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SCATTERING FOR THE K^-MESON FROM THE DEUTERON*

by

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A simple model is constructed to study the scattering of the K^- meson from the deuteron. This model does not treat the nucleon as heavy and takes into account the multiple scattering, the binding energy corrections and the contribution from the off-energy shell scattering. The scattering problem is investigated by using the Watson multiple scattering expansion of the transition operator $t$. Considering the multiple scattering up to double order only, the $t$-matrix is written as

$$t = t_p + t_n + t_c + t_I$$

where $t_p$ and $t_n$ correspond to the single scattering of the K^- meson from the proton and the neutron in the deuteron. $t_c$ corresponds to the bound state contribution and $t_I$ corresponds to the continuum state contribution of the K^- meson from the deuteron.

The interaction K^-P and the K^-N is taken as a point interaction and is of the form

$$t_p = \chi_p \tilde{f}(r - \ell/2)$$
$$t_n = \chi_n \tilde{f}(r + \ell/2)$$

where $\chi$ and $\ell/2$ are the relative co-ordinates of the K^- meson and the nucleon in the center of mass system of the K^-d. $\chi_p$ and $\chi_n$ are taken as constants and are determined by using Dalitz solutions I and II given by Ross and Humphrey.
The potential between the nucleons in the deuteron is taken as a separable non-local potential which is such that the Hulthen wave function satisfies the Schrödinger equation for the bound state of the deuteron.

The matrix elements corresponding to the bound state and the continuum state are calculated only for forward scattering. For any other scattering angle triple integrals are obtained and need too much numerical calculation.

The double scattering contribution is compared with the Brueckner model and also the correspondence between the two models is studied.

The forward differential, elastic and total cross sections are calculated.

The correction to the cross sections due to the charge exchange in the intermediate state is calculated.

The results of this model are compared with the experimental data of the K^-d scattering cross sections to study the predictions of this model and to find the favorable Dalitz solution.
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CHAPTER I.
INTRODUCTION

As neutron is not available in a free state, the scattering of
the mesons from neutron has always been studied indirectly mainly
through the study of the scattering of the meson from the deuteron.
The deuteron is taken as the target because of its simple structure and
also because it is a loosely bound system i.e. the average separation
of its constituents is large compared to the range of the two-body in-
teraction and its binding energy per particle is small. It would be
expected that each of the nucleons in the deuteron would scatter the in-
cident particle in a manner not much different from the way a free
nucleon would scatter the incident particle.

The problems of nucleon-deuteron, pion-deuteron, and kaon-
deuteron scattering are three-body problems; they have not been solved
exactly. Aside from this basic fact, many of the physical details -
spin dependence of nucleon-nucleon interaction, the presence of tensor
forces and exchange forces in the nucleon-nucleon interaction - in-
volved contribute to their complexity. Various approximations (the
(1) resonating group structure method , the Born approximation , the

1 H.S. Massey, Prog. in Nucl. Phys., 2, 235 (1953).
G. F. Chew, ibid., 74, 809 (1948).
F. de Hoffman, ibid., 78, 216 (1950).
The high energy approximation, the impulse approximation, several variational procedures have been applied to these problems.

One of the approximations that has been applied to this kind of problem is the impulse approximation given by Chew et al. In this approximation, the incident particle is viewed as scattering once from either of the two nucleons in the deuteron; each of these scatterings is viewed as the scattering from a free nucleon whose momentum distribution is that of the actual bound nucleon. The only role played by the intra-deuteron potential is the determination of this momentum distribution. This approximation indeed leads to an expression (for example) for the elastic neutron-deuteron scattering cross section and a form factor for the deuteron structure. But this approximation neglects "potential" effects, multiple scattering effects, i.e., effects due to the incident particle's scattering more than once from the individual


5 E. Clemental, Nuovo Cimento 8, 185 (1951).

target nucleons, and the diminution in amplitude of the incident wave in crossing the nucleus.

The multiple scattering can be taken into account by Brueckner model. It is supposed in this model that the scattering taking place is S-wave scattering from the two point potentials. The wave function outside the potentials is written as the sum of the incident plane wave, a wave scattered from potential one, and a wave scattered from potential two, i.e.

$$\psi(z) = e^{i \frac{q}{\hbar} z} + A e^{i \frac{q}{\hbar} |z - R_1|} + B e^{i \frac{q}{\hbar} |R - R_2|}$$

(1.1)

The outgoing amplitude $A$ is given in terms of the total amplitude at $R_1$, by

$$A = \eta_p [e^{i \frac{q}{\hbar} r} + \frac{B e^{i q f}}{p}]$$

(1.2)

$\eta_p$ is the $K^{-} p$ scattering amplitude (in the deuteron).

Similarly

$$B = \eta_n [e^{i \frac{q}{\hbar} r} + A e^{i \frac{q}{\hbar} f}]$$

(1.3)

$\eta_n$ is the $K^{-} n$ scattering amplitude (in the deuteron).

Solving for $A$ and $B$ from Eqs. (1.2) and (1.3) and substituting these values in Eq. (1.1), we get an expression for the scattering amplitude

$$F_p(q', q) = \left(1 + \eta_p \eta_n \frac{e^{i q f}}{p^2}\right) \left[\eta_p e^{i q (q' - q) \cdot \mathbf{R}} + \eta_n e^{i q (q' - q) \cdot \mathbf{R}_2}ight]

+ \eta_p \eta_n \frac{e^{i q f}}{p^2} \left(e^{i (q' \cdot \mathbf{R} - q' \cdot \mathbf{R}_2) + e^{i (q' \cdot \mathbf{R} - q' \cdot \mathbf{R}_2)}}\right)$$

(1.4)

K.A. Brueckner, Phys. Rev., 89, 834 (1953); 90, 715 (1953).
The first two terms in the numerator of Eq. (1.4) are the single scattering (or impulse approximation) terms. The next two terms are the double scattering terms. The denominator represents all higher-order multiple scatterings. The propagator in the intermediate state is \( e^{i \sqrt{q^2}} \) which is very large for the region \( q^2 < 1 \).

In this approximation, the nucleons are supposed to be infinitely heavy compared to the incident particle. Therefore, the recoil of the nucleons is neglected. This method has been used to study the pion-deuteron as well as the kaon-deuteron scattering. The validity of this method is questionable for pions and even more for heavy particles.

Drell and Verlet consider the multiple-scattering corrections to the impulse approximation in the calculation of the scattering of \( \pi^\pm \) mesons by deuterons.

They calculated the scattering amplitude for the meson-deuteron scattering problem by using \( t \)-matrix formalism. In this treatment the binding energy, motion of the sources and the absorption of the mesons are neglected. Three models are given:

1. Brueckner model of point scatterer with propagator \( e^{i \sqrt{q^2}} \).

This model has the difficulties mentioned above.

2. We consider the scatterers to be of finite extent and scattering from a separable potential of the type given by Yamaguchi.

\[ \text{References} \]

8 S. D. Drell and L. Verlet, Phys Rev., 92, 849 (1955)

In this model, we approximate the Schrödinger equation describing the scattering of a particle by a potential, by replacing the wave function which appears in the interaction term by its average over the potential.

(3) The third model of momentum variation of the scattering amplitude considers only scattering on the energy shell in the intermediate state. In this approximation the intermediate wave propagates as $\langle \frac{\sin \varphi}{\varphi} \rangle$. This is smooth for $\varphi < 1$. Therefore it does not matter whether the sources are assumed to be of zero or of finite extent.

Using these models, it is found that the double scattering correction to the elastic cross section $n^f \rightarrow D + \pi^f$ is of the order of 10 percent or less and is model dependent. For the total cross sections (elastic, inelastic and absorption) as deduced from the imaginary part of the scattering amplitude in the forward direction, the correction is of the order of 10 percent or less and is quite model dependent.

(10) Fulton and Schwed have applied the Born and impulse approximations to calculate the nucleon-deuteron differential elastic cross sections. In order to carry out the impulse approximation calculation in complete detail, including, in particular, contributions from off-energy shell two particle matrix elements, the assumption is made that the two particle scattering is completely described by effective range theory. The results of the Born and impulse approximation differ

considerably. Their analysis shows that the experimental results are described better by the impulse approximation, including off-energy shell effects.

The Brueckner model has been used by Day, Snow and Sucher to calculate the $K^-D$ elastic and total cross sections. The calculations show that the impulse approximation does not give sensible results, and, therefore, the multiple scattering contribution is important.

One of the purposes of this thesis is to construct a simple model to study the scattering of the $K^-\text{meson}$ from the deuteron, which does not treat the nucleon as heavy. This model takes into account the multiple scattering, the binding energy corrections and the contribution from the off-energy shell scattering. The scattering problem is investigated by using Watson's multiple scattering expansion of the transition-operator $t$. The expansion is given by

$$t = t_p + t_n + t_p \frac{q}{j} t_n + t_n \frac{q}{j} t_p + t_p \frac{q}{j} t_n \frac{q}{j} t_p + \ldots \ldots$$  \hspace{1cm} (1.5)$$

where

$$q_j = (E - \frac{1}{2} - U + i\epsilon)^{-1}$$

and

$$t_p = t_{11} = v_1 + v_1 (E - \frac{1}{2} - U + i\epsilon)^{-1} t_1 \ldots \ldots (1.6)$$

---

and
\[ \tau_1 = \tau_2 = V_2 + V_2 \left( E - K - U + c \epsilon \right)^{-1/2} \tau_2 \]  \hspace{1cm} (1.7)

where \( U \) is the potential between the nucleons in the deuteron.

The other symbols are obvious.

In our calculation, we shall ignore the effects of the binding energy corrections on \( t_1 \) and \( t_2 \) due to the potential \( U \).

The interaction between \( K^- \) meson and the nucleon is taken as a point interaction i.e.
\[ t_1 = t_1^0 \delta \left( \xi - \xi_1 / 2 \right) \]  \hspace{1cm} (1.8)
\[ t_2 = t_2^0 \delta \left( \xi + \xi_2 / 2 \right) \]  \hspace{1cm} (1.9)

where \( \xi \) is the meson position vector and \( \xi = \xi_1 - \xi_2 \).

The \( t_1^0 \) and \( t_2^0 \) are taken as constants. No further approximations will be made in the calculations.

The magnitude of the \( t_1^0 \) and \( t_2^0 \) are determined with certain modifications by using the Dalitz solutions of scattering lengths at low energies, defined by
\[ \sqrt{\Gamma} \Delta \xi = \sqrt{\lambda} \Lambda \]  \hspace{1cm} (1.10)

where \( \Gamma = \) isotopic spin.

For \( K^- \)-P system \( I = 0 \), and \( 1 \)

For \( K^- \)-N system \( I = 1 \)

\[ \text{13} \]
The scattering lengths are determined by the experimental data of the scattering of the $K^-$-meson from the free nucleons. These solutions are given by Ross and Humphrey.

The interaction between the nucleons in the deuteron is supposed to be given by a separable non-local potential, which is such that in the case of the bound state of the deuteron, the Schrödinger equation is satisfied by the Hulthen wave function

$$
\varphi(p) = N \left( e^{-Np} - e^{-Bp} \right) / p
$$

where

$$
N = \phi_{2\pi} \cdot \frac{1}{60^\circ}
$$

Using the potential deduced, the wave function for the continuum state is obtained by solving the Schrödinger equation.

The double scattering of the $K^-$-meson is shown in the Feynmann diagram (Fig. 1).

Figure 1.

---

16 See ref. 9.
The matrix elements corresponding to the bound state and the continuum state are calculated up to second order only for forward scattering and put in a suitable form for numerical calculations. For any other angle the matrix elements can be expressed in terms of triple integrals which are not very easy to handle numerically. The forward differential, elastic and total cross sections are evaluated at three energies. The total cross section is calculated using the optical theorem,

\[ \sigma_T = \frac{4\pi}{q^2} \Im f(\theta') \]  

(1.12)

The correction to the elastic scattering due to charge exchange in the intermediate state is also calculated. The Feynmann diagram for this is shown in Fig. 2.

![Figure 2](image)

The results of this model are compared with Brueckner model of S-wave scattering from two point potentials and also with the experimental data of the K-D scattering cross sections to study the predictions of this model and to find the favorable Dalitz's solution at low energies.
CHAPTER II

K\textsuperscript{−}-D SCATTERING

Let K\textsuperscript{−} be indicated by index '0', proton and neutron by '1' and '2'.

The initial and final wave functions for the K\textsuperscript{−}-D system can be written as

\[ \Psi_0 = c_1 \phi_{K,0} \phi_{D,0} \]

\[ \Psi_f = c_2 \phi_{K,1} \phi_{D,1} \]  \hspace{5cm} (2.1)

where \( q_1 \) and \( q_1' \) are the initial and final momenta of the K-meson; K and F that of the deuteron.

\( \tau_0, \tau_1, \tau_2 \) are the co-ordinates of the K-meson, proton and neutron, and

\[ \bar{\tau} = \frac{1}{2}(\tau_1 + \tau_2) \]

Introduce,

\[ \xi = \tau_1 - \tau_2 \]

\[ \tau_2 = \tau_1 - \bar{\tau} \]

\[ \xi = \frac{m_K \tau_2 + m_D \bar{\tau}}{m_K + m_D} \]

\[ m_D = m_p + m_n \]

where \( m_K \) and \( m_D \) are the masses of the K-meson and the deuteron.

\[ \xi = \frac{m_K \xi + m_D \bar{\tau}}{m_K + m_D} \]

\[ \bar{\tau} = \xi - \frac{m_K}{m_K + m_D} \xi \]  \hspace{5cm} (2.4)

\[ \xi = \frac{m_K \xi + m_D \bar{\tau}}{m_K + m_D} \]

or

\[ \bar{\tau} = \xi - \frac{m_K}{m_K + m_D} \xi \]  \hspace{5cm} (2.4)
The $T$ matrix for the $K^-D$ scattering problem is given by

$$ T = V + V(E - H + i\epsilon)^{-1} V $$

where $V = V_1 + V_2$

$V_1$ is the potential between the $K^-$-meson and the proton.

$V_2$ is the potential between the $K^-$-meson and the neutron.

The Hamiltonian for the system is given by

$$ H = \hat{H}_0 + \hat{H}_1 + \hat{H}_2 + U + V $$

where $U$ is the potential between the proton and the neutron.

The matrix element for the scattering of the $K^-$-meson from the deuteron is

$$ M = \langle \psi_f | T | \psi_i \rangle = \int \phi^*(p) T \phi(p) e^{i \frac{(q^2 + K^2)}{2}} d\tau_3 d\tau_4 d\xi d\sigma $$

Now $d\tau_3 d\tau_4 = df d\xi$

$$ M = \int e^{-\frac{(q^2 + K^2)}{2}} \phi^*(p) T \phi(p) d\tau_3 df d\xi $$

Using Eqs. (2.3) and (2.4), we get

$$ \frac{1}{2} r_m + K \cdot \xi = \frac{1}{2} (r_m + \xi) + K \cdot \xi $$

$$ \frac{1}{2} r_m + \xi = \frac{1}{2} (r_m + K) \cdot \xi $$

$$ \frac{1}{2} r_m + \xi = \frac{1}{2} (r_m + K) \left( \xi - \frac{m_K}{m_K + m_d} \right) $$

$$ = \left( \frac{1}{2} - \frac{m_K}{m_K + m_d} (q + K) \right) \cdot \xi $$

$$ = \frac{m_d \left( \frac{1}{2} - m_k \xi \right)}{m_K + m_d} \cdot \xi $$

$$ = \frac{m_d \frac{1}{2} - m_k \xi}{m_K + m_d} \cdot \xi $$

$$ = q \cdot \xi + (q + K) \cdot \xi $$

where $q_\xi = - \frac{m_d \frac{1}{2} - m_k \xi}{m_K + m_d} = \frac{1}{2} - \frac{m_k (q + K)}{m_K + m_d}$

and $d\tau_3 d\tau_4 = d\xi d\xi$
In the C. M. system of the $K^-$-meson and the deuteron, the T matrix becomes

$$t = V + V_1 \left[ \frac{q^2}{2\mu_{kd}} + \epsilon_b - \frac{\mu_{kd}}{2\mu_{kd}} \right] V$$

(2.9)

where

$$\mu_{kd} = \frac{m_{K^-} m_d}{m_{K^-} + m_d}$$

is the reduced mass of the $K^-$-meson and the deuteron.

$$\mu_d = \frac{m_p m_n}{m_p + m_n}$$

is the reduced mass of the deuteron.

Using Eqs. (2.7), (2.8) and (2.9), the matrix element becomes

$$M = \int e^{i(q'_0 - q_0) \cdot \vec{z}} \cdot e^{i(q'_0 + q'_0 - q_0) \cdot \vec{z}} = \phi^*(\vec{r}) t \phi(\vec{r}) \, df \, ds \, dl \, dz$$

$$= (2\pi)^3 \delta(k' - k + q'_0 - q_0) \cdot \int e^{i(q'_0 - q_0) \cdot \vec{z}} \cdot \phi^*(\vec{r}) t \phi(\vec{r}) \, df \, ds \, dl \, dz$$

$$= (2\pi)^3 \delta(k' - k + q'_0 - q_0) \cdot \langle q'_0, \phi | t | \phi, q_0 \rangle$$

gives the conservation of momenta.

Matrix element in the center of mass is given by

$$M = \langle q'_0, \phi | t | \phi, q_0 \rangle$$

(2.10)

We can write

$$t = V + V_1 \cdot V$$

(2.11)

where,

$$V = V_1 + V_2$$

and

$$V_1 = \left[ \frac{q^2}{2\mu_{kd}} + \epsilon_b - \frac{\mu_{kd}}{2\mu_{kd}} \right] V$$

From the above equations the multiple expansion of $t$ may be easily obtained in the form.
\[ t = \sum_{i} t_{i} + \sum_{j} t_{j} G t_{j} + \sum_{i,j} t_{i} t_{j} G t_{k} + \cdots \] \tag{2.12}

where \( t_{i} \) is defined by
\[ t_{i} = V_{i} + V_{i} G t_{i} \] \tag{2.13}

and \( G \) is given by
\[ G = \left[ \frac{p_{/2}}{2} \frac{\hbar}{\hbar} - E_{K} - \hbar^2 - U + i \epsilon \right]^{-1} \] \tag{2.14}

We now separate \( t \) into a "coherent" part \( t_{c} \) corresponding to the process in which the deuteron is always in its ground state while the \( K^- \text{-meson} \) is scattered from one nucleon to the other and a remaining part which will be called "incoherent". Thus we introduce \( P = |\Psi \rangle \langle \Psi| \) the projection operator on the deuteron ground state and write
\[ G_{c} = G P + G' \] \tag{2.15}

where
\[ G_{c} = \left[ \frac{p_{/2}}{2} \frac{\hbar}{\hbar} - \hbar^2 + i \epsilon \right]^{-1} \] \tag{2.16}

and
\[ (\hbar + U) \psi = E_{G} \Psi \] \tag{2.17}

On substituting Eq. (2.15) in Eq. (2.12) we get
\[ t = t_{c} + t_{2} + t_{c} + t_{1} \]

where
\[ t_{c} = \sum_{i} t_{i} G_{c} t_{j} \] \tag{2.18}

and
\[ t_{1} = \sum_{i} t_{i} G' t_{j} \] \tag{2.19}

In the multiple expressions of \( t \), we shall confine ourselves up to the double scattering. Therefore \( t = t_{2} + t_{c} + t_{1} \) where \( t_{d} \) represents the term contributing to the double scattering only.
In this thesis we restrict our attention to the effects of single and double scattering only. In the next chapter we consider a model which allows us to compute the matrix elements of coherent part

\[ t_1 G_0 P t_2 + t_1 G_1 P t_1 \]  

(2.20)

and incoherent or continuum part,

\[ t_1 G' t_2 + t_2 G' t_1 \]  

(2.21)

in the form which allows numerical evaluation.

Using Eqs. (2.12), (2.15) in Eq. (2.10), we get

\[ M = M_1 + M_2 + M_{12}^C + M_{12}^F \]  

(2.22)

where

\[ M_1 = \langle q', \varphi | t_1 | ^{\varphi, 0} \rangle \]  

(2.23)

\[ M_{12}^C = \langle q', \varphi | t_1 G_1 P t_2 | ^{\varphi, 0} \rangle \]  

(2.24)

\[ M_{12}^F = \langle q', \varphi | t_1 G' t_2 | ^{\varphi, 0} \rangle \]  

(2.25)
CHAPTER III

MODEL OF THE K\(^{-}\) MESON AND DEUTERON SCATTERING

A. Assumptions. 1. We shall suppose that the interaction between the K\(^{-}\)-meson and the nucleon is a point interaction and therefore we can write

\[ t_p = t_1 = t_1^0 \delta \left( r \frac{1}{2} \right) \]  

\[ t_n = t_2 = t_2^0 \delta \left( r + \frac{1}{2} \right) \]  

(3.1) (3.2)

where \( t_1^0 \) and \( t_2^0 \) are constants. (For the physical K\(^{-}\)-meson scattering problem, \( t_1^0 \) and \( t_2^0 \) will be determined by the Dalitz solutions for the scattering lengths for zero effective range approximation for K\(^{-}\) scattered by free nucleon - See Chapter VII).

2. We shall suppose that the potential between the proton and the neutron in the deuteron is a separable non-local potential \( U \) which is such that the solution of the Schrödinger equation for bound state is the Hulthen wave function.

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This approximation corresponds to a K\(^{-}\)-nucleon force range so short compared to the separation \( r \) of the nucleon in the deuteron and to the neglect of the binding interactions while the K\(^{-}\)-meson interacts repeatedly with the same nucleon. This approximation should not be too bad for \( r_o \approx 1.5 \) and \( r = 5.5 \).
B. Separable Potential and Ground State Wave Function of the Deuteron. We introduce a separable non-local potential $U$ which is such that the solution of Schrödinger equation

$$\left(\frac{\hbar^2}{2\mu_d} + U\right)\psi(r) = \varepsilon_0 \psi(r) \quad (3.3)$$

for deuteron in the ground state is the Hulthen wave function

$$\psi(r) = N \left( e^{-\frac{2\alpha}{3\hbar}} - e^{-\frac{2\beta}{3\hbar}} \right) \quad (3.4)$$

where

$$N^2 = \frac{\alpha}{2\pi} \frac{1}{1 - 2\alpha} = \frac{\alpha}{2\pi} \cdot \frac{1}{606} \quad (3.5)$$

and

$$\beta = 6.2 \cdot 2 \cdot \alpha \quad (3.6)$$

$\varepsilon_0 = -\frac{\alpha^2}{2\mu_d}$ is the binding energy of the deuteron.

Schrödinger equation becomes

$$\left(\frac{\hbar^2}{2\mu_d} - \nabla^2\right)\psi(r) = -U\psi(r) \quad (3.7)$$

Assume

$$U\psi(r) = -\psi(r) \int \nu(p')\psi(p') dp' \quad (3.8)$$

Now

$$\nabla^2\psi(r) = \nabla^2 \cdot \frac{2}{\beta^2} \left(\frac{\hbar^2}{2\mu_d} \right)\psi(r)$$

$$\frac{\partial}{\partial p} \psi(r) = N \left[ e^{\alpha p} - e^{-\beta p} - \frac{\alpha e^{\alpha p}}{\hbar^2} + \frac{\beta e^{-\beta p}}{\hbar^2} \right]$$

$$\left(\frac{\partial^2}{\partial p^2} \psi(r)\right) = N \left[ -e^{\alpha p} + e^{-\beta p} - \alpha e^{\alpha p} \frac{\beta e^{\beta p}}{\hbar^2} + \beta e^{-\beta p} \right]$$

$$\nabla^2 \psi(r) = N \left[ \frac{\alpha^2}{\hbar^2} e^{\alpha p} - \frac{\beta^2}{\hbar^2} e^{-\beta p} \right]$$

$$\therefore \quad \nabla^2 \psi(r) = N \left( \frac{\alpha^2 e^{\alpha p}}{\hbar^2} - \frac{\beta^2 e^{-\beta p}}{\hbar^2} \right)$$
Eq. (3.7) becomes

\[ \frac{\alpha^2}{2\mu_d} \left( \frac{e^{-\beta_1}}{\rho} - \frac{e^{-\beta_2}}{\rho} \right) = \frac{N}{2\mu_d} \left( \frac{\alpha^2 e^{-\beta_1} - \alpha^2 e^{-\beta_2}}{\rho} \right) = u(t) \int u(p') \varphi(p') dp' \]

or

\[ \frac{N}{2\mu_d} (\alpha^2 - \alpha^2) \frac{e^{-\beta_1}}{\rho} = u(t) \int u(p') \varphi(p') dp' \quad (3.9) \]

Assume

\[ u(p) = \frac{A}{\rho} \frac{e^{-\beta_1}}{\rho} \quad (3.10) \]

R.H.S.

\[ = \frac{N}{2\mu_d} \cdot \frac{A^2 e^{-\beta_1}}{\rho} \int e^{\beta_1} e^{-\beta_1} \left( \frac{e^{-\beta_1}}{\rho} - \frac{e^{-\beta_2}}{\rho} \right) \cdot 4\pi \rho' \frac{d\rho'}{\rho} \]

\[ = \frac{N}{2\mu_d} \cdot \frac{A^2 e^{-\beta_1}}{\rho} \cdot 4\pi \int e^{-\beta_1} \left( \frac{1}{\rho + \beta} \right) \frac{d\rho'}{\rho} \]

\[ = \frac{N}{2\mu_d} \cdot \frac{4\pi A^2 e^{-\beta_1}}{\rho} \left( \frac{1}{\rho + \beta} - \frac{1}{2\beta} \right) \]

\[ = \frac{N}{2\mu_d} \cdot \frac{4\pi A^2 (\beta - \alpha)}{2\beta (\alpha + \beta)} \cdot \frac{e^{-\beta_1}}{\rho} \]

Eq. (3.9) becomes

\[ (\beta^2 - \alpha^2) \frac{e^{-\beta_1}}{\rho} = A^2 \frac{e^{-\beta_1}}{\rho} \cdot \frac{2\pi (\beta - \alpha)}{\beta (\alpha + \beta)} \]

\[ \therefore \quad A^2 = \frac{\beta (\alpha + \beta)^2}{2\pi} \quad (3.11) \]
\[ \therefore U(q, q') = - \frac{\alpha^2}{2\mu d} \frac{e^{-\beta p}}{p} \frac{e^{-\beta p'}}{p'} \]

\[ = - \frac{\beta (\alpha + \beta)^2}{4\pi \mu d} \frac{e^{-\beta p}}{p} \frac{e^{-\beta p'}}{p'} \quad (3.12) \]

Therefore

\[ U(p) = \frac{\beta (\alpha + \beta)^2}{\sqrt{4\pi \mu d}} \frac{e^{-\beta p}}{p} \quad (3.13) \]

Eq. (3.12) gives the potential in the deuteron in the co-ordinate space, we shall express the same potential in the momentum space by taking the Fourier transform of Eq. (3.13).

Therefore, we get

\[ U(p) = \frac{1}{(2\pi)^{3/2}} \int U(q) e^{i\mathbf{p} \cdot \mathbf{r}} \, d^3q \quad (3.14) \]

\[ = \sqrt{\frac{\beta (\alpha + \beta)^2}{4\pi \mu d}} \frac{2\pi}{(2\pi)^{3/2}} \int_0^\infty \left( e^{i\mathbf{p} \cdot \mathbf{r}} - e^{i\mathbf{p} \cdot \mathbf{r}} \right) \frac{e^{-\beta p}}{p} \rho^2 d\rho \]

\[ = \sqrt{\frac{\beta (\alpha + \beta)^2}{4\pi \mu d}} \frac{1}{(2\pi)^{3/2}} \frac{1}{4p} \left[ \frac{1}{p - i\rho} - \frac{1}{p + i\rho} \right] \]

\[ = \sqrt{\frac{\beta (\alpha + \beta)}{4\pi \mu d}} \frac{2}{(2\pi)^{3/2} (p^2 + \rho^2)} \quad (3.15) \]
In the momentum space we shall define

\[ U(p, p') = -U(p) \cdot U(p') \]  \hspace{1cm} (3.16)

From Eq. (3.15), we get

\[
U(p, p') = - \frac{\beta(\omega + \beta)^2}{4\pi \mu_d} \cdot \frac{1}{2\pi} \cdot \frac{2}{p^2 + \beta^-} \cdot \frac{2}{p'^2 + \beta^-} \\
= - \frac{\beta(\omega + \beta)^2}{\pi^2} \cdot \frac{1}{2\mu_d} \cdot \frac{i}{p^2 + \beta^-} \cdot \frac{i}{p'^2 + \beta^-} \\
= - \frac{B^2}{2\mu_d} \cdot \frac{1}{p^2 + \beta^-} \cdot \frac{1}{p'^2 + \beta^-} \]  \hspace{1cm} (3.17)

where

\[
B^2 = \frac{\beta(\omega + \beta)^2}{\pi^2} \]  \hspace{1cm} (3.18)
C. Continuum Wave Function. Now we find the continuum solutions of the Schrödinger equation.

\[ \psi_\xi(p) = \phi_\xi(p) + c_0 \cup \psi_\kappa(p) \]  \hspace{1cm} (3.19)

where

\[ \phi_\kappa(p) = \frac{e^{-\frac{x^2}{2}}} {(2\pi)^{3/2}} \]  \hspace{1cm} (3.20)

and

\[ c_0 = \sqrt{E - H_0 + \epsilon} \]  \hspace{1cm} (3.21)

Eq. (3.19) is equivalent to

\[ (E - H_0 + \epsilon) \psi_\kappa(p) = i \psi_\kappa(p) \]

Substituting Eq. (3.16) in the above equation in momentum space, we get

\[ \left( \frac{k^2}{2\hbar^2} - \frac{\varepsilon}{\hbar^2} \right) \psi(p) = -i \hbar \frac{\partial}{\partial x} \psi(p) \]

Using Eq. (3.17) in Eq. (3.22) and multiplying by \( (2\pi)^3 \) throughout, we get

\[ (k^2 - p^2 + \epsilon) \psi(p) = - \frac{\hbar^2}{2m} \int \frac{1}{p^2 + \beta^2} \psi(p') dp' \]  \hspace{1cm} (3.23)

we shall suppose that

\[ \psi(p) = \delta(p - \kappa) - \frac{k^2 + \beta^2}{2\pi^2} \cdot f(k) \cdot \frac{1}{p^2 + \beta^2} \cdot \frac{1}{(k^2 - p^2 + \epsilon)} \]  \hspace{1cm} (3.24)

Substituting Eq. (3.24) in Eq. (3.23) and solving for \( f(k) \), we find that

\[ f(k) = \frac{1}{(k^2 + \beta^2)^{3/2}} \int \psi(p) e^{ik \cdot p} dp \]  \hspace{1cm} (3.26)
Substituting Eq. (3.24) in Eq. (3.26), we get

\[
\psi(t) = \frac{e^{-\mathbf{k}\cdot\mathbf{f}}}{(2\pi)^{3/2}} - \frac{k^2 + \beta^2}{2\pi^2} \cdot \mathcal{F}(\mathbf{k}) \cdot \int \frac{1}{p^2 + \beta^2} \cdot \frac{e^{-\mathbf{k}\cdot\mathbf{f}}}{k^2 - p^2 + i\epsilon} \, dp
\]

\[
= \frac{e^{-\mathbf{k}\cdot\mathbf{f}}}{(2\pi)^{3/2}} - \frac{k^2 + \beta^2}{2\pi^2} \cdot \mathcal{F}(\mathbf{k}) \cdot I \quad (3.27)
\]

where

\[
I = \int \frac{1}{p^2 + \beta^2} \cdot \frac{e^{-\mathbf{k}\cdot\mathbf{f}}}{k^2 - p^2 + i\epsilon} \, dp
\]

\[
= 2\pi \int_{-\infty}^{\infty} \frac{e^{-\mathbf{k}\cdot\mathbf{f}}}{p + \beta} \cdot \frac{1}{p^2 + \beta^2} \cdot \frac{p^2 \, dp}{k^2 - p^2 + i\epsilon}
\]

\[
= \frac{2\pi}{\mathbf{k} \cdot \mathbf{f}} \int_{-\infty}^{\infty} \frac{e^{-\mathbf{k}\cdot\mathbf{f}}}{p + \beta} \cdot \frac{1}{p^2 + \beta^2} \cdot \frac{p \, dp}{k^2 - p^2 + i\epsilon}
\]

\[
= \frac{2\pi}{\mathbf{k} \cdot \mathbf{f}} \cdot 2\pi i \times \text{(sum of residues)}
\]

The poles are at \( p = -\beta \) and \( p = k + i\epsilon \).

Residue at \( p = -\beta \) is \( \frac{e^{-\mathbf{k}\cdot\mathbf{f}}}{\mathbf{k} \cdot \mathbf{f}} \cdot \frac{\mathbf{f}}{k^2 + \beta^2} = \frac{e^{-\beta\mathbf{f}}}{2(k^2 + \beta^2)} \).

Residue at \( p = k + i\epsilon \) is \( \frac{-e^{-\mathbf{k}\cdot\mathbf{f}}}{(k^2 + \beta^2)} \cdot \frac{\mathbf{f}}{2k} = -\frac{e^{-\mathbf{k}\cdot\mathbf{f}}}{2(k^2 + \beta^2)} \).

\[
I = \frac{2\pi^2}{(k^2 + \beta^2)} \cdot \frac{e^{-\beta\mathbf{f}} - e^{-\mathbf{k}\cdot\mathbf{f}}}{\mathbf{f}} \quad (3.28)
\]
Substitute Eq. (3.28) in Eq. (3.27), we get (See ref. 9)

$$
\psi_c(\theta) = \frac{e^{-i k \frac{\phi}{\rho}}}{i \pi \gamma_\pi} - \frac{f(\kappa)}{(2 \pi)^2} \frac{e^{i \rho \frac{\kappa}{\pi^2}}}{\kappa^2 \rho^2} \cdot \frac{2 \pi^2}{\kappa^2 \rho^2} \cdot \frac{e^{i \rho \frac{\phi}{\rho}} - e^{-i \rho \frac{\phi}{\rho}}}{\rho}
$$

$$
= \int_{(2 \pi)^2} \left[ e^{i \frac{\pi}{\rho} \rho} + f(\kappa) \left( e^{i \rho \frac{\phi}{\rho}} - e^{-i \rho \frac{\phi}{\rho}} \right) \right]
$$

(3.29)

Eq. (3.29) is the required continuum wave function in the presence of the separable potential $U$.

In the next sections, we shall compute the single and double scattering of the $K^-$-meson from the deuteron.
D. Single Scattering. For single scattering, the matrix element is

\[ M_1 = \langle \frac{q'}{2}, \varphi (r) | t_1 | \varphi (r), \frac{q}{2} \rangle \]

\[ q_1 = \frac{q}{2} \] dropping the index.

Using Eq. (3.4), we get

\[ M_1 = N^2 \int e^{i \left( \frac{q}{2} \cdot r - \frac{q'}{2} \cdot r' \right)} \left( \frac{e^{- \alpha r} - e^{- \beta r}}{r} \right)^2 d\varphi d\tau \]

Using Eq. (3.1), we get

\[ M_1 = N^2 \int e^{i \left( \frac{q}{2} - \frac{q'}{2} \right) \cdot \tau} \left( \frac{e^{- \alpha \tau} - e^{- \beta \tau}}{\tau} \right)^2 d\varphi d\tau \]

\[ = N^2 \int_0^\infty e^{i Q \rho} \left( \frac{e^{- \alpha \rho} - e^{- \beta \rho}}{\rho} \right)^2 2\pi d\rho d\tau \]

\[ = \frac{4\pi N^2 t_1}{Q} \int_0^\infty \frac{\sin Q \rho}{\rho} \left( e^{- \alpha \rho} - e^{- \beta \rho} \right)^2 d\rho \]

where

\[ \rho = \left| \frac{q}{2} - \frac{q'}{2} \right| .\]
Using
\[ \int_0^\infty \frac{S_{xy}^2}{y} e^{-\lambda y} \, dy = \tan^{-1} \left( \frac{y}{\lambda} \right) \] (3.30)

we get

\[ M_1 = \frac{4\pi N^2 t^0}{Q} \left[ \int_0^\infty \frac{S_{xy}^2}{y} \left[ e^{-\frac{2\pi}{Q} y} + e^{-\frac{2\pi}{Q} y} - 2 e^{-\frac{(y+\theta)}{Q} y} \right] \right] \]

\[ = \frac{4\pi N^2 t^0}{Q} \left[ \tan^{-1} \left( \frac{Q}{2\lambda} \right) + \tan^{-1} \left( \frac{Q}{2\lambda} \right) - 2 \tan^{-1} \left( \frac{Q}{\lambda + \beta} \right) \right] \] (3.31)

where \( y = \theta / \beta \)

Similarly, we get

\[ M_2 = \frac{4\pi N^2 t^0}{Q} \left[ \tan^{-1} \left( \frac{Q}{2\lambda} \right) + \tan^{-1} \left( \frac{Q}{2\lambda} \right) - 2 \tan^{-1} \left( \frac{Q}{\lambda + \beta} \right) \right] \] (3.32)

From Eqs. (3.31) and (3.32), we get

\[ M_1 + M_2 = \frac{4\pi N^2 (\xi_0 + \xi_1)}{Q} \left[ \tan^{-1} \left( \frac{Q}{2\lambda} \right) + \tan^{-1} \left( \frac{Q}{2\lambda} \right) - 2 \tan^{-1} \left( \frac{Q}{\lambda + \beta} \right) \right] \] (3.33)

\[ \theta = q \cdot S_w(\theta_2) \quad \text{for} \quad |y| = |\xi_1| \] (3.34)

\[ M_1 + M_2 = (t_1 + t_2) \quad \text{for} \quad \theta = 0^\circ \text{ or } \theta = 0 \] (3.35)
E. Double Scattering: Bound State or Coherent Scattering Contribution. Using Eq. (2.20), coherent scattering contribution is given by

\[ M_{i_{1}2}^{C} = \langle \vec{q}', \Phi(\vec{k}) | t_{1} | \frac{\vec{q}_{4} - \vec{q}_{3}}{2M_{dk}} + i\epsilon \rangle P_{t_{2}} | \Phi(\vec{k}), \vec{k} \rangle + (1 \leftrightarrow 2) \]  

(3.36)

Introducing

\[ \sum_{2} \langle \xi_{2} | \xi_{1} \rangle = \int \frac{d\xi}{(2\pi)^{3/2}} \langle \xi_{2} | \xi_{1} \rangle \langle \xi_{1} | \xi_{2} \rangle \]

and \( P = \) \( | \Phi(\vec{k}) \rangle \langle \Phi(\vec{k}) | \)

in Eq. (3.36), we get in an obvious notation

\[ M_{i_{1}2}^{C} = \int \frac{d\vec{q}_{3}}{(2\pi)^{3/2}} \left[ \langle \vec{q}', \Phi(\vec{k}) | t_{1} | \Phi(\vec{k}), \vec{k} \rangle \left( \frac{\vec{q}_{4} - \vec{q}_{3}}{2M_{dk}} + i\epsilon \right) \right. \]

\[ \times \left. \langle \vec{k}, \Phi(\vec{k}) | t_{2} | \Phi(\vec{k}), \vec{q} \rangle \right] + (1 \leftrightarrow 2) . \]  

(3.37)

Now

\[ \langle \vec{q}', \Phi(\vec{k}) | t_{1} | \Phi(\vec{k}), \vec{k} \rangle = \int \int e^{-i\vec{q}' \cdot \vec{r}} | \Phi(\vec{k}) \rangle e^{i\vec{k} \cdot \vec{r}} \, d\vec{r} \, d\vec{k} \]

Using Eq. (3.1), we get

\[ \langle \vec{q}', \Phi(\vec{k}) | t_{1} | \Phi(\vec{k}), \vec{k} \rangle = \int \int e^{-i(\vec{k} - \vec{q}') \cdot \vec{r}} | \Phi(\vec{k}) \rangle e^{i\vec{k} \cdot \vec{r}} \, d\vec{r} \, d\vec{k} \]

\[ = \int e^{-i(\vec{k} - \vec{q}') \cdot \vec{r}} | \Phi(\vec{k}) \rangle \, d\vec{r} \]

\[ = \int e^{-i(\vec{k} - \vec{q}') \cdot \vec{r}} N \left( e^{-\vec{q}' \cdot \vec{r}} - e^{\vec{q}' \cdot \vec{r}} \right) \, d\vec{r} \, d\vec{k} \]

\[ = 4\pi N^{2} e^{i(\vec{k} - \vec{q}') \cdot \vec{r}} \int_{0}^{\infty} \left| \frac{k_{z}^{2} - q_{z}^{2}}{k_{z}^{2} - q_{z}^{2}} \right| \left( e^{\vec{q}' \cdot \vec{r}} - e^{-\vec{q}' \cdot \vec{r}} \right) \, d\vec{r} \, d\vec{k} \]
Let
\[ \left( \mathbf{b} - \mathbf{e}_2 \right) \mathbf{p}_2 = \mathbf{p}_1 \]and\n\[ \mathbf{p}_2 = \frac{\mathbf{e}_2}{\mathbf{b} - \mathbf{e}_2} \]
and
\[ \phi(\mathbf{p}) = \frac{2\pi N^2 t_0}{\mathbf{b} - \mathbf{e}_2} \int_0^1 dy \left[ \begin{array}{c} -\mathbf{e}_2 \gamma \mathbf{v} \\ \mathbf{b} - \mathbf{e}_2 \end{array} \right] + \mathbf{e}_2 \left( \mathbf{b} - \mathbf{e}_2 \right)^{-1} \gamma \mathbf{v} \]

Using Eq. (3.30), we get
\[ \left\langle \psi', \phi(\mathbf{p}) | \mathbf{t}_1 | \phi(\mathbf{p}), \psi \right\rangle = \frac{8\pi N^2 t_0}{\mathbf{b} - \mathbf{e}_2} \left[ \begin{array}{c} \tan \left( \frac{\mathbf{e}_2}{\mathbf{v}_b} \right) \\ + 4a_0 \left( \frac{\mathbf{e}_2}{\mathbf{v}_b} \right) \left( 2 + \mathbf{e}_2 \mathbf{e}_2 \right) \end{array} \right] \]

Let \( T(\mathbf{b} - \mathbf{e}_2) = 4a_0 \left( \frac{\mathbf{e}_2}{\mathbf{v}_b} \right) \left( 2 + \mathbf{e}_2 \mathbf{e}_2 \right) \)

Similarly
\[ \left\langle \psi', \phi(\mathbf{p}) | \mathbf{t}_1 | \phi(\mathbf{p}), \psi \right\rangle = \frac{8\pi N^2 t_0}{\mathbf{b} - \mathbf{e}_2} \left( \mathbf{b} - \mathbf{e}_2 \right) \]

 Substitute Eq. (3.39) and Eq. (3.40) in Eq. (3.37), we get
\[ M_{12} = \frac{\left( 8\pi N^2 \right)^2 t_0 t_2}{(2\pi)^3} \int \frac{d\mathbf{b}}{(2\pi)^3} \left( \frac{T(\mathbf{b} - \mathbf{e}_2)}{\mathbf{b} - \mathbf{e}_2} \right) \left( \frac{1}{(\mathbf{b} - \mathbf{e}_2 \mathbf{e}_2)^2} \right) \left( \frac{T(\mathbf{b} - \mathbf{e}_2)}{\mathbf{b} - \mathbf{e}_2} \right) \]

\[ = \frac{\left( 8\pi N^2 \right)^2 t_0 t_2}{(2\pi)^3} \left( \frac{T(\mathbf{b} - \mathbf{e}_2)}{\mathbf{b} - \mathbf{e}_2} \right) \left( \frac{1}{(\mathbf{b} - \mathbf{e}_2 \mathbf{e}_2)^2} \right) \left( \frac{T(\mathbf{b} - \mathbf{e}_2)}{\mathbf{b} - \mathbf{e}_2} \right) \]

(3.41)
For any arbitrary angle between $\theta$ and $\theta'$, the above integral reduces to a triple integral which is not very convenient for numerical evaluation. Therefore, we shall evaluate it for the forward scattering only i.e. we shall take $\theta = 0$. Substituting $\theta = \theta'$ in Eq. (3.41), we get

\[
M_{12}^{(c)} = \frac{(8\pi N t)^2 t^0 t^0 \mu_{12}}{(2\pi)^3} \int_0^{\infty} \int_0^{2\pi} \frac{2\pi L^2 dL d\phi}{(9^2 - L^2 + \xi^2)} \left( \frac{T(L^2 - \xi)}{|L - \xi|} \right)^2
\]

\[
= \left( 64 N^4 t^0 t^0 \mu \right) \int_0^{\infty} L^2 dL \int_0^{2\pi} d\phi \frac{1}{(9^2 - L^2 + \xi^2)} \left( \frac{T(L^2 - \xi)}{|L - \xi|} \right)^2
\]

Let

\[
|L - \xi| = \xi
\]

or \(L^2 = \xi^2 - 2L\xi \phi = \xi^2 - 2L\xi \theta \)

or

\[
\frac{d\xi}{\xi} = -\frac{1}{L\xi} \frac{d\xi}{\xi}
\]

Therefore

\[
M_{12}^{(c)} = \left( 64 N^4 t^0 t^0 \mu \right) \int_0^{\infty} \left( \frac{L^2 - \xi}{\xi^2} \right) \frac{1}{(9^2 - L^2 + \xi^2)} \left( \frac{T(9^2)}{\xi} \right)^2
\]

\[
= \left( 64 N^4 t^0 t^0 \mu \right) \int_0^{\infty} \frac{\xi^2 d\xi}{\xi} \int_0^{2\pi} \frac{d\phi}{9^2 - L^2 + \xi^2} \frac{T(9^2)}{\xi} \left( \frac{1}{9^2 - L^2 + \xi^2} \right)
\]
Let 
\[
\frac{\partial f}{\partial y} = \ell
\]

\[
\therefore M_{12}^c = (6\mu N^t\xi_0 t^0 \mu_0) \int_0^\infty \frac{y^2}{2} \frac{d\xi}{\xi} \int_0^{q \ln + \gamma} \frac{(T(\xi))^2}{\xi} \frac{1}{(2(1-y^2 + i\ell))}
\]

\[
= (6\mu N^t\xi_0 t^0 \mu_0) \int_0^\infty \frac{y^2}{2} \frac{d\xi}{\xi} \int_0^{q \ln + \gamma} \frac{(T(\xi))^2}{\xi} \frac{1}{(1-y^2 + i\ell)}
\]

Now
\[
\frac{1}{1-y^2 + i\ell} = \frac{1}{1-y^2} - i\pi \delta(1-y^2)
\]

\[
= \frac{1}{1-y^2} - i\pi \frac{\delta(1-y) + \delta(1+y)}{2|y|}
\]

We get
\[
M_{12}^c = \left[ (6\mu N^t\xi_0 t^0 \mu_0) \int_0^\infty \frac{y^2}{2} \frac{d\xi}{\xi} \int_0^{q \ln + \gamma} \frac{(T(\xi))^2}{\xi} \frac{1}{(1-y^2)} - i\pi \int_0^\infty \frac{y^2}{2} \frac{d\xi}{\xi} \int_0^{q \ln + \gamma} \frac{(T(\xi))^2}{\xi} \frac{\delta(1-y) + \delta(1+y)}{2|y|} \right]
\]

\[
= \left[ \frac{1}{2} \int_0^\infty \frac{y^2}{2} \frac{d\xi}{\xi} \int_0^{q \ln + \gamma} \frac{(T(\xi))^2}{\xi} \frac{1}{(1-y^2)} - \frac{\ell}{2} \int_0^{2q} \frac{d\xi}{\xi} \frac{(T(\xi))^2}{\xi} \right]
\]
Therefore,

\[ M_{12} = \left( \frac{\delta_{4Nt^0t^0}}{q^2} \right) \left[ \frac{1}{q^2} \left( \frac{1}{q^2} \right)^2 + \frac{1}{q^2} \right] \]  

For \( q = 0 = q' \), from Eq. (3.41), we get

\[ M_{12} = -\left( \frac{2\pi^2}{(2\pi)^3} \right) \int d\ell \cdot \frac{(T(\ell))^2}{\ell^2} \frac{1}{(\ell - \omega)} \]

\[ = -\left( \frac{2\pi^2}{(2\pi)^3} \right) \int d\ell \cdot \frac{(T(\ell))^2}{\ell^2} \quad \ell \rightarrow 0 \]

where

\[ T(\ell) = \tan^{-1} \left( \frac{\ell}{q_0} \right) + i \omega^{-1} \left( \frac{\ell}{q_0} \right) - 2 \tan^{-1} \left( \frac{\ell}{2(\pi + \beta)} \right) \]
F. Double Scattering: Continuum State Scattering

**Contribution.** The double scattering in case of the continuum state is given by

\[
M_{12}^T = \langle \frac{q'}{2} , \varphi | t_1 | \frac{q}{2} , \varphi \rangle \left( \frac{q'^2}{4} + \varepsilon_b - \varepsilon_d - \varepsilon_a + U + i \varepsilon \right) \times
\]

\[
\langle t_2 | \varphi , \frac{q}{2} \rangle + (1 \leftrightarrow 2). \tag{3.45}
\]

We shall use Eq. (3.29) for the continuum state of the deuteron in the intermediate state. Since the Eq. (3.29) includes the effects of the potential U of the nucleon in the deuteron, we have

\[
\sum_{l \neq l'} | \psi_{l'} \rangle \langle \psi_{l'} | + \sum_{l} | \phi \rangle \langle \phi | = 1 \tag{3.46}
\]

and

\[
| \psi_{l'} \rangle (1 - P) = | \psi_{l} \rangle; \quad | \phi \rangle (1 - P) = 0 \tag{3.47}
\]

We get, on introducing

\[
\sum_{l} | \psi_{l'} \rangle \langle \psi_{l'} | = \int \frac{d|l\rangle}{(2\pi)^3} \langle l | \psi_{l'} \rangle \langle \psi_{l'} | \langle l \rangle
\]

and

\[
\sum_{l} | \psi_{l} \rangle \langle \psi_{l} | = \int \frac{d|l\rangle}{(2\pi)^3} \langle l | \psi_{l} \rangle \langle \psi_{l} | \langle l \rangle
\]

in Eq. (3.45),

\[
M_{12}^T = \int \int \frac{d|l\rangle}{(2\pi)^3} \langle l, \psi_{l'} | t_1 | l, \psi_{l} \rangle \left( \frac{q'^2}{4} + \varepsilon_b - \varepsilon_d - \varepsilon_a + U + i \varepsilon \right) \times
\]

\[
\langle l, \psi_{l} | t_2 | \phi, \frac{q}{2} \rangle + (1 \leftrightarrow 2) \tag{3.48}
\]

where

\[
E_l = \frac{q'^2}{2} \mu_{xd}
\]

(3.49)

and

\[
E_{l'} = \frac{q^2}{2} \mu_{ad}
\]

(3.50)
we can write Eq. (3.29) as

$$\Psi_e(t) = \frac{1}{(2\pi)^3} \left( \frac{1}{2} \right) + \chi_e(t)$$

(3.51)

where

$$\chi_e(t) = f(t) \left( e^{i\beta t} - e^{-i\beta t} \right)$$

(3.52)

and

$$f(t) = \frac{1}{\sqrt{\beta(\omega^2 + \beta^2)^2}} \cdot e^{ict} + \frac{e^{-i\beta t}}{2\beta}$$

(3.53)

Substituting Eq. (3.52) in Eq. (3.48), we get

$$M_{12} = M_{12}^{(1)} + M_{12}^{(2)} + M_{12}^{(3)} + M_{12}^{(4)} + \text{ (equiv 2)}$$

where

$$M_{12}^{(1)} = \int \frac{d\xi d\eta}{(2\pi)^6} \langle q', \phi | t_1, l_1, \xi \rangle \langle q, \phi | t_2, l_2 \rangle (q_1^2 + \beta^2) - E_1 - q_1 - q_2 \rangle$$

(3.54)

$$M_{12}^{(2)} = \int \frac{d\xi d\eta}{(2\pi)^6} \langle q', \phi | t_1, l_1, \xi \rangle \langle q, \phi | t_2, l_2 \rangle (q_1^2 + \beta^2) - E_1 - q_1 + i\epsilon \rangle$$

(3.55)

$$M_{12}^{(3)} = \int \frac{d\xi d\eta}{(2\pi)^6} \langle q', \phi | t_1, l_1, \xi \rangle \langle q, \phi | t_2, l_2 \rangle (q_1^2 + \beta^2) - E_1 - q_1 + i\epsilon \rangle$$

(3.56)

$$M_{12}^{(4)} = \int \frac{d\xi d\eta}{(2\pi)^6} \langle q', \phi | t_1, l_1, \xi \rangle \langle q, \phi | t_2, l_2 \rangle (q_1^2 + \beta^2) - E_1 - q_1 + i\epsilon \rangle$$

(3.57)
1. Computation of $\langle q_i | 1 \rangle$. We shall compute the matrix element $\langle q_i | 1 \rangle$ first.

\[
\langle q_i | 1 \rangle = \int \frac{d\ell}{(2\pi)^{12}} \langle \ell | 1 \rangle \langle q_i | \ell \rangle \langle \ell | \Phi \rangle 
\]

The Fourier transform of Eq. (3.4) is

\[
\Phi(\ell) = \pi \int e^{i \ell \cdot \Phi} \left( \frac{1}{\ell^2 + d^2} - \frac{1}{\ell^2 + \beta^2} \right)
\]

where

\[
\pi = \sqrt{2\pi}
\]

Now

\[
\langle q', \Phi | t_1 | q, \ell \rangle = \int \frac{d\ell}{(2\pi)^{12}} e^{i \ell \cdot q'} \Phi(\ell) t_1 e^{i \ell \cdot q} e^{i \ell \cdot \Phi}
\]

\[
= t_1 \int \frac{d\ell}{(2\pi)^{12}} e^{i (\ell - \ell') \cdot q} \Phi'(\ell) \delta(\ell - \xi) e^{i \ell \cdot \Phi}
\]

\[
= t_1 \int \frac{d\ell}{(2\pi)^{12}} e^{i (\xi + \ell') \cdot \ell} \Phi'(\ell)
\]

Using Eq. (3.58), we get

\[
\langle q', \Phi | t_1 | q, \ell \rangle = \pi t_1 \int \frac{d\ell}{(2\pi)^{12}} e^{i (\xi' + \ell + \frac{t}{2} \cdot \xi) \cdot \ell} \left( \frac{1}{\ell^2 + d^2} - \frac{1}{\ell^2 + \beta^2} \right)
\]

\[
= \pi t_1 \int \frac{d\ell}{(2\pi)^{12}} e^{i (\xi + \ell + \frac{t}{2} \cdot \xi) \cdot \ell} \left( \frac{1}{\ell^2 + d^2} - \frac{1}{\ell^2 + \beta^2} \right)
\]

\[
= \pi t_1 \int \frac{d\ell}{(2\pi)^{12}} e^{i (\xi' + \ell + \frac{t}{2} \cdot \xi) \cdot \ell} \left( \frac{1}{\ell^2 + d^2} - \frac{1}{\ell^2 + \beta^2} \right)
\]

\[
= \pi t_1 \int \frac{d\ell}{(2\pi)^{12}} e^{i (\xi' + \ell + \frac{t}{2} \cdot \xi) \cdot \ell} \left( \frac{1}{\ell^2 + d^2} - \frac{1}{\ell^2 + \beta^2} \right)
\]

where

\[
\xi' = \xi + \frac{t}{2} (q - q')
\]
Similarly

\[
\langle \xi, \xi | t_2 | \Phi, \gamma \rangle = (2\pi)^3 \bar{N} t_2^0 \left( \frac{1}{p^+ d^2} - \frac{1}{p^+ \beta^2} \right) \quad (3.60)
\]

where

\[
p = t - \gamma_0 (\epsilon - q) \quad (3.66)
\]

Using Eqs. (3.59) and (3.60) in Eq. (3.54), we get

\[
\begin{align*}
\langle l \rangle_{M_{12}} &= (\bar{N})^2 \bar{t} \bar{t}^0 \int d^4 \xi d^4 \xi' \left[ \left( \frac{1}{p^+ d^2} - \frac{1}{p^+ \beta^2} \right) \left( \frac{1}{p'^+ d^2} - \frac{1}{p'^+ \beta^2} \right) \right] \\
&\quad \times \left( q^+ + E - E' - E_{\xi} + \epsilon \right)^{-1} \\
&= (\bar{N})^2 \bar{t} \bar{t}^0 \int d^4 \xi d^4 \xi' \left[ \left( \frac{1}{p^+ d^2} - \frac{1}{p^+ \beta^2} \right) \left( \frac{1}{p'^+ d^2} - \frac{1}{p'^+ \beta^2} \right) \right] \\
&\quad \times \left( -2 \mu \right) \left( q - \frac{\mu \alpha \xi}{\mu \alpha} + \frac{\mu \beta \xi}{\mu \beta} \right)^{-1} \\
&\quad \times \left( q + \frac{\mu \alpha \xi}{\mu \alpha} + \frac{\mu \beta \xi}{\mu \beta} \right)^{-1} \quad (3.61)
\end{align*}
\]

since

\[
\xi = - \frac{\omega^2}{2 \mu d} \quad (18)
\]

We shall put

\[
\langle l \rangle_{M_{12}} = M_{12}(\alpha d) + M_{12}(\beta \beta) - M_{12}(\alpha \beta) - M_{12}(\beta \alpha) \quad (3.62)
\]

where

\[
M_{12}(\alpha \beta) = (\bar{N})^2 \bar{t} \bar{t}^0 \int d^4 \xi d^4 \xi' \left[ \left( p^+ + \alpha^2 \right) \left( p'^+ + \beta^2 \right) \right] \\
\quad \times \left( q^2 - \mu d (q - \xi) \right) \left( q^2 - \mu d (q - \xi) \right)^{-1} \quad (3.63)
\]

This separation is only for computational convenience and introduces spurious divergences which are cancelled at the end.
We combine denominators in order to make the denominator a function of $t^2$ and $k^2$ only, by using the Feynmann formula

$$\frac{1}{8\pi^2} = \int \int_{0}^{1} \frac{2udu dv}{[\bar{a}(u) + luv + \bar{c}u(1-u)]^3}$$  \hspace{1cm} (3.64)$$

Let

$$\bar{a} = t^2 - \frac{1}{\mu^2}(q^2 - l^2) + k^2 - i\varepsilon$$

$$\bar{b} = t^2 + t\cdot(k - q') + \left(\frac{t^2}{\mu^2}\right)^2 + A^2$$

$$\bar{c} = t^2 - t\cdot(k - q') + \left(\frac{t^2}{\mu^2}\right)^2 + B^2$$

we can write

$$\bar{a}(1-u) + \bar{b}uv + \bar{c}u(1-u) = (\varepsilon + E)^2 + D^2$$

where

$$E = k_2 \left[ (k-q')uv - (k-q)u(1-u) \right]$$

$$D = -k_2 \left[ (k-q')uv - (k-q)u(1-u) \right]^2 + \left( k_2 \cdot (k-q')^2 + A^2 \right)uv$$

$$+ \left( \frac{t^2}{\mu^2} \right)^2 + B^2 \right)u(1-u) - \left( \frac{1}{\mu^2} (q^2 - l^2) + i\varepsilon \right)(1-u) + \alpha^2 (1-u)$$

or

$$D = c\bar{a}^2 + \bar{b} \cdot \bar{c} + \alpha$$

$$= c \left( \bar{a} - \frac{\bar{a}^2}{4c} \right) + 4ac - \frac{\bar{a}^2}{4c}$$  \hspace{1cm} (3.65)$$
where

\[
\begin{align*}
\alpha &= -\gamma^3 \frac{w^2}{4} - 9 \frac{q^2}{4} \frac{w(a-w)}{4} + 2 \frac{q^2}{4} \frac{w(a-w)}{4} \\
&+ \left( \frac{q^2}{4} + A^2 \right) w^2 + \left( \frac{q^2}{4} + B^2 \right) w(a-w) - \frac{1}{4} \frac{w^2}{4} \\
&+ a^2 (1-u) - i \epsilon (1-u). \\
(3.66)
\end{align*}
\]

\[
\beta = \left( \frac{q^2}{4} (v^2 - \frac{v}{2}) + \frac{q}{4} (\gamma - 1) \gamma + \frac{q}{4} (1-4) \right) w + \left( \frac{q}{4} - \frac{q}{2} \right) \gamma^2 \frac{w}{4} \\
(3.67)
\]

\[
\begin{align*}
\gamma &= w^2 (v^2 - \frac{v}{2}) + \frac{q}{4} (1-4) \gamma + \frac{q}{4} \gamma \\
&= (3.68)
\end{align*}
\]

\[
M_{12}^{\alpha}(AB) = \nabla t \nabla^2 l_{i} (-2 \mu d) \delta t \delta l \int_{0}^{1} \int_{0}^{1} \frac{2u du dv}{(t + E)^2 + D^2} \\
\]

Making the transformation \( t \to t - F \), we get

\[
M_{12}^{\alpha}(AB) = \nabla t \nabla^2 l_{i} (-2 \mu d) \delta t \delta l \int_{0}^{1} \int_{0}^{1} \frac{2u du dv}{(t + E)^2 + D^2} \\
\]

Substituting for \( D \) from Eq. (3.65), we get

\[
M_{12}^{\alpha}(AB) = \nabla t \nabla^2 l_{i} (-2 \mu d) \delta t \delta l \int_{0}^{1} \int_{0}^{1} \frac{2u du dv}{(t + E)^2 + D^2} \\
\]

Next, make the transformation

\[
\sqrt{c} \left( \frac{t^2}{2} - \frac{t/2c}{2c} \right) \to \beta \\

We get

\[
M_{12}^{\alpha}(AB) = \nabla t \nabla^2 l_{i} (-2 \mu d) \delta t \delta l \int_{0}^{1} \int_{0}^{1} \frac{2u du dv}{(t + E)^2 + D^2} \\
(3.69)
\]
We introduce the six dimensional space given by

$$ R^2 = t^2 + h^2 $$

so that

$$ d\xi d\eta = d^6 R = R^5 dR d\Omega. $$

We can show that in 6-dimensions

$$ \int d\Omega = \pi^3 $$

Let

$$ \lambda^2 = \frac{4a c - \beta^2}{4c} $$

Using Eqs. (3.70), (3.71), (3.72) and (3.73) in Eq. (3.69), we get

$$ M_1^I(AB) = N^2 t_1 t_0 (-4\mu) \int_{0}^{1} \psi d\psi \int_{0}^{\infty} \frac{R^5 dR d\Omega}{(R^2 + \lambda^2)^3} $$

or

$$ M_1^I(AB) = N^2 t_1 t_0 (-4\mu) \int_{0}^{1} \psi d\psi \int_{0}^{\infty} \frac{R^5 dR}{(R^2 + \lambda^2)^3}. $$

Now

$$ \frac{R^5}{(R^2 + \lambda^2)^3} = \frac{R}{R^2 + \lambda^2} - \frac{2R A^L}{(R^2 + \lambda^2)^2} + \frac{R A^4}{(R^2 + \lambda^2)^3} $$

$$ = \frac{R}{R^2 + \lambda^2} + \frac{\alpha_1}{dR} \left[ \frac{A^2}{R^2 + \lambda^2} - \frac{1}{4} \frac{A^4}{(R^2 + \lambda^2)^2} \right] $$

$$ = \frac{1}{2} \log |R^2 + \lambda^2|^{\infty} - \frac{3}{4} $$

$$ = \left( \log \left( \frac{h_0}{h_0 - t_0} R^2 \right) - \frac{3}{4} \right) - \frac{1}{2} \log |A^2| $$

$$ = \text{Div} R - \frac{3}{4} - \frac{1}{2} \log |A^2| $$

where

$$ \text{Div} R = \lim_{R \to \infty} \frac{\partial}{\partial X}. $$
Since \( \mathcal{M}_{12} \) is given by Eq. (3.62), the factor \( \text{Div} R - \frac{1}{4} \) occurs twice with positive and negative signs. Therefore it does not contribute.

We shall prove for \( Re \, \lambda^2 < 0 \) and \( Im \, \lambda^2 < 0 \):

\[
\int_0^\infty \frac{R^2 dR}{\sqrt{\lambda^2 + \lambda^2}} = -\frac{1}{2} \log |\lambda^2| + i \frac{\lambda^2}{2} + \text{Div} R
\]

Let

\[
\lambda^2 = - (\lambda^2 + i \varepsilon(1 - \omega))
\]

\[
\therefore \int_0^\infty R^2 dR = \int_0^\infty \frac{R^2 dR}{(R^2 + \lambda^2)(R + A + \lambda^2)}
\]

\[
= \left[ \int_0^\infty \frac{dR}{R + A + \lambda^2} \right]^{\lambda^2} \left[ \int_0^\infty \frac{dR}{R - A - \lambda^2} \right]^{\lambda^2}
\]

\[
= \left[ \frac{1}{2} \log \frac{(R - A)^2 + \varepsilon^2(1 - \omega)}{(R + A)^2 + \varepsilon^2(1 - \omega)} \right]^{\lambda^2} - \frac{R - A}{\varepsilon^2(1 - \omega)}
\]

\[+ \frac{1}{2} \log \frac{(R + A)^2 + \varepsilon^2(1 - \omega)}{(R - A)^2 + \varepsilon^2(1 - \omega)} - \frac{R + A}{\varepsilon^2(1 - \omega)} \]

\[
= \left[ \frac{1}{2} \log \frac{(R - A)^2}{(R + A)^2} - \log \frac{|\lambda^2 + \varepsilon^2(1 - \omega)|}{|\lambda^2|} \right]^{\lambda^2}
\]
\[ + i \left( \tau_2 - \tan\frac{A}{\xi_2(-u)} \right) - i \left( \frac{\tau}{2} - \tan\frac{A}{\xi_2(-u)} \right) \]

\[ = \frac{i}{2} \left[ \text{Lt}_{R \to \infty} \left( \log R^2 \right) - \log \left| A^2 + \xi'_2(-u) \right| + 2 i \tan\frac{A}{\xi_2(-u)} \right] \]

\[ = \frac{i}{2} \left[ \text{Lt}_{R \to \infty} \left( \log R^2 \right) - \log |A|^2 + 2 i \frac{\tau}{2} \right], \quad \varepsilon \to 0 \]

\[ = - \frac{i}{2} \log |A|^2 + i \frac{\tau}{2} + \text{Div} R \]

\[ = - \frac{i}{2} \log |A|^2 + i \frac{\tau}{2}. \]

Omitting \( \text{Lt}_{R \to \infty} \left( \frac{1}{2} \log R^2 \right) \) as it will cancel with other terms as discussed above.

We can write

\[ \int_{\infty}^{0} \frac{R^2 dR}{(R^2 + A^2)^3} = - \frac{i}{2} \left[ \log |A|^2 - i \pi \Theta(-A^2) - \frac{\tau}{2} \right] \quad (3.75) \]

where

\[ \Theta(-x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases} \quad (3.76) \]

From Eqs. (3.74) and (3.75), we get

\[ M_{12}^{(A)} = \left( R_t L_3 \cdot 2 \mu d \pi^3 \right) \int_{0}^{\infty} \int_{0}^{\infty} w dudv \left[ \log |A|^2 - i \pi \Theta(-A^2) + \frac{\tau}{2} \right] \]
Using Eq. (3.62), we get

\[
\begin{align*}
(9) \quad M_{12} & = M_{12}^{\text{ind}} + M_{12}^{\text{d}}(\beta \beta) - M_{12}^{\text{d}}(\sigma \beta) - M_{12}^{\text{d}}(\beta \sigma) \\
& = (N^2 t_i t_j \cdot 2 \mu d \pi^3) \int \int_0^1 u du d\nu \left[ \frac{\log c_z}{c_{3/2}} \left| \frac{A'(x \eta) A'(\beta \eta)}{A'(x \eta) A'(\beta \eta)} \right| \right. \\
& \quad \left. - i \pi \Theta(-A'(x \eta)) - i \pi \Theta(-A'(\beta \eta)) \right] \\
& \quad + i \pi \Theta(-A'(x \eta)) + i \pi \Theta(-A'(\beta \eta))
\end{align*}
\]

where

\[
A'^2 = \frac{4 \alpha c - \beta^2}{4c}
\]

\[
= X + Y \cos (\theta, \theta')
\]

where

\[
X = \left[ \frac{u^2}{4} (u^2 - \nu^2 + 1) + \frac{u^2}{4} - \mu d (1 - \omega) \right] \frac{q^2}{2} + \left[ \frac{u^2}{4} (u^2 - \nu^2) \right] - \left( u^2 - \nu^2 \right) \left( \frac{\nu^2}{2} - \nu + \lambda_4 \right) + 2 u^3 x \\
\left\{ - \frac{u^2}{2} (u^2 - \nu^2) - (1 - \nu)(\nu - \xi) (\nu - 1) \right\} \frac{q^v}{4c} + \\
(\lambda^2 \nu + B^2 (1 - \nu)) u + d^2 (1 - \nu),
\]

(3.79)
\[ Y = \frac{\nu}{\sqrt{2}} \nu \left(1 - \nu\right) \hat{q}^2 - \left[ 2 e^\nu \left( \nu^2 - \nu \nu \right) \left( \frac{\nu}{2} - \frac{3}{2} \nu + \nu^2 \right) + \right. \\
\left. \frac{\nu}{\sqrt{2}} \cdot \nu \left(1 - \nu\right) \right] \times \frac{\hat{q}^2}{\sqrt{4c}} \tag{3.80} \]

\[ C = -\nu^2 \left( \nu - \frac{1}{2} \right)^2 + \nu \nu \left(1 - 4 \frac{\mu_4}{\mu} \right) + \frac{\mu^2}{\mu_4}, \tag{3.81} \]

and

\[ I = \int_0^1 \int_0^1 \frac{u \, dU \, dV}{c^2} \left[ \log \left| \begin{array}{c} A^2_{\nu \alpha} \cdot A^2_{\nu \beta} \\
A^2_{\nu \alpha} \cdot A^2_{\nu \beta} 
\end{array} \right| - i \pi \Theta(-A^2_{\nu \alpha}) \\
i \pi \Theta(-A^2_{\nu \beta}) + i \pi \Theta(-A^2_{\nu \beta}) + i \pi \Theta(-A^2_{\nu \beta}) \right] \tag{3.82} \]
2. Computation of $\langle M_1, \mathcal{I} \rangle$ and $\langle M_2, \mathcal{I} \rangle$. Having calculated $\langle M_1, \mathcal{I} \rangle$, we shall calculate $\langle M_2, \mathcal{I} \rangle$ from Eq. (3.55).

Using Eq. (3.52), we get

$$
\langle \xi', \varphi(t_1, t, x_1) \rangle = \int d\xi \int d\rho \ e^{-i \varphi(t_1, t, x_1)} \varphi(t, x_1) \frac{e^{-i \varphi(t, x_1)} - e^{-i \varphi(t, x_1)}}{\rho} 
$$

Using Eq. (3.1), we get

$$
\langle \xi', \varphi(t_1, t, x_1) \rangle = \int d\xi \int d\rho \ e^{-i \varphi(t_1, t, x_1)} \varphi'(\rho) \xi(t_1, x_1 - \xi) \xi(t, x_1) \frac{e^{-i \varphi(t, x_1)} - e^{-i \varphi(t, x_1)}}{\rho} 
$$

$$
= t_1 \int d\rho \ e^{i (\xi - \xi')} \varphi'(\rho) \xi(t) \frac{e^{-i \varphi(t)} - e^{-i \varphi(t)}}{\rho} 
$$

Substituting for $\varphi'(\rho)$, we have

$$
\langle \xi', \varphi(t_1, t, x_1) \rangle = (Nt^0 2\pi) \int d\rho \int dx \ e^{i (\xi - \xi')} \int_{-\infty}^{\infty} \xi(t) \frac{e^{-i \varphi(t)} - e^{-i \varphi(t)}}{\rho} \varphi'(\rho) 
$$

$$
\times e^{i (\xi - \xi')} \int_{-\infty}^{\infty} \xi(t) \frac{e^{-i \varphi(t)} - e^{-i \varphi(t)}}{\rho} \varphi'(\rho) 
$$

$$
\times e^{i (\xi - \xi')} \int_{-\infty}^{\infty} \xi(t) \frac{e^{-i \varphi(t)} - e^{-i \varphi(t)}}{\rho} \varphi'(\rho) 
$$

$$
= (Nt^0 2\pi) \int d\rho \left( \frac{e^{i (\xi - \xi') \rho_2} - e^{i (\xi - \xi') \rho_2}}{1 + (\xi - \xi') \rho_2} \right) \left( e^{-i \varphi(t)} - e^{-i \varphi(t)} \right) \varphi'(\rho) 
$$

$$
\times f(t) \left( e^{i \rho_2} - e^{-i \rho_2} \right) 
$$

$$
= (Nt^0 2\pi) \int d\rho \left( \frac{e^{i (\xi - \xi') \rho_2} - e^{i (\xi - \xi') \rho_2}}{1 + (\xi - \xi') \rho_2} \right) \left( e^{-i \varphi(t)} - e^{-i \varphi(t)} \right) \varphi'(\rho) 
$$

$$
\times f(t) \left( e^{i \rho_2} - e^{-i \rho_2} \right) 
$$
Introduce $\int_0^\infty \varphi_0 \, dy = \frac{1}{\rho}$.

We get

$$\langle q', \varphi | t, | 0, x \rangle = \left( \frac{\text{const}}{2\pi} \right)^{\frac{d}{2}} \int_0^\infty \int_0^\infty \left[ e^{-(\sigma + y)} \rho + i(t + i\Delta' - i\rho) \rho \right]$$

$$+ e^{-(\sigma + y)} \rho - \left( \text{terms with } \frac{k^2 - 1}{2} \rightarrow -\frac{k^2 - 1}{2} \right) \times \rho(t)$$

$$= \left( \frac{\text{const}}{2\pi} \right)^{\frac{d}{2}} \int_0^\infty \int_0^\infty \left[ (\sigma + y - i(t + i|k| - i\rho))^{-1} - (\sigma + y - i(t - |k| - i\rho))^{-1} \right]$$

$$- (\sigma + y - i(t + |k| - i\rho))^{-1} + (\sigma + y - i(t - |k| - i\rho))^{-1}$$

$$+ (\sigma + \beta + y + i(t - |k| - i\rho))^{-1} - (\sigma + \beta + y + i(t + |k| - i\rho))^{-1}$$

$$+ (\sigma + \beta + y + i(t - |k| - i\rho))^{-1} - (\sigma + \beta + y + i(t + |k| - i\rho))^{-1} \right] \rho(t)$$

$$= \left( \frac{\text{const}}{2\pi} \right)^{\frac{d}{2}} \int_0^\infty \left[ (\sigma + y - i\sigma)^2 + (k^2 - 1)^2 \right]^{-1}$$

$$- (\sigma + y - i\sigma + (k^2 - 1)^2)^{-1} - (\sigma + \beta + y)^2 + (k^2 - 1)^2$$

$$+ (\sigma + \beta + y)^2 + (k^2 - 1)^2 \right]$$

(3.83)
We have already proved that

\[ \langle \xi', t | t_2 | \phi, q \rangle = (2\pi)^3 N t_2^0 \left( \frac{1}{\beta' \rho_1} - \frac{1}{\beta^2 \rho_2} \right) \]  

(3.60)

\[ \beta' = t - \frac{1}{2} (\xi - q) \]  

(3.60')

**Forward Scattering, \( \beta' = \frac{1}{2} \)**

Using Eqs. (3.83, 3.60) in Eq. (3.55), we get

\[
(3.83) \quad M_{12}^{(3)} = \frac{4\pi N N t_2^0 e^{i(3.83)}}{(2\pi)^3} \int \frac{d\xi}{i2\pi} \int_0^\infty dy \left[ \left( \gamma + i\gamma_{-i\xi} \right)^{1/2} \left( \gamma_{-i\xi} - 1 \right)^{1/2} - \left( \gamma + i\gamma_{-i\xi} \right)^{1/2} \left( \gamma_{-i\xi} - 1 \right)^{-1/2} \right]
\]

\[
\times \left( \left( \gamma_{-i\xi} - 1 \right)^{1/2} \left( \gamma_{-i\xi} - 1 \right)^{-1/2} \right) \left( \gamma_{-i\xi} + i\gamma_{-i\xi} \right) x^i(t)
\]

Note: \( i\xi \) has \( - \) sign before it.

Using

\[
\frac{1}{\xi - \xi_0} = \rho(\xi_0) + \pi \delta(\xi)
\]

we can write

\[
M_{12}^{(3)} = R e M_{12}^{(3)} + i m M_{12}^{(3)}
\]
where

\[
\text{Re} \: \tilde{M}_2^{(6)} = \frac{4 \pi N N \tilde{E}_0 (2 \pi)^3}{(2 \pi)^3} \int \int \int \int \eta(t, \tilde{z}, \tilde{z}) \left\{ \left[ \left( \delta_{\tilde{z}} - c_\eta \right)^2 + \left( \frac{k - q}{2} \right)^2 \right]^{-1} - \\
\left( \left( b + \gamma - \xi \right)^2 + \left( \frac{k - q}{2} \right)^2 \right)^{-1} - \left( \left( d + \beta + \gamma \right)^2 + \left( \frac{k - q}{2} \right)^2 \right)^{-1} + \\
\left( \left( 2 \beta + \gamma \right)^2 + \left( \frac{k - q}{2} \right)^2 \right)^{-1} \right\} \left[ \left( \tilde{t} - \frac{k - q}{2} \right)^2 + \alpha^2 \right]^{-1} - \left( \tilde{t} - \frac{k - q}{2} \right)^2 + \beta^2 \right\} \\
x \left( \tilde{t}^2 + \frac{1}{\mu_{k\gamma}} \left( \tilde{k} - \gamma \gamma \right)^2 + \alpha^2 \right)^{-1} \times f(t) \right\}
\]

(3.84)

\[
\text{Im} \: \tilde{M}_2^{(6)} = -\frac{4 \pi N N \tilde{E}_0 (2 \pi)^3}{(2 \pi)^3} \int \int \int \int \eta(t, \tilde{z}, \tilde{z}) \left\{ \left[ \left( \delta_{\tilde{z}} - c_\eta \right)^2 + \left( \frac{k - q}{2} \right)^2 \right]^{-1} - \\
\left( \left( b + \gamma - \xi \right)^2 + \left( \frac{k - q}{2} \right)^2 \right)^{-1} - \left( \left( d + \beta + \gamma \right)^2 + \left( \frac{k - q}{2} \right)^2 \right)^{-1} + \\
\left( \left( 2 \beta + \gamma \right)^2 + \left( \frac{k - q}{2} \right)^2 \right)^{-1} \right\} \left[ \left( \tilde{t} - \frac{k - q}{2} \right)^2 + \alpha^2 \right]^{-1} - \\
\left( \tilde{t} - \frac{k - q}{2} \right)^2 + \beta^2 \right\} \\
x \left( \tilde{t}^2 + \frac{1}{\mu_{k\gamma}} \left( \tilde{k} - \gamma \gamma \right)^2 + \alpha^2 \right)^{-1} \times f(t) \times \delta \left( \tilde{t}^2 + \frac{1}{\mu_{k\gamma}} \left( \tilde{k} - \gamma \gamma \right)^2 + \alpha^2 \right) \right\}
\]

(3.85)
From Eq. (3.56), we have

\[
R_e^{(3)\Gamma} \mathcal{M}_{12} = R_e^{(3)\Gamma} \mathcal{M}_{12} + g_m^{(3)\Gamma} \mathcal{M}_{12}^T
\]

\[
R_e^{(3)\Gamma} \mathcal{M}_{12} = \frac{4\pi\left(t^2 - 2\mu \tau\right)^{\frac{1}{3}}}{(2\pi)^{\frac{3}{2}}} \int \frac{d\nu}{\nu} \int_0^\infty d\xi \left[ \left( (\xi + \gamma + \nu \tau)^2 + (\frac{\nu^2}{2})^2 \right)^{-1} - \left( (\xi + \gamma + \nu \tau)^2 + \left( \frac{\nu^2}{2} \right)^2 \right)^{-1} \right] \times \left[ \left( (\xi + \frac{\gamma}{2} + \frac{\nu^2}{2})^2 + \xi \right)^{-1} - \left( (\xi + \frac{\gamma}{2} + \frac{\nu^2}{2})^2 + \xi \right)^{-1} \right] \left( \xi^2 + \frac{\nu^2}{\xi^2} (\xi^2 - q^2) + \nu \right) \mathcal{f}^*(t) \right]
\]

Comparing it with Eq. (3.84), we see that

\[
\frac{R_e^{(3)\Gamma} \mathcal{M}_{12}}{t^0 t_1^0} = \frac{(R_e^{(3)\Gamma} \mathcal{M}_{12})^*}{t^0 t_1^0}
\]

Also

\[
g_m^{(3)\Gamma} \mathcal{M}_{12} = \frac{4\pi\left(t^2 - 2\mu \tau\right)^{\frac{1}{3}}}{(2\pi)^{\frac{3}{2}}} \int \frac{d\nu}{\nu} \int_0^\infty d\xi \left[ \left( (\xi + \gamma + \nu \tau)^2 + (\frac{\nu^2}{2})^2 \right)^{-1} - \left( (\xi + \gamma + \nu \tau)^2 + (\frac{\nu^2}{2})^2 \right)^{-1} \right] \times \left[ \left( (\xi + \frac{\gamma}{2} + \frac{\nu^2}{2})^2 + \xi \right)^{-1} - \left( (\xi + \frac{\gamma}{2} + \frac{\nu^2}{2})^2 + \xi \right)^{-1} \right] \left( \xi^2 + \frac{\nu^2}{\xi^2} (\xi^2 - q^2) + \nu \right) \mathcal{f}^*(t) \right]
\]

Comparing it with Eq. (3.85), we find that

\[
g_m^{(3)\Gamma} \mathcal{M}_{12} = \frac{(g_m^{(3)\Gamma} \mathcal{M}_{12})^*}{t^0 t_1^0}
\]
It is obvious from Eq. (3.84) that if we make the transformation
\[ l \rightarrow l + a \]  
(3.86)
we shall have the angular dependence in the integrand only in one of the denominators, and there is no angular integration with respect to \( t \).

Eq. (3.84) becomes
\[
\Re e M_n = \frac{4 \pi N \tilde{N} \tilde{N}^* (-2\mu a)}{(2\pi)^3} \int \frac{d\xi d\bar{\mu}}{\xi^2} \left\{ \frac{a}{(\xi + \bar{\mu})^2 + \lambda^2} \right\} -
\]
\[
(\lambda + \bar{\mu})^{-1} + \left( \frac{a}{(\xi + \bar{\mu})^2 + \lambda^2} \right)^{-1} + \left( \frac{a}{(\xi + \bar{\mu})^2 + \lambda^2} \right)^{-1}
\]
\[
\times \left[ \left( \frac{a}{(\xi + \bar{\mu})^2 + \lambda^2} \right)^{-1} - \left( \frac{a}{(\xi + \bar{\mu})^2 + \lambda^2} \right)^{-1} \right] \times
\]
\[
\left( \frac{a}{(\xi + \bar{\mu})^2 + \lambda^2} \right)^{-1} \times f(t)
\]  
(3.87)
Further, we can simplify the integration over \( t \) by the following transformation

\[
\begin{align*}
\begin{bmatrix}
\xi_1 \\
\xi_2 \\
\xi_3 \\
\xi_4
\end{bmatrix}
&= \begin{bmatrix}
l & l' \\
\gamma & \gamma' \end{bmatrix}
\end{align*}
\]
(3.88)

Eq. (3.87) becomes
\[
\Re e M_n = \frac{4 \pi N \tilde{N} \tilde{N}^* (-2\nu a)}{(2\pi)^3} \int \frac{d\xi d\bar{\mu}}{\xi^2} \left\{ \frac{a}{(\xi + \bar{\mu})^2 + \lambda^2} \right\} -
\]
\[
- \left( (x + y - i)^2 + \frac{1}{y_{+}} \right)^{-1} - \left( (x + y - i)^2 + \frac{1}{y_{-}} \right)^{-1} - \left( (x + y - i)^2 + (y_{+} + y_{-}) \right)^{-1}
\]
\[
\times \left[ \left( 1 + \frac{1}{y_{+}} - \frac{1}{\gamma} \gamma + \delta_{1}^2 \right)^{-1} - \left( 1 + \frac{1}{y_{-}} - \frac{1}{\gamma} \gamma + \delta_{2}^2 \right)^{-1} \right] \times 
\]
\[
\left( 1 + \delta_{1}^2 + \frac{1}{\mu_{d}} \left( \delta_{1}^2 + 2 \frac{q_{+} q_{-}}{\gamma} \gamma \right) \right)^{-1} \times f(s) \right) \right] 
\]

In the evaluation of \( R_{\gamma}^{(2)} \), we need the following results

\[
\int_{0}^{\infty} \frac{dy}{(x+y-i) + \frac{1}{y_{+}}} = \frac{1}{2} \ln \left| \frac{(1 + \delta_{1})^2 + \delta_{1}^2}{(1 - \delta_{1})^2 + \delta_{1}^2} \right| + \frac{T(s, \ell)}{\ell} \quad (3.89)
\]

where

\[
T(s, \ell) = \tan^{-1} \left| \frac{\delta_{1}'}{\delta_{1}} \right| - \tan^{-1} \left| \frac{\delta_{1}''}{\delta_{1}} \right|, \quad \ell \leq 2 \quad (3.90a)
\]

\[
= \pi - \tan^{-1} \left| \frac{\delta_{1}'}{\delta_{1}} \right| - \tan^{-1} \left| \frac{\delta_{1}''}{\delta_{1}} \right|, \quad \ell \geq 2 \quad (3.90b)
\]

\[
\int_{0}^{\infty} \frac{dy}{(x+y)^2 + \frac{1}{y_{+}}} = \frac{3\delta_{1}'}{2} \left[ \frac{\delta_{1}'}{\delta_{1}} - \tan^{-1} 2 \delta_{1} \right] \quad (3.91)
\]

\[
\int \frac{dn}{1 + \frac{y_{+}}{\gamma_{+}} + \gamma_{+} \delta_{2}^2} = \frac{2\delta_{2}}{\ell} \left| \frac{(1 + \delta_{1})^2 + \delta_{2}^2}{(1 - \delta_{1})^2 + \delta_{2}^2} \right| \quad (3.92)
\]
\[
\int \frac{dN}{1 + \eta^2 + \eta'N(e^2 + 2\eta \eta' \xi)} = \pi \frac{t_2}{g_l} \log \left( \frac{1 + \delta_2 + \mu_N e^2 + 2\eta \eta' \xi}{1 + \delta_1 + \mu_N e^2 - 2\eta \eta' \xi} \right) 
\]

(3.93)

See Appendix (1) for Eq. (3.89)

To evaluate \( R_e^{\tilde{M}_{12}} \), first perform the integration over \( y' \), using Eqs. (3.89) and (3.91), we write

\[
R_e^{\tilde{M}_{12}} = 4\pi NN \int \frac{d\xi'd\xi}{(\xi')^3} \left[ \frac{\log \left( \frac{1 + \delta_2 + \delta_1^2}{(1 - \delta_2)^2 + \delta_1^2} \right)}{\xi'} + \frac{\bar{T}(\xi)}{\xi'} \right] 
\]

\[
\quad - \frac{\delta_1}{(1 - \delta_2)^2 + \delta_1^2} \frac{T(\xi')}{\xi'} - \frac{2}{\xi'} \left( \frac{\xi}{2} - \frac{\delta_1}{\xi} - \frac{2\delta_1}{\xi} \right) 
\]

\[
\quad + \frac{2}{\xi'} \left( \frac{\xi}{2} - \frac{\delta_1}{\xi} + \frac{2\delta_1}{\xi} \right) \times \left( 1 + \frac{\delta_1}{\xi} - \xi' \xi' + \delta_1^2 \right) 
\]

\[
\quad - \left( 1 + \frac{\delta_1}{\xi} \right) \xi' \xi' \xi' \xi' + \delta_1^2 \right) \times T(\xi) \right] 
\]

Let

\[
T(\xi_1, \xi_2, \xi_3, \xi_4, \xi) = T(\xi_1, \xi) - T(\xi_2, \xi) + 2 \left( e^{\xi_1} \frac{i}{\xi} - e^{\xi_2} \frac{i}{\xi} \right) 
\]

(3.94)
We get

\[ \text{Re} \left( \frac{a}{i} M_{12} \right) = \frac{4 \pi N L_{12} L_{21} e^{2} (-2 \mu_{d})}{(2 \pi)^{3}} \int_{0}^{\infty} \int_{0}^{\infty} \left[ \frac{i}{2} \log \left| \frac{(l + \ell_{x}^{2} + \ell_{y}^{2} + \ell_{z}^{2}) (l - \ell_{x}^{2} + \ell_{y}^{2} + \ell_{z}^{2})}{(l - \ell_{x}^{2} + \ell_{y}^{2} + \ell_{z}^{2}) (l + \ell_{x}^{2} + \ell_{y}^{2} + \ell_{z}^{2})} \right| \right] \right. \\
+ \left. T(\delta_{1}, \delta_{2}, \delta_{3}, \delta_{4}, \ell) \right] \cdot \int d\epsilon \left[ (1 + \ell'_{x}^{2} - \ell'_{x}^{2} + \ell'_{z}^{2})^{-1} \right. \\
\left. - (1 + \ell'_{x}^{2} - \ell'_{x}^{2} + \ell'_{z}^{2})^{-1} \right] \cdot \int d\epsilon \frac{n_{\epsilon}}{1 + \delta_{1}^{2} + \delta_{2}^{2} \mu_{d} (\ell'_{x}^{2} + \ell'_{y}^{2} + \ell'_{z}^{2})} \cdot f(t) \]

Using Eqs. (3.92) and (3.93), we get

\[ \text{Re} \left( \frac{a}{i} M_{12} \right) = \frac{4 \pi N L_{12} L_{21} e^{2} (-2 \mu_{d})}{(2 \pi)^{3}} \int_{0}^{\infty} \int_{0}^{\infty} \left[ \frac{i}{2} \log \left| \frac{(l + \ell_{x}^{2} + \ell_{y}^{2} + \ell_{z}^{2}) (l - \ell_{x}^{2} + \ell_{y}^{2} + \ell_{z}^{2})}{(l - \ell_{x}^{2} + \ell_{y}^{2} + \ell_{z}^{2}) (l + \ell_{x}^{2} + \ell_{y}^{2} + \ell_{z}^{2})} \right| \right. \\
+ \left. T(\delta_{1}, \delta_{2}, \delta_{3}, \delta_{4}, \ell) \right] \cdot \frac{2 \pi L_{d}}{\ell} \left. \left| \frac{(l + \ell_{x}^{2} + \ell_{y}^{2} + \ell_{z}^{2}) (l - \ell_{x}^{2} + \ell_{y}^{2} + \ell_{z}^{2})}{(l - \ell_{x}^{2} + \ell_{y}^{2} + \ell_{z}^{2}) (l + \ell_{x}^{2} + \ell_{y}^{2} + \ell_{z}^{2})} \right| \right. \\
\left. \right. \times \left. \left. \frac{n_{\epsilon}}{1 + \delta_{1}^{2} + \delta_{2}^{2} \mu_{d} (\ell_{x}^{2} + \ell_{y}^{2} + \ell_{z}^{2})} \cdot f(t) \right. \right. \right. \\
\left. \right. \left. \right. \left. \right. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \left. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \r...
Define

\[ L(t) = \left| \frac{\left( + \varepsilon_3 \right)^2 + \varepsilon_2}{\left( - \varepsilon_3 \right)^2 + \varepsilon_2} \right| \frac{\left( + \varepsilon_3 \right)^2 + \varepsilon_2}{\left( - \varepsilon_3 \right)^2 + \varepsilon_2} \]  
(3.95)

\[ g(t) = \int_0^\infty \left( \frac{d \phi}{\phi} \right) \left( \frac{d \phi}{\phi} \right)^2 \frac{d \phi}{\phi} \frac{1}{\varepsilon_3^2 + \frac{\varepsilon_3}{\mu}} \left( \frac{1}{\varepsilon_3^2 + \frac{\varepsilon_3}{\mu}} \right) \]  
(3.96)

\[ \varphi(t) = \int_0^\infty \left( \frac{d \phi}{\phi} \right) \left( \frac{d \phi}{\phi} \right)^2 \frac{d \phi}{\phi} \frac{1}{\varepsilon_3^2 + \frac{\varepsilon_3}{\mu}} \left( \frac{1}{\varepsilon_3^2 + \frac{\varepsilon_3}{\mu}} \right) \]  
(3.97)

\[ G(t) = \frac{1}{2} g(t) + \varphi(t) \]  
(3.98)

Using Eqs. (3.95), (3.96), (3.97) and (3.98), we get

\[ \text{Re} \left( M_{12} \right) = - \frac{t_1}{t_2} N^2 \int_0^\infty dt \cdot G(t) \cdot \varphi(t) \]  
(3.99)

From Eq. (3.84), we have

\[ \text{Re} \left( M_{12} \right) = - \frac{t_1}{t_2} N^2 \int_0^\infty dt \cdot G(t) \cdot \varphi(t) \]  
(3.99)
From Eq. (3.53), we have

\[ f(t) = \frac{1}{A + a t^2 - i t - \lambda} \]

where

\[ A = \frac{1}{2 \beta (\alpha + \beta)^2} \]
\[ a = \frac{\beta}{(\alpha + \beta)^2} + \frac{1}{2 \beta} \]
\[ \lambda = \beta^2 - \frac{1}{2} \beta (\alpha + \beta)^2 \]

We have,

\[ f(t) - f^*(t) = \frac{2 i t}{(A t^4 + a t^2 - \lambda)^2 + t^2} \]
\[ f(t) + f^*(t) = \frac{2 (A t^4 + a t^2 - \lambda)}{(A t^4 + a t^2 - \lambda)^2 + t^2} \]  

(3.100)

Adding up Eqs. (3.99) and (3.99) and using Eq. (3.100), we get

\[ \Re \left[ (\theta \mathbf{M}_1 + \bar{\mathbf{M}}_1^T) \right] = -t^2 \int_0^\infty \left[ \frac{\beta(t)}{(A t^4 + a t^2 - \lambda)^2 + t^2} + \frac{2 t \left( A t^4 + a t^2 - \lambda \right) \beta(t)}{(A t^4 + a t^2 - \lambda)^2 + t^2} \right] \]  

(3.100)
We shall find now
\[ \lim_{\theta \to 0} \text{Re} \left( q_{H_{10}} + q_{M_{10}} \right) \]

Using \[ \lim_{\theta \to 0} \frac{\log(1+\theta)}{\theta} = \frac{1}{K} \]
for \( \theta < 1 \), we get

\[ \lim_{\theta \to 0} \frac{\log(1+\theta)}{\theta} \left( \frac{1}{1+e^{-A/\mu - 2\Delta f/c}} \right) = \lim_{\theta \to 0} \frac{\log(1+2\mu_d/\mu \cdot J/c \cdot \frac{1}{(1+\Delta f + \mu_d f/c^2)^2})}{\theta} \]

From Eqs. (3.96) and (3.101), we have

\[ \mathcal{G}(t) = \int_0^\infty \frac{4\mu_d \cdot \frac{1}{t} \cdot \frac{1}{(1+\Delta f + \mu_d f/c^2)^2}} \]

where

\[ \mathcal{G}_1(t) = \int_0^\infty \frac{1}{(1+\Delta f + \mu_d f/c^2)^2} \]

From Eqs. (3.97) and (3.101), we have

\[ \mathcal{F}(t) = \int_0^\infty \frac{4\mu_d \cdot \frac{1}{t} \cdot \frac{1}{(1+\Delta f + \mu_d f/c^2)^2}} \]

where

\[ \mathcal{F}_1(t) = \int_0^\infty \frac{1}{(1+\Delta f + \mu_d f/c^2)^2} \]
where

\[ h_i(t) = \int_0^\infty d\ell \int_0^\infty d\ell' T(\xi, \Omega, \omega, \Omega, \ell) \left( \frac{1}{(1 + \xi^2 + \mu d d')^2} \right) \]  

(3.103)

For \( \nu = 0 \), substituting Eqs. (3.102) and (3.103) for Eq. (3.100)

\[ R_{d}^{(\nu)}(M_{12}, M_{12}) = -4N_{\eta}^{2} \left( t_0 t_{2} \right) \int_0^\infty \left[ -t \frac{g_i(t) + 2(At^2 + \alpha t^2 - \lambda) \hat{h}_i(t)}{(At^2 + \alpha t^2 - \lambda)^2 + t^2} \right] dt \]

\[ = -4 \pi i \]  

(3.104)

where

\[ \pi = \mu d \left( t_0 t_{2} \right) \left( f_0 \right) \]  

(3.105)

and

\[ I = \int_0^\infty \left[ -t \frac{g_i(t) + 2(At^2 + \alpha t^2 - \lambda) \hat{h}_i(t)}{(At^2 + \alpha t^2 - \lambda)^2 + t^2} \right] dt \]

(3.106)

Imaginary part of \( \left( \frac{\partial}{\partial \nu} M_{12} + \frac{\partial}{\partial \nu} M_{12} \right) \)

Referring to Eq. (3.84)

\[ \delta(t^2 + \mu d (q^2 - q_i^2) + \lambda^2) = \frac{\delta \left( t^2 + \frac{\mu d (q^2 - q_i^2) - \lambda^2}{2} \right)}{2 \sqrt{\mu d (q^2 - q_i^2) - \lambda^2}} \]

only \( t = \sqrt{\mu d (q^2 - q_i^2) - \lambda^2} \) contributes

Since \( t \) cannot be negative, maximum \( \lambda \) is given by

\[ \frac{\mu d (q^2 - q_i^2) - \lambda^2}{2 \mu d} = 0 \]  

or

\[ \lambda = \sqrt{q^2 - \frac{\mu d q_i^2}{\mu d}} = q_i \]
Therefore, \( \ell \) varies from 0 to \( \ell_1 \).

Let
\[
\ell = \frac{\ell - \ell_0}{2}
\]

We get from Eq. (3.85)

\[
g_m \overline{\mathcal{M}_{12}} = (\mu \pi^2 \frac{\mathcal{N}_\pi}{(2\pi)^3}) \int_0^\infty d\psi \int_0^\infty \frac{d\mathcal{L}}{\mathcal{L}} \left[ \left( \left( \frac{\mathcal{L} + \mathcal{L} + \mathcal{L} + L^2}{} \right)^{-1} - \left( (\beta + \gamma + \lambda^2 + \lambda^2 + \lambda^2) \right)^{-1} + (\lambda^2 + \lambda^2 + \lambda^2) \right) \right]
\]

Using Eq. (3.58) and the limits of \( \ell \), we have

\[
g_m \overline{\mathcal{M}_{12}} = (\mu \pi^2 \frac{\mathcal{N}_\pi}{(2\pi)^3}) \int_0^\infty d\psi \int_0^\infty \int d\mathcal{L} \int d\mathcal{N} \int_0^\infty \mathcal{L} \left[ \left( \left( \frac{\mathcal{L} + \mathcal{L} + \mathcal{L} + \lambda^2}{} \right)^{-1} - \left( (\beta + \gamma + \lambda^2 + \lambda^2 + \lambda^2) \right)^{-1} + (\lambda^2 + \lambda^2 + \lambda^2) \right) \right]
\]

\[
\cdot \left[ \left( (\ell - \ell_0)^2 + \lambda^2 \right)^{-1} - \left( (\ell - \ell_0)^2 + \lambda^2 \right)^{-1} \right] \cdot f(t) \cdot t \left( \frac{\mu d (\mathcal{L}^2 - \mathcal{L}^2)}{\mu d (\mathcal{L}^2 - \mathcal{L}^2)} \right)
\]

In the above expression \( t = \sqrt{\frac{\mu d (\mathcal{L}^2 - \mathcal{L}^2)}{\mu d (\mathcal{L}^2 - \mathcal{L}^2)}} \).
Using Eq. (3.92), we have

\[ \int_{\mathcal{M}_0} \frac{d^3v}{(2\pi)^3} \rho_{\mathcal{M}_0} t_1 t_2 \int_0^{\infty} d\gamma \int_0^{2\pi} d\phi \left\{ \left[ (\beta + \gamma - \omega t)^2 + \lambda^2 \right]^{-1} - \left[ (\beta + \gamma - \omega t)^2 + \lambda^2 \right]^{-1} \right\} \]

\[ \times \frac{\pi}{t} \frac{1}{L} \log \left| \frac{(t-L)^2 + \lambda^2}{(t-L)^2 + \lambda^2} \frac{(t+L)^2 + \beta^2}{(t+L)^2 + \beta^2} \right| \]

\[ = - \frac{\pi}{(2\pi)^3} \frac{1}{L} \int_0^{2\pi} d\phi \int_0^{\infty} d\gamma \int_0^{\infty} d\phi \left\{ \left[ (\beta + \gamma - \omega t)^2 + \lambda^2 \right]^{-1} - \left[ (\beta + \gamma - \omega t)^2 + \lambda^2 \right]^{-1} \right\} \]

\[ \times \frac{1}{L} \log \left| \frac{(t-L)^2 + \lambda^2}{(t-L)^2 + \lambda^2} \frac{(t+L)^2 + \beta^2}{(t+L)^2 + \beta^2} \right| \]

Now

\[ \frac{1}{(t+\beta - \omega t)^2 + \lambda^2} = \frac{1}{2iL} \left[ \frac{1}{\gamma + \alpha - i(t+L)} - \frac{1}{\gamma + \alpha - i(t-L)} \right] \]

We get

\[ \int_0^{\infty} dy \left[ y + i(t+L) \right]^{\infty} + \gamma \left( \frac{1}{L} + \tan^{-1} \frac{d}{t+L} \right) \]

and
\[ \int_0^\infty \frac{dy}{y^2 + \frac{(t-L)^2}{(t-L)^2 + (t-L)^2}} = \frac{1}{2L} \log \left| \frac{(y+\omega)^2 + (t-L)^2}{(y+\omega)^2 + (t+L)^2} \right| + \frac{(\gamma_2 - \omega^2 |\gamma_2|)}{2L} \]

Therefore

\[ \int_0^\infty \frac{dy}{y^2 + \frac{(t-L)^2}{(t-L)^2 + (t+L)^2}} = \frac{1}{2L} \left[ \frac{1}{2} \log \left| \frac{(y+\omega)^2 + (t-L)^2}{(y+\omega)^2 + (t+L)^2} \right| + \frac{(\gamma_2 - \omega^2 |\gamma_2|)}{2L} \right] \]

\[ = \frac{1}{2L} \log \left| \frac{(y+\omega)^2 + (t-L)^2}{(y+\omega)^2 + (t+L)^2} \right| + \frac{T(\omega,L)}{2L} \tag{3.107} \]

where

\[ T(\omega,L) = (\gamma_2 - \omega^2 |\gamma_2|) - \frac{\omega - L}{\omega - L} (\gamma_2 - \omega^2 |\gamma_2|) \tag{3.108} \]

Using Eqs. (3.107) and (3.108), we get by integrating over $y$:

\[ g_m^2 \gamma_2 = -\frac{\omega^2}{(2\pi)^3} \mu_d \frac{t^2}{t^2 + \omega^2} \int_{|\omega|}^{\infty} d\omega \left\{ \frac{1}{2} \log \left| \frac{(\omega + \omega)^2 + (t-L)^2}{(\omega + \omega)^2 + (t+L)^2} \right| + \frac{T(\omega,L)}{2L} \right\} \]

\[ - \left( \frac{\gamma_2 - \omega^2 |\gamma_2|}{\omega - \omega} \right) + \frac{T(\omega,L)}{2L} - \frac{1}{2} \left( \gamma_2 - \omega^2 |\gamma_2| \right) \]

\[ + \frac{\gamma_2 (\gamma_2 - \omega^2 |\gamma_2|)}{L} \log \left| \frac{(\omega + \omega)^2 + (t-L)^2}{(\omega + \omega)^2 + (t+L)^2} \right| \times f(t) \]

\[ = \int_{|\omega|}^{\infty} \frac{\omega^2}{2\pi^2} \frac{t^2}{t^2 + \omega^2} \left( \frac{\gamma_2 - \omega^2 |\gamma_2|}{\omega - \omega} \right) d\omega \]
where

\[ M = \left| \frac{\alpha^2 + (\beta + \gamma)^2}{\alpha^2 + (\beta - \gamma)^2} \right| \] \tag{3.109}

Let

\[ T = T(d, L) - T(\beta, L) + 2 \left( + \alpha' |\beta + \gamma| - \alpha' |\beta - \gamma| \right) \] \tag{3.110}

Using Eq. (3.110), we get

\[ \mathfrak{Im} \left( M_n \right) = -\frac{i N^2}{(2\pi)^2} \left( \frac{\beta \pm i}{2} \right)^2 \int_0^q \left[ \frac{d \beta / d t}{|t^2|} \left[ \frac{1}{2} \log M + T \right] (\log M) \times f(t) \right] \] \tag{3.111}

Comparing Eqs. (3.85) and (3.85)', we find

\[ \mathfrak{Im} \left( M_{12} \right) = -\frac{i N^2}{(2\pi)^2} \left( \frac{\beta \pm i}{2} \right)^2 \int_0^q \left[ \frac{d \beta / d t}{|t^2|} \left[ \frac{1}{2} \log M + T \right] (\log M) \times f(t) \right] \] \tag{3.111}

We get the imaginary part from the last two expressions,

\[ \mathfrak{Im} \left( \frac{1}{M_n} + \frac{1}{M_{12}} \right) = -\frac{i N^2}{(2\pi)^2} \left( \frac{\beta \pm i}{2} \right)^2 \int_0^q \left[ \frac{d \beta / d t}{|t^2|} \left[ \frac{1}{2} \log M + 2 \left( \frac{\alpha^2 + \beta^2 - \gamma^2}{2} \right) T \right] \right] \times (\log M) \] \tag{3.111}

where, we have used the results of Eq. (3.100)
3. Computation of \( M_{12}^F \). We shall calculate \( M_{12}^F \) now for \( q = q' \). Referring to Eq. (3.57), we have

\[
M_{12}^F = \int \frac{d\xi dx}{(2\pi)^3} \langle \phi | \xi | \phi' \rangle \times (\phi')^* \langle \phi' | q' \rangle
\]

Using Eq. (3.83), we get

\[
M_{12}^F = \frac{\xi^2 N^2 \pi^2 \xi^2}{(2\pi)^6} \int \frac{d\xi dx}{(2\pi)^3} \int \frac{dq}{q} \left[ \left( \beta + \gamma_2 + \langle \xi^2 \rangle \right)^{-1} - \left( \beta + \gamma_2 + \langle \xi^2 \rangle \right)^{-1} \right]
\]

\[
\times \int dq \left[ \left( \beta + \gamma_2 + \langle \xi^2 \rangle \right)^{-1} - \left( \beta + \gamma_2 + \langle \xi^2 \rangle \right)^{-1} \right]
\]

\[
\left( \beta + \gamma_2 + \langle \xi^2 \rangle \right)^{-1} + \left( \beta + \gamma_2 + \langle \xi^2 \rangle \right)^{-1}
\]

\[
= \Re M_{12}^F + \Im M_{12}^F
\]

where,

\[
\Re M_{12}^F = -\frac{\xi^2 N^2 \pi^2 \xi^2}{(2\pi)^6} \int \frac{d\xi dx}{(2\pi)^3} \int \frac{dq}{q} \left[ \left( \beta + \gamma_2 + \langle \xi^2 \rangle \right)^{-1} - \left( \beta + \gamma_2 + \langle \xi^2 \rangle \right)^{-1} \right]
\]

\[
\times \int dq \left[ \left( \beta + \gamma_2 + \langle \xi^2 \rangle \right)^{-1} - \left( \beta + \gamma_2 + \langle \xi^2 \rangle \right)^{-1} \right]
\]

\[
\left( \beta + \gamma_2 + \langle \xi^2 \rangle \right)^{-1} + \left( \beta + \gamma_2 + \langle \xi^2 \rangle \right)^{-1}
\]

\[
= \frac{f(t) \tilde{f}(t)}{\left( \beta + \gamma_2 + \langle \xi^2 \rangle \right)^{-1}}
\]
and

\[ \gamma_m^{(0)} \mathcal{M}_{k_1} \equiv \left( \frac{-\gamma_{\mu} \gamma_{\nu} N^2 t^2 \xi_k^2}{(2 \pi)^4} \right) \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dy \left[ \left( (\xi + \gamma t)^2 + (\xi + \gamma t)^2 \right) \right] - \left( \left( (\xi + \gamma t)^2 + (\xi + \gamma t)^2 \right) \right] - \left( \left( (\xi + \gamma t)^2 + (\xi + \gamma t)^2 \right) \right] - \left( \left( (\xi + \gamma t)^2 + (\xi + \gamma t)^2 \right) \right] \times \delta(t^2 + \mu_k^4(x^2 + y^2)) \nonumber \]

We shall compute now, \( R \mathcal{M}_{k_1}^{(0)} \).

As for \( \mathcal{M}_{k_1}^{(0)} \), make the transformation \( \xi + \xi + \xi \) in Eq. (3.112) and let

\[ \xi = \xi' + \xi' ; \quad \xi = \xi' + \xi' \]

\[ \xi = \xi' + \xi ' + \xi ' ; \quad \xi ' = \xi ' + \xi ' + \xi ' \]

in Eq. (3.112)

\[ R \mathcal{M}_{k_1}^{(0)} \equiv \left( \frac{-\gamma_{\mu} \gamma_{\nu} N^2 t^2 \xi_k^2}{(2 \pi)^4} \right) \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dy \left[ \left( (\xi + \gamma t)^2 + (\xi + \gamma t)^2 \right) \right] - \left( \left( (\xi + \gamma t)^2 + (\xi + \gamma t)^2 \right) \right] - \left( \left( (\xi + \gamma t)^2 + (\xi + \gamma t)^2 \right) \right] - \left( \left( (\xi + \gamma t)^2 + (\xi + \gamma t)^2 \right) \right] \times \delta(t^2 + \mu_k^4(x^2 + y^2)) \nonumber \]
Using Eqs. (3.89), (3.91) and (3.93), we get

\[
\text{Re} \left( e_{\delta_2} \right) L = - \frac{1}{2\pi^2} \int_0^\infty \int_0^\infty \frac{dk'}{k'} \left[ \frac{1}{2} \left( \log L(k') + T(c_1, c_2, c_3, c_4) \right) \right] \times \\
\left[ - \frac{1}{2} \left( \log L(k') + T(c_1, c_2, c_3, c_4) \right) \right] \times \\
\left\{ \frac{1}{q} \log \left| \frac{1 + c_1^2 + k_2^2 c_1 (c_2^2 + 2 c_1 q_2)}{1 + c_1^2 + k_2^2 c_1 (c_2^2 - 2 c_1 q_2)} \right| \times f(t) f'(t) \right\}
\]

Using Eq. (3.94), we get

\[
\text{Re} \left( e_{\delta_2} \right) L = - \frac{1}{2\pi^2} \int_0^\infty \int_0^\infty \frac{dk'}{k'} \left[ \frac{1}{2} \left( \log L(k') + T(c_1, c_2, c_3, c_4) \right) \right] \times \\
\left[ - \frac{1}{2} \left( \log L(k') + T(c_1, c_2, c_3, c_4) \right) \right] \times \\
\left\{ \frac{1}{q} \log \left| \frac{1 + c_1^2 + k_2^2 c_1 (c_2^2 + 2 c_1 q_2)}{1 + c_1^2 + k_2^2 c_1 (c_2^2 - 2 c_1 q_2)} \right| \times f(t) f'(t) \right\}
\]
Using Eq. (3.96), we get

\[ \Re \left( \frac{\text{Eq. (3.96)}_{M_{12}}}{\pi L} \right) = -\frac{\mu_k N^2}{2 \pi^3} \int_{0}^{\infty} dt \left[ \frac{1}{4} \text{Re} \left( g(t) + K(t) \right) \right] \frac{t^2}{(\omega^2 + \omega^2 - \omega^2 + \omega^2 + k^2) + k^2} \quad (3.114) \]

where

\[ K(t) = \int_{0}^{\infty} \frac{dt'}{\epsilon'} \left( \tau(\epsilon', \epsilon, \epsilon, \epsilon') \right)^2 \frac{1}{2} \log \left[ \frac{1 + \omega^2 + \frac{4 \pi}{\omega} (\omega^2 + 2 \omega + \omega')}{1 + \omega^2 + \frac{4 \pi}{\omega} (\omega^2 - 2 \omega + \omega')} \right] \quad (3.115) \]

We shall compute now \( g_{m_{12}}^{(3)} \).

For Eq. (3.113) using the calculations given in \( g_{m_{12}}^{(3)} \), we get

\[ g_{m_{12}}^{(3)} = \frac{-\mu_k \frac{N^2}{2 \pi^3} e^8}{2} \int dL \int dL' \left\{ \left[ -\frac{1}{4L} \log M + \frac{T}{2L} \right] \right. \]

\[ \times \left[ \frac{1}{4L} \log M + \frac{T}{2L} \right] \delta(t) \delta(t') \delta(t^2 + \frac{4 \pi}{\omega} (\omega^2 + q^2 + \omega')) \]

where \( M \) and \( T \) are defined by Eqs. (3.109) and (3.110).

i.e. we have used the result

\[ \int_{0}^{\infty} dy \left[ \left( \sigma + \gamma - i t \right)^2 + L^2 \right]^{-1} = \left( \sigma + \gamma - i t \right)^2 + L^2 \right]^{-1} + \left( \sigma + \gamma + i t \right)^2 + L^2 \right]^{-1} + \left( \sigma + \gamma - i t \right)^2 + L^2 \right]^{-1} = -\frac{\pi}{L} \log M + T_{2L} \]
Using the properties of $\mathcal{B}$-function, we get

\[
q_m(M, t) = -\frac{\nu \cdot \eta \cdot \xi \cdot t^2}{(2\pi)^3} \int \frac{d\xi \cdot d\eta \cdot d\nu}{L^2} \left[ \frac{1}{\xi} \left( \log \nu \right)^2 + T^\prime \right] \times f(t) f^*(t) \times
\]

\[
\frac{\delta(t - \sqrt{\nu \cdot \eta \cdot (q^2 - t^2)}) + \delta(t + \sqrt{\nu \cdot \eta \cdot (q^2 - t^2)})}{2\nu \cdot \eta \cdot (q^2 - t^2) - t^2}
\]

\[
= -\frac{\nu \cdot \eta \cdot \xi \cdot t^2}{(2\pi)^3} \int \frac{d\xi \cdot d\eta \cdot d\nu}{L^2} \left[ \frac{1}{\xi} \left( \log \nu \right)^2 + T^\prime \right] f(t) f^*(t) \times \frac{t}{\nu \cdot \eta \cdot (q^2 - t^2) - t^2}
\]

\[
el \text{ varies from } 0 \text{ to } q = \sqrt{q^2 - \frac{\nu \cdot \eta \cdot \xi \cdot t^2}{L^2}}, \text{ as before}
\]

\[
q_m(M, t) = -\frac{\nu \cdot \eta \cdot \xi \cdot t^2}{(2\pi)^3} \int \frac{d\xi \cdot d\eta \cdot d\nu}{L^2} \left[ \frac{1}{\xi} \left( \log \nu \right)^2 + T^\prime \right] \times
\]

\[
\frac{t}{(\xi^2 + \eta^2 + \nu^2) - t^2} \times \frac{t}{\nu \cdot \eta \cdot (q^2 - t^2) - t^2} \quad (3.116)
\]
We shall find now 
\[ \lim_{q \to 0} \mathbb{P}_q \mathbb{M}_{1,2} \, \mathbb{I} \, \rho, \]

Using Eq. (3.101) in Eq. (3.115), we find

\[
K(t) = \int_{0}^{\infty} \frac{d\ell'}{\ell'} \frac{(\mathcal{T}(\hat{s}_1, \hat{s}_2, \hat{s}_3, \hat{s}_4, \ell'))^2}{1 + \delta^2 + \frac{1}{2} \mu \ell'} \frac{\mu d}{\mu t} \]

\[
= \frac{4 \frac{\mu d}{\mu t}}{1} \cdot K_1(t) \]

where

\[
K_1(t) = \int_{0}^{\infty} \frac{d\ell'}{\ell'} \frac{(\mathcal{T}(\hat{s}_1, \hat{s}_2, \hat{s}_3, \hat{s}_4, \ell'))^2}{1 + \delta^2 + \frac{1}{2} \mu \ell'} \tag{3.117} \]

Using Eqs. (3.102) and (3.117), we get for \( q = 0 \)

\[
\Re \left( \mathbb{M}_{12}^x \right) = - \frac{\mu N^2}{\pi^2} t^0 \int_{0}^{\infty} \left( \mathcal{T} j(t) + K_1(t) \right) \frac{t}{(\ell t + \ell t')^2 + \ell^2} \tag{3.118} \]
G. Summary of the Calculations.

For $q = 0$, $M_{i12}^c$ is given by Eq. (3.42): 

$$M_{i12}^c = \{1\} M_{i12}^{(1)} + \{3\} M_{i12}^{(3)} + \{3\} M_{i12}^{(3)} + \{4\} M_{i12}^{(4)} + (1 \leftrightarrow 2)$$

$(1)$ $M_{i12}^{(1)}$ is given by Eq. (3.77): 

$$\text{Re} \left( \beta_{i1}^{(1)} + \gamma_{i1}^{(1)} \right)$$ is given by Eq. (3.100) 

$$\text{Im} \left( \beta_{i1}^{(1)} + \gamma_{i1}^{(1)} \right)$$ is given by Eq. (3.111) 

$$\text{Re} \left( \beta_{i1}^{(4)} + \gamma_{i1}^{(4)} \right)$$ is given by Eq. (3.114) 

$$\text{Im} \left( \beta_{i1}^{(4)} + \gamma_{i1}^{(4)} \right)$$ is given by Eq. (3.116)

For $q = 0$

$$\text{Re} \left( \beta_{i1}^{(0)} + \gamma_{i1}^{(0)} \right)$$ is given by Eq. (3.104) 

$$\text{Re} \left( \beta_{i1}^{(0)} + \gamma_{i1}^{(0)} \right)$$ is given by Eq. (3.118)

The matrix element for the scattering of the $K^-$-meson from the deuteron is given by:

$$M = M_1 + M_2 + M_{i12} + M_{i12}^{\Gamma} \quad (2.22)$$
H. Correction to Elastic Scattering Due to Charge Exchange

in the Intermediate State. We shall calculate now the contribution due to charge exchange

1. \( K^- + \pi \rightarrow \bar{K}^0 + \eta \)
2. \( \bar{K}^0 + \eta \rightarrow K^- + \pi \)

We shall suppose that

\[
\tilde{t}_1^{e^+} = \tilde{t}_1^{e^+}(\xi_t - \xi_{A_1}) \quad \text{for the process 1.} \tag{3.119}
\]

\[
\tilde{t}_2^{e^+} = \tilde{t}_2^{e^+}(\xi_t + \xi_{A_2}) \quad \text{for the process 2.} \tag{3.120}
\]

where \( \tilde{t}_1^{e^+} \) and \( \tilde{t}_2^{e^+} \) are constants and will be determined in Chapter VII. There is no direct charge exchange of the \( K^- \)-meson with the neutron.

The Feynmann diagram for the charge exchange in the intermediate state is shown in Fig. 3.

![Feynmann diagram](image)

Figure 3.

The matrix element for this process is given by

\[
M_{12}^{e^+} = \langle \psi_f | \tilde{t}_2^{e^+} \tilde{t}_1^{e^+} | \psi_i \rangle \tag{3.121}
\]
Unn is the separable non-local potential between the two neutrons in the intermediate state.

\[ \Psi_{c} \text{ and } \Psi_{f} \text{ are defined in Eqs. (2.1) and (2.2).} \]

Introducing
\[ \sum_{\frac{q_{i}}{2}} |q_{i}\rangle\langle q_{i}| = \int \frac{d^{3}q_{i}}{(2\pi)^{3}} |q_{i}(\tau_{q})\rangle\langle q_{i}(\tau_{q})| \]
\[ \sum_{\frac{q_{i}}{2}} |\Psi_{q_{i}}\rangle \langle \Psi_{p_{i}}| = \int df_{i} |\Psi_{q_{i}}(\tau_{q})\rangle \langle \Psi_{p_{i}}(\tau_{p})| \]

and similarly for \( \Psi_{q_{i}} \) in Eq. (3.122), we get

\[ M_{\alpha\beta} = \int \int \int \langle \Psi_{f} | t^{\alpha\beta} | q_{i}, \Psi_{q_{i}}(\tau_{q}), \Psi_{p_{i}}(\tau_{p}) \rangle \langle E - E_{q_{i}} - E_{p_{i}} - E_{q_{i}} + i\epsilon | \Psi_{f} \rangle \]
\[ \cdot \int \frac{d^{3}q_{i}}{(2\pi)^{3}} \frac{d^{3}p_{i}}{(2\pi)^{3}} df_{i} \]

In writing the above expression, we do not have to consider the Pauli exclusion principle, since the additional terms obtained thereby, are already included as part of the single scattering. Fig. 4 represents such a diagram.
This actually represents impulse approximation contribution.

There is no double scattering taking place at all as the $K^-$-meson is being scattered from only one particle.

Using Eqs. (2.3) and (2.4), we can write Eq. (3.121) in the center of mass system. The matrix element in the center of mass system is given by

$$M_{12}^{ex} = \langle q', \varphi | t_L^{ex} G_{mn} t_L^{ex} | \varphi, q \rangle$$

(3.123)

where $q_L$ and $q'_L$ are defined in center of mass system.

In the plane wave approximation, we have

$$G_{nn} = G \text{ with } U = 0$$

Proceeding exactly as in Chapter II, we can write

$$M_{12}^{ex} \simeq \left( \frac{\hbar}{M_{12}^e} \right)^{ex}$$

(3.124)

Comparing this expression with Eq. (2.25), we can write

$$M_{12}^{ex} \simeq \frac{e_k}{2} \left( \frac{M_{12}^e + i\pi \omega \lambda}{1 - \frac{\pi \omega \lambda}{M_{12}^e}} \right)$$

(3.125)

The factor of $\frac{1}{2}$ appears in the above expression because in Fig. 3, we can not exchange the proton and the neutron while the expressions for $(M_{12}^e + i\pi \omega \lambda)$ includes the exchange of the proton and the neutron as shown in Eq. (2.25).

We can define the total contribution by

$$M^T = M + M_{12}^{ex}$$

(3.126)

The treatment for charge exchange is not exact, but does give an order of magnitude of the correction to the elastic scattering cross sections.
CHAPTER IV

NUMERICAL EVALUATION

Single Scattering. For any $q_L$ and $\theta = 0$

$$M_1 + M_2 = t_1^0 + t_2^0 \quad (3.25)$$

Double Scattering. The calculations have been done at three energies $q_L = 0 \text{ MeV}, 105 \text{ MeV}$ and $194 \text{ MeV}$.

Bound State. The results for the bound state forward scattering are given in Table I.

**TABLE I**
Values of the matrix element $M_{12}^c$

<table>
<thead>
<tr>
<th>$q_L$ (MeV)</th>
<th>$q (q^{'})$</th>
<th>$M_{12}^c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.42</td>
<td>-27.4 - 12.9</td>
</tr>
<tr>
<td>105</td>
<td>0.78</td>
<td>-16.4 - 113.9</td>
</tr>
<tr>
<td>194</td>
<td>0.78</td>
<td>-10.1 - 113.9</td>
</tr>
</tbody>
</table>

where $z = \mu_1^{1/2} \cdot N_1 t_1^0 \cdot t_2^0 \quad (3.105)$
Continuum State. The results for the continuum state forward scattering are given in Table II.

### TABLE II
Values of the matrix element $M_{12}^\uparrow$

<table>
<thead>
<tr>
<th>$q_L (M_{12})$</th>
<th>$(0)<em>{M</em>{12}/2}$</th>
<th>$(\langle \frac{M_{12}}{2} + \frac{M_{12}}{2} \rangle)/2$</th>
<th>$(\nu)<em>{M</em>{12}/2}$</th>
<th>$M_{12}/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-5.0 -1 1 0</td>
<td>18.2 - 1 0</td>
<td>-6.1 -1 0</td>
<td>+14.1 -1 0</td>
</tr>
<tr>
<td>105</td>
<td>-5.9 -1 1.7</td>
<td>21.1 - 1 1.8</td>
<td>-5.9 -1.34</td>
<td>18.6 -13.4</td>
</tr>
<tr>
<td>194</td>
<td>-3.8 -1 2.7</td>
<td>16.2 - 1 1.0</td>
<td>-5.1 -11.1</td>
<td>14.6 -19.5</td>
</tr>
</tbody>
</table>

Now

The results for $M$ are given in Table III.

### TABLE III
Values of the matrix element $M$

<table>
<thead>
<tr>
<th>$q_L (M_{12})$</th>
<th>$M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$t_1^r + t_2^r - \bar{Z} (13.3 -1 0)$</td>
</tr>
<tr>
<td>105</td>
<td>$t_1^r + t_2^r - \bar{Z} (2.2 -116.3)$</td>
</tr>
<tr>
<td>194</td>
<td>$t_1^r + t_2^r - \bar{Z} (4.4 -123.3)$</td>
</tr>
</tbody>
</table>
**Charge Exchange.** In the plane wave approximation, the total contribution due to charge exchange scattering is given in Table IV.

**TABLE IV**

Values of the matrix element $M_{12}^{ex}$.

<table>
<thead>
<tr>
<th>$q_4 (M_{12})$</th>
<th>$M_{12}^{ex} / z'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$- (5.0 + i 1.0)$</td>
</tr>
<tr>
<td>105</td>
<td>$- (5.9 + i 1.7)$</td>
</tr>
<tr>
<td>194</td>
<td>$- (3.8 + i 2.7)$</td>
</tr>
</tbody>
</table>

where $z' = \frac{\hbar d}{\mathcal{M}} \cdot N^2 t^0_{ex} t^0_{ex}$

Now total matrix element is

$$M_T = M + M_{12}^{ex}$$

which is given in Table V obtained from Tables III and IV.

**Table V**

Values of the matrix element $M_T$.

<table>
<thead>
<tr>
<th>$q_4 (M_{12})$</th>
<th>$M_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$t_1^0 + t_2^0 + z(13.3 - 1.0) - z'(5.0 + i 1.0)$</td>
</tr>
<tr>
<td>105</td>
<td>$t_1^0 + t_1^0 + z(2.2 - 116.3) - z'(5.9 + 12.7)$</td>
</tr>
<tr>
<td>194</td>
<td>$t_1^0 + t_1^0 + z(4.4 - 123.3) - z'(3.8 + 12.7)$</td>
</tr>
</tbody>
</table>
CHAPTER V

BRÜCKNER MODEL

We have shown in Chapter I, that the scattering amplitude for the $K^-$-meson is given by

$$F_\rho (q^1, q^2) = (1 - \eta_p \eta_n \frac{2 \cdot 9 \rho}{F_\rho})^{-1} \left[ \eta_p e^{i(q \cdot x) \cdot r_1} + \eta_n e^{i(q \cdot x) \cdot r_2} \right]$$

$$\eta_p \eta_n \frac{2 \cdot 9 \rho}{F_\rho} \left[ e^{i(q \cdot r_1 - q' \cdot r_2)} + e^{i(q \cdot r_2 - q' \cdot r_1)} \right]$$

where

$$\rho = |r_1 - r_2|$$

We can write

$$R_1 = \frac{\rho}{2} \quad \text{and} \quad R_2 = -\frac{\rho}{2}.$$

We get

$$F_\rho (q^1, q^2) = \frac{1}{D} \left[ \eta_p e^{i(q \cdot x) \cdot \frac{\rho}{2}} + e^{-i(q \cdot x) \cdot \frac{\rho}{2}} \eta_n \frac{2 \cdot 9 \rho}{F_\rho} \left( e^{i(q \cdot r_1 - q' \cdot r_2)} + e^{i(q \cdot r_2 - q' \cdot r_1)} \right) \right]$$

(5.1)
where

\[
\mathcal{D} = \left(1 - \eta_p \eta_n \frac{\epsilon_k}{\rho} \right)^{-1}
\]  \hspace{1cm} (5.2)

Since the scattering of the K\textsuperscript{−}-meson from the deuteron can take place whatever the separation \( \rho \) and the direction of \( \rho \) may be, we have to average over \( \rho \) and the direction of \( \rho \).

Anticipating the spherically symmetrical wave function for the deuteron, we can average over the angles between the \( \rho \) and \( \gamma' \) and \( \rho \) and \( \gamma' \), we get

\[
f(\theta, \rho) = \frac{N}{\mathcal{D}}
\]  \hspace{1cm} (5.3)

where

\[
N = \left( \eta_p + \eta_n \right) \frac{S_{-} \rho_{L} \rho_{T}}{\rho_{T}^2} + 2 \eta_p \eta_n \frac{\epsilon_k}{\rho} \frac{S_{-} \rho_{L} \rho_{P}}{\rho P}
\]  \hspace{1cm} (5.4)

\[
\rho = 2 \eta \frac{S_{-} \theta}{\rho_{T}}, \quad P = \eta \frac{Q \theta}{\rho_{T}}
\]  \hspace{1cm} (5.5)

\( \theta \) is the angle of scattering of the meson in the K\textsuperscript{−}-d center of mass system.

Here, \( \frac{Q}{\rho} \) is the final K\textsuperscript{−} momentum in the K\textsuperscript{−}-d center of mass system. In the above equations, \( \eta_p \) and \( \eta_n \) are the K\textsuperscript{−}-P and K\textsuperscript{−}-n scattering amplitudes (in the deuteron) and are related to the free

---

19
See ref. 11
$K^-$-nucleon amplitudes ($f_p$ and $f_n$) by

$$\eta_p = \frac{\mu_{K\Lambda}}{\mu_{K\Lambda}} f_p \quad , \quad \eta_n = \frac{\mu_{K\Lambda}}{\mu_{K\Lambda}} f_n \quad (5.6)$$

Where $\mu_{\Lambda}$ are appropriate reduced masses, $f_p$ and $f_n$ are given in terms of the scattering lengths $A_0$ and $A_1$ by

$$f_p = \frac{1}{2} \left( \frac{A_1}{1 - q_0 A_1} + \frac{A_2}{1 - q_0 A_2} \right)$$

$$f_n = -q_0 A_1 \quad (5.7)$$

$q_0$ is the $K^-$-momentum in the $K^-$-nucleon center of mass system.

$$q_0 = \frac{m_p}{m_K + m_p} q_L \quad (5.8)$$

where $q_L$ is the $K^-$-momentum in the laboratory system.

The scattering amplitude from a deuteron is given by averaging over all $f$. Therefore, the scattering amplitude is

$$f(\theta) = \langle f(\theta, \rho) \rangle = \int|\rho(\rho)|^2 f(\theta, \rho) \, d\rho \quad (5.9)$$
We consider the contributions up to double scattering only.

Double scattering approximation.

\[ f(\theta) = f_{mp}(\theta) + 2 \eta_p \eta_n \langle \frac{e^{i\varphi}}{p} \frac{S_{mn} pp}{p^2} \rangle \]

\[ = f_{mp}(\theta) + 2 \eta_p \eta_n \cdot g(\theta) \quad (5.10) \]

where

\[ f_{mp}(\theta) = (\eta_p + \eta_n) \langle \frac{S_{mn} pp}{p^2} \rangle \quad (5.11) \]

\[ g(\theta) = \Re g(\theta) + i \Im g(\theta) \]

\[ \Re g(\theta) = \frac{1}{P(1-a^2)} \left[ -\alpha + \alpha^{-1} \frac{P+2}{2a} - \beta + \alpha^{-1} \frac{P+q}{2a} + (e^+ \beta) \right] \]

\[ \times \left[ \frac{\alpha^{-1} P+q}{d+\beta} + \frac{P+2}{4} \log \left| \frac{(q+\beta)^2 + (P+q)^2}{4\beta^2 + (P+q)^2} \right| \right] \]

\[ + \text{similar terms with } q \rightarrow -q \quad (5.12) \]
\[ \text{Im } g(\theta) = \frac{2\pi}{P(1-\alpha)} \left[ -\frac{\partial}{\partial \epsilon} \left| \frac{4\omega^2 + (r+q)^2}{r^2 + (r+q)^2} \right| + \frac{\epsilon}{2} \left( \frac{1 - \alpha^2}{4\omega^2 - q^2 + \epsilon^2} \right) \right] + \epsilon \left( 2\alpha \beta \gamma \right) \left( \frac{1 - \alpha^2}{4\omega^2 - q^2 + \epsilon^2} \right) \]

+ similar terms with \( \alpha \to \beta \)

\[ g(0) = 0 \]

In Eq. (5.13),

\[ \epsilon(\alpha, \beta, \gamma) = 1 \]

\[ \alpha^2 - \beta^2 + \gamma^2 < 0 \]

\[ \epsilon(\alpha, \beta, \gamma) = 0 \]

\[ \alpha^2 - \beta^2 + \gamma^2 > 0 \]

The values of \( g(0) \) are given in Table VI.

**TABLE VI**

Values of the double scattering contribution \( g(0) \)

<table>
<thead>
<tr>
<th>( q_\perp (\text{Mev}) )</th>
<th>( g(0^\circ) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.57 + i 0</td>
</tr>
<tr>
<td>105</td>
<td>0.38 + i 0.24</td>
</tr>
<tr>
<td>194</td>
<td>0.21 + i 0.27</td>
</tr>
</tbody>
</table>
A. COMPARISON OF BRUECKNER MODEL WITH THE MODEL GIVEN IN CHAPTER III

To compare the two models we should express $t_1^0$ and $t_2^0$ in terms of $\eta_f$ and $\eta_q$. 

Consider the scattering of the $K^-$-meson from proton. The scattering amplitude is given by

$$f_p = -\frac{\lambda_{KP}}{2\pi} \left< q' | t_1^0 | q \right>$$

where $q$ and $q'$ are the momenta of the $K^-$-meson in the center of mass system of $K^-P$ system.

$$t_1 = t_1^0 \delta(z_2 - z_1) = t_1^0 \delta(z_2 - z_1) \quad \text{(Using Eq. (2.3))}$$

Therefore,

$$f_p = -\frac{\lambda_{KP}}{2\pi} t_1^0 \left< q' | \delta(z_2 - z_1) | q \right>$$

$$= -\frac{\lambda_{KP}}{2\pi} t_1^0 \quad \text{(6.1)}$$

If the proton is bound in the deuteron, then the effective amplitude of scattering is

$$\eta_p = \frac{\lambda_{KP}}{\sqrt{k_{KP}}} \cdot f_p \quad \text{(5.6)}$$
since the amplitude $f_p$ is proportioned to the reduced mass of the scattering particles, we get

$$\eta_p = -\frac{\mu_{kP}}{2\pi} t_2^0$$

or

$$t_2^0 = -\frac{2\pi}{\mu_{kP}} \eta_p$$  \hspace{1cm} (6.2)

Similarly

$$t_2^e = -\frac{2\pi}{\mu_{k\eta}} \eta_\eta$$  \hspace{1cm} (6.3)

Substituting Eqs. (6.2) and (6.3) in Eq. (3.105), we get

$$Z = \mu_{g\eta}^2 \sqrt{2} t_2^0 t_2^e$$

$$= \frac{2\mu_{k\eta}^2}{\mu_{kP}} \frac{N^e}{\mu_{k\eta}} \left(2\eta_{\eta} \eta_{\eta}\right)$$  \hspace{1cm} (6.4)

The scattering amplitude of the $K^-d$ scattering is defined by

$$f_{k'd} = -\frac{\mu_{k'd}}{2\pi} M$$  \hspace{1cm} (6.5)

Considering only the double scattering without charge exchange in the intermediate state, the scattering amplitude is given by

$$f_{k'd} = -\frac{\mu_{k'd}}{2\pi} \left(M_{1} + M_{2}^*\right)$$  \hspace{1cm} (6.6)
Making use of the results for $M_i^C$ and $M_i^V$ given in Tables I and II in Eqs. (6.4) and (6.6), we can find the values of $f_{\delta_d}^{d}$ at various energies. These values are given in Table VII along with the values of $\mathcal{G}(0^*)$ (from Table VI) which represents the corresponding expression in the Brueckner model. Neglecting the continuum state contribution, we calculate the scattering amplitude $f_{\delta_d}^{d}$ for double scattering due to the bound state of the deuteron. These values are also given in Table VII.

**TABLE VII.**

Comparison of the double scattering contribution given by Brueckner model and the model given in Chapter III.

<table>
<thead>
<tr>
<th>$\gamma_L (\frac{M_{dN}}{c})$</th>
<th>$\frac{f_{\delta_d}^{d}}{\gamma \eta_1 \eta_2}$</th>
<th>$\frac{f_{\delta_d}^{d}}{\gamma \eta_1 \eta_2}$</th>
<th>$\mathcal{G}(0^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.93 + 1 0</td>
<td>.45 + 1 0</td>
<td>.57 + 1 0</td>
</tr>
<tr>
<td>105</td>
<td>.56 + 1.44</td>
<td>-.07 + 1.55</td>
<td>.38 + 1.24</td>
</tr>
<tr>
<td>194</td>
<td>.36 + 1.47</td>
<td>-.15 + 1.79</td>
<td>.21 + 1.27</td>
</tr>
</tbody>
</table>

We conclude, on comparing the 3rd and 4th columns, that the Brueckner model is quite unreliable for $\gamma > 100 \frac{M_{dN}}{c}$, especially for what would be the real part of $f_{\delta_d}^{d}$ if $\eta_1$ and $\eta_2$ were real. Comparison with the 2nd column shows that in some sense the Brueckner model approximates the effect of the continuum states very badly, agreeing much better with just the bound state contribution. This result seems reasonable from a physical point of view.
B. CORRESPONDENCE BETWEEN THE BRUECKNER MODEL AND THE MODEL GIVEN IN CHAPTER II.

We can deduce the Brueckner model from the model given in Chapter II. The $K^{-}d$ scattering amplitude is given by

$$ f_{kl} = -\frac{k_{rl}}{2\pi} \langle \mathcal{Y}, \mathcal{P} | t_{1} + t_{2} + \ldots | \mathcal{Y}, \mathcal{P} \rangle $$ \hspace{1cm} (6.7)

Using Eq. (3.1), we get

$$ \langle \mathcal{Y}, \mathcal{P} | t_{1} | \mathcal{Y}, \mathcal{P} \rangle = \int e^{i\mathbf{q} \cdot \mathbf{r}} \langle \mathcal{Y}, \mathcal{P} | t_{1} | \mathcal{Y}, \mathcal{P} \rangle_1 \mathcal{A} \rangle \mathcal{A} \rangle $$ \hspace{1cm} (6.8)

where $\mathcal{A}$ is defined in Eq. (3.34).

Similarly

$$ \langle \mathcal{Y}, \mathcal{P} | t_{2} | \mathcal{Y}, \mathcal{P} \rangle = \langle \mathcal{Y}, \mathcal{P} | t_{2}^{0} e^{-i\mathbf{q} \cdot \mathbf{r}} | \mathcal{Y}, \mathcal{P} \rangle $$ \hspace{1cm} (6.9)

Now

$$ \langle \mathcal{Y}, \mathcal{P} | t_{1} + t_{2} | \mathcal{Y}, \mathcal{P} \rangle = \int \frac{d\mathbf{k}}{(2\pi)^{3}} \int \frac{d\mathbf{k}'}{(2\pi)^{3}} \langle \mathcal{Y}, \mathcal{P} | t_{1} + t_{2} | \mathcal{Y}, \mathcal{P} \rangle $$ \hspace{1cm} (6.10)

where $E_{k}$ and $E_{l}$ are defined by Eqs. (3.49 and 3.50).

Let us suppose that $E_{l}$ is replaced by some average value $\bar{E}_{l}$ and

$$ \langle \mathcal{Y}, \mathcal{P} | t_{1} + t_{2} | \mathcal{Y}, \mathcal{P} \rangle = \langle \mathcal{Y}, \mathcal{P} | t_{1} + t_{2} | \mathcal{Y}, \mathcal{P} \rangle $$ \hspace{1cm} (6.11)

Taking $U = 0$
Using Eq. (6.11) in Eq. (6.10), we get

$$\langle q', \varphi | t, \varphi | \frac{\delta}{\delta \varphi} \rangle = \int \frac{dE}{(2\pi)^3} \left( \frac{1}{\beta^2 (\beta^2 g + 1)} \right) ^{\frac{1}{2}} \left| \begin{array}{c} \frac{1}{2} \mu_{\varphi} - E + i\varepsilon \\ E \end{array} \right| \left( \delta(q - q') \right) \left( \langle \varphi | t \varphi | q, \varphi \rangle \right).$$

Using

$$\int \frac{dE}{(2\pi)^3} \left( \frac{1}{\beta^2 (\beta^2 g + 1)} \right) = -\frac{\mu_{\varphi} d}{2\pi} \left( \frac{E}{2} \right),$$

we get

$$\langle q', \varphi | t, \varphi | \frac{\delta}{\delta \varphi} \rangle = \int \frac{dE}{(2\pi)^3} \left( \frac{1}{\beta^2 (\beta^2 g + 1)} \right) ^{\frac{1}{2}} \left| \begin{array}{c} \frac{1}{2} \mu_{\varphi} - E + i\varepsilon \\ E \end{array} \right| \delta(q - q') \left( \langle \varphi | t \varphi | q, \varphi \rangle \right).$$

$$= \int df e^{\frac{i}{2\pi} \left( \frac{1}{2} \mu_{\varphi} \cdot q + \frac{i}{2} \varphi' \right)} \left( \frac{1}{2\pi} \int \frac{dE}{(2\pi)^3} \left( \frac{1}{\beta^2 (\beta^2 g + 1)} \right) \right) \left( \langle \varphi | t \varphi | q, \varphi \rangle \right).$$

$$= \langle \varphi | e^{\frac{i}{2\pi} \left( \frac{1}{2} \mu_{\varphi} \cdot q + \frac{i}{2} \varphi' \right)} \left( \frac{1}{2\pi} \int \frac{dE}{(2\pi)^3} \left( \frac{1}{\beta^2 (\beta^2 g + 1)} \right) \right) | \varphi \rangle \quad (6.12)$$
Similarly
\[ \langle \Phi \mid t, \Phi \rangle = \langle \Phi \mid e^{-iQ \cdot \Phi} \cdot e^{iQ \cdot \Phi} \rangle \]

Substituting Eqs. (6.8), (6.9), (6.12) and (6.13) in Eq. (6.7), we get
\[ f_{kd} = -\frac{\mu_{kd}}{2\pi} \langle \Phi \mid t, \Phi \rangle = \frac{\mu_{kd}}{2\pi} \langle \Phi \mid t, \Phi \rangle \]

Using Eqs. (6.1) and (5.6), we get
\[ -\frac{\mu_{kd}}{2\pi} t_0 = \gamma_p \]
and
\[ -\frac{\mu_{kd}}{2\pi} t_0 = \gamma_n \]

Substituting Eqs. (6.15) and (6.16) in Eq. (6.14), we get
\[ f_{kd} = \langle \Phi(0) \mid \eta_p e^{iQ \cdot \Phi} + \eta_n e^{iQ \cdot \Phi} + \eta_p \eta_n e^{iQ \cdot \Phi} \]
\[ = \langle \Phi(0) \mid e^{iQ \cdot \Phi} + e^{i(Q+\Phi) \cdot \Phi} \rangle \]

It is obvious that Eq. (6.17) is identical to the scattering amplitude obtained by using the Brueckner model except \( \Phi \) is replaced by \( \Phi' \).

If \( \vec{E}_B = 0 \), we get \( \Phi' = \chi \) and two models give the same scattering amplitudes. It seems likely that some choice of \( \Phi' \), such that \( \Phi' \neq 1 \), might give better agreement with the results of our model calculation (See Table VII, Columns 3rd and 4th). This would be interesting to pursue a further investigation.
CHAPTER VII

COMPARISON OF THE MODEL GIVEN IN CHAPTER III
WITH THE EXPERIMENTAL DATA.

A. Computation of Cross Sections. In this chapter, we shall try to compute the differential, elastic and total cross sections. As mentioned before in Chapter III, the matrix elements for any angle between \( q \) and \( q' \) refer to the triple integrals which are hard to evaluate, the differential cross section is available only for the forward scattering. Consequently a rough estimation of the elastic cross section is obtained by using the relation

\[
\sigma_{21} \sim \frac{(d\sigma/d\Omega)_{\theta=0^\circ} \int (d\sigma'/d\Omega) d\Omega}{(d\sigma'/d\Omega)_{\theta=0^\circ}}
\]  

(7.1)

where \( d\sigma'/d\Omega \) is the differential cross section in the impulse approximation.

The total cross section is calculated by using the optical theorem

\[
\sigma_T = \frac{4\pi}{\varphi} \Im \mathcal{M}(0')
\]

(7.2)
The scattering lengths $f_P$ and $f_n$ defined by Eq. (5.7) are calculated by using the Dalitz Solutions for the scattering lengths in the zero range approximation. The scattering lengths are:

Sol. I
$A_1 = .02 + i .38$
$A_o = -.22 + i .74$

Sol. II
$A_1 = 1.20 + i .56$
$A_o = -.59 + i .96$

The values of $\eta_p$ and $\eta_n$ defined by Eq. (5.6) are given in Table VIII.

**TABLE VIII.**

Values of $\eta_p$ and $\eta_n$.

<table>
<thead>
<tr>
<th>$q_m (\text{MeV})$</th>
<th>Sol.</th>
<th>$\eta_p (f)$</th>
<th>$\eta_n (f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>105</td>
<td>I</td>
<td>-.03 + i 1.05</td>
<td>.02 + i .41</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>.26 + i .91</td>
<td>.90 + i .88</td>
</tr>
<tr>
<td>194</td>
<td>I</td>
<td>-.01 + i .78</td>
<td>.02 + i .37</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>.17 + i .81</td>
<td>.59 + i .83</td>
</tr>
</tbody>
</table>

\[21\]

See refs. 14 and 15.
Using the definitions of $t_1^0$ and $t_2^0$ given in Eqs. (6.2) and (6.3) and the results given in Tables III and VIII, we can find the scattering amplitudes $f_{kl}$. The results are given in Table IX.

**TABLE IX.**

Numerical values of the scattering amplitudes.

<table>
<thead>
<tr>
<th>$Q_n (\text{MeV})$</th>
<th>Sol.</th>
<th>$f_{kl} (f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>105</td>
<td>I</td>
<td>0.05 + 1.29</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>0.33 + 1.28</td>
</tr>
<tr>
<td>194</td>
<td>I</td>
<td>0.08 + 1.93</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>0.12 + 1.89</td>
</tr>
</tbody>
</table>

The differential, elastic and total cross sections are given in Table X. The elastic cross section is calculated by using Eq. (7.1).

**TABLE X.**

Values of the elastic and total cross sections.

<table>
<thead>
<tr>
<th>$Q_n (\text{MeV})$</th>
<th>Sol.</th>
<th>$\frac{d\sigma}{d\Omega} (\text{cm}^2)$</th>
<th>$\sigma_{el}$</th>
<th>$\sigma_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>105</td>
<td>I</td>
<td>16.7</td>
<td>110</td>
<td>383</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>20.1</td>
<td>132</td>
<td>410</td>
</tr>
<tr>
<td>194</td>
<td>I</td>
<td>8.7</td>
<td>35</td>
<td>150</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>8.0</td>
<td>31</td>
<td>142</td>
</tr>
</tbody>
</table>

As the coulomb interaction is significant at low energies, the cross sections are not calculated at $0 \text{ MeV}$.
B. Contribution Due to Charge Exchange. The scattering amplitude for the process

\[ \bar{K}^+ + p \rightarrow \bar{K}^+ + n \]

is given by

\[ f_{\text{ex}} = \frac{1}{2} \left[ \frac{A_1}{1 - i\tau A_1} - \frac{A_2}{1 - i\tau A_2} \right] \]

It is obvious that for the inverse process

\[ \bar{K}^+ + n \rightarrow K^+ + p \]

we have

\[ f_{\text{ex}} = f_{\text{ex}}' \]

The corresponding expressions for \( \eta_{\text{ex}} \) are given by Eq.(5.6) with \( f_{\text{ex}} \) and \( f_{\text{ex}}' \) replaced by \( \eta_{\text{ex}} \) and \( \eta_{\text{ex}}' \).

The order of magnitude of the contribution due to charge exchange in the intermediate state can be estimated easily. We can write the elastic cross section as

\[ \sigma_{\text{el}} \sim \left| M_{\text{imp}} + M_{\text{ad}} + M_{\text{ex}} \right|^2 \]

\[ = \left| M_{\text{imp}} + M_{\text{ad}} \right|^2 \left[ 1 + 2 \text{Re} \left( \frac{M_{\text{ex}}}{M_{\text{imp}} + M_{\text{ad}}} \right) + \ldots \right] \quad (7.3) \]

where

\[ M_{\text{ad}} = M_{\text{ad}}^C + M_{\text{ad}}^I \]

and

\[ M_{\text{ex}} = M_{\text{ex}}^C \]
Using plane wave approximation in the evaluations of $M_{12}^{\pi X}$, the values of $2 \Re \frac{M_{\text{imp}} + M_{\text{ord}}}{M_{\text{imp}} + M_{\text{ord}}}$ at $\theta = 0$ are calculated by using the Tables III and IV. The results are given in Table XI.

**TABLE XI.**

Ratio of the charge exchange scattering to ordinary scattering.

<table>
<thead>
<tr>
<th>$\gamma$ [MeV]</th>
<th>Sol.</th>
<th>$2 \Re \frac{M_{\text{imp}} + M_{\text{ord}}}{M_{\text{imp}} + M_{\text{ord}}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>105</td>
<td>I</td>
<td>- .24</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>.25</td>
</tr>
<tr>
<td>194</td>
<td>I</td>
<td>- .17</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>.18</td>
</tr>
</tbody>
</table>

It is obvious from Table XI, that the contribution due to charge exchange in the intermediate state is small compared to the ordinary scattering contribution. However, if anything approaching say 20% accuracy in computing cross sections is desired, these effects would have to be included.

If we take the plane wave approximation in the charge exchange contribution, we can calculate the cross sections using the Table V. The results are given in Table XII.
**TABLE XII.**

The cross sections including virtual charge exchange.

<table>
<thead>
<tr>
<th>$q_L (\text{MeV})$</th>
<th>Sol.</th>
<th>$f_{\pi^0}(t)$</th>
<th>$\frac{d\sigma(0)}{dt}$</th>
<th>$\sigma_{el}$</th>
<th>$\sigma_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>105</td>
<td>I</td>
<td>-.28 + 1.15</td>
<td>14.0</td>
<td>92</td>
<td>342</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>.82 + 1.44</td>
<td>27.5</td>
<td>180</td>
<td>428</td>
</tr>
<tr>
<td>194</td>
<td>I</td>
<td>-.01 + 1.86</td>
<td>7.4</td>
<td>29</td>
<td>138</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>.21 + 1.26</td>
<td>9.7</td>
<td>38</td>
<td>154</td>
</tr>
</tbody>
</table>
C. Break up Cross Sections and Comparison with the Experimental Results. A crude estimation of the absorption cross section of the $K^-$-nucleon is made and the results are given in Tables XIII. We shall take

$$\sigma_{\alpha\ell} \sim \sigma_{p\ell} + \sigma_{n\ell}$$

where

$$\sigma_{p\ell} = \sigma_{\text{tot}} - \sigma_{p\ell}$$

and

$$\sigma_{n\ell} = \sigma_{\text{tot}} - \sigma_{n\ell}$$

Using the Table XII, the results for $(\sigma_T - \sigma_{\alpha\ell})$ are also given in Table XIII.

**TABLE XIII.**

Values of $(\sigma_T - \sigma_{\alpha\ell})$.

<table>
<thead>
<tr>
<th>$q_L(N_0)$</th>
<th>Sol.</th>
<th>$\sigma_{p\ell}$</th>
<th>$\sigma_{n\ell}$</th>
<th>$\sigma_{\alpha\ell}$</th>
<th>$\sigma_T - \sigma_{\alpha\ell}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>105</td>
<td>I</td>
<td>181</td>
<td>106</td>
<td>287</td>
<td>55</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>155</td>
<td>125</td>
<td>280</td>
<td>148</td>
</tr>
<tr>
<td>194</td>
<td>I</td>
<td>67</td>
<td>48</td>
<td>115</td>
<td>23</td>
</tr>
<tr>
<td></td>
<td>II</td>
<td>66</td>
<td>44</td>
<td>110</td>
<td>44</td>
</tr>
</tbody>
</table>

We have

$$\sigma_T = \sigma_{\alpha\ell}(K^-d \rightarrow K^+d) + \sigma(K^-d \rightarrow K^+n+p) + \sigma(K^-d \rightarrow \bar{K}^+\pi^-\pi^+)$$

$$+ \sigma(\text{Hyperon Production})$$

Therefore

$$\sigma_T - \sigma_{\alpha\ell} = \sigma_{\alpha\ell}(K^-d \rightarrow K^+d) + \sigma(K^-d \rightarrow K^+n+p) + \sigma_{\alpha\ell}$$

(7.4)
The experimental data\(^{(23)}\) for the total cross sections for the reactions
\[ \kappa^+ + d \rightarrow \kappa^- + \Delta \]
and
\[ \kappa^- + d \rightarrow \kappa^- + \gamma + p \]
is given in Table XIV.

**TABLE XIV.**

The experimental data and the theoretical values for the total cross sections for the reactions \( \kappa^+ + d \rightarrow \kappa^- + \Delta \) and \( \kappa^- + d \rightarrow \kappa^- + \gamma + p \)

<table>
<thead>
<tr>
<th>( q_L ) (MeV)</th>
<th>( \sigma_{\text{exp.}} )</th>
<th>( \sigma^- - \sigma_{\Delta}^T )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Sol. I</td>
</tr>
<tr>
<td>105</td>
<td>mB</td>
<td>55</td>
</tr>
<tr>
<td>125</td>
<td>145 ± 35</td>
<td></td>
</tr>
<tr>
<td>175</td>
<td>55 ± 15</td>
<td></td>
</tr>
<tr>
<td>194</td>
<td>23</td>
<td></td>
</tr>
<tr>
<td>210</td>
<td>95 ± 25</td>
<td></td>
</tr>
</tbody>
</table>

---

D. **Conclusions.** A study of Table XIV shows that the Dalitz Solution II is more favorable compared to the Solution I.

It should be emphasized that some of the estimates made to get the comparison with the data have been extremely crude. They have been made only for illustration purposes. A more serious test and application of the model in question would require the computations for other angles than 0° of the elastic amplitude including a more careful estimate of the virtual charge exchange - and this could certainly be done by machine calculation - as well as a direct computation of the break-up cross section 

\[ \sigma(k^+ d \rightarrow k^+ n + p) \]

within the framework of this model. Again, this seems feasible and the results obtained in this thesis would seem to justify further work along this line. Our conclusion that Solution II is favored by the deuterium data is the same as that reached recently by Chand\(^{(24)}\) and Dalitz who have studied the same problem by an entirely different method which is, however, closely related to the Brueckner model, binding and recoil corrections being neglected. Our results (Table VII) for the double scattering part of the amplitude in a model in which such corrections\(^{(25)}\) have been included do not support the conjecture of Chand that corrections of this kind are unlikely to be important.

\(^{(24)}\) Private communication.

\(^{(25)}\) R. Chand: Thesis on \(K^-\)-Deuterium Interactions - University of Chicago.
Appendix

Evaluation of the integral (3.89)

We shall evaluate the integral

\[ I = \int_0^\infty \frac{dy}{(y + \delta - \zeta) + \xi_y} \]

Since

\[ \frac{1}{(y + \delta - \zeta)^2 + \xi_y^2} = \frac{1}{(y + \delta - \zeta - i\xi_y)(y + \delta - \zeta + i\xi_y)} \]

\[ = \frac{1}{\xi_y} \left[ \frac{1}{y + \delta - \zeta - i\xi_y} - \frac{1}{y + \delta - \zeta + i\xi_y} \right] \]

Therefore, we get

\[ \int_0^\infty \frac{dy}{y + \delta - \zeta - i(1 + \xi_y)} = \int_0^\infty \frac{y + \delta + \zeta(1 + \xi_y)}{(y + \delta)^2 + (1 + \xi_y)^2} \]

\[ = \left[ \log \left| \frac{y + \delta}{1 + \xi_y} \right| \right]_0^\infty + \zeta \left( \frac{1 + \xi_y}{1 + \xi_y} \right)^{-1} \left| \frac{y + \delta}{1 + \xi_y} \right|_0^\infty \]

\[ = \left[ \log \left| \frac{y + \delta}{1 + \xi_y} \right| \right]_0^\infty + \zeta \left( \frac{1}{1 + \xi_y} \right) - \frac{1}{\xi_y} \log \left| \frac{y + \delta}{1 + \xi_y} \right|_0^\infty \]

\[ = \left[ \log \left( \frac{1 + \xi_y}{y + \delta} \right) \right]_0^\infty + \zeta \left( \frac{\pi}{1 + \xi_y} - \frac{1}{\xi_y} \right) \left( \frac{1 + \xi_y}{y + \delta} \right) \]  \hspace{1cm} \text{(A.1)}
For $l < 2$, we get

\[
\int_0^\infty \frac{dy}{(y^2 + \delta^2 + \alpha + \beta + \gamma + \lambda)} = \int_0^\infty \frac{(y^2 + \delta^2) + i((\alpha + \beta) + \gamma + \lambda)}{(y^2 + \delta^2 + \alpha + \beta + \gamma + \lambda)} \, dy
\]

\[
= \left[ \frac{k}{2} \log \left| (y^2 + \delta^2) + (\alpha + \beta) + \gamma + \lambda \right| \right]_0^\infty + \imath \left( \frac{\alpha + \beta}{\alpha} \right) \log \left| \frac{y^2 + \delta^2}{y^2 + \delta^2 + \alpha + \beta + \gamma + \lambda} \right|_0^\infty
\]

From Eqs. (A.1) and (A.2), we get for $l < 2$

\[
\int_0^\infty \frac{dy}{(y^2 + \delta^2 + \alpha)^l} = \frac{i}{\imath l} \left[ -\frac{k}{2} \log \left| (y^2 + \delta^2) + \gamma + \lambda \right| \right] + \imath \left( \frac{\alpha}{\alpha + \beta} \right) \log \left| \frac{y^2 + \delta^2}{y^2 + \delta^2 + \alpha + \beta + \gamma + \lambda} \right| + \frac{T(\gamma, \lambda)}{\alpha} \quad (3.89)
\]

For $l > 2$, we get

\[
\int_0^\infty \frac{dy}{(y^2 + \delta^2 + \alpha - (\alpha + \beta) - \gamma - \lambda)} = \int_0^\infty \frac{(y^2 + \delta^2) - i((\alpha + \beta) - \gamma - \lambda)}{(y^2 + \delta^2 + \alpha - (\alpha + \beta) - \gamma - \lambda)} \, dy
\]

\[
= \left[ \frac{k}{2} \log \left| (y^2 + \delta^2) + (\alpha + \beta) + \gamma + \lambda \right| \right]_0^\infty + \imath \left( \frac{(\alpha + \beta) - \gamma - \lambda}{\alpha - \beta} \right) \log \left| \frac{y^2 + \delta^2}{y^2 + \delta^2 + \alpha - (\alpha + \beta) - \gamma - \lambda} \right|_0^\infty
\]

\[
= \left[ \frac{k}{2} \log \left| (y^2 + \delta^2) + (\alpha + \beta) + \gamma + \lambda \right| \right]_0^\infty - \imath \left( \frac{(\alpha + \beta) - \gamma - \lambda}{\alpha - \beta} \right) \log \left| \frac{y^2 + \delta^2}{y^2 + \delta^2 + \alpha - (\alpha + \beta) - \gamma - \lambda} \right|_0^\infty \quad (A.3)
\]
From Eqs. (A.1) and (A.3), we get for $l > 2$

\[
\int_{0}^{\infty} \frac{dy}{(y + \delta - \xi)^{2} + \frac{\delta^2}{\xi}} = \frac{\nu}{2\pi} \left[ \frac{\log \left( \frac{(1 - \xi \alpha)^{2} + \delta^{2}}{(1 - \xi \beta)^{2} + \delta^{2}} \right)}{\xi \alpha - \xi \beta} \right] + \frac{T(\xi, \xi)}{\lambda}
\]

where $T(\xi, \xi)$ is defined in Eqs. (3.90a) and (3.90b)
Selected Bibliography

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R. H. Dalitz and S. F. Tuan, Annals of Physics, 5, 100 (1959)
L. H. Schick, Rev. of Modern Physics, 33, 608 (1961)