousness is especially apparent in the coating method. It is probably due mainly to neglecting terms of higher than 8th order: it is still stronger if we use only terms up to 5th order as Table 7.2. shows. It seems, however, to be hardly worth-while to go up higher than to 8th order, in view of the increasing complexity of the formulas for Qₙ and qₙ and the uncertainty of the numerical material for cₙ. Since the higher order terms can be expected to influence mainly the dips, it is justifiable to "bridge" them empirically in order to get a smooth monotonous function.

More recent values for the cₙ can be obtained from [Kaula, 1961] and from [Uotila, 1962]. Both give a development of the gravity anomaly field in fully normalized spherical harmonics. Writing (7.20) in fully normalized harmonic coefficients we get [Kaula, 1959, eq. (33)]

\[ c_n = \frac{\Delta g}{\Delta g_n} = \sum_{m=0}^{n} (a_{nm}^2 + b_{nm}^2). \]

By this equation, the cₙ can be computed.

From [Uotila, 1962, Table 5] we get (flattening 1/297):

\[ c_2 = 22, \quad c_3 = 42, \quad c_4 = 32. \]

Only values up to 4th order are given. The almost perfect agreement
with the $c_n$ of [Kaula, 1959] is accidental.

From [Kaula, 1961] we get for the flattening $1/298.24$:

$$c_2 = 0.7, \quad c_3 = 19, \quad c_4 = 11, \quad c_5 = 6, \quad c_6 = 11, \quad c_7 = c_8 = 5;$$

if we use the flattening $1/297$, then $a_2$ is changed to $a_2 = 17$. The influence of the distant zones, according to these values, is given in Table 7.3 and in Fig. 7.2. We see that the effect of a wrong flattening is much stronger in the direct than in the coating method. This can be expected since, as Tables 7.1.a and 7.1.b show, $q_2$ is smaller and changes sign more often than $q_6$. 

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Table 7.3.

RMS influence of the zones beyond a radius $y$ on $\delta_N$, $\delta_M$, $\delta_L$, for two different flattenings, based on a spherical harmonics development obtained by a combined adjustment of gravimetric, astro-geodetic, and satellite data [Kaula, 1961] (summation up to 8th order).

<table>
<thead>
<tr>
<th>$y$</th>
<th>Direct</th>
<th>Coating</th>
<th>Direct</th>
<th>Coating</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Delta \delta$</td>
<td>$\Delta \delta$</td>
<td>$\Delta \delta$</td>
<td>$\Delta \delta$</td>
</tr>
<tr>
<td></td>
<td>N</td>
<td>M</td>
<td>N</td>
<td>M</td>
</tr>
<tr>
<td></td>
<td>mgal</td>
<td>mgal</td>
<td>mgal</td>
<td>mgal</td>
</tr>
<tr>
<td>5°</td>
<td>4.3</td>
<td>5.7</td>
<td>0.9</td>
<td>5.0</td>
</tr>
<tr>
<td>8.99°(1000km)</td>
<td>3.3</td>
<td>4.1</td>
<td>0.6</td>
<td>3.0</td>
</tr>
<tr>
<td>10°</td>
<td>3.1</td>
<td>3.8</td>
<td>0.5</td>
<td>2.6</td>
</tr>
<tr>
<td>13.49°(1500km)</td>
<td>2.5</td>
<td>3.1</td>
<td>0.3</td>
<td>1.9</td>
</tr>
<tr>
<td>17.99°(2000km)</td>
<td>2.0</td>
<td>2.5</td>
<td>0.3</td>
<td>2.2</td>
</tr>
<tr>
<td>20°</td>
<td>1.8</td>
<td>2.4</td>
<td>0.3</td>
<td>2.4</td>
</tr>
<tr>
<td>25°</td>
<td>1.4</td>
<td>1.9</td>
<td>0.4</td>
<td>2.7</td>
</tr>
<tr>
<td>30°</td>
<td>1.2</td>
<td>1.5</td>
<td>0.5</td>
<td>2.9</td>
</tr>
<tr>
<td>40°</td>
<td>1.1</td>
<td>1.3</td>
<td>0.6</td>
<td>3.0</td>
</tr>
<tr>
<td>50°</td>
<td>1.2</td>
<td>1.4</td>
<td>0.5</td>
<td>2.4</td>
</tr>
<tr>
<td>60°</td>
<td>1.2</td>
<td>1.5</td>
<td>0.3</td>
<td>1.6</td>
</tr>
<tr>
<td>70°</td>
<td>1.4</td>
<td>1.9</td>
<td>0.3</td>
<td>1.5</td>
</tr>
<tr>
<td>80°</td>
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<td>2.1</td>
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</tr>
<tr>
<td>90°</td>
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<td>1.9</td>
<td>0.3</td>
<td>1.7</td>
</tr>
<tr>
<td>100°</td>
<td>1.3</td>
<td>1.7</td>
<td>0.3</td>
<td>1.6</td>
</tr>
<tr>
<td>110°</td>
<td>1.1</td>
<td>1.3</td>
<td>0.2</td>
<td>1.2</td>
</tr>
<tr>
<td>120°</td>
<td>1.0</td>
<td>1.2</td>
<td>0.2</td>
<td>0.9</td>
</tr>
<tr>
<td>130°</td>
<td>1.2</td>
<td>1.4</td>
<td>0.2</td>
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<tr>
<td>140°</td>
<td>1.3</td>
<td>1.7</td>
<td>0.2</td>
<td>1.1</td>
</tr>
<tr>
<td>150°</td>
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<td>0.2</td>
<td>1.0</td>
</tr>
<tr>
<td>160°</td>
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<td>0.1</td>
<td>0.7</td>
</tr>
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<td>0.3</td>
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<tr>
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<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>
Figure 7.2

Graphical representation of $\Delta \bar{\delta}_M = \Delta \bar{\delta}_L$ of Table 7.3.
Finally we consider the spherical harmonics expansion of Zhongolovich quoted and used in [Molodenskii et al., 1962, pp. 145-6]. Here,

\[ c_2 = 61, \quad c_3 = 96, \quad c_4 = 12, \quad c_5 = 8, \quad c_6 = 14, \quad c_7 = 5, \quad c_8 = 8 \text{ mgal} \]

(computed by \( c_n = \Delta g_n^2 \)). With these coefficients we find values considerably higher than those given above (Table 7.4). They are, however, probably too high since the development of Zhongolovich is already out of date (it was computed in 1952).
Table 7.4
RMS influence of the zones beyond a radius $\gamma$ on $\delta_N$, $\delta_M$, $\delta_L$, based on the spherical harmonics development of Zhongolovich, 1952 (summation up to 8th order).

<table>
<thead>
<tr>
<th>$\gamma$ (degrees)</th>
<th>Direct $\Delta\delta_N$</th>
<th>Direct $\Delta\delta_M = \Delta\delta_L$</th>
<th>Coating $\Delta\delta_N$</th>
<th>Coating $\Delta\delta_M = \Delta\delta_L$</th>
</tr>
</thead>
<tbody>
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<td>16.0</td>
<td>3.5</td>
<td>13.9</td>
</tr>
<tr>
<td>8.99° (1000km)</td>
<td>14.2</td>
<td>13.6</td>
<td>2.6</td>
<td>10.1</td>
</tr>
<tr>
<td>10°</td>
<td>13.8</td>
<td>13.0</td>
<td>2.4</td>
<td>9.2</td>
</tr>
<tr>
<td>13.49° (1500km)</td>
<td>12.3</td>
<td>11.5</td>
<td>1.8</td>
<td>6.6</td>
</tr>
<tr>
<td>-17.99° (2000km)</td>
<td>10.7</td>
<td>9.9</td>
<td>1.0</td>
<td>4.2</td>
</tr>
<tr>
<td>20°</td>
<td>10.2</td>
<td>9.3</td>
<td>0.8</td>
<td>3.7</td>
</tr>
<tr>
<td>25°</td>
<td>9.1</td>
<td>8.2</td>
<td>0.7</td>
<td>4.0</td>
</tr>
<tr>
<td>30°</td>
<td>8.4</td>
<td>7.5</td>
<td>1.1</td>
<td>5.3</td>
</tr>
<tr>
<td>40°</td>
<td>8.1</td>
<td>7.2</td>
<td>1.7</td>
<td>7.4</td>
</tr>
<tr>
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<td>7.3</td>
<td>1.9</td>
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</tr>
<tr>
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<td>7.2</td>
<td>1.8</td>
<td>6.4</td>
</tr>
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<tr>
<td>80°</td>
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<td>6.5</td>
<td>1.0</td>
<td>4.3</td>
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<tr>
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<td>0.8</td>
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</tr>
<tr>
<td>120°</td>
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</tr>
<tr>
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</tr>
<tr>
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<td>1.5</td>
</tr>
<tr>
<td>170°</td>
<td>0.6</td>
<td>0.8</td>
<td>0.1</td>
<td>0.5</td>
</tr>
<tr>
<td>180°</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>
Conclusion. On the whole, the values of Fig. 1a,b, based on [Kaula, 1959] and confirmed by [Uotila, 1962], might be the most realistic estimates, apart from the dips. We should like to propose the smoothed values of Table 7.5.

Table 7.5

RMS influence of the zones beyond a radius $\psi_o$ on $\delta_N'$, $\delta_M'$, $\delta_L'$; proposed values.

<table>
<thead>
<tr>
<th>$\psi_o$</th>
<th>Direct $\Delta\delta_N'$</th>
<th>$\Delta\delta_M' = \Delta\delta_L'$</th>
<th>Coating $\Delta\delta_N'$</th>
<th>$\Delta\delta_M' = \Delta\delta_L'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.99° (1000km)</td>
<td>8</td>
<td>8</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>13.49° (1500km)</td>
<td>7</td>
<td>7</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>17.99° (2000km)</td>
<td>6</td>
<td>6</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>20°</td>
<td>6</td>
<td>6</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>25°</td>
<td>5</td>
<td>5</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>30°</td>
<td>5</td>
<td>5</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>45°</td>
<td>5</td>
<td>5</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>60°</td>
<td>4</td>
<td>5</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>90°</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>120°</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>150°</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>
These values are referring to a flattening in the neighborhood of 1/297. If we use a better flattening, then these values should be somewhat less.

We see that for \( \gamma_o > 20^\circ \) or \( 30^\circ \) the influence of the distant zones decreases very slowly. It is, therefore, impractical to extend the integration much farther than \( 20^\circ \) (coating method) or \( 30^\circ \) (direct method), unless it is extended over the whole earth.

The effect of the distant zones on \( \delta_M \) and \( \delta_L \) is somewhat less in the coating than in the direct method. The effect on \( \delta_N \) is much less. By using another method which will be described in the next section, however, the influence of the distant zones on \( \delta_N \) can be made still smaller.

8. PRACTICAL COMPUTATION OF THE GRAVITY DISTURBANCES

We shall describe now a method for the practical computation of gravity disturbances. Since the geoid undulations \( N \) have been computed for large parts of the world—the accuracy is as good as the existing gravity material—we can assume them to be known. Therefore, the coating method can be applied which has certain advantages over the direct method: simplicity, less influence of the distant zones, etc.

**Upward Continuation Method.** For the vertical component \( \delta_n \), there is, however, a method which is still better than the coating method.
Compute the vertical component at ground (or at the geoid) by

\[ (8.1) \quad \delta_n^o = \Delta g + \frac{2y}{R^N}. \]

Since \( \delta_n = -r \frac{\partial \psi}{\partial r} \) is a harmonic function, Poisson's integral for harmonic functions can be applied to give

\[ r \delta_n = \frac{R^2 - R^2}{4\pi R} \int_0^{2\pi} \frac{\delta_0 \phi}{D_s^3} \cdot R^2 d\sigma \]

where

\[ r = R + H, \]
\[ D_s = \sqrt{R^2 + r^2 - 2Rr \cos \psi}. \]

Setting as before

\[ t = \frac{R}{r} = \frac{R}{R + H}, \]
\[ D = \sqrt{1 - 2t \cos \psi + t^2} \]

we have

\[ D_s = rD. \]

Hence

\[ (8.2) \quad \delta_n^o = \frac{t^2(1-t^2)}{4\pi} \int_0^{2\pi} \frac{\delta_0 \phi}{D_s^3} d\sigma \]

where \( \delta_n^o \) is given by (8.1).

This formula (8.2) is called "upward continuation integral"; it can also be used for the upward continuation of the gravity anomalies \( \Delta g \), since \( r \Delta g \) is also a harmonic function. Here,
however, it is used for the upward continuation of the gravity
disturbances.

If we compare (8.2) to the corresponding formula for the coating
method,

\[(8.3) \quad \delta_n = \frac{1}{2\pi} \int_{\Omega} \mu \cdot \frac{t^2}{D^3} (1 - t \cos \psi) \, d\sigma \]

where

\[\mu = \Delta g + \frac{3\gamma}{2R} N,\]

then we find that the integrand in (8.3) decreases, with increasing
distance \(D\), like \(1/D\), whereas in the upward continuation integral
(8.2) it decreases much faster, like \(1/D^3\). It can, therefore, be
expected that the influence of the distant zones in the upward con-
tinuation method is much smaller than in the coating method.

Indeed, if we put \(H = 0\), then \(\delta_n = \delta_n^0\), i.e., the gravity
disturbance is equal to the point value in \(P\) itself and it does not
at all depend on the other values and on the distant zones. Thus,
for \(H = 0\) the influence of the distant zones is rigorously zero in
the upward continuation method whereas, as we have seen in section 7,
this is not at all the case in the coating method (neither in the direct
method, as a matter of fact). From this we can conclude that, even
for higher elevations, only the nearest surroundings of \(P\) will be of
some influence.
Therefore, we can replace the terrestrial sphere by its tangential plane at P. If in this plane we assume a rectangular coordinate system x,y (origin P; the x-axis pointing northward, the y-axis pointing eastward), we have instead of (8.2) the plane formula

\[ (8.4) \quad \delta_n = \frac{H}{2\pi} \int_0^\infty \int_{-\infty}^{\infty} \frac{\delta_n^o}{(H^2 + x^2 + y^2)^{3/2}} \, dx \, dy. \]

In order to estimate the influence of the distant zones beyond a certain distance s from P we introduce polar coordinates s, \( \alpha \) by

\[ x = s \cos \alpha, \]
\[ y = s \sin \alpha, \]

getting

\[ \delta_n = \frac{H}{2\pi} \int_0^{2\pi} \int_0^\infty \frac{\delta_n^o}{(H^2 + s^2)^{3/2}} \, s \, ds \, d\alpha. \]

The influence \( \Delta \delta_n \) of the zones beyond the distance s is given by

\[ \Delta \delta_n = \frac{H}{2\pi} \int_{\alpha=0}^{2\pi} \int_{s=s_0}^\infty \frac{\delta_n^o}{(H^2 + s^2)^{3/2}} \, s \, ds \, d\alpha. \]

Since for large s, \( \sqrt{H^2 + s^2} \approx s \), we can simplify this to

\[ \Delta \delta_n = H \int_{\delta_n^o = s_0}^\infty \delta_n^o \, ds \]

where \( \delta_n^o \) is the average of \( \delta_n^o \) over the circle with radius s.

If M is the upper limit of the absolute amount of \( \delta_n^o \),

\[ \delta_n^o < M, \]
we get
\[ \Delta \delta_n < HM \int_0^\infty \frac{ds}{s^2} = \frac{H}{s_0} M. \]

To get some kind of average effect \( \overline{\Delta \delta_n} \) rather than this maximum estimate, we may replace \( M \) by the average disturbance \( \overline{M} \) outside the radius \( s_0 \), which is certainly close to 0:

\[ (8.5) \quad \overline{\Delta \delta_n} = \frac{H}{s_0} \overline{M}. \]

From this, we see immediately that, to get the same error for different elevations, \( s_0 \) must be proportional to \( H \). If, e.g., \( s_0 = 10 H \), then

\[ \overline{\Delta \delta_n} = 0.1 M. \]

Since \( \overline{M} \) will hardly exceed 10 mgal, \( \overline{\Delta \delta_n} \), in this case, will be smaller than 1 mgal. So, it should, in general, be sufficient to go as far as 10 times the elevation. For larger \( H \), conditions are even more favorable since, the larger \( s_0 \) is, the smaller can \( \overline{M} \) be expected to be.

Plane formulas analogous to (8.4) also hold for the horizontal components \( \delta_m \) and \( \delta_\ell \):

\[ (8.6) \quad \left\{ \begin{aligned} \delta_m &= \frac{H}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(H^2 + x^2 + y^2)^{3/2}} \left\{ \begin{aligned} \delta_m^o \\ \delta_\ell^o \end{aligned} \right\} dx dy, \\
\delta_\ell &= \frac{H}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(H^2 + x^2 + y^2)^{3/2}} \left\{ \begin{aligned} \delta_m^o \\ \delta_\ell^o \end{aligned} \right\} dx dy. \end{aligned} \right. \]

But, since \( \delta_m^o = \gamma \xi \), \( \delta_\ell^o = \gamma \eta \), the use of these formulas would require the knowledge of the components \( \xi, \eta \) of the deflection of the vertical in the entire neighborhood of \( P \). If we know \( \xi \) and \( \eta \), then
these formulas can indeed be applied with advantage. (To compute $\delta$ and $\eta$ by differentiating the N-field will not, in general, be accurate enough.) In the following, however, we shall regard only $\Delta g$ and $N$ as known, so that $\delta_m$ and $\delta_\ell$ must be computed by the coating method.

Formulas and Approximations. For easy reference we put the relevant formulas together:

\begin{align*}
(8.7.a) \quad \delta_n &= \frac{t^2(1-t^2)}{4\pi} \int_\sigma \frac{\delta \sigma^o}{D^3} d\sigma, \\
(8.7.b) \quad \left\{ \begin{array}{l}
\delta_m \\
\delta_\ell
\end{array} \right\} = \frac{t^3}{2\pi} \int_\sigma \frac{\mu}{D^3} \sin \left\{ \begin{array}{l}
\cos \alpha \\
\sin \alpha
\end{array} \right\} d\sigma
\end{align*}

where

\begin{align*}
t &= \frac{R}{R+H}, \quad D = \sqrt{1 - 2t \cos \Psi + t^2}, \quad d\sigma = \sin \Psi d\Psi d\alpha; \\
\delta^o &= \Delta g + \frac{2\Psi}{R} N, \\
\mu &= \Delta g + \frac{3\Psi}{2R} N.
\end{align*}

At least for the nearest neighborhood of $P$, the spherical formulas can be approximated by plane ones:

\begin{align*}
(8.8.a) \quad \delta_n &= \frac{H}{2\pi} \int x \frac{\delta \sigma^o}{D^3} dx dy, \\
(8.8.b) \quad \left\{ \begin{array}{l}
\delta_m \\
\delta_\ell
\end{array} \right\} = \frac{1}{2\pi} \int \frac{\mu}{D^3} \left\{ \begin{array}{l}
x \\
y
\end{array} \right\} dx dy
\end{align*}
where

\[(8.9) \quad D_0 = \sqrt{H^2 + x^2 + y^2}.\]

They follow from (8.7.a,b) by a series development with respect to \(Y = \frac{s}{R}\) and to \(\frac{H}{R}\) if we put \(s \cos \alpha = x, s \sin \alpha = y\).

In order to get an estimate on the validity of these approximations, we develop \(1/D^3\) or, which is equivalent,

\[
\frac{1}{D^3} = \frac{1}{(R+H)^3D^3} - \frac{1}{(R^2+H^2-2Rr \cos Y)^{3/2}}
\]

in this manner, finding

\[(8.10) \quad \frac{1}{D^3} = \frac{1}{D_0^3} \left( 1 - \frac{3}{2} \frac{H s^2}{R D_0^2} + \frac{1}{8} \frac{s^4}{R^3 D_0^2} + \ldots \right).\]

Here \(1/D^3\) is spherical and \(1/D_0^3\) is the equivalent plane quantity (8.9); and \(s^2 = x^2 + y^2\).

The first small quantity in the bracket is

\[
\frac{3}{2} \frac{H s^2}{R D_0^2} \approx \frac{3H}{2R},
\]

independent of the distances. If \(H = 63.7\) km, this is 0.015 or 1.5\%; if \(H = 637\) km it is 15\%. Since the higher up we go the smaller the gravity disturbances are, this can be neglected anyway. The second term is

\[
\frac{s^4}{8R^3 D_0^2} \approx \frac{s^2}{8R^2}.
\]

For \(s = 1000\) km this is 0.003 or 0.3\%; for \(s = 2000\) km it is 0.01 or 1\%. This can be neglected, too; the more so as \(1/D_0^3\) itself becomes smaller with increasing distance.
We see that up to $20^\circ$ distance from $P$ we can, in general, use the plane formulas (8.8.a,b) instead of the spherical ones (8.7.a,b). In particular this holds practically always for the upward continuation integral (8.8.a), except for very high elevations ($>250$ km, say).

It is also useful in the case that in the coating formulas (8.7.b) we extend the integration only up to $20^\circ$ (as we have seen, it is impractical to go much farther unless the integration is extended over the whole earth, which of course is necessary for highest accuracy).

Now we have to show how the $x, y$ in (8.8.a,b) are to be computed. The simplest way is

$$x = R(\phi - \phi_P), \quad (8.11)$$
$$y = R \cos \phi_P (\lambda - \lambda_P);$$

another way,

$$x = R(\phi - \phi_P), \quad (8.12)$$
$$y = R \cos \phi (\lambda - \lambda_P).$$

The $x$'s are the same in both; the $y$'s differ in that in (8.11) we have the factor $\cos \phi_P$ and in (8.12), the factor $\cos \phi$. In our latitudes there is, for large $|\phi - \phi_P|$, a big difference between both formulas. Take $\phi_P = 40^\circ$, $|\phi - \phi_P| = 20^\circ$, i.e., $\phi = 20^\circ$ or $60^\circ$. Then

$$\cos 20^\circ = 0.940,$$
$$\cos 40^\circ = 0.766,$$
$$\cos 60^\circ = 0.500.$$
The differences between \( \cos \varphi \) and \( \cos \varphi_0 \) are 22% and 35%.

It is easily seen that (8.12) is preferable for larger distances; only if we use this formula rather than (8.11) is the reasoning following (8.10) applicable. The disadvantage of (8.12) is that the 5'\( \times \) 5', 10'\( \times \) 10', etc., blocks are not rectangles in the system \( x, y \) but trapezoids. In (8.11) these blocks are represented by rectangles.

Evaluation of the Integrals. In practice, the integrals must be evaluated by sums. The formulas (8.7.a,b) and 8.8.a,b) are of the type

\[
\begin{align*}
\delta = \int \int \mu \cdot f(\varphi, \lambda) \, dq,
\end{align*}
\]

\[
\begin{align*}
\delta = \int \int \mu \cdot f(x, y) \, dq.
\end{align*}
\]

In \( \delta \) we have \( \delta_n \) instead of \( \mu ; \) \( \Psi, \alpha, \Omega \) in (8.7,a,b) are, of course, functions of the geographical coordinates \( \varphi, \lambda \); \( dq \) is the element of area, defined by

\[
\begin{align*}
dq &= R^2 d\sigma = R^2 \cos \varphi \, d\varphi \, d\lambda, \\
dq &= dx \, dy,
\end{align*}
\]

respectively.

The preceding integrals can be approximated by sums:

\[
\begin{align*}
\delta = \sum \mu_k \, c_k
\end{align*}
\]

where

\[
\begin{align*}
\mu_k &= f(\varphi_k, \lambda_k) \cdot q_k, \\
\text{or} \\
\mu_k &= f(x_k, y_k) \cdot q_k,
\end{align*}
\]

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where \( q_k \) is the area of a certain block \( B_k \) (5' \( \times \) 5' or 1° \( \times \) 1°, say), and \( \phi_k, \lambda_k \) or \( x_k, y_k \) are the coordinates of the center of the block \( B_k \).

Especially for low elevations and for the nearest neighborhood of \( P \) it is better to compute \( c_k \) in the integrated form.

We limit ourselves to rectangular coordinates \( x, y \). Then (Fig. 8.1) we can compute \( c_k \) by

\[
(8.16) \quad c_k = F(1) - F(2) + F(3) - F(4)
\]

where

\[
(8.17) \quad F(x,y) = \iint f(x,y) \, dx \, dy
\]

is the indefinite double integral of \( f(x,y) \) so that

\[
f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}
\]
and, e.g., \( F(1) \) is the value of \( F(x, y) \) in point \( 1 \).

As a matter of fact, this is equivalent to computing \( c_k \) by

\[
(8.18.a) \quad c_k = \int \int_{B_k} f(x, y) \, dx \, dy
\]

rather than by

\[
(8.18.b) \quad c_k = f(x_k, y_k) \cdot q_k.
\]

If \( f(x, y) \) varies little in the block \( B_k \) considered, then (8.18.a) and (8.18.b) are practically equal. If not, then (8.18.a), i.e., (8.16), is to be preferred: if \( \mu \) is constant throughout the block, then (8.18.a) yields the correct result whereas (8.18.b) does not.

(8.18.b) is in this case, therefore, subject to a systematical error.

Let us now return to formulas (8.8.a,b). For (8.8.a),

\[
f_n(x, y) = \frac{H}{2\pi D^3} = \frac{H}{2\pi(x^2 + y^2 + z^2)^{3/2}};
\]

for (8.8.b),

\[
f_m(x, y) = \frac{x}{2\pi D^3} = \frac{x}{2\pi(x^2 + y^2 + z^2)^{3/2}};
\]

\[
f_L(x, y) = \frac{y}{2\pi D^3} = \frac{y}{2\pi(x^2 + y^2 + z^2)^{3/2}}.
\]

Integration by (8.17) yields

\[
(8.19.a) \quad F_n(x, y) = \frac{1}{2\pi} \tan^{-1} \frac{xy}{HD}.
\]
(8.19.b) \( F_m(x,y) = -\frac{1}{2\pi} \ln (y + D_0) \),

(8.19.c) \( F_n(x,y) = -\frac{1}{2\pi} \ln (x + D_0) \).

Then, the coefficients \( c_k \) for \( \delta_n, \delta_m, \delta_\ell \) can be computed by (8.16).

In order to be able to apply (8.16), the figure 1234 must be an exact rectangle. Therefore, \( x, y \) must in this case be computed by (8.11) instead of (8.12). Therefore, we are limited to the nearest surroundings of \( P \) where (8.11) and (8.12) are practically equal, say to a rectangle \( 3^\circ \times 4^\circ \) in the center of which \( P \) is situated (Region A, p. 25). Outside this rectangle, \( c_k \) can very well be computed by the simpler formula (8.18.b).

For the computation of the contribution of the innermost zones it is necessary to use as small blocks as possible. The smallest size is usually \( 5^\prime \times 5^\prime \). Even so, it might be necessary to take the deviation of the gravity anomalies from the mean anomaly of the respective block into account.

For this purpose we consider the gravity anomaly to be a linear function throughout the block. Then,

(8.20) \( \Delta g = a_0 + a_1 x + a_2 y \).

Inserting this in (8.8.a) and (8.8.b) and integrating over one block \( B_k \) only we get

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\[ \Delta \delta_n = H(a_j + a_1 j_1 + a_2 j_2) , \]
\[ \Delta \delta_m = a_j j_1 + a_1 j_{11} + a_2 j_{12} , \]
\[ \Delta \delta_k = a_j j_2 + a_1 j_{12} + a_2 j_{22} , \]

where \( \Delta \delta_n, \Delta \delta_m, \Delta \delta_k \) are the contributions of the block \( B_k \) to \( \delta_n \), \( \delta_m, \delta_k \) and

\( \text{8.21) } J = F(1) - F(2) + F(3) - F(4) \)

where

\[ F_0(x,y) = \frac{1}{2\pi} \iint \frac{dx\,dy}{D_0} = \frac{1}{2\pi} \tan(-1) \frac{xy}{HD_0} = \frac{1}{H} F_n(x,y) , \]
\[ F_1(x,y) = \frac{1}{2\pi} \iint \frac{x\,dx\,dy}{D_0} = -\frac{1}{2\pi} \ln(y+D_0) = F_m(x,y) , \]
\[ F_2(x,y) = \frac{1}{2\pi} \iint \frac{y\,dx\,dy}{D_0} = -\frac{1}{2\pi} \ln(x+D_0) = F_k(x,y) , \]

\( \text{8.22) } \)
\[ F_{11}(x,y) = \frac{1}{2\pi} \iint \frac{x^2\,dx\,dy}{D_0} = \frac{y}{2\pi} \ln(x+D_0) - \frac{H}{2\pi} \tan(-1) \frac{xy}{HD_0} , \]
\[ F_{12}(x,y) = \frac{1}{2\pi} \iint \frac{xy\,dx\,dy}{D_0} = -\frac{D_0}{2\pi} , \]
\[ F_{22}(x,y) = \frac{1}{2\pi} \iint \frac{y^2\,dx\,dy}{D_0} = \frac{x}{2\pi} \ln(y+D_0) - \frac{H}{2\pi} \tan(-1) \frac{xy}{HD_0} . \]

Consider now the contribution of the innermost block. Let the sides of the block be 2a, 2b, so that the points 1, 2, 3, 4 have the coordinates:

1(-a, -b) \hspace{2cm} 2(-a, +b)

3(+a, +b) \hspace{2cm} 4(+a, -b) .
Then,
\[ J_1 = J_2 = J_{12} = 0 \]
and

(8.23.a) \[ \Delta \delta_n = a \frac{H J}{J} \],
(8.23.b) \[ \Delta \delta_m = a_1 J_{11} \],
(8.23.c) \[ \Delta \delta_k = a_2 J_{22} \],

where

\[
J_0 = \frac{2}{\pi H} \tan (-1) \frac{ab}{HD_{0,1}},
\]

\[
J_{11} = \frac{b}{\pi} \ln \frac{D_{0,1} + a}{D_{0,1} - a} - \frac{2H}{\pi} \tan (-1) \frac{ab}{HD_{0,1}},
\]

(8.24)

\[
J_{22} = \frac{a}{\pi} \ln \frac{D_{0,1} + b}{D_{0,1} - b} - \frac{2H}{\pi} \tan (-1) \frac{ab}{HD_{0,1}} ;
\]

\[
D_{0,1} = \sqrt{a^2 + b^2 + H^2}.
\]

We see that for the central block \( \Delta \delta_n \) is the same as if we would use formula (8.16). For \( \Delta \delta_m \) and \( \Delta \delta_n \) (8.16) yields zero whereas the correct values are given by (8.23.b,c).

In order to estimate numerically the contribution of the innermost compartment, \( \Delta \delta_m \) and \( \Delta \delta_k \), we take it as a square with sides \( s \), so that

\[ a = b = s ; \quad D_{0,1} = \sqrt{2s^2 + H^2} . \]

Since, then,
\[ J_{11} = J_{22} \]

it is sufficient to consider \( \Delta \delta_m \) only.
We have

\[(8.25) \quad J_{11} = \frac{s}{\pi} \ln \frac{D_{0,1} + s}{D_{0,1} - s} - \frac{2H}{\pi} \tan^{-1} \left( \frac{s^2}{HD_{0,1}} \right) ; \]

we take

\[s = 4 \text{ km} \]

which corresponds to the average size of 5' x 5' blocks. Table 8.1 gives the values of \( J_{11} \) for several elevations \( H \).

Table 8.1

<table>
<thead>
<tr>
<th>( H , \text{km} )</th>
<th>( J_{11} , \text{km} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.24</td>
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<td>0.23</td>
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<td>0.04</td>
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<td>50</td>
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</tr>
<tr>
<td>100</td>
<td>0.00</td>
</tr>
</tbody>
</table>

If we have a north-south gradient \( a_1 = 1 \text{ mgal per } 1 \text{ km} \), then \( J_{11} \) is numerically equal to the effect \( \Delta \delta_m \), mgal; e.g., for \( H = 0 \) we have

\[\Delta \delta_m = 2.2 \text{ mgal} .\]
Since also gradients of 2 or even 3 mgal/km are possible, the
effect of the gradient on the horizontal components $\Delta \delta_m$ and $\Delta \delta_l$
can be considerable for lower elevations, $H < 5$ km. For $H > 5$ km,
the effect of the gradient is negligible.

We have also to investigate the influence of the eight blocks
adjacent to the central one. Since the effect is a maximum for
$H = 0$, we consider this case only. We again take the block to be a
square, but with arbitrary sides. We find that the effect of the
innermost block is

\[(8.26) \quad \Delta \delta_m = 0.561 a_1 s\]

and the combined effect of the eight surrounding blocks is only

\[(8.27) \quad \Delta' \delta_m = -0.071 a_1 s .\]

If the gradient $a_1 = 1$ mgal/km and $s = 4$ km ($5 \times 5$ blocks), then

\[\Delta' \delta_m = -0.3 \text{ mgal}.\]

These values hold for $H = 0$, but even here the effect of the gradient
in the surrounding blocks will be negligible in most cases, even if
it must be taken into account for the central block (for $H < 5$ km).

These considerations are equally valid for the component $\delta_l$;
we have only to replace $a_1$ by the east-west gradient $a_2$. For the
vertical component $\delta_n$, the influence of the gradients $a_1$ and $a_2$
will always be negligible.
An easy way of computing the gradients $a_1$ and $a_2$ which can also be used for high-speed computations is (Fig. 8.2):

\[ a_1 = \frac{\Delta g_1 - \Delta g_3}{4a}, \quad a_2 = \frac{\Delta g_2 - \Delta g_4}{4b}. \]

$\Delta g_1$ through $\Delta g_4$ are the gravity anomalies in the centers of the adjacent blocks 1 to 4, which we call substations. These point anomalies can also be replaced by the mean anomalies of the respective blocks, provided the gravity anomalies are sufficiently linear (this is presupposed in (8.28)).

**Station and Substations.** For reasons of symmetry the station $P$ at which $\delta_n, \delta_m, \delta_l$ are computed must be at the center of the central block. If we, however, use given 5 x 5 mean anomalies for which the division into blocks is fixed a priori, it is best to select 5 substations (centers of adjacent blocks) 0, 1, 2, 3, 4 (Fig. 8.2)
and compute the gravity disturbance in those three substations which are the corners of a triangle containing P in its interior (0, 1, 4 in Fig. 8.2). The value for P is then interpolated between these three substations. This procedure is particularly necessary for elevations smaller than about 70 km.

Summary of Formulas and Outline of Computation. In the following we shall give a computational method valid for any elevation from 0 to several hundred kilometers. It is, therefore, suitable for the computation of the gravity vector along rocket trajectories.

According to p. 25 we use the following regions

A \( \varphi_2 - \varphi_1 = 3^\circ \)  
\( \lambda_2 - \lambda_1 = 4^\circ \)  
5 \( \times \) 5 blocks,

B \( \varphi_2 - \varphi_1 = 7^\circ \)  
\( \lambda_2 - \lambda_1 = 9^\circ \)  
outside A, 20 \( \times \) 20 blocks,

C \( \varphi_2 - \varphi_1 = 25^\circ \)  
\( \lambda_2 - \lambda_1 = 30^\circ \)  
outside B, 1\(^\circ\) \( \times \) 1\(^\circ\) blocks,

D outside C, 5\(^\circ\) \( \times \) 5\(^\circ\) blocks.
1. For each block compute

\[ \mu = \Delta g + 0.231 \, N , \quad \delta_n = \Delta g + 0.308 \, N ; \]

\( \Delta g \): mean free-air anomalies, \( N \): geoid undulations; \( \Delta g, \mu, \delta_n \) in mgal, \( N \) in meters.

2. For each station \( P \) select 3 substations, according to Fig. 8.2. One substation, \( 0 \), is the center of the block in which \( P \) lies, the other two are centers of adjacent blocks so that the three substations form the corners of a triangle containing \( P \). The elevation of each substation is equal to the elevation \( H \) of \( P \).

The gravity disturbances are computed at each substation and are interpolated for \( P \).

In the following, only the substations will be considered; they will be denoted by \( Q \).

3. Now we consider one definite substation \( Q \) only.

Zone A.

Compute for all grid points ( = corners of blocks):

\[ x = R(\phi - \phi_Q) , \]

\[ y = R \cos \phi_Q (\lambda - \lambda_Q) ; \]

\[ D = \sqrt{x^2 + y^2 + H^2} ; \]
\[ F_n = \frac{1}{2\pi} \tan(-1) \frac{xy}{HD_0} , \]

\[ F_m = -\frac{1}{2\pi} \ln \left( \frac{y + D}{O} \right) , \]

\[ F_\ell = -\frac{1}{2\pi} \ln \left( \frac{x + D}{O} \right) . \]

Compute for each block

\[ c_n = F_n(1) - F_n(2) + F_n(3) - F_n(4) , \]

\[ c_m = F_m(1) - F_m(2) + F_m(3) - F_m(4) , \]

\[ c_\ell = F_\ell(1) - F_\ell(2) + F_\ell(3) - F_\ell(4) . \]

Be sure that the grid points 1, 2, 3, 4 are arranged as in Fig. 8.1.

For elevations \( H > 40 \) km the coefficients \( c \) can also be computed as in zone B.

**Zone B**

Compute for all block centers

\[ x = R(\varphi - \varphi_Q) , \]

\[ y = R \cos \varphi(\lambda - \lambda_Q) ; \]

\[ D_0 = \sqrt{x^2 + y^2 + H^2} ; \]

\[ c_n = \frac{H}{2\pi D_0^3} \cdot q , \]

\[ c_m = \frac{x}{2\pi D_0^3} \cdot q , \]

\[ c_\ell = \frac{y}{2\pi D_0^3} \cdot q ; \]

\( q \) is the area of the block.
Zone C can be treated either like Zone B or like Zone D.

Zone D

If many points are to be computed then the following computations need not be performed for all points. In this case it is sufficient to compute the effect of Zone D for a few points only and interpolate between these.

Compute:

\[ Y = \cos^{-1}\left( \sin \phi_Q \sin \varphi + \cos \phi_Q \cos \varphi \cos (\lambda - \lambda_Q) \right), \]

\[ \alpha = \text{ctn}^{-1}\left( \frac{\cos \phi_Q \sin \varphi - \sin \phi_Q \cos \varphi \cos (\lambda - \lambda_Q)}{\cos \varphi \sin (\lambda - \lambda_Q)} \right); \]

\[ t = \frac{R}{R + H}, \]

\[ D = \sqrt{1 - 2t \cos Y + t^2}; \]

\[ c_n = \begin{cases} 0, & H \leq 200 \text{ km} \\ \frac{t^3(1-t^2)}{4\pi D^3}, & Q, \ H > 200 \text{ km} \end{cases}; \]

\[ c_m = \frac{t^3}{2\pi D^3} \sin Y \cos \alpha \cdot q, \]

\[ c_L = \frac{t^3}{2\pi D^3} \sin Y \sin \alpha \cdot q. \]

4. For the substation Q considered compute by summation over all blocks i:
\[ \delta_n = \sum_{i} c_{n,i} \delta_n^i \]
\[ \delta_m = \sum_{i} c_{m,i} \mu_i \]
\[ \delta_l = \sum_{i} c_{l,i} \mu_i \]

For elevations < 5 km we have to add, to \( \delta_m \) and to \( \delta_l \), the terms \( \Delta \delta_m \) and \( \Delta \delta_l \), respectively:

\[ \Delta \delta_m = a_1 \cdot \left[ \frac{b}{\pi} \ln \frac{D_1+a}{D_1-a} - \frac{2H}{\pi} \tan \left( \frac{1}{ab} \right) \right] \]
\[ \Delta \delta_l = a_2 \cdot \left[ \frac{a}{\pi} \ln \frac{D_1+b}{D_1-b} - \frac{2H}{\pi} \tan \left( \frac{1}{ab} \right) \right] \]

where \( 2a,2b \) are the sides of the central 5' x 5' block and

\[ D_1 = \sqrt{a^2 + b^2 + H^2} \]
\[ a_1 = \frac{\Delta g_1 - \Delta g_3}{4a}, \quad a_2 = \frac{\Delta g_2 - \Delta g_4}{4b} \]

These formulas for \( a_1 \) and \( a_2 \) are valid for the central substation 0 (Fig. 8.2); for the other substations they are to be modified in an evident way.

5. Now denoting by \( x, y, z \) the coordinates in the geocentric system defined on p. 2 (z-axis = axis of rotation of the earth) compute the components of the gravity disturbance in this system by
\[
\begin{align*}
\delta_x &= - \cos \varphi \cos \lambda \cdot \delta_n - \sin \varphi \cos \lambda \cdot \delta_m - \sin \lambda \cdot \delta_z, \\
\delta_y &= - \cos \varphi \sin \lambda \cdot \delta_n - \sin \varphi \sin \lambda \cdot \delta_m + \cos \lambda \cdot \delta_z, \\
\delta_z &= - \sin \varphi \cdot \delta_n + \cos \varphi \cdot \delta_m.
\end{align*}
\]

These components \( \delta_x', \delta_y', \delta_z' \) are positive away from the earth's center; \( \varphi, \lambda \) refer to the substation \( Q \).

6. This procedure must be repeated for all substations and the values of \( \delta_x', \delta_y', \delta_z' \) for the station \( P \) are interpolated.

For elevations \( H > 70 \) km, it is in general sufficient to use the central substation \( O \) only and to put the values at \( P \) equal to the values at \( O \).

7. Computation of the components of the gravity vector:

\[
\begin{align*}
\gamma_x &= \gamma_x - \delta_x, \\
\gamma_y &= \gamma_y - \delta_y, \\
\gamma_z &= \gamma_z - \delta_z.
\end{align*}
\]

\( \gamma_x, \gamma_y, \gamma_z; \gamma_x', \gamma_y', \gamma_z' \) are positive toward the earth's center.
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Methods for computing the gravity vector outside the earth are presented and compared. For the purpose of computation, gravity is split up into normal gravity and gravity disturbance. The report first gives a practical method for computing normal gravity. For gravity disturbances, different methods are described and compared by means of accuracy studies (on influence of distant zones, etc.); finally a practical computation procedure is given.
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