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Stochastic Duels--II

Trevor Williams

13 September 1963

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SYSTEM DEVELOPMENT CORPORATION, SANTA MONICA, CALIFORNIA

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Trevor Williams
System Development Corporation
Santa Monica, California

ABSTRACT

This report continues the project of systematically developing the elements of a stochastic theory of combat begun in an earlier work. The mean, variance, third central moment, etc., of a marksman's time to hit a passive target are evaluated in terms of the corresponding parameters of his time to fire a single round. Next, the solution of the simple duel, in the case where each protagonist's time-to-kill is distributed as a gamma-variate, is obtained as the cumulative distribution of a certain binomial variate, and this result is employed to furnish an approximate solution to the general simple duel. An expansion of the moment-generating function of the marksman's time-to-kill in powers of his kill probability is next derived and found to provide a good approximation to the solution of the simple duel; various properties of the expansion are also considered. We then examine a stochastic battle in which all men on both sides are at all times able to participate in the action: it is deduced that for large numbers of opposing forces the lead-order behavior of the system is deterministic and reduces to Lanchester's square law, the exchange rate being suitably interpreted. Finally, we entertain a stochastic battle where the two forces can only be brought into play at a single point of contact: we find that the lead-order behavior is now given by Lanchester's linear law, but in this case we are able to go much further toward obtaining a complete solution.

1. INTRODUCTION

> After a tiresom harangue in his ordinary style, he took down from his book shelves a number of musty volumes on the subject of the duello, and entertained me for a long time with their contents...
>
> --- Poe (Mystification)

The present report is a continuation and extension of the project, begun by Williams and Ancker in an earlier paper, of developing the elements of a stochastic theory of combat by means of idealized mathematical models whose implications can be thoroughly traced out. Free use is made of the results of that prior work, which we henceforth refer to as Stochastic Duels--I: an equation cited here as, say, "(I-22)" means the 22nd equation of that paper, and the citation "I-Figure 3" means the third figure of that paper. The term "stochastic duel" is a convenient one and we now broaden it to include any situation in which two enemy forces engage in conflict; in this we have a certain etymological authority since the Latin "bellum" is itself a variant form of the older "duellum," which meant a contest between two parties, and gives us our own "duel" by way of the Italian "duello." The discipline of model-building in any application of mathematics to the physical world requires certain qualifications if it is to be successful. First of all, significant aspects of the problem must be ferreted out for consideration and in so doing we must hew away irrelevancies. There are many instances in the literature of military operations research where the argument is cluttered up with minute details which lend a spurious verisimilitude to the model but have no bearing on the fundamental issues and render the mathematics completely intractable. Having sorted out the key features of the system, we must weigh what we might like to do against

what we can reasonably expect to be able to do. An idealized and simplified model which we can solve explicitly and completely is of much greater utility in fostering insight than a very realistic model which we cannot solve. Finally, we must build up the bits and pieces of information so obtained into more and more comprehensive patterns in order to approach the complexity of the world we live in. This report represents a modest attempt to take a few such steps.

2. CONNECTION BETWEEN CUMULANTS OF TIME-TO-FIRE AND TIME-TO-KILL

> And a certain man drew a bow at a venture, and smote the king of Israel between the joints of the harness...
>
> ---I Kings 22:34

We assume that a marksman fires repeatedly at a target until he hits it, his probability of doing so on any given round being p, where p is constant from round to round. The frequency function of his time to fire we call f(t) and that of his time to score a hit, h(t). The corresponding moment-generating functions we write as

$$\varphi(z) = \int_0^\infty e^{zt} f(t)dt \quad \text{and} \quad \Phi(z) = \int_0^\infty e^{zt} h(t)dt \ . \tag{2.1}$$

Assuming successive firing-times to be statistically independent, we have shown (I-14) that

$$\Phi(z) = \frac{p\varphi(z)}{1-q\varphi(z)} \ . \tag{2.2}$$

Strictly speaking, this result was derived for the characteristic function rather than the moment-generating function, but since one may be obtained from the other by the subsituation z=iu, i.e., a rotation through 90°, the distinction is of no importance analytically.

The cumulant-generating function for the time-to-fire will be called

$$\theta(z) \equiv \log \varphi(z) \equiv \sum_{n=1}^{\infty} \kappa_n \frac{z^n}{n!} , \tag{2.3}$$

where κ_1 is the mean time-to-fire, κ_2 the variance, etc. Similarly, for the cumulant-generating function of the time-to-kill we write

$$\Theta(z) = \log \Phi(z) = \sum_{n=1}^{\infty} K_n \frac{z^n}{n!} . \tag{2.4}$$

Then (2.2) yields

$$\Theta(z) = \log \frac{p\varphi}{1-q\varphi} = \log \frac{pe^{\theta}}{1-qe^{\theta}} , \tag{2.5}$$

upon invoking (2.3); in other words,

$$\Theta = \theta - \log \frac{1-qe^{\theta}}{p} = \theta - \log[1 - \frac{q}{p}(e^{\theta}-1)] . \tag{2.6}$$

For convenience writing

$$t = q/p , \tag{2.7}$$

(2.6) becomes

$$\Theta = \theta + \sum_{\nu=1}^{\infty} \frac{t^\nu}{\nu} (e^\theta - 1)^\nu = \theta + \sum_{\nu=1}^{\infty} \frac{t^\nu}{\nu!} \sum_{\mu=o}^{\nu} (-)^{\nu-\mu} \binom{\nu}{\mu} e^{\mu\theta}$$

$$= \theta + \sum_{\nu=1}^{\infty} \frac{t^\nu}{\nu} \sum_{\mu=o}^{\nu} (-)^{\nu-\mu} \binom{\nu}{\mu} \sum_{n=o}^{\infty} \frac{\mu^n \theta^n}{n!}$$

$$= \theta + \sum_{n=o}^{\infty} \frac{\theta^n}{n!} \sum_{\nu=1}^{\infty} \frac{t^\nu}{\nu} \sum_{\mu=o}^{\nu} (-)^{\nu-\mu} \binom{\nu}{\mu} \mu^n \ . \tag{2.8}$$

But, we recall that, by definition

$$\Delta u_k \equiv u_{k+1} - u_k \tag{2.9}$$

and

$$\Delta^\nu u_k = \Delta(\Delta^{\nu-1} u_k) \qquad (\nu = 2, 3, \ldots) \tag{2.10}$$

from which it follows that

$$\Delta^\nu u_k = \sum_{\mu=o}^{\nu} (-)^{\nu-\mu} \binom{\nu}{\mu} u_{k+\mu} \ . \tag{2.11}$$

(See, e.g., Whittaker and Robinson.) Hence, for the sequence $u_k = k^n$ we have

$$\sum_{\mu=o}^{\nu} (-)^{\nu-\mu} \binom{\nu}{\mu} \mu^n = \Delta^\nu k^n \Big]_{k=o} = \Delta^\nu 0^n \ , \tag{2.12}$$

the latter form being the conventional abbreviation. Since differences of a polynomial of higher order than its degree vanish, we have

$$\Delta^{\nu} 0^{n} = 0 \qquad (\nu > n) \tag{2.13}$$

and substituting this result and (2.11) into (2.8), we obtain

$$\Theta = \theta + \sum_{n=o}^{\infty} \frac{\theta^{n}}{n!} \sum_{\nu=1}^{n} \frac{t^{\nu}}{\nu} \Delta^{\nu} 0^{n} . \tag{2.14}$$

We note that the term for n=o is a null sum and may therefore be suppressed. Furthermore, we have

$$\Delta^{\nu} 0^{n} = \nu(\Delta^{\nu} 0^{n-1} + \Delta^{\nu-1} 0^{n-1}) . \tag{2.15}$$

For, in consequence of (2.9) and (2.10),

$$\Delta^{\nu} 0^{n-1} = \Delta^{\nu-1} 1^{n-1} - \Delta^{\nu-1} 0^{n-1} ; \tag{2.16}$$

hence the right-hand side of (2.15) is

$$\nu \cdot \Delta^{\nu-1} 1^{n-1} = \nu \sum_{\mu=o}^{\nu-1} (-)^{\nu-1-\mu} \binom{\nu-1}{\mu} (\mu+1)^{n-1} , \tag{2.17}$$

by (2.11), and this last may be re-written as

$$\nu \sum_{\mu=1}^{\nu} (-)^{\nu-\mu} \binom{\nu-1}{\mu-1} \mu^{n-1} = \sum_{\mu=1}^{\nu} (-)^{\nu-\mu} \frac{\nu}{\mu} \binom{\nu-1}{\mu-1} \mu^{n}$$

$$= \sum_{\mu=1}^{\nu} (-)^{\nu-\mu} \binom{\nu}{\mu} \mu^{n} = \Delta^{\nu} 0^{n} , \tag{2.18}$$

when we refer back to (2.12) and note that the term for μ=o contributes nothing

to the sum. Hence (2.15) is established; when we employ it in (2.14) we obtain

$$\Theta = \theta + \sum_{n=1}^{\infty} \frac{\theta^n}{n!} \sum_{\nu=1}^{n} t^{\nu} (\Delta^{\nu} 0^{n-1} + \Delta^{\nu-1} 0^{n-1})$$

$$= \theta + \sum_{n=1}^{\infty} \frac{\theta^n}{n!} \left(\sum_{\nu=1}^{n} t^{\nu} \Delta^{\nu} 0^{n-1} + \sum_{\nu=o}^{n-1} t^{\nu+1} \Delta^{\nu} 0^{n-1} \right) . \tag{2.19}$$

Now, of the sums within the parentheses, the latter may be extended up to ν=n, since this entails merely the introduction of a term $t^n \Delta^n 0^{n-1}$ which vanishes by virtue of (2.13); also the former may be extended down to ν=o, introducing a term 0^{n-1}, which also vanishes provided $n > 1$. Hence we may rewrite (2.19) as

$$\Theta = \theta + \theta t + \sum_{n=2}^{\infty} \frac{\theta^n}{n!} (1+t) \sum_{\nu=o}^{n} t^{\nu} \Delta^{\nu} 0^{n-1}$$

$$= (1+t) \sum_{n=1}^{\infty} \frac{\theta^n}{n!} \sum_{\nu=o}^{n} t^{\nu} \Delta^{\nu} 0^{n-1} , \tag{2.20}$$

upon noting that the term for n=1 precisely compensates for the term in front of the summation. From (2.7),

$$1+t = 1/p , \tag{2.21}$$

and thus, writing (2.20) out at some length, in terms of the well-known values of the "differences of zero," we obtain, finally,

$$\Theta = \frac{1}{p} \theta + \frac{q}{p^2} \frac{\theta^2}{2!} + \frac{q}{p^2} (1+2 \frac{q}{p}) \frac{\theta^3}{3!} + \frac{q}{p^2} (1+6\frac{q}{p} + 6\frac{q^2}{p^2}) \frac{\theta^4}{4!} + \cdots . \tag{2.22}$$

Going back now to (2.3) and (2.4) and substituting these expansions into (2.22) and comparing coefficients of powers of z, we readily derive the following relationships between the K's, which are the cumulants of the time-to-kill, and the κ's, which are the cumulants of the time-to-fire:

$$K_1 = \frac{1}{p}\kappa_1 , \tag{2.23}$$

$$K_2 = \frac{1}{p^2}(p\kappa_2+q\kappa_1^2) , \tag{2.24}$$

$$K_3 = \frac{1}{p^3}[p^2\kappa_3+3pq\kappa_1\kappa_2+q(1+q)\kappa_1^3] , \tag{2.25}$$

and

$$K_4 = \frac{1}{p^4}[p^3\kappa_4+p^2q(4\kappa_1\kappa_3+3\kappa_2^2)+6pq(1+q)\kappa_1^2\kappa_2+q(p^2+6pq+6q^2)\kappa_1^4] . \tag{2.26}$$

These are the desired results. A convenient check on the algebra is afforded by the fact that for exponentially-distributed firing time with rate of fire r, we have

$$\varphi(z) = \int_0^\infty e^{zt}re^{-rt}dt = \frac{1}{1-z/r} \tag{2.27}$$

by (2.1) and

$$\Phi(z) = \frac{p}{1-\frac{z}{r}-q} = \frac{1}{1-z/rp} , \tag{2.28}$$

by (2.2). Also, from (2.3) and (2.27) it follows that

$$\sum_{n=1}^{\infty} K_n \frac{z^n}{n!} = -\log(1 - \frac{z}{r}) = \sum_{n=1}^{\infty} \frac{z^n}{nr^n} , \tag{2.29}$$

whence

$$K_n = \frac{(n-1)!}{r^n} , \tag{2.30}$$

and similarly,

$$K_n = \frac{(n-1)!}{r^n p^n} . \tag{2.31}$$

Substitutions into (2.23)-(2.26) yields

$$\frac{1}{rp} = \frac{1}{p} \cdot \frac{1}{r} \tag{2.32}$$

$$\frac{1}{r^2 p^2} = \frac{1}{p^2}(p \frac{1}{r^2} + q \frac{1}{r^2}) \tag{2.33}$$

$$\frac{2}{r^3 p^3} = \frac{1}{p^3}[p^2 \cdot \frac{2}{r^3} + 3pq \frac{1}{r^3} + q(1+q) \frac{1}{r^3}] \tag{2.34}$$

and

$$\frac{6}{r^4 p^4} = \frac{1}{p^4}[p^3 \frac{6}{r^4} + p^2 q(\frac{8}{r^4} + \frac{3}{r^4}) + 6pq(1+q) \frac{1}{r^4}$$

$$+q(p^2+6pq+6q^2)\frac{1}{r^4}] . \tag{2.35}$$

The first two of these are obvious identities; the latter pair reduce to

$$2p^2+3pq+q(1+q) = 2 \ , \tag{2.36}$$

and

$$6p^3+11p^2q+6pq(1+q)+q(p^2+6pq+6q^2) = 6 \ , \tag{2.37}$$

which may be verified by replacing p by 1-q throughout.

3. SOLUTION OF THE SIMPLE DUEL WHEN THE TIMES-TO-KILL ARE GAMMA-VARIATES

Laertes. My lord, I'll hit him now.
King. I do not think't.

---Hamlet (V,ii)

The solution of the simple duel which we have already exhibited (I-20) becomes, in terms of moment-generating functions,

$$P(A) = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} \Phi_A(-z)\Phi_B(z)\frac{dz}{z} \ . \tag{3.1}$$

As we might expect, the only difference in the result is that the path of integration has been rotated through 90° in the complex plane. Now, we define a $\gamma(k,\alpha)$ variate as one having as frequency function

$$f(t) = \frac{t^{k-1}}{(k-1)!\alpha^k} e^{-t/\alpha} \ . \tag{3.2}$$

The MGF (moment-generating function) is thus

$$\Phi(z) = \int_0^\infty e^{zt} \frac{t^{k-1}}{(k-1)!\alpha^k} e^{-t/\alpha}dt = \frac{1}{(1-\alpha z)^k} \ . \tag{3.3}$$

This result is obvious when we recall that a $\gamma(k,\alpha)$ variate is the sum of k independent exponential variates, each with mean α.

Let us now assume that A's time to kill is distributed as $\gamma(k,\alpha)$, while B's is $\gamma(\ell,\beta)$. Then we have

$$P(A) = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{1}{(1+\alpha z)^k(1-\beta z)^\ell} \cdot \frac{dz}{z} . \tag{3.4}$$

This is, of course, merely $\text{Prob}\{\gamma(k,\alpha) \leq \gamma(\ell,\beta)\}$. Writing

$$\psi = \alpha/\beta , \tag{3.5}$$

and replacing z by z/β in (3.4), we find

$$P(A) = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{1}{(1+\psi z)^k(1-z)^\ell} \cdot \frac{dz}{z} . \tag{3.6}$$

Strictly speaking, the ϵ in (3.6) is not the same as that in (3.4), but all we need be concerned with is the fact that it is some positive number. We complete the contour in (3.6) into the right half-plane since by doing so we pick up one residue (namely the one at z=1) rather than two. Since the contour encircles the pole in the negative sense, we obtain

$$P(A) = -\operatorname*{res}_{z=1} \frac{1}{(1+\psi z)^k(1-z)^\ell z} . \tag{3.7}$$

Writing

$$z = 1+w , \tag{3.8}$$

this becomes

$$P(A) = - \underset{1/w}{\text{coeff}} \frac{1}{(1+\psi+\psi w)^k (-w)^\ell (1+w)}$$

$$= \frac{(-1)^{\ell-1}}{(1+\psi)^k} \underset{w^{\ell-1}}{\text{coeff}} \frac{1}{(1+\frac{\psi}{1+\psi}w)^k (1+w)}$$

$$= \frac{(-1)^{\ell-1}}{(1+\psi)^k} \sum_{\nu=o}^{\ell-1} \binom{-k}{\nu} \left(\frac{\psi}{1+\psi}\right)^\nu (-1)^{\ell-\nu-1}$$

$$= \frac{1}{(1+\psi)^{k+\ell-1}} \sum_{\nu=o}^{\ell-1} (-)^\nu \binom{-k}{\nu} \psi^\nu (1+\psi)^{\ell-\nu-1}$$

$$= \frac{1}{(1+\psi)^{k+\ell-1}} \sum_{\nu=o}^{\ell-1} (-)^\nu \binom{-k}{\nu} \psi^\nu \sum_{\mu=o}^{\ell-\nu-1} \binom{\ell-\nu-1}{\mu} \psi^\mu$$

$$= \frac{1}{(1+\psi)^{k+\ell-1}} \sum_{\nu=o}^{\ell-1} (-)^\nu \binom{-k}{\nu} \sum_{\mu=\nu}^{\ell-1} \binom{\ell-\nu-1}{\mu-\nu} \psi^\mu$$

$$= \frac{1}{(1+\psi)^{k+\ell-1}} \sum_{\mu=o}^{\ell-1} \psi^\mu \sum_{\nu=o}^{\mu} (-)^\nu \binom{-k}{\nu} \binom{\ell-\nu-1}{\mu-\nu} . \tag{3.9}$$

The inner summation here may be evaluated by means of two elementary and well-known identities, namely,

$$\binom{x}{\nu} = (-)^\nu \binom{\nu-x-1}{\nu} \tag{3.10}$$

and

$$\sum_{\nu=0}^{\mu}\binom{x}{\nu}\binom{y}{\mu-\nu} = \binom{x+y}{\mu} . \tag{3.11}$$

The first of these is self-evident, and the second, known as the "factorial binomial theorem," and due to the mathematican Vandermonde, may be obtained by equating coefficients of like powers of t in the identity

$$(1+t)^x(1+t)^y = (1+t)^{x+y} . \tag{3.12}$$

Using (3.10) twice and (3.11) once, the inner sum in (3.9) may be written

$$(-)^\mu \sum_{\nu=0}^{\mu}\binom{-k}{\nu}\binom{\mu-\ell}{\mu-\nu} = (-)^\mu\binom{-k+\mu-\ell}{\mu} = \binom{k+\ell-1}{\mu} \tag{3.13}$$

and so we arrive at the result

$$P(A) = \frac{1}{(1+\psi)^{k+\ell-1}} \sum_{\mu=0}^{\ell-1}\binom{k+\ell-1}{\mu}\psi^\mu$$

$$= \frac{1}{(\alpha+\beta)^{k+\ell-1}} \sum_{\mu=0}^{\ell-1}\binom{k+\ell-1}{\mu}\alpha^\mu\beta^{k+\ell-\mu-1} , \tag{3.14}$$

upon invoking (3.5). In other words, if we set

$$p = \frac{\alpha}{\alpha+\beta} , \quad q = \frac{\beta}{\alpha+\beta} , \quad \text{and} \quad n=k+\ell-1 \tag{3.15}$$

and define μ to be a binomial variate,

$$\mu \sim B(n,p) \tag{3.16}$$

then

$$P(A) = \text{Prob}\{\mu \leq \ell-1)\} \tag{3.17}$$

may be looked up in the standard tables. We may summarize the result of our calculation in the single identity

$$\text{Prob}\{\gamma(k,\alpha) \leq \gamma(\ell,\beta)\} = \text{Prob}\{B(k+\ell-1, \frac{\alpha}{\alpha+\beta}) \leq \ell-1\} \ . \tag{3.18}$$

We may combine this result with those of the first section to derive an approximate solution to the simple duel. It follows from (3.3), and is of course well-known, that for a $\gamma(k,\alpha)$ variate the mean is $k\alpha$ and the variance $k\alpha^2$. Identifying these with corresponding cumulants of the time-to-kill in (2.4), we have

$$\kappa_1 = k\alpha \quad \text{and} \quad \kappa_2 = k\alpha^2 \ , \tag{3.19}$$

from which it follows that

$$\alpha = \kappa_2/\kappa_1 \quad \text{and} \quad k = \kappa_1^2/\kappa_2 \ . \tag{3.20}$$

We now make use of (2.23) and (2.24) and make the fact that the quantities k and α refer to side A explicit by putting a subscript A on all the symbols:

$$\alpha = (p_A\sigma_A^2+q_A\mu_A^2)/p_A\mu_A \quad \text{and} \quad k = \mu_A^2/(p_A\sigma_A^2+q_A\mu_A^2) \ . \tag{3.21}$$

Here we have also replaced κ_1 and κ_2 by the commoner μ and σ^2. Likewise we have the pair of relations

$$\beta = (p_B\sigma_B^2+q_B\mu_B^2)/p_B\mu_B \quad \text{and} \quad \ell = \mu_B^2/(p_B\sigma_B^2+q_B\mu_B^2) \ . \tag{3.22}$$

Utilizing (3.21) and (3.22) in (3.15)-(3.17) gives rise to the approximation alluded to. Of course, k and ℓ are no longer integers here and interpolation in the tables of the binomial distribution becomes necessary.

Certain limiting cases are of interest. First of all, we may dismiss the situation where $p_A \to 1$, for then (3.21) merely falls back into (3.20) and indeed all the K's become equal to the corresponding κ's, as they must, since the time-to-kill becomes identical with the time-to-fire. The same is true when $p_B \to 1$. When, however, p_A and $p_B \to o$ we find from (3.21) and (3.22) that

$$\alpha \sim \frac{\mu_A}{p_A} \ , \quad \beta \sim \frac{\mu_B}{p_B} \ , \quad k \to 1, \ \ell \to 1 \quad (p_A, p_B \to o) \ . \tag{3.23}$$

Now, in (I-1), we defined

$$r_A = 1/\mu_A \quad \text{and} \quad r_B = 1/\mu_B \ . \tag{3.24}$$

Hence in this limiting case we have, from (3.17),

$$P(A) = \text{Prob}\{B(1, \frac{\alpha}{\alpha+\beta}) \leq 0\} = \frac{\beta}{\alpha+\beta} = \frac{p_A r_A}{p_A r_A + p_B r_B} \ , \tag{3.25}$$

which is the same as (I-6). Indeed the left-hand side of (3.18) tells us that we are dealing with precisely the same problem, since, with $k=\ell=1$, both gamma-variates are merely exponential-variates. Equation (3.25) suggests that we might expect our isoprobs to radiate from the origin of the (p_A, p_B)-plane the

same way as in Figure 2 of Stochastic Duels-I, and we shall subsequently find further confirmation of this.

Another case which recommends itself is that where the coefficients of variation of A's and B's time-to-fire are small; we might at least hope that this would be a reasonable assumption in practice. We then find that

$$\alpha \sim \frac{q_A \mu_A}{p_A}, \quad \beta \sim \frac{q_B \mu_B}{p_B}, \quad k \to \frac{1}{q_A}, \quad \ell \to \frac{1}{q_B} \qquad (\mu_A \gg \sigma_A, \mu_B \gg \sigma_B) \tag{3.26}$$

where the symbol "≫" carries the customary meaning of "is much greater than." If we further assume that q_A and q_B are small, as many military men would insist, then (3.16) approaches normality and (3.17) may be evaluated accordingly.

An illustration may help to clarify the relations between the approximations we have obtained. In (I-24) we have derived the solution to the simple duel with firing-times distributed as gamma-variates with k=2. For this case we have

$$\sigma_A^2 = \frac{1}{2}\mu_A^2 \quad \text{and} \quad \sigma_B^2 = \frac{1}{2}\mu_B^2 . \tag{3.27}$$

Hence (3.21) and (3.22) become

$$\alpha = \frac{1+q_A}{2r_A p_A}, \quad \beta = \frac{1+q_B}{2r_B p_B}, \quad k = \frac{2}{1+q_A} \quad \text{and} \quad \ell = \frac{2}{1+q_B} . \tag{3.28}$$

For simplicity, take

$$p_A = p_B = \frac{2}{3} \quad \text{and} \quad r_A = 2r_B ; \tag{3.29}$$

then the exact solution, (I-24), yields

$$P(A) = \frac{28}{39} = .71795 \ , \tag{3.30}$$

whereas (3.17) leads to

$$P(A) \approx \text{Prob}\{B(2,\tfrac{1}{3}) \leq \tfrac{1}{2}\} \ . \tag{3.31}$$

Now, when ℓ is an integer, all terms up to <u>and</u> including ℓ-1 are to be taken in evaluating (3.17); hence we may display the complete cumulative probability distribution implicit in (3.31) as follows:

ℓ	C.D.F	Δ	Δ^2	Δ^3
-1	0			
		4/9		
0	4/9		0	
		4/9		-1/3
1	8/9		-1/3	
		1/9		
2	1			

<u>Table 1</u>.

We have here taken the range of ℓ to be such that the first column of differences consists of the <u>entire</u> binomial distribution of order 2, with p = 1/3; we thus retain all the information at our disposal. First of all, linear interpolation in Table 1 yields

$$P(A) \approx \frac{2}{3} \ , \qquad 7.14\% \text{ smaller than true value.} \tag{3.32}$$

It is interesting to remark that this is the same answer we would get upon substituting (3.29) into (I-6), the solution of the simple duel with exponential firing-times. That this might provide a fair approximation may be seen from the fact that the isoprobs of I-Figure 5 are all reasonably straight. Applying the Newton-Bessel interpolation formula to Table 1, on the other hand, we find

$$P(A) \approx \frac{2}{3} + \frac{\frac{1}{2}(-\frac{1}{2})}{2} \cdot \frac{1}{2}(-\frac{1}{3}) = \frac{11}{16} = .6875 \ , \quad 4.24\% \text{ too small.} \tag{3.33}$$

Hence in this case the use of the more refined approximation has not quite cut the error in half, and furthermore remains on the same side of the true value. It seems likely, however, that duels for which the isoprobs display greater curvature would give rise to a more consistently favorable behavior of (3.17) over that of (I-6).

4. APPROXIMATION TO THE SOLUTION OF THE SIMPLE DUEL IN TERMS OF MOMENTS OF THE TIMES-TO-FIRE

Round numbers are always false.

---Dr. Johnson

Looking at (2.23)-(2.26) we see that if we define a new variate

$$\tau \equiv pt \ , \tag{4.1}$$

where t is the time-to-kill, then

$$\kappa_n(\tau) = p^n K_n \to (n-1)!\,\kappa_1^n \quad (p \to o) \ , \tag{4.2}$$

where $\kappa_n(\tau)$ is the $n^{\underline{th}}$ cumulant of the new variate τ and κ_1 without any argument is the mean of the time-to-fire. But, it follows from (4.2) that τ is exponentially distributed, with mean κ_1, i.e.,

$$\tau \sim \gamma(1,\kappa_1) \ , \tag{4.3}$$

or, in virtue of (4.1),

$$t \sim \gamma(1, \frac{\kappa_1}{p}) \ . \tag{4.4}$$

Hence, when p_A and p_B are both small, the times-to-kill are, in the limit, exponentially distributed, with means μ_A/p_A and μ_B/p_B, respectively. The solution therefore reduces to the one for exponential firing-times, for which we saw (I-2) that the situation just described holds rigorously.

This, then, yields the zeroth-order behavior of the solution and raises the natural question of whether one can push the analysis further to obtain higher-order corrections, thereby taking into account the curvature of the isoprobs for all but the case of exponential firing-times. The multiplication of the time-to-kill by p in equation (4.1) results in a random variable with MGF

$$\Phi_\tau(z) \equiv E(e^{\tau z}) = E(e^{ptz}) = \Phi_t(pz) \ . \tag{4.5}$$

Hence, by (2.2), we are led to consider

$$\Phi(pz) \equiv \frac{p\varphi(pz)}{1-q\varphi(pz)} . \tag{4.6}$$

Employing ℓ'Hôpital's rule we find

$$\lim_{p\to o} \Phi(pz) = \varphi(o) \lim_{p\to o} \frac{p}{1-(1-p)\varphi(pz)} = \lim_{p\to o} \frac{1}{\varphi(pz)-(1-p)z\varphi'(pz)}$$

$$= \frac{1}{1-z\varphi'(o)} = \frac{1}{1-\mu z} , \tag{4.7}$$

where μ is the mean time-to-fire. We recognize (4.7) at once as the MGF of an exponential variate with mean μ, and thus we have a corroboration of (4.3). Prompted by the last result we now write

$$\Phi(pz) = \sum_{k=o}^{\infty} \psi_k(z) \frac{p^k}{k!} , \tag{4.8}$$

and recall that

$$\varphi(pz) = \sum_{k=o}^{\infty} \mu_k \frac{p^k z^k}{k!} , \tag{4.9}$$

where we use μ_k to designate the $k^{\underline{th}}$ moment about the origin, of the time-to-fire. Equation (4.6) then yields

$$\left\{1-(1-p)\sum_{k=o}^{\infty} \mu_k \frac{p^k z^k}{k!}\right\} \sum_{\nu=o}^{\infty} \psi_\nu(z) \frac{p^\nu}{\nu!} = p \sum_{k=o}^{\infty} \mu_k \frac{p^k z^k}{k!} . \tag{4.10}$$

The term within the accolades is merely

$$1 - \sum_{o}^{\infty} \mu_k \frac{p^k z^k}{k!} + \sum_{o}^{\infty} \mu_k \frac{p^{k+1} z^k}{k!} = \sum_{1}^{\infty} p^k \left[\mu_{k-1} \frac{z^{k-1}}{(k-1)!} - \mu_k \frac{z^k}{k!}\right] , \tag{4.11}$$

and so (4.10) becomes

$$\sum_{k=1}^{\infty}(k\mu_{k-1} - z\mu_k)z^{k-1}\frac{p^k}{k!}\sum_{\nu=o}^{\infty}\psi_\nu(z)\frac{p^\nu}{\nu!} = \sum_{k=1}^{\infty}\mu_{k-1}\frac{z^{k-1}p^k}{(k-1)!} \quad . \tag{4.12}$$

But the left-hand side of this equation may be rewritten as

$$\sum_{k=1}^{\infty}(k\mu_{k-1}-z\mu_k)z^{k-1}\frac{p^k}{k!}\sum_{\nu=1}^{\infty}\psi_{\nu-k}(z)\frac{p^{\nu-k}}{(\nu-k)!}$$

$$= \sum_{\nu=1}^{\infty}\frac{p^\nu}{\nu!}\sum_{k=1}^{\nu}\binom{\nu}{k}z^{k-1}(k\mu_{k-1}-z\mu_k)\psi_{\nu-k}(z) \ , \tag{4.13}$$

upon inverting the order of summation. Comparing the coefficients of like powers of p in (4.12) and (4.13), we find

$$\sum_{k=1}^{\nu}\binom{\nu}{k}z^{k-1}(k\mu_{k-1}-z\mu_k)\psi_{\nu-k}(z) = \nu z^{\nu-1}\mu_{\nu-1} \qquad (\nu=1,2,3,\ldots) \ . \tag{4.14}$$

Repeated application of this formula enables us to solve successively for the ψ's. We readily find

$$\psi_o(z) = \frac{1}{1-\mu z} \ , \tag{4.15}$$

$$\psi_1(z) = \frac{(\mu_2-2\mu^2)z^2}{2(1-\mu z)^2} \ , \tag{4.16}$$

$$\psi_2(z) = \frac{1}{3(1-\mu z)^2}\{(\mu_3-6\mu\mu_2+6\mu^3)z^3 + \frac{1}{2}(3\mu_2^2-2\mu\mu_3)z^4\} \ , \tag{4.17}$$

and

$$\psi_3(z) = \frac{1}{4(1-\mu z)^3}\{(\mu_4-8\mu\mu_3 + 36\mu^2\mu_2-6\mu_2^2-24\mu^4)z^4$$

$$+(8\mu^2\mu_3+4\mu_2\mu_3-12\mu\mu_2^2-2\mu\mu_4)z^5$$

$$+(3\mu_2^3-4\mu\mu_2\mu_3+\mu^2\mu_4)z^6\} \quad . \tag{4.18}$$

From here on the complications become well-nigh insuperable. It is very questionable, however, whether we would ever want to go any further than ψ_3, since ψ_4 introduces the fifth moment of the time-to-fire; all we ever really know about these moments are statistical estimates and we recall that the sampling variance of the $k^{\underline{th}}$ moment entails moments up to and including the $(2k)^{\underline{th}}$. Hence, it is extremely difficult to get any kind of hold on the higher-order moments of a distribution unless the sample size is very large indeed.

A check on the algebra of the last four equations is afforded by considering once again the case of exponential firing-times. There we have

$$\mu_n = n!\mu^n \qquad (n=0,1,2,\ldots) \ , \tag{4.19}$$

and when we substitute these values into (4.15)-(4.18), we see that ψ_1, ψ_2 and ψ_3 vanish identically. This, of course, re-affirms equation (I-2).

We shall have a little more to say about the ψ's later in this section, but for the present it will suffice to note that their general form is obviously

$$\psi_n(z) = \frac{1}{(1-\mu z)^{n+1}} \sum_{\nu=o}^{n} \psi_{n\nu} z^{n+\nu} \quad . \tag{4.20}$$

(The doubly-subscripted ψ will not be confused with the singly-subscripted and the

dual use of ψ presents a needless proliferation of the notation.) We note that ψ_{no} is just a Kronecker delta:

$$\psi_{no} = 1(n=o) \ , = 0 \ (\text{otherwise}). \tag{4.21}$$

The following ψ's are, by (4.16)-(4.18),

$$\psi_{11} = \frac{1}{2}(\mu_2 - 2\mu^2) \ , \tag{4.22}$$

$$\psi_{21} = \frac{1}{3}(\mu_3 - 6\mu\mu_2 + 6\mu^3) \ , \tag{4.23}$$

$$\psi_{22} = \frac{1}{6}(3\mu_2^2 - 2\mu\mu_3) \ , \tag{4.24}$$

$$\psi_{31} = \frac{1}{4}(\mu_4 - 8\mu\mu_3 + 36\mu^2\mu_2 - 6\mu_2^2 - 24\mu^4) \ , \tag{4.25}$$

$$\psi_{32} = 2\mu^2\mu_3 + \mu_2\mu_3 - 3\mu\mu_2^2 - \frac{1}{2}\mu\mu_4 \ , \tag{4.26}$$

and

$$\psi_{33} = \frac{1}{4}(3\mu_2^3 - 4\mu\mu_2\mu_3 + \mu^2\mu_4) \ . \tag{4.27}$$

Let us now specify in (4.8) and (4.20) which duelist we are referring to, as follows,

$$\Phi_A(p_A z) = \sum_{k=o}^{\infty} \psi_{kA}(z) \frac{p_A^k}{k!} \ , \tag{4.28}$$

$$\Phi_B(p_B z) = \sum_{k=o}^{\infty} \psi_{kB}(z) \frac{p_B^k}{k!} \ , \tag{4.29}$$

$$\psi_{kA}(z) = \frac{1}{(1-\mu_A z)^{k+1}} \sum_{\nu=o}^{k} \psi_{k\nu}(A) z^{k+\nu} \ , \tag{4.30}$$

and

$$\psi_{kB}(z) = \frac{1}{(1-\mu_B z)^{k+1}} \sum_{\nu=o}^{k} \psi_{k\nu}(B) z^{k+\nu} . \tag{4.31}$$

Here, e.g., we clearly have

$$\psi_{11}(A) = \frac{1}{2}(\mu_{2A} - 2\mu_A^2) , \tag{4.32}$$

where μ_{2A} is the second moment about the origin of A's time-to-fire; and so on.

Substituting into (3.1) we obtain, upon invoking (4.28) and (4.29),

$$P(A) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \sum_{k=o}^{\infty} \psi_{kA}(-\frac{z}{p_A}) \frac{p_A^k}{k!} \sum_{\ell=o}^{\infty} \psi_{\ell B}(\frac{z}{p_B}) \frac{p_B^\ell}{\ell!} \frac{dz}{z} . \tag{4.33}$$

Writing for convenience

$$\alpha = \mu_A/p_A \quad \text{and} \quad \beta = \mu_B/p_B , \tag{4.34}$$

(4.30) and (4.31) now yield

$$P(A) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \sum_{k=o}^{\infty} \frac{p_A^k}{k!} \frac{1}{(1+\alpha z)^{k+1}} \sum_{\ell=o}^{\infty} \frac{p_B^\ell}{\ell!} \frac{1}{(1-\beta z)^{\ell+1}}$$

$$\cdot \sum_{\nu=o}^{k} \psi_{k\nu}(A) (-\frac{z}{p_A})^{k+\nu} \sum_{\lambda=o}^{\ell} \psi_{\ell\lambda}(B) (\frac{z}{p_B})^{\ell+\lambda} \frac{dz}{z} . \tag{4.35}$$

If now we write

$$I_{\nu\lambda}^{k\ell} = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{z^{k+\ell+\nu+\lambda-1} dz}{(1+\alpha z)^{k+1} (1-\beta z)^{\ell+1}} , \tag{4.36}$$

then (4.35) becomes

$$P(A) = \sum_{k=o}^{\infty} \frac{1}{k!} \sum_{\ell=o}^{\infty} \frac{1}{\ell!} \sum_{\nu=o}^{k} (-)^{k+\nu} \frac{\psi_{k\nu}(A)}{p_A^{\nu}} \sum_{\lambda=o}^{\ell} \frac{\psi_{\ell\lambda}(B)}{p_B^{\lambda}} I_{\nu\lambda}^{k\ell} . \tag{4.37}$$

Now, when we attempt to evaluate this we are immediately confronted by the unpleasant fact that the integral in (4.36) only converges when $\nu + \lambda \leq 1$. It is easy to calculate by residue theory that

$$I_{oo}^{oo} = \frac{\beta}{\alpha+\beta} , \quad I_{10}^{10} = \frac{1}{(\alpha+\beta)^2} , \quad \text{and} \quad I_{01}^{01} = - \frac{1}{(\alpha+\beta)^2} , \tag{4.38}$$

but as soon as we consider the next case, where $k+\ell=2$, we run into divergent terms, namely, I_{20}^{20}, I_{11}^{11} and I_{02}^{02}, although the remaining two terms that do not vanish, namely, I_{10}^{20} and I_{01}^{02}, are convergent. Hence, all that we can rigorously deduce from this expansion is the first-order correction to the linear isoprobs. From equations (4.21), (4.32), (4.34), (4.37) and (4.38) we are led finally to

$$P(A) \approx \frac{\mu_B/p_B}{\frac{\mu_A}{p_A} + \frac{\mu_B}{p_B}} + \frac{1}{2\left(\frac{\mu_A}{p_A} + \frac{\mu_B}{p_B}\right)^2} \left\{ \frac{\sigma_A^2-\mu_A^2}{p_A} - \frac{\sigma_B^2-\mu_B^2}{p_B} \right\} . \tag{4.39}$$

Of course, the lead term is again merely the familiar (I-6) in a slightly different form.

As a test of (4.39) we use the same example we have considered hitherto, viz., (3.27), (3.29) and (3.30); we find

$$P(A) \text{ of eqn. } (4.39) = \frac{2}{3} + \frac{1}{27} \left\{ - \frac{1}{2} + 2 \right\} = \frac{2}{3} + \frac{1}{18} = \frac{13}{18} = .72222 . \tag{4.40}$$

The error here is, comparing this result with (3.30), positive and precisely one part in 168, or, approximately 0.6%. This is, needless to say, a great improvement over any of the approximations we have so far derived.

The success of equation (4.39) in this case makes one wish that one might extend it at least to the next order, which would, as we remarked below (4.38) entail five new terms. The divergences already cited, however, seem for the present effectively to block this. Still, it is very difficult to feel that (4.37) does not really contain all the stochastic information inherent in the duel, provided a suitable interpretation of (4.36) is found. One might, for example, call upon Hadamard's notion of the "finite part" of an integral; or one might attempt to confine attention to contributions of residues in the right half-plane alone. At this writing, however, our efforts in this direction have come to nothing.

We close this section with a brief examination of a couple of properties of the $\psi_{n\nu}$ of equation (4.20). Substituting the latter into (4.14) and reversing the order of summation, we find

$$\sum_{k=o}^{\nu-1}\binom{\nu}{k}z^{\nu-k-1}\left[(\nu-k)\mu_{\nu-k-1}-z\mu_{\nu-k}\right]\frac{1}{(1-\mu z)^{k+1}}\sum_{\lambda=o}^{k}\psi_{k\lambda}z^{k+\lambda}$$

$$= \nu z^{\nu-1}\mu_{\nu-1}\,, \tag{4.41}$$

which becomes, when we clear of fractions,

$$\sum_{k=o}^{\nu-1}\binom{\nu}{k}\left[(\nu-k)\mu_{\nu-k-1}-z\mu_{\nu-k}\right](1-\mu z)^{\nu-k-1}\sum_{\lambda=o}^{k}\psi_{k\lambda}z^{\lambda}$$

$$= \nu\mu_{\nu-1}(1-\mu z)^{\nu}\,. \tag{4.42}$$

The highest power of z which occurs in this identity is the $\nu^{\underline{th}}$, and when we equate its coefficient on both sides of the equation, we find

$$\sum_{k=o}^{\nu-1}(-)^k\binom{\nu}{k}\mu^{\nu-k-1}\mu_{\nu-k}\psi_{kk} = \nu\mu^{\nu}\mu_{\nu-1} \ . \tag{4.43}$$

Letting ν=1,2,3 and 4 in this, we are easily led to equations (4.21), (4.22), (4.24) and (4.27) again.

Consider next the generating function

$$g(x) \equiv \sum_{\nu=o}^{\infty}(-)^{\nu}\psi_{\nu\nu}\frac{x^{\nu}}{\nu!} \ ; \tag{4.44}$$

then (4.43) yields

$$\sum_{\nu=1}^{\infty}\frac{x^{\nu}}{\nu!}\nu\mu^{\nu}\mu_{\nu-1} = \mu x\sum_{\nu=o}^{\infty}\frac{x^{\nu}\mu^{\nu}}{\nu!}\mu_{\nu} = \mu x\varphi(\mu x)$$

$$= \sum_{\nu=1}^{\infty}\frac{x^{\nu}}{\nu!}\sum_{k=o}^{\nu-1}(-)^k\binom{\nu}{k}\mu^{\nu-k-1}\mu_{\nu-k}\psi_{kk}$$

$$= \sum_{k=o}^{\infty}(-)^k\frac{\psi_{kk}}{k!}\sum_{\nu=k+1}^{\infty}\frac{x^{\nu}\mu^{\nu-k-1}\mu_{\nu-k}}{(\nu-k)!}$$

$$= \frac{1}{\mu}\sum_{k=o}^{\infty}(-)^k\frac{\psi_{kk}}{k!}x^k\sum_{\nu=1}^{\infty}\mu_{\nu}\frac{(\mu x)^{\nu}}{\nu!} = \frac{1}{\mu}g(x)\left[\varphi(\mu x)-1\right], \tag{4.45}$$

from which it follows that

$$\sum_{\nu=o}^{\infty}(-)^{\nu}\psi_{\nu\nu}\frac{x^{\nu}}{\nu!} = \frac{\mu^2 x\varphi(\mu x)}{\varphi(\mu x)-1} \ . \tag{4.46}$$

Here again we readily check the validity of the equations listed at the end of the previous paragraph. We note, furthermore, that for an exponential firing-time, $\varphi(x) = 1/(1-\mu x)$ and the right-hand side of (4.46) becomes identically unity, confirming the remark made in the paragraph containing (4.19).

We may also examine similarly the behavior of the coefficients at the "opposite end," so to speak. The constant term in (4.42) yields

$$\sum_{k=o}^{\nu-1} \binom{\nu}{k} (\nu-k)\mu_{\nu-k-1}\psi_{ko} = \nu\mu_{\nu-1} , \tag{4.47}$$

which is trivial, in light of (4.21). The coefficient of z gives, on the other hand,

$$\sum_{k=o}^{\nu-1} \binom{\nu}{k} \psi_{ko}[-\mu_{\nu-k}-(\nu-k-1)\mu(\nu-k)\mu_{\nu-k-1}]$$

$$+\sum_{k=o}^{\nu-1} \binom{\nu}{k} \psi_{k1}(\nu-k)\mu_{\nu-k-1} = -\nu^2\mu\mu_{\nu-1} , \tag{4.48}$$

which becomes, by virtue of (4.21),

$$-\mu_\nu-\nu(\nu-1)\mu\mu_{\nu-1} +\sum_{k=o}^{\nu-1}(\nu-k)\binom{\nu}{k}\mu_{\nu-k-1}\psi_{k1} = -\nu^2\mu\mu_{\nu-1} , \tag{4.49}$$

or

$$\sum_{k=o}^{\nu-1}\nu\binom{\nu-1}{k}\mu_{\nu-k-1\ k1} = \mu_\nu-\nu\mu\mu_{\nu-1} , \tag{4.50}$$

whence, on replacing ν by $\nu+1$,

$$\sum_{k=o}^{\nu} \binom{\nu}{k} \mu_{\nu-k}\psi_{k1} = \frac{\mu_{\nu+1}}{\nu+1} - \mu\mu_\nu . \tag{4.51}$$

Here, again, setting ν=0,1,2 and 3, in succession, we obtain verification of (4.21)-(4.23), and (4.25).

Let us now consider the generating function

$$G(x) = \sum_{\nu=o}^{\infty} \psi_{\nu 1} \frac{x^{\nu}}{\nu!} ; \tag{4.52}$$

(4.51) then yields

$$\sum_{\nu=o}^{\infty} \frac{x^{\nu}}{\nu!} \left(\frac{\mu_{\nu+1}}{\nu+1} - \mu\mu_{\nu}\right) = \sum_{\nu=1}^{\infty} \mu_{\nu} \frac{x^{\nu-1}}{\nu!} - \mu \sum_{\nu=o}^{\infty} \mu_{\nu} \frac{x^{\nu}}{\nu!}$$

$$= \frac{1}{x} [\varphi(x) - 1] - \mu\varphi(x)$$

$$= \sum_{\nu=o}^{\infty} \frac{x^{\nu}}{\nu!} \sum_{k=o}^{\nu} \binom{\nu}{k} \mu_{\nu-k} \psi_{k1} = \sum_{k=o}^{\infty} \frac{\psi_{k1}}{k!} \sum_{\nu=k}^{\infty} \frac{\mu_{\nu-k}}{(\nu-k)!} x^{\nu}$$

$$= \sum_{k=o}^{\infty} \frac{\psi_{k1}}{k!} x^{k} \sum_{\nu=o}^{\infty} \frac{\mu_{\nu}}{\nu!} x^{\nu} = G(x)\varphi(x) , \tag{4.53}$$

thus showing that

$$\sum_{\nu=o}^{\infty} \psi_{\nu 1} \frac{x^{\nu}}{\nu!} = \frac{1}{x} - \mu - \frac{1}{x\varphi(x)} . \tag{4.54}$$

Once again we have a verification of equations (4.21)-(4.23), and (4.25); and this time an exponential firing-time causes the right-hand side of (4.54) to vanish identically, which is as it should be.

There is a last identity which we shall point out. From (4.44) and (4.46) we have

$$\frac{1}{g(x)} = \frac{1}{\mu^2 x} - \frac{1}{\mu^2 x\varphi(\mu x)} \quad , \tag{4.55}$$

and from (4.52) and (4.54),

$$G(\mu x) = \frac{1}{\mu x} - \mu - \frac{1}{\mu x\varphi(\mu x)} \quad . \tag{4.56}$$

It follows that

$$\frac{1}{g(x)} = 1 + \frac{1}{\mu} G(\mu x) \quad , \tag{4.57}$$

$$g(x)G(\mu x) = \mu[1-g(x)] \quad . \tag{4.58}$$

From this we obtain the curious convolution identity,

$$\begin{aligned} \sum_{\lambda=o}^{k}(-)^{\lambda}\binom{k}{\lambda}\mu^{k-\lambda}\psi_{(k-\lambda)1}\psi_{\lambda\lambda} &= (-)^{k-1}\psi_{kk}\mu \qquad (k\neq o) \\ &= 0 \qquad (k=o) \quad . \end{aligned} \tag{4.59}$$

5. A STOCHASTIC BATTLE

When, without stratagem,
But in plain shock and even play of battle,
Was ever known so great and little loss
On one part and on the other?

---King Henry V. (IV, viii)

We assume that two opposing forces, A and B, consist of α and β men, respectively. All men on side A have the same rate-of-fire, r_A, and the same kill probability, p_A, and their time-to-fire is exponentially distributed.

Then we have seen that their time-to-kill will be exponential, with rate-of-kill equal to $r_A p_A$. Making the letters A and B do double duty for simplicity of notation, we write

$$A \equiv r_A p_A \ , \quad B \equiv r_B p_B \ . \tag{5.1}$$

Since all α men on side A are firing simultaneously, the time to A's first kill of one of B's men is the smallest in a sample of size α from an exponential variate of mean 1/A. By the well-known formula from order statistics, therefore, the probability that A's first kill occurs at time $\geq$ t is equal to

$$H_A(t) = [e^{-At}]^{\alpha} = e^{-A\alpha t} \ , \tag{5.2}$$

and so his time-to-first-kill is distributed as

$$h_A(t) = A\alpha e^{-A\alpha t} \ , \tag{5.3}$$

i.e., it too is exponential, the rate-of-first-kill being merely $A\alpha$. A similar result holds for B:

$$h_B(t) = B\beta e^{-B\beta t} \ . \tag{5.4}$$

Hence, the probability that the first kill is inflicted by A on B (rather than by B on A) is, just as in the case of the simple duel with exponential firing-times:

$$\text{Prob. A first kills a B} = \frac{A\alpha}{A\alpha+B\beta} \quad . \tag{5.5}$$

If we consider a point in the (α,β)-plane of coordinates, this means that precisely two transitions are possible, and that their respective probabilities are

$$\text{Prob}\{(\alpha,\beta) \rightarrow (\alpha,\beta-1)\} = \frac{A\alpha}{A\alpha+B\beta} \quad , \tag{5.6}$$

and

$$\text{Prob}\{(a,\beta) \rightarrow (\alpha-1,\beta)\} = \frac{B\beta}{A\alpha+B\beta} \quad . \tag{5.7}$$

At this juncture it might be argued that, once a kill has occurred on one side or the other, the situation becomes very complicated, since we must now wait to see whether the second kill for the successful side comes before or after the still-awaited first kill for the other side and the developing pattern of events ramifies out more and more. That this is not the case it is quite easy to see. For, an exponential distribution describes a constant-risk situation, and is inherently Markovian. Thus, to say that A's time to his first kill is distributed as (5.3) is the same as to say that in every time interval dt the probability that A scores his first kill is $A\alpha dt$, regardless of what may have happened prior; the only assumption necessary is that A <u>does</u> have α men during the interval dt. Of course, if B gets in first with a kill, then A's probability of a kill immediately drops to $A(\alpha-1)dt$.

Thus, equations (5.6) and (5.7) fully characterize the course of the engagement as it unfolds. They are of course merely a description of a two-

dimensional random walk in which only downward and leftward steps are permissible and for which the transition probabilities are not constant but vary from lattice point to lattice point. Clearly the walk can never pass either of the axes since the number of combatants on either side is necessarily positive. Now, in practice, a military unit which is suffering a sufficiently high casualty rate will break and run, but for purposes of simplicity we shall assume that both sides stick it out until one side or the other is utterly annihilated. It would be simple enough to treat the problem on the assumption of a built-in breaking point, but the present discussion will be adequate for a first treatment. Thus, the random walk must with probability unity ultimately be absorbed into either the α-axis or the β-axis, the former event representing a victory by B, the latter a victory by A. A moment's consideration will show that a tie is impossible since the origin could only be reached from either the point (1,0) or the point (0,1), both of which are absorbing states. We should like to determine the probability that A wins <u>and</u> the distribution of the number of his survivors, given that he wins.

We define $p_\nu(\alpha,\beta)$ as the probability that A wins starting with α men opposing β, and that he has precisely ν men left at the termination of the battle. In other words,

$$p_\nu(\alpha,\beta) = \text{prob. of absorption at } (\nu,0), \text{ starting from } (\alpha,\beta) \ . \tag{5.8}$$

It then immediately follows from (5.6) and (5.7) that

$$p_\nu(\alpha,\beta) = \frac{B\beta}{A\alpha+B\beta}\, p_\nu(\alpha-1,\beta) + \frac{A\alpha}{A\alpha+B\beta}\, p_\nu(\alpha,\beta-1) \ , \tag{5.9}$$

for his path to (ν,o) <u>must</u> pass through one or the other of the points $(\alpha-1,\beta)$ or $(\alpha,\beta-1)$. We note also that

$$p_\nu(\alpha,\beta) = 0 \qquad (\nu > \alpha) , \tag{5.10}$$

since rightward steps are prohibited. Clearly the over-all probability that A wins is given by

$$P(A) = \sum_{\nu=1}^{\alpha} p_\nu(\alpha,\beta) . \tag{5.11}$$

We note that here we are using our conventional notation for the simple duel in using A as an "argument" on the left-hand side of the equation; the α and β on the other side of the equation are arguments in the customary mathematical usage of the term.

The random variable ν is unnormalized, because of the possibility of absorption of the process into the β-axis. Even for an unnormalized variate, however, we can define moments by means of the usual formula for the $k^{\underline{th}}$ moment about the origin,

$$\mu_k(\alpha,\beta) = \sum_{\nu=1}^{\alpha} \nu^k p_\nu(\alpha,\beta) . \tag{5.12}$$

The notation makes explicit the dependence of the moment on the initial point. Plainly we have

$$P(A) = \mu_o(\alpha,\beta) , \tag{5.13}$$

and in order to convert to moments in the traditional meaning we have only to divide (5.12) by (5.13). By virtue of (5.9) we now have

$$\begin{aligned}(A\alpha+B\beta)\mu_k(\alpha,\beta) &= \sum_{\nu=1}^{\alpha}\nu^k\{B\beta p_\nu(\alpha-1,\beta) + A\alpha p_\nu(\alpha,\beta-1)\} \\ &= \alpha^k B\beta p_\alpha(\alpha-1,\beta) + B\beta\sum_{\nu=1}^{\alpha-1}\nu^k p_\nu(\alpha-1,\beta) + A\alpha\sum_{\nu=1}^{\alpha}\nu^k p_\nu(\alpha,\beta-1) .\end{aligned} \tag{5.14}$$

But the first term on the right vanishes because of (5.10), and (5.12) evaluates what remains on the right; we find

$$(A\alpha+B\beta)\mu_k(\alpha,\beta) = B\beta\mu_k(\alpha-1,\beta) + A\alpha\mu_k(\alpha,\beta-1) . \tag{5.15}$$

This equation is of precisely the same form as (5.9), with the subscript ν dropped. It is interesting to remark that (5.15) does not depend at all upon k itself. This is not to say, of course, that $\mu_k(\alpha,\beta)$ is independent of k, since the latter enters into the boundary conditions.

In this respect we note that specifying the values of μ_k along the (positive) α and β axes serves to determine the solution of (5.15) uniquely throughout (all the lattice points of) the first quadrant. For, all we need to know, by (5.15), to find the value of μ_k at a given point are its values at the lattice points next left and next below. We can therefore build up the solution from left to right a tier at a time. As for the appropriate boundary conditions we note that (5.8) patently implies that

$$\begin{aligned} p_\nu(\alpha,0) &= 1 \qquad (\nu=\alpha) , \\ &= 0 \qquad (\nu\neq\alpha) , \end{aligned} \tag{5.16}$$

and

$$p_\nu(0,\beta) = 0 \quad (\text{all } \nu) \ . \tag{5.17}$$

Putting these values into (5.12), we obtain

$$\mu_k(\alpha,0) = \alpha^k \ , \tag{5.18}$$

and

$$\mu_k(0,\beta) = 0 \ . \tag{5.19}$$

In a recent paper, Brown has considered the special case where k=0; in other words he has confined his attention to consideration of P(A). He obtained for it an expression which we unknowingly duplicated; the reader is referred to his paper in the bibliography for the details. The expression follows readily enough from successive applications of (5.15), together with (5.18) and (5.19); it is an alternating series containing exactly α terms and is of limited practical applicability for large α because the individual terms become enormous and a very great cancellation of significant figures results, so that in order to obtain meaningful answers one would have to carry an inordinate number of decimal places. Equation (5.15) itself involves no such loss of accuracy and is quite suitable for α and β not too large, and could very readily be programmed for a digital computer. Here again, however, for increasing α and β one finds that the computation time mounts rapidly. What we require is a serviceable asymptotic expansion. We shall now present merely the zeroth-order behavior of the solution for large α and β; we have not yet succeeded in

extending the analysis any further (notwithstanding repeated attempts), but the result which emerges seems to us interesting, unexpected, and indicative of the desirability of pursuing the matter.

We note first the Taylor's expansions

$$\mu_k(\alpha-1,\beta) = \mu_k(\alpha,\beta) - \frac{\partial}{\partial\alpha}\mu_k(\alpha,\beta) + - \ldots \quad , \tag{5.20}$$

and

$$\mu_k(\alpha,\beta-1) = \mu_k(\alpha,\beta) - \frac{\partial}{\partial\beta}\mu_k(\alpha,\beta) + - \ldots \quad . \tag{5.21}$$

Since we are looking for a "stabilized" asymptotic behavior for large α and β, it is reasonable to ignore the higher-order terms here; the two Taylor's series are in powers of unity, which we assume small compared with α and β. Substituting into (5.15), we find

$$B\beta \frac{\partial\mu_k}{\partial\alpha} + A\alpha \frac{\partial\mu_k}{\partial\beta} = 0 \ ; \tag{5.22}$$

we here suppress the arguments (α,β) since they are the same as the independent variables. Now, the subsidiary equations necessary for the solution of (5.22) -- cf. Forsyth -- are

$$\frac{d\alpha}{B\beta} = \frac{d\beta}{A\alpha} = \frac{d\mu_k}{0} \ , \tag{5.23}$$

the last two of which yield

$$d\mu_k = 0 \ , \quad \mu_k = C_1 \ , \tag{5.24}$$

C_1 being the constant of integration. Similarly, the first two equations yield

$$A\alpha d\alpha = B\beta d\beta \ , \quad A\alpha^2 - B\beta^2 = C_2 \ . \tag{5.25}$$

Hence the most general solution of (5.22) is

$$C_1 = \Psi(C_2) \ , \tag{5.26}$$

where Ψ is a completely arbitrary function, soon to be specified by the boundary conditions. Employing (5.24) and (5.25), (5.26) becomes

$$\mu_k = \Psi(A\alpha^2 - B\beta^2) \ . \tag{5.27}$$

But, from (5.18) we have

$$\Psi(A\alpha^2) = \alpha^k \ . \tag{5.28}$$

When we write

$$u = A\alpha^2 \ , \quad \alpha = (u/A)^{1/2} \ , \tag{5.29}$$

(5.28) becomes

$$\Psi(u) = (u/A)^{k/2} \ , \tag{5.30}$$

and using this in (5.27) we have, finally,

$$\mu_k(\alpha,\beta) = (\alpha^2 - \frac{B}{A}\beta^2)^{k/2} \ . \tag{5.31}$$

These are, of course, moments about the origin, and the cases k=1 and k=2 provide us with the mean and variance, namely,

$$\mu(\alpha,\beta) = \sqrt{\alpha^2 - \frac{B}{A}\beta^2} \tag{5.32}$$

and

$$\sigma^2(\alpha,\beta) = \mu_2 - \mu^2 = 0 \ . \tag{5.33}$$

It is the last result which comes as a surprise, but looking at (5.31) we see that it is indeed precisely the situation which obtains when we have a "random" variable which is in fact constant and therefore identically equal to the value given by (5.32). Now, it is clear that for (5.32) to be a meaningful statement we next have

$$\alpha \geq \sqrt{\frac{B}{A}}\beta \ . \tag{5.34}$$

When this holds, then (5.13) and (5.31) show that

$$P(A) = 1 \ . \tag{5.35}$$

In other words in the wedge between the α axis and the line $\alpha = \sqrt{\frac{B}{A}}\beta$, α <u>always</u> wins (assuming that α and β are both large) and he always finds himself at the end of the fray with precisely the number of survivors given by (5.32). Of course, a complementary situation obtains in the remainder of the first quadrant.

We remark that (5.32) is exactly what we get from Lanchester's "square" law (Morse and Kimball) if we set A's efficiency relative to B equal to

$$E = A/B \tag{5.36}$$

and put the number of combatants on side B equal to zero. Thus, we have established that Lanchester's square law, which is based on strictly deterministic reasoning, is in fact valid and, furthermore, valid in a strictly deterministic sense (i.e., with no random component to a first approximation), when large numbers face one another. But, more than that we have, by (5.1) and (5.36), been able to identify his relative efficiency in a very simple and natural manner.

It is of interest to remark that the solution of (5.22) calls upon only one of the boundary conditions, namely, (5.18), whereas the solution of (5.15) calls upon both (5.18) and (5.19). It is this that accounts for the discontinuity in the solution of the former. Equation (5.35) is less anomalous when we consider that (5.18) implies that P=1 along the α-axis, and hence $\partial P/\partial A = 0$ there. But, setting β=0 in (5.22) shows that $\partial P/\partial\beta = 0$ also, along the α-axis. Thus P must be constant and hence equal to its value, unity, along the α-axis. The overwhelming effect of superiority of numbers has long been recognized, and reference to the index of Clausewitz's On War will show the reader that that work is laced through with injunctions as to the importance of this factor. The author inveighs against quixotic generals who have deliberately failed to bring the full force of their troops to bear upon the enemy, although as a practical man of affairs he occasionally tempers his position with a warning against the over-simplification of assuming that there are no other factors involved in the outcome of a battle besides superiority in numbers.

The fact that k does not appear explicitly in (5.15) means we can rewrite that

equation very simply in terms of the moment-generating function. Define

$$\varphi(z;\alpha,\beta) \equiv \sum_{k=o}^{\infty} \mu_k(\alpha,\beta) \frac{z^k}{k!} . \tag{5.37}$$

Then (5.15) yields at once

$$(A\alpha+B\beta)\varphi(z;\alpha,\beta) = B\beta\varphi(z;\alpha-1,\beta) + A\alpha\varphi(z;\alpha,\beta-1) , \tag{5.38}$$

and the approximate analysis follows through exactly as before, leading to the counter-part of (5.27), viz.,

$$\varphi(z;\alpha,\beta) = \Psi(A\alpha^2-B\beta^2) . \tag{5.39}$$

The boundary condition is now, by virtue of (5.18) and (5.37), given by

$$\varphi(z;\alpha,0) = \sum_{k=o}^{\infty} \alpha^k \frac{z^k}{k!} = e^{\alpha z} . \tag{5.40}$$

In equation (5.38), z plays the role of a parameter, just as k did in (5.15). Equations (5.39) and (5.40), together with (5.29) now yield

$$\varphi(z;\alpha,\beta) = \exp\{z\sqrt{\alpha^2 - \frac{B}{A}\beta^2}\} , \tag{5.41}$$

thereby checking (5.31). It seems to be pretty much of a toss-up whether this analysis or the former is any simpler in this case, although this does not preclude the possibility that one of them might, in a more fully-blown treatment, turn out to be preferable to the other.

We assumed at the beginning of this section that firing-times were exponential and noted in (5.3) that this implied that time-to-first-kill was

also exponential. It makes sense to ask what happens when we drop this restriction. The extreme-value distribution, of the algebraically smallest value in a sample of large size, is known to follow a Fisher-Tippett distribution (cf. Cramer), provided certain conditions are met, one of which is that the distribution being sampled from have an infinite tail in the negative direction. This is plainly not the case with the inherently positive firing-times we have to deal with, and we may expect that result to break down; it does. For concreteness let us take the time-to-fire to be distributed as a gamma-variate with parameters 2 and ν:

$$f(t) = \frac{\nu^2 t}{2} e^{-\nu t} . \tag{5.42}$$

This will obviously be a more realistic assumption than the exponential. It follows that

$$F(t) \equiv \int_t^\infty f(t)dt = (1+\nu t)e^{-\nu t} , \tag{5.43}$$

upon integrating by parts. Hence the right-sided cumulative distribution function of the time-to-first-kill is, analogously to (5.2),

$$H(t) = (1+\nu t)^\alpha e^{-\alpha\nu t} , \tag{5.44}$$

where α is still the number of men on side A. Thus, the frequency function is

$$h(t) = -H'(t) . \tag{5.45}$$

Now, for <u>any</u> positive random variable, and, in particular, in the present case, the mean is given by

$$\mu = \int_0^\infty h(t)\,t\,dt = -\int_0^\infty H'(t)\,t\,dt = -\,tH(t)\Big]_0^\infty + \int_0^\infty H(t)\,dt$$

$$= \int_0^\infty H(t)\,dt\ , \tag{5.46}$$

provided $H(t)$ goes to zero as $t \to \infty$ faster than $1/t$, i.e., provided that

$$H(t) = o\,(1/t) \qquad (t \to \infty)\ . \tag{5.47}$$

Therefore the mean time-to-first-kill is, in the case <u>sub judice</u>,

$$\mu = \int_0^\infty (1+\nu t)^\alpha e^{-\alpha\nu t}\,dt\ , \tag{5.48}$$

on account of (5.44) and (5.46). Consequently,

$$\mu = \int_0^\infty \sum_{k=o}^{\alpha} \binom{\alpha}{k} \nu^k t^k e^{-\alpha\nu t}\,dt = \sum_{k=o}^{\alpha} \binom{\alpha}{k} \nu^k \int_0^\infty t^k e^{-\alpha\nu t}\,dt$$

$$= \sum_{k=o}^{\alpha} \binom{\alpha}{k} \nu^k \frac{k!}{(\alpha\nu)^{k+1}} = \frac{\alpha!}{\alpha\nu} \sum_{k=o}^{\alpha} \frac{1}{(\alpha-k)!\,\alpha^k}$$

$$= \frac{\alpha!}{\nu\cdot\alpha^{\alpha+1}} \sum_{k=o}^{\alpha} \frac{\alpha^k}{k!}\ , \tag{5.49}$$

upon reversing the order of summation. Now, the terms in the summation are proportional to the probabilities in a Poisson distribution of parameter α. This has mean α and standard deviation $\sqrt{\alpha}$, and approaches normality when $\alpha \to \infty$.

The terms for $k=\alpha-1$ and $k=\alpha$ are readily seen to be equal, and the last argument shows that the totality of terms contributing appreciably to the sum lie within a few multiples of $\sqrt{\alpha}$ on either side of these terms. Thus,

$$\sum_{k=o}^{\alpha} \frac{\alpha^k}{k!} \sim \frac{1}{2}\sum_{k=o}^{\infty} \frac{\alpha^k}{k!} = \frac{1}{2}e^{\alpha} \qquad (\alpha \to \infty) \ . \tag{5.50}$$

Substituting this, and Stirling's formula, into (5.49) we get

$$\mu \sim \frac{1}{\nu}\, \alpha^{-\alpha-1} \cdot \alpha^{\alpha+\frac{1}{2}} \sqrt{2\pi}\, e^{-\alpha} \cdot \frac{1}{2}\, e^{\alpha} \quad (\alpha \to \infty) \ , \tag{5.51}$$

which is to say that

$$\mu \sim \sqrt{\frac{\pi}{2\alpha}} \cdot \frac{1}{\nu} \quad (\alpha \to \infty) \ . \tag{5.52}$$

We therefore introduce a new random variable,

$$\tau \equiv \sqrt{\alpha}\, \nu t \ , \tag{5.53}$$

for which the expectation is

$$E(\tau) \to \sqrt{\frac{\pi}{2}} \quad (\alpha \to \infty) \ . \tag{5.54}$$

Its RSCDF (right-sided cumulative distribution function) is, by (5.44),

$$H(\tau) = (1 + \frac{\tau}{\sqrt{\alpha}})^{\alpha} e^{-\sqrt{\alpha}\tau} \ , \tag{5.55}$$

which yields

$$\log H = -\sqrt{\alpha}\tau + \alpha \log\left(1 + \frac{\tau}{\sqrt{\alpha}}\right)$$

$$= -\sqrt{\alpha}\tau + \alpha\left(\frac{\tau}{\sqrt{\alpha}} - \frac{1}{2}\frac{\tau^2}{\alpha} + \frac{1}{3}\frac{\tau^3}{\alpha\sqrt{\alpha}} + \cdots\right)$$

$$\to -\frac{1}{2}\tau^2 \qquad (\alpha \to \infty) . \tag{5.56}$$

Thus,

$$H(\tau) \sim \exp\left(-\frac{1}{2}\tau^2\right) \qquad (\alpha \to \infty) , \tag{5.57}$$

and, by (5.45),

$$h(\tau) \sim \tau e^{-\frac{1}{2}\tau^2} \qquad (\alpha \to \infty) . \tag{5.58}$$

It is a simple matter now to check the validity of (5.54).

We see from this last analysis that the time-to-first-kill behaves quite differently from the way it did before; contrast (5.58) and (5.53) with (5.3). However, although the battle commences differently in this case, we may legitimately anticipate that it will soon shift over to the situation described earlier. For, consider first of all α men firing at a target, each with mean time-to-fire μ and variance σ^2. Then the α first rounds fired by them will produce a near-volley which is a statistical image of the firing-time; the α second rounds fired will be spaced out a little more perceptibly, having mean 2μ and variance $2\sigma^2$; and so on. The $n^{\underline{th}}$ round will thus be a sample of size α from a distribution with mean $n\mu$ and variance $n\sigma^2$; and the $(n+1)^{\underline{st}}$ round will have mean $(n+1)\mu$ and variance $(n+1)\sigma^2$. Thus successive rounds will begin to overlap

and, provided α is not too small, this overlapping will ultimately result in a complete jumbling of the rounds, so that the timing of the successive shots as they actually occur will degenerate into a Poisson process. The fact that B is simultaneously firing at A is, to be sure, an added complication, but it does not alter the conclusion that (5.6) and (5.7), with A and B suitably intepreted, will be valid for the bulk of the process.

6. ANOTHER STOCHASTIC BATTLE

> Or what king, going to make war against another king, sitteth not down first, and consulteth whether he be able with ten thousand to meet him that cometh against him with twenty thousand?
>
> ---Luke 14:31

It is evident that the difficulties intrinsic to the discussion presented in the last section are immediately traceable to the fact that the transition probabilities (5.6) and (5.7) are functions of α and β. Let us therefore replace these equations by

$$\text{Prob}\{(\alpha,\beta) \rightarrow (\alpha,\beta-1)\} = p, \tag{6.1}$$

and

$$\text{Prob}\{(\alpha,\beta) \rightarrow (\alpha-1,\beta)\} = q\ , \tag{6.2}$$

where p and q are constant and, of course,

$$p + q = 1 . \tag{6.3}$$

This battle may be conceived of as a series of single contests between A's and B's men, there being constant probability p that A wins any one of these contests. A graphic way of bearing in mind the difference between the two battle schemata is as follows: in the former battle the opposing forces were deployed facing one another along parallel lines and all surviving men on both sides were engaged in conflict at all times; in the present battle the opposing forces are deployed in two columns which meet head-on in a narrow defile and combat is confined to the men heading the two columns of survivors. As before we were led to a justification of Lanchester's square law, here we might well expect to vindicate his linear law, and this we shall indeed find to be the case.

We retain the notation of the last section and begin by skimming through those parts of the analysis which carry over with trivial alterations. Equation (5.9) becomes

$$p_\nu(\alpha,\beta) = qp_\nu(\alpha-1,\beta) + pp_\nu(\alpha,\beta-1) , \tag{6.4}$$

which leads to

$$\mu_k(\alpha,\beta) = q\mu_k(\alpha-1,\beta) + p\mu_k(\alpha,\beta-1) , \tag{6.5}$$

analogous to (5.15), and

$$q \frac{\partial\mu_k}{\partial\alpha} + p \frac{\partial\mu_k}{\partial\beta} = 0 \qquad (\alpha \text{ and } \beta \text{ both large}) , \tag{6.6}$$

analogous to (5.22); all the rest of the intervening equations (except, of

course, for (5.14)) remain unchanged. The subsidiary equations (5.23) now become

$$\frac{d\alpha}{q} = \frac{d\beta}{p} = \frac{d\mu_k}{0} \quad , \tag{6.7}$$

yielding (5.24) and, in place of (5.25),

$$p\alpha - q\beta = C_2 \ . \tag{6.8}$$

Hence the most general solution of (6.6) is

$$\mu_k = \Psi\,(p\alpha - q\beta) \ , \tag{6.9}$$

analogous to (5.27). Boundary condition (5.18) now yields

$$\Psi\,(p\alpha) = \alpha^k \ , \tag{6.10}$$

or, setting

$$u = p\alpha \ , \qquad \alpha = u/p \ , \tag{6.11}$$

$$\Psi\,(u) = (u/p)^k \ . \tag{6.12}$$

Substituting these results into (6.9), we find

$$\mu_k(\alpha,\beta) = (\alpha - \frac{q}{p}\,\beta)^k \ , \tag{6.13}$$

which implies that the random variable ν implied in (6.4) is in fact identically equal to $\alpha - \frac{q}{p}\,\beta$

$$\nu \equiv \alpha - \frac{q}{p}\beta \qquad (\alpha \text{ and } \beta \text{ both large}) \ . \tag{6.14}$$

Thus, analogous to (5.34) and (5.35), we have

$$P(A) = 1 \qquad (\alpha > \frac{q}{p}\beta) \ . \tag{6.15}$$

Hence A always wins over B provided $\alpha > \frac{q}{p}\beta$ (assuming α and β both large) and his victory costs him precisely $\frac{q}{p}\beta$ of his men. If we set the number of men on side B equal to zero, and here define the exchange rate to be

$$E = p/q \tag{6.16}$$

we are led to (6.14).

But, we said at the outset that we might expect a detailed analysis to be more tractable for this schema than for the former and we shall now see that this is perfectly true. For, the process may now be visualized as a two-dimensional random walk with downward transition probabilities all equal to p and leftward transition probabilities all equal to q. For A to wipe out B, with a <u>loss</u> of precisely ν of his men, requires, when he starts from the point (α,β), that he take exactly ν steps to the left and exactly β steps down, which introduces a factor of $q^{\nu}p^{\beta}$. (To be sure, the order in which these leftward and downward steps are taken will vary from one realization to the next, but the total numbers of the two different kinds of steps clearly must always be ν and β, respectively.) Now, of these steps, the nature of the last alone is dictated; it must be downward, from $(\nu,1)$ to $(\nu,0)$, for, since the α-axis is an absorbing barrier, the

step $(\nu+1,0)$ to $(\nu,0)$ is prohibited. Hence the total number of permissible paths is the number of ways that ν leftward steps can be chosen out of a total of $\beta+\nu-1$ steps. We therefore see that

$$p_{\alpha-\nu}(\alpha,\beta) = \binom{\beta+\nu-1}{\nu} q^{\nu} p^{\beta} . \tag{6.17}$$

It is a simple matter to check that this satisfies the recurrence (6.4) and the boundary conditions (5.16) and (5.17). It follows from (6.17) that

$$P(A) = p^{\beta} \sum_{\nu=0}^{\alpha-1} \binom{\beta+\nu-1}{\nu} q^{\nu} , \tag{6.18}$$

and combining this with the corresponding result for P(B) we are led to the interesting and non-trivial identity

$$p^{\beta} \sum_{\nu=0}^{\alpha-1} \binom{\beta+\nu-1}{\nu} q^{\nu} + q^{\alpha} \sum_{\nu=0}^{\beta-1} \binom{\alpha+\nu-1}{\nu} p^{\nu} = 1 . \tag{6.19}$$

For the purpose of clarity, solution (6.18) is set forth explicitly for the first few cases in the following table.

$\beta = 3$	0	p^3	$p^3(1+3q)$	$p^3(1+3q+6q^2)$
2	0	p^2	$p^2(1+2q)$	$p^2(1+2q+3q^2)$
1	0	p	$p(1+q)$	$p(1+q+q^2)$
0	$\boxed{X}$	1	1	1
	$\alpha=0$	1	2	3

Table 2. P(A) <u>qua</u> function of α and β.

The symbolic argument A in P(A) is of no utility analytically and so henceforth we write

$$P(\alpha,\beta) \equiv P(A) \ , \tag{6.20}$$

understanding that the probability of win, P, refers to side A, and displaying explicitly its dependence upon the initial numbers of forces. Reading up along any given column in Table 2 we see that, for fixed α, $P(\alpha,\beta)$ goes from 1 to 0 as β goes from 0 to ∞; in other words it behaves like a RSCDF with β as RV (random variable). This suggests that we might hope to get a firmer hold on (6.15) by looking at the moments of β; thus, we would anticipate that its mean should fall somewhere near $p\alpha/q$, and that, since P apparently drops precipitately thereabouts, the standard deviation of β ought to be small, relative to its mean. We proceed to follow up this clue.

For a discrete positive RV, β, with frequency distribution, f_β, we define the RSCDF to be

$$F_\beta \equiv \sum_{b=\beta}^{\infty} f_b \qquad (\beta=0,1,2,\ldots) \ , \tag{6.21}$$

which makes

$$F_o = 1 \tag{6.22}$$

and

$$f_\beta = F_\beta - F_{\beta+1} \qquad (\beta=0,1,2,\ldots) \ . \tag{6.23}$$

Thus the $k^{\underline{th}}$ factorial moment is

$$E[\beta(\beta-1)\cdots(\beta-k+1)] = \sum_{\beta=0}^{\infty}\beta(\beta-1)\cdots(\beta-k+1)f_\beta$$

$$= \sum_{\beta=k}^{\infty}\beta(\beta-1)\cdots(\beta-k+1)(F_\beta-F_{\beta+1})$$

$$= \sum_{\beta=k}^{\infty}\beta\cdots(\beta-k+1)F_\beta - \sum_{\beta=k+1}^{\infty}(\beta-1)\cdots(\beta-k)F_\beta$$

$$= \sum_{\beta=k}^{\infty}(\beta-1)\cdots(\beta-k+1)[\beta-(\beta-k)]F_\beta$$

$$= k\sum_{\beta=k}^{\infty}(\beta-1)(\beta-2)\cdots(\beta-k+1)F_\beta \qquad (\beta=1,2,3,\ldots) \ . \tag{6.24}$$

We have, in the discussion preceding (5.8), remarked the inaccessibility of the point (0,0) to the recurrence equation, a fact which has already been duly indicated in Table 2. We now want to set up a generating function for $P(\alpha,\beta)$, and so, keeping this difficulty in mind, we write

$$g(x,y) = \sum_{\alpha=1}^{\infty}x^\alpha\sum_{\beta=1}^{\infty}y^{\beta-1}P(\alpha,\beta) \ . \tag{6.25}$$

The reason for the exponent on y is that it makes

$$\left.\frac{\partial^{k-1}g}{\partial y^{k-1}}\right]_{y=1} = \sum_{\alpha=1}^{\infty}x^\alpha\sum_{\beta=1}^{\infty}(\beta-1)(\beta-2)\cdots(\beta-k+1)P(\alpha,\beta) \ ; \tag{6.26}$$

the first k terms in the inner summation vanish and, in consequence, (6.24) and (6.26) yield

$$E[\beta(\beta-1)\cdots(\beta-k+1)] = k \underset{x^\alpha}{\text{coeff}} \left.\frac{\partial^{k-1}}{\partial y^{k-1}} g(x,y)\right]_{y=1} , \tag{6.27}$$

where the notation introduced carries the construction:

$$\text{if } f(x) = \sum_{\alpha=0}^{\infty} C_\alpha x^\alpha , \qquad \text{then} \qquad \underset{x^\alpha}{\text{coeff}}\ f(x) \equiv C_\alpha . \tag{6.28}$$

It follows from (6.18) and (6.25) that

$$g(x,y) = \sum_{\alpha=1}^{\infty} x^\alpha \sum_{\beta=1}^{\infty} y^{\beta-1} p^\beta \sum_{\nu=o}^{\alpha-1} \binom{\beta+\nu-1}{\nu} q^\nu$$

$$= \sum_{\beta=1}^{\infty} p^\beta y^{\beta-1} \sum_{\alpha=o}^{\infty} x^\alpha \sum_{\nu=o}^{\alpha-1} \binom{\beta+\nu-1}{\nu} q^\nu , \tag{6.29}$$

since, β being always greater than 0 now, the inclusion of the term for $\alpha=0$ introduces only a zero into the summation over α. Inverting the order of summation in the innermost pair of sums, (6.29) becomes

$$g(x,y) = \sum_{\beta=1}^{\infty} p^\beta y^{\beta-1} \sum_{\nu=o}^{\infty} \binom{\beta+\nu-1}{\nu} q^\nu \sum_{\alpha=\nu+1}^{\infty} x^\alpha$$

$$= \frac{x}{1-x} \sum_{\beta=1}^{\infty} p^\beta y^{\beta-1} \sum_{\nu=o}^{\infty} \binom{\beta+\nu-1}{\nu} q^\nu x^\nu , \tag{6.30}$$

which reference to (3.10) enables us to write as

$$g(x,y) = \frac{x}{1-x} \sum_{\beta=1}^{\infty} p^\beta y^{\beta-1} \sum_{\nu=o}^{\infty} (-)^\nu \binom{-\beta}{\nu} q^\nu x^\nu$$

$$= \frac{x}{1-x} \sum_{\beta=1}^{\infty}{}' p^{\beta} y^{\beta-1} \frac{1}{(1-qx)^{\beta}} = \frac{x}{1-x} \frac{p}{1-qx} \frac{1}{1 - \frac{py}{1-qx}} , \tag{6.31}$$

or,

$$g(x,y) = \frac{px}{(1-x)[1-(qx+py)]} . \tag{6.32}$$

It is now easy to see that

$$\left.\frac{\partial^{k-1} g}{\partial y^{k-1}}\right]_{y=1} = \frac{px}{1-x} \left.\frac{(k-1)!\,p^{k-1}}{[1-(qx+py)]^{k}}\right]_{y=1}$$

$$= \frac{x}{1-x} \frac{(k-1)!\,p^{k}}{(q-qx)^{k}} = (k-1)!\left(\frac{p}{q}\right)^{k} \frac{x}{(1-x)^{k+1}} , \tag{6.33}$$

and, expanding this into a binomial series, we obtain

$$\left.\frac{\partial^{k-1} g}{\partial y^{k-1}}\right]_{y=1} = (k-1)!\left(\frac{p}{q}\right)^{k} \sum_{\alpha=o}^{\infty}{}'(-)^{\alpha} \binom{-k-1}{\alpha} x^{\alpha+1}$$

$$= (k-1)!\left(\frac{p}{q}\right)^{k} \sum_{\alpha=o}^{\infty} \binom{\alpha+k}{\alpha} x^{\alpha+1} , \tag{6.34}$$

upon yet another invocation of (3.10). Equations (6.27) and (6.34) now yield

$$E[\beta(\beta-1)\cdots(\beta-k+1)] = k!\left(\frac{p}{q}\right)^{k} \binom{\alpha+k-1}{\alpha-1}$$

$$= k!\left(\frac{p}{q}\right)^{k} \binom{\alpha+k-1}{k} , \tag{6.35}$$

when we call to mind the fact that

$$\binom{n}{\nu} = \binom{n}{n-\nu} . \tag{6.36}$$

Therefore we have, finally,

$$E[\beta(\beta-1)\cdots(\beta-k+1)] = \alpha(\alpha+1)\cdots(\alpha+k-1)\left(\frac{p}{q}\right)^k . \tag{6.37}$$

The first significant case here is that for which k=1; we find

$$E(\beta) = \frac{p}{q}\alpha . \tag{6.38}$$

This is really rather more than we dared desire, since it says that the handicap point described by (6.15), which marks a cross-over from a victory by A to a victory by B, and which was derived on the assumption that α and β are both large, holds, in a mean sense, for <u>all</u> α and β, and, furthermore, holds precisely. Letting k=2, we see from (6.37) that

$$E(\beta^2-\beta) = \left(\frac{p}{q}\right)^2\alpha(\alpha+1) , \tag{6.39}$$

whence

$$E(\beta^2) = \left(\frac{p}{q}\right)^2\alpha(\alpha+1) + \frac{p}{q}\alpha . \tag{6.40}$$

Thus, the variance of β is given by

$$\begin{aligned} \text{var}\ (\beta) &= \left(\frac{p}{q}\right)^2\alpha(\alpha+1) + \frac{p}{q}\alpha - \left(\frac{p}{q}\right)^2\alpha^2 \\ &= \frac{p}{q}\left(\frac{p}{q} + 1\right)\alpha = \frac{p}{q^2}\alpha . \end{aligned} \tag{6.41}$$

Equations (6.38) and (6.41) are very informative, indeed, since they begin to suggest that β may actually be the sum of α independent selections of the same random variable. Upon this hint we speak: if this is truly the case, then the RV being sampled can only be that for which $\alpha=1$, i.e., that whose RSCDF is the first column of Table 2. But that is of course merely a RV distributed according to a geometric distribution whose frequency distribution is, by (6.23) and (6.18) with $\alpha=1$,

$$f_\beta = p^\beta - p^{\beta+1} = q \cdot p^\beta . \tag{6.42}$$

Now, the factorial-moment-generating function of any RV β is by definition

$$\pi(z) = E(z^\beta) , \tag{6.43}$$

which reduces, in this case, to

$$\pi_o(z) = q \sum_{\beta=o}^{\infty} p^\beta z^\beta = \frac{q}{1-pz} . \tag{6.44}$$

But plainly the MGF with argument log z is in general given by

$$\varphi(\log z) = E(e^{\beta \log z}) = E(z^\beta) = \pi(z) , \tag{6.45}$$

according to (6.43). The well-known properties of the MGF show that if β is the sum of α independent RV's, each of which has MGF $\varphi_o(z)$, then the MGF of β is

$$\varphi(z) = [\varphi_o(z)]^\alpha . \tag{6.46}$$

Calling the FMGF (factorial-moment-generating function) of β $\pi(z)$, we thus have

$$\pi(z) = \varphi(\log z) = [\varphi_o(\log z)]^{\alpha} = [\pi_o(z)]^{\alpha} , \tag{6.47}$$

i.e., the FMGF has exactly the same properties as the MGF. Here $\pi_o(z)$ is the FMGF of the RV being sampled, and the hypothesis we are entertaining in the present paragraph assumes that this is given by (6.44). Hence the conjecture before us states in effect, by virtue of (6.44) and (6.47), that the FMGF of β is

$$\pi(z) = \left(\frac{q}{1-pz}\right)^{\alpha} . \tag{6.48}$$

Now, it follows from (6.43) that

$$\left.\frac{d^k\pi}{dz^k}\right]_{z=1} \equiv \pi^{(k)}(1) = E[\beta(\beta-1)\cdots(\beta-k+1)] . \tag{6.49}$$

Applying this to (6.48) produces the result

$$E[\beta(\beta-1)\cdots(\beta-k+1)] = q^{\alpha} \left.\frac{\alpha(\alpha+1)\cdots(\alpha+k-1)\,p^k}{(1-pz)^{\alpha+k}}\right]_{z=1}$$

$$= \alpha(\alpha+1)\cdots(\alpha+k-1)\,\frac{q^{\alpha}p^k}{(1-p)^{\alpha+k}} = \alpha(\alpha+1)\cdots(\alpha+k-1)\left(\frac{p}{q}\right)^k , \tag{6.50}$$

which is identical with (6.37). The conjecture is therefore established; namely, to state it explicitly: $P(\alpha,\beta)$ is the RSCDF of a RV which is the sum of α independent selections of a variate geometrically distributed according to (6.42).

This is truly rather a remarkable result, since it affirms that the device introduced just following (6.20) seemingly as a deus ex machina actually leads to an exact and rigorous conclusion. By (6.45) and (6.48), the MGF of β is equal to

$$\varphi(z) = \pi(e^z) = \left(\frac{q}{1-pe^z}\right)^{\alpha} , \tag{6.51}$$

and so its CGF (cumulant-generating-function) is, by (2.3),

$$\theta(z) = \alpha \log \left(\frac{q}{1-pe^z}\right) = \alpha\left[-z+\log \left(\frac{qe^z}{1-pe^z}\right)\right] , \tag{6.52}$$

from which we can identify the successive cumulants from the right-hand side of (2.3). But, we have come full circle, for no calculations whatsoever are necessary when we now call upon (2.5) and (2.22)! To do so it is only necessary to interchange p and q and to write z for θ: we find, from (6.52),

$$\kappa_1 = \alpha(-1 + \frac{1}{q}) = \frac{p}{q}\alpha , \tag{6.53}$$

$$\kappa_2 = \frac{p}{q^2}\alpha , \tag{6.54}$$

and

$$\kappa_3 = \frac{p}{q^2}(1+2\frac{p}{q})\alpha , \tag{6.55}$$

the first two of these equations agreeing with (6.38) and (6.41). The last two give the skewness,

$$\gamma_1 \equiv \frac{\kappa_3}{\kappa_2^{3/2}} = \frac{p}{q^2}\,\frac{1+p}{q}\alpha\,\frac{q_3}{(p\alpha)^{3/2}}\ , \tag{6.56}$$

or,

$$\gamma_1 = \frac{1+p}{\sqrt{p\alpha}}\ . \tag{6.57}$$

The proposition immediately following (6.50), together with the central-limit theorem, enables us to state that the variate

$$\xi \equiv \frac{q\beta - p\alpha}{\sqrt{p\alpha}} \sim N(0,1) \qquad (\alpha \to \infty)\ , \tag{6.58}$$

i.e., it is distributed as a standardized normal variate for large α. Since P is the RSCDF of β, the last result shows that if β exceeds $p\alpha/q$ by appreciably more than $\sqrt{\alpha}$, then A is virtually certain to lose, whereas if β falls short of $p\alpha/q$ by appreciably more than $\sqrt{\alpha}$, then A is virtually certain to win. The tangible values may be read from the tables of the normal distribution.

These arguments serve to validate and delimit (6.15) and, in so doing, to add increased credibility to (5.34) and (5.35), which were derived by a completely analogous chain of reasoning. The way is now open for a thorough-going examination of the "linear" battle, along the lines just initiated; we hope to be able to present further results in a sequel.

7. SUMMARY

We have examined in this report a number of aspects which go to make up a stochastic theory of duels. We have considered the manner in which a

marksman's time to hit a passive target may be statistically characterized in terms of his time to fire a single round, and have used this information to obtain approximate solutions to the duel. We have worked out one case of some generality in which the solution of the duel may be written down in a simple form. We have described two battle schemata which lead in the limit of large numbers of opposing forces to Lanchester's square and linear laws; for these we have broadened the stochastic output to include not only the probability of a given side's winning, but also the cost of victory in the number of his forces sacrificed. We have found the latter schema to be the mathematically more tractable situation and have found there a convenient and rather remarkable expression for the probability that a given side wins, noting that the techniques there employed give promise of providing a full solution to that duel.

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Unclassified report

DESCRIPTORS: Sampling (Mathematics).
Statistical Analysis (Mathematics).

Continues the project of systematically developing the elements of a stochastic theory of combat begun in an earlier work.

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Evaluates the mean, variance, third central moment, etc., of a marksman's time to hit a passive target in terms of the corresponding parameters of his time to fire a single round. Obtains the solution of the simple duel, in the case where protagonist's time-to-kill is distributed as a gamma-variate, and this result is employed to furnish an approximate solution to the general simple duel. Derives an expansion of the moment-generating function of the marksman's time-to-kill in powers of his kill probability. Considers various properties of the expansion. Examines a stochastic battle in which all men on both sides are at all times able to participate in the action. Entertains a stochastic battle where the two forces can only be brought into play at a single point of contact.

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