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A Bibliography on Functional Equations.

First Version

Annual Summary Report No. 1 - Contract ~~No.~~

AF 61 (052) - 602 ~~30.4.1963~~

György I. Targonski,
Institut de Physique Théorique de l'Université
de Genève

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A Bibliography on Functional Equations.

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Introduction

This is a bibliography on the literature of functional equations, up to approximately the middle of 1962. It consists of two parts - a list of papers and books up to the end of 1945, and a similar list from 1946 on, with remarks on the contents.

This sub-division is due to the fact that the author was not able to devote more than a fraction of his time to this work. It is planned that the first part shall be re-edited with commentaries similar to the second, and finally the whole bibliography shall be streamlined into a single.

In order to compile such a bibliography, it was necessary first to define a Functional Equation. This has been done in the past in several ways, some of which appear too wide, others too narrow.

In the most general case, a Functional Equation is an equation which serves to determine one, or more unknown functions, or classes of such functions. In this sense, every differential,-difference, and integral equation is also a Functional Equation ; to compile a bibliography on such a wide class is almost certainly impossible and most certainly unnecessary.

Differential-difference - and integral equations have been described in a very large number of excellent textbooks, easily accessible to everyone interested ; they also have a vast literature of publications, growing at a speed difficult even to follow. Let us then describe as "Functional equations proper" those functional equations which are neither differential, nor difference, nor integral equations, nor a mixture of those, nor do they contain at all differential-, difference or integral operators. This is the present author's definition of a functional equation ; a "historical" definition rather than an axiomatic one, and a definition he should be most eager to see replaced by a better one. It should be mentioned here that narrower definitions exist, which have an axiomatic character ; the reader is referred to the remarks on Aczél's book, one of the very few ever to be published on functional equations. This narrower definition is well-founded, but one still feels that it is descriptive rather than essential, and the definition mentioned above rules out the famous Abel equation,

(1) $$f\left[\varphi(x)\right] = f(x) + 1$$

the first functional equation to become known and probably the most important of them all. The reason for this exclusion is interesting. The current "axiomatic" definition mentioned above tends to rule out functional equations in which the number of variables is not higher than the number of variables in the unknown function. The reason is that such equations are rather difficult to solve, and their solution requires quite different methods. Let us, in contrast to (1), see a functional equation which is "admissible" :

(2) $$f(x+y) = F\left[f(x), f(y)\right]$$

i.e. a so-called addition theorem for the unknown function $f(x)$. Here, the number of variables in the equation is 2, while the unknown function is of one variable. We have, so to speak, one degree of freedom we can utilize in many ways; we can put e.g. $x=y$, $y=0$, $y=-x$ and obtain from (2) three equations:

$$(3) \qquad (a) \quad f(2x) = F\left[f(x), f(x)\right]$$

$$(b) \quad f(x) = F\left[f(x), f(0)\right]$$

$$(c) \quad f(0) = F\left[f(x), f(-x)\right]$$

which can also be combined among themselves, e.g. (3) (b) and (3) (c); all of them are separately equations of the type of (1); it is no wonder that literature on the equations of the type of (2) is overwhelmingly larger than that of type (1) and that some definitions, as said, tend to rule out type (1); this latter, however, is the more fundamental and, in the long run, more important; in any case, they are included in our "historical definition" and thus in this Bibliography. We excluded systems of functional equations, a limitation one has to admit to be a little arbitrary. Equally arbitrary was the way the line had to be drawn towards the fields of mathematics bordering on that of functional equations. Let us describe the way the line was drawn in each case.

The most close ties link the theory of functional equations to iteration theory; in fact, it is unnatural and even impossible to consider them as two different theories. Iteration methods are indispendable in solving equations of type (1); on the other hand, for the solution of the problem of a bitrary iteration index, one of the most important problems of iteration theory, this has to rely on the "translation equation":

$$(4) \qquad F\left[F(x, v), u\right] = F\left(x, u+v\right)$$

which, in its turn, leads to Abel's equation (1).

Still, in practice, it was not too difficult to find a dividing line; clearly practically all numerical approximation methods are iteration methods, the practical value of which has been immensely increased since digital electronic computers are available; there was no question of including any part of the vast literature of those methods. Iteration theory was included in those cases where functional equations are involved in an essential and explicite way.

Another field intimately related to functional equations is that of calculability and nomographability. To quote a simple example: in order to calculate a function $w = f(x, y, z)$ by a nomogram of points in alignment, one needs first a "simple" nomogram consisting of two curves, linking x to y, and then a second such nomogram to link the result to z. A necessary condition for the nomographability is thus that solutions g, h should exist to the functional equation

$$(5) \qquad f(x, y, z) = g\left[h(x, y), z\right]$$

Such topics have been included but only if the nomographic side is not predominant. It is planned to add to the final version of this bibliography a special list of works on what one could name "theoretical nomography".

As said, equations containing differential operators were excluded, so was the theory of geometrical objects, which forms a separate branch of mathematics. On the borderline between functional equations and algebra, the functional equations of distributivity, associativity etc..., had to be included ; in fact, the contributions of Abel to this subject were among the first results of the theory. Deeper-going investigations however, such as the theory of continuous groups, form a domain by themselves and had to be omitted. Similar consideration prompted us to leave out the theory of stochastic processes, firmly rooted in probability theory ; one or two papers related to the common generalization of exponential and Poisson distributions were included due to their exclusive reliance on the appropriate functional equation.

Number theoretical functions, strongly multiplicative functions etc..., were excluded on the ground that these are "functions" of positive integers only ; again we are facing an established part of number theory. This is in line with our tendency to collect the "floating" material of the functional equations, not firmly or exclusively attached to any established branch of mathematics, and- as one hopes- to be organized into one of the most interesting branches of our science.

Functional inequalities do not, as a matter of fact, enter this bibliography ; due to their very general character, they determine properties of functions rather than functions or even "classes" of them.

Thus we have drawn the borderline separating the subject of this bibliography from other branches of mathematics ; even this brief survey shows the central position of this discipline within mathematics - a view, one may add, perhaps correct only seen with the eyes of those interested mainly in functional equations !

The comment on the paper, given in part II , does not tend to describe in detail the contents, this should have multiplied by a factor of ten the size of this bibliography. The aim was not even to describe the main theorem, but rather to inform the reader about the problem which is solved, or treated in the paper in question, and, more often than not, to give the equation which is solved. This, it is hoped, shall be useful not so much to the mathematician, but to the physicist etc..., facing a functional equation and trying to locate literature on it.

On last remark : this bibliography does not pretend to be complete ; the author shall be grateful to colleagues pointing out errors or omissions, he shall equally welcome lists of publications, reprints, and other material which may render the final version of this bibliography more useful.

Some technical remarks.

The notation FE (s) stands for Functional Equation (s).

The title of a book is given in the language in which it is written, and an English translation is added. The title of a paper is always given in English, and an abbreviation points out the language in which it is written.

The abbreviations are the following :

C	Czech	I	Italian
D	Dutch	J	Japanese
Da	Danish	P	Polish
E	English	Po	Portuguese
Es	Esperanto	R	Romanian
F	French	Ru	Russian
G	German	S	Spanish
H	Hungarian	Se	Serb, Croat

The books and papers are given in the alphabetical order of the author, and, for one author, in the order of publication.

Within the alphabetical order, ß has been taken as ss, etc...
For languages using an alphabet other than the latin, the usual phonetical transcription has been used, in case of doubt, the name figures in both forms, one with a reference to the other form, e.g. "Wilner, see Vilner".

In the case of papers written by several authors, the paper figures under the name of the author first in alphabetical order, in the rare case, however, where the alphabetical order was not followed in the title of the paper, the name figures under the name of the first author, irrespective of alphabetical order. The name of the other authors figures of course also in the list, with a reference to the first.

Some papers were added to the bibliography at the last moment and for these there is no comment, only the reference, they are denoted by an *.

Part I

A list of publications until the end of the year 1945

Abel, N.H.

A general method to find a function of one variable if a property of this function is expressed by an equation in two variables (F).
Mag. Naturvidenskab. 1 (1823) reprinted in Oeuvres (Ed. Sylow et Lie) I, 1-10.

Determination of a function by means of an equation which contains one variable only (F).
Oeuvres complètes II (1824) 36-39.

Investigation of functions of two independent variables x and y such that $f(x,y)$ has the property that $f[Z, f(x,y)]$ is a symmetric function of z,x and y (G).
J. f. Math. 1, 1 (1826) Oeuvres (Ed. Sylow et Lie) 1, 61-65.

Investigations on the series $1 + \frac{m}{1}x + \frac{m(m-1)}{2}x^2$. (G.
Oeuvres Complètes I (1826) 219-250.

On functions satisfying the equation $\varphi x + \varphi y = \psi(xfy + yfx)$. (G)
Oeuvres complètes I (1827) 389-398.

Alaci, V.

Pseudo-homogenous functions (R).
Rev. mat. Timisoara 3, no. 1 (1923) 3-4.

Pseudo-homogenous functions and a new class of differential and partial differential equations (F).
Bull. sci. Ecole Polyt. Timisoara 11 (1943-1944) 6-13.

The analytic solution of a functional system (F).
Bull. sci Ecole Polyt. Timisoara 11 (1943-1944) 174-178.

On two FEs (F).
Mathematica Cluj Timisoara 19 (1943) 23-25.

Alexiewicz, A. and Orlicz, W.

Remarks on the FE $f(x+y) = f(x) + f(y)$ (F)
Fundamenta math. 13 (1945) 314-315.

Alt, W.

On real functions of one real variable possessing a rational addition theorem (G).
Deutsche Math. 5 (1940) 1-12.

Amaldi, V.

(See Pincherle, S.)

Andrade, J.

On Poisson's FE (F).
Bull. Soc. math. France 28 (1900) 59-63

Andreoli, G.

On a simple and well-known FE (I)
R.C. Accad. Napoli (3) 29 (1923) 12-14.

Angelesco, A.

On a functional property of conics (F).
C.R. Paris 175 (1922) 666-668.

On a functional property common to the circle and the logarithmic spiral (R).
Gaz. mat. Buc. 29 (1924) 364-368.

On a FE
(R).
Gaz. mat. Buc. 32 (1927) 281-286.

Anghelutza, Th.

On a FE characterizing the polynomials (F).
Bul. Soc. Sti Cluj 6, (1931) 139-145.

On a FE
(F).
C.R. Paris 194 (1932) 420-422.

On the integration of a FE
(F).
Mathematica Cluj 10 (1935) 99-116.

On a FE defining polynomials in several variables (F).
Bull. Sci. math. (2) 61 (1937) 357-360.

On a FE
(F).
Bull. sci. Ecole Polyt. Timisoara 11 (1943-1944) 42-44.

Circular transformations characterized by a FE. (R).
Gaz. mat. Buc. 51 (1945) 94-98.

Appel, P.

Forming a function possessing the property
$F[\varphi(x)] = F(x)$. (F).
C.R. Acad. Sci. 88 (1879) 807-810.

On functions such that $F(\sin \pi/2 \cdot x) = F(x)$. (F).
C.R. Acad. Sci. 88 (1879) 1022-1024.

On linear differential equations the integrals of which satisfy relation of the form $F[\varphi(x)] = \Psi(x)\ F(x)$. (F).
C.R. Acad. Sci. 93 (1881) 699-701.

Appell, P.

On linear differential equations which can be transformed into themselves by a change of function and variable (F).
Acta Math. 15 (1891) 231-315.

On the integral $\int_{x_0}^{y} f(y)\, d\, f(x)$ where x and y are symmetrically related (F).
Acta math. 44 (1923) 213-215.

Aumann, G. — Constructing mean values of several variables II (G). Math. Ann. 111 (1935) 713-730.

Baer, R. — A Theory of Crossed Characters. (E). Trans. Amer. math. Soc. 54 (1943) 103-170.

Babbage, Ch. — Algebraical analysis of FEs (F). Ann. de mat. pur appl. 12 (1821/22) 73-103.

Ballantine, J.P. — On a Certain Functional Condition (E). Bull. Amer. math. Soc. 32 (1926) 153-155.

Banach, S. — On the FE $f(x + y) = f(x) + f(y)$. (F). Fundamenta math. 1 (1920) 123-124.

Banach, S., Ruziewicz, S. — On the solutions of a FE by J.Cl. Maxwell (F). Bull. Acad. polon. Sci. (A) (1922) 1-8.

Beetle, R.D. — On the Complete Independence of Schimmack's Postulates for the Arithmetic Mean (E). Math. Ann. 76 (1915) 444-446.

Behrbohm, H. — On the algebraicity of the meromorphisms of an elliptic function field (G). Nachr. Ges. Wiss. Göttingen (2) 1 (1934-1940) 131-134.

Bell, E.T. — A Partial Isomorph of Trogonometry (E). Bull. Amer. math. Soc. 25 (1918-1919) 311-321.

Algebraic Arithmetic (E). Colloquium Publications of the American Mathematical Society 7. (1927) New York.

Possible Types of Multiplication Series (E). Amer. math. Monthly 37 (1930) 484-485.

FEs of Totients (E). Bull. Amer. Math. Soc. 37 (1931) 14.

Distributivity of Associative Polynomial Compositions (E). Ann. Math. 37 (1936) 368-373.

A FE in Arithmetic (E). Trans. Amer. math. Soc. 39 (1936) 341-344.

Bemporad, G. — On the arithmetic mean (I). R.C. Accad. Lincei (6) 3 (1926) 87-91.

The significance of the arithmetic mean (I). R.C. Accad. Lincei (6) 11 (1930) 789-794.

Bernstein, B.A. — Postulates for Abelian Groups and Fields in Terms of Non-associative Operations (E). Trans. Amer. math. Soc. 43 (1938) 1-6.

Bieberbach, L. Remarks on the Thirteenth Problem of Hilbert (G).
J.reine angew. Math. 165 (1931) 89-92.

Addendum to the paper "Remarks on the Thirteenth Problem of Hilbert" (G).
J. reine angew. Math. 170 (1937) 242.

Blumberg, H. Non-measurable Functions Connected with Certain Functional Equations (E).
Ann. Math. (2) 27 (1926) 199-208.

Bohnenblust, A. An axiomatic Characterization of L_p- Spaces (E).
Duke math. J. 6 (1940) 627-640.

Borel, E., Deltheil, R., and Frattini, G. A problem of extension by isomorphism in the theory of relativity (I).
Atti. Accad. Nuovi Lincei 76 (1923) 94-98.

Bouriet, C. On operations in general and on linear differential equations of infinite order (F).
Ann. sci. Ecole norm. sup. (3) 14 (1897) 133-150.

On certain equations analogous to differential equations (With a remark by P. Appel) (F).
C.R. Acad. Sci. 124 (1897) 1431-1434.

On the problem of iteration (F).
Ann. Fac. Sci. Toulouse (1) 12 no.3 (1898) 1-12.

Broggi, U. On the principle of arithmetic means (F).
Enseign. math 11 (1909) 14-17.

Brouwer, L. E. J. The theory of finite continuous groups independent of the Lie axioms (G).
Math. Ann. 67 (1909) 246-267.

Burstin, C. On a special class of real periodic functions (G).
Mh. Math. Phys. 26 (1915) 229-262.

Burstin, C., and Mayer, W. Distributive Groups (G).
J. reine angew. Math 160 (1929) 111-130.

Burstin, C. A contribution to the theory of functions of two variables (G).
Tôhoku math. J. 31 (1929) 300-311.

Caccioppoli, R. On the FE $f(x+y) = f(x) + f(y)$. (I).
Boll. Unione mat. Ital. 5 (1926) 227-228.

The FE $f(x+y) = F\left[f(x), f(y)\right]$ (I).
Giorn. Mat. Battaglini 66 (1928) 69-74.

Cantor, M. FEs with three independent variables (G).
Z. Math. Phys. 41 (1896) 161-163.

Carmichael, R.D. — On certain FEs (E).
Amer. math. Monthly 16 (1909) 180-183.

A Generalization of Cauchy's FE. (E).
Bull. Amer. math. Soc. 18 (1911-1912) 164.

della Casa, L. — Relations of heterogeneous quantities (I).
Atti. Accad. Torino 51 (1915-1916) 1175-1193.

Cayley, A. — On a Theorem of Abel, Note (F).
Collected Math. Papers, IV (1857) 5-6.

On a FE.
(E).
Quart. J. pure appl. math. 15 (1878) 319-325.
Reprinted in Coll. math. papers X, 298-306.

Certaine, J. — The Ternary Operation $(abc) = ab^{-1}c$ of a Group (E).
Bull. Amer. math. Soc. 49 (1943) 869-877.

Chini, M. — On a FE which gives rise to two remarkable formulae of finance mathematics
(I).
Period. Mat. (3) 4 (1907) 254-270.

Cioranescu, N. — On the functional definition of polynomials and on some "two and three level" formulae (F).
Bull. math. Soc. Roum. Sci. (1933) 33-34, 39-47.

Some FEs characterizing the linear function (F).
Bull. Sec. sci. Acad. Roum. 15 (1932-1933) 87-92.

Colucci, A. — On the FE $f(x+y) = f(x) + f(y)$. (I).
Giorn. Mat. Battaglini 64 (1926) 222-223.

van der Corput, J. G. — Goniometric functions characterized by a FE (D).
Euclides 17 (1940) 55-75.

Cremer, H. — On Schröder's FE and the Schwarz problem of mapping "corners".
Ber. Math. Phys. Klasse der Sächs. Akad. Leipzig (13.6.1932)
84 (1932).

Darboux, G. — On the fundamental theorem of projective geometry (F).
Math. Ann. 17 (1880) 55-61.

Deltheil, R. — (see Borel, E.)

Deslisle, A. — Determination of the most general function satisfying the FE of the σ function (G).
Math. Ann. 30 (1887) 91-119.

Dickson, L.E. An extension of the Theory of Numbers by Means of Correspondences between Fields (E).
Bull. Amer. math. Soc. 23 (1916-1917) 109-111.

Homogenous Polynomials with a Multiplication Theorem (E).
C.R. Congrès. Int. de Math. Strasbourg (1920) 215-230.

Composition of polynomials (F).
C.R. Paris 172 (1921) 636-640.

Dienes, P. Reality and Mathematics (M).
Budapest 1914.

Dodd, E.L. The Chief Characteristic of Statistical Means (E).
Colorado College Publ. 21 (1936) 89-92.

Some Elementary Means and Their Properties (E).
Colorado College Publ. 21 (1936) 85-89.

Ermolowa, O.W. On the separation of variables in an equation of any number of variables (Ru).
Uchenie zapiski Mosk. Gos. Univ. nom. 28 (1939) 43-54.

Falk, M. On the principal properties of analytic functions of one variable possessing addition theorems (G).
Nova Acta Soc. Sci. Upsal. (4) 1_2 No. 8 (1907) 1-78.

Farkas, J. On iterative functions (F).
J. de Math. 10 (1884) 101-108.

Fatou, P. On the uniform solutions of certain FEs (F).
C.R. Acad. Sci. 143 (1906) 546-548.

On rational substitutions (F).
C.R. Acad. Sci. 165 (1917) 992-995.

On FEs and the Properties of certain boundaries (F).
C.R. Acad. Sci. 166 (1918) 204-206.

On FEs (second note) (F).
Bull. Soc. Math. France 48 (1920) 33-94.

Favre, A. On homogenous functions (F).
Nouv. Ann. Math. (4) 17 (1917) 426-428.

de Finetti, B. On the notion of the mean (I).
Giorn. Ist. Ital. Attuarii 2 (1931) 369-396.

Formenti, C. On problems of Abel (I).
Reale Ist. Lomb. Rend. (2) 8 (1875) 276-282.

Franklin, P. Two FEs with Integral Arguments (E).
Amer. math. Monthly 38 (1931) 154–157.

Frattini, G. (See Borel, E.)

Fréchet, M. A Functional Definition of Polynomials (F).
Nouv. Ann. Math. (4) 9 (1909) 145–162.

Every continuous functional can be developed into a series of functionals of integral order (F).
C.R. Paris 148 (1909) 155–156.

On the FE $f(x+y) = f(x) + f(y)$. (Es).
Enseign. math. 15 (1913) 390–393.

On an article about the FE
$f(x+y) = f(x) + f(y)$. (F).
Enseign. math. 16 (1914) 136.

Fréchet, M., and Roullet, H. (Nomography).(F).
Nomographie
Ed. Librairie Armand Colin, Paris (1928).

Fréchet, M. On the most general continuous solution of a FE in the theory of probability chains (F).
C.R. Paris 195 (1932) 639–641.

The most general continuous solution of a FE in the theory of probability chains (F).
Bull. Soc. Math. France 60 (1932) 242–277.

The most general continuous solution of a FE in the theory of probability chains. Supplement (F).
Bull. Soc. Math. France 61 (1933) 182–185.

Fridman, A.A. On the question of the proof of the parallelogram of forces (G).
Journal Soc. phys.-math. Univ. Permi 1 (1922) 33–43.

Galbura, G. On a certain functional equation .(I)

Acta Pont. Acad. Sci. 5 (1941) 7–41.

Gerretsen, J.C.H. Characterization of the goniometric functions by means of a FE (D).
Euclides 16 (1939) 92–99.

Ghermanescu, M. On some FEs of M.D. Pompeiu (F).
Bull. Sec. Sci. Acad. Roum. 26 (1943–1944) 582–585.

On a FE characterizing the polynomials (F).
Mathematica Cluj-Timisoara 19 (1943) 148–158.

(Ghermanescu, M.) On some FEs (F).
Bull. sci. Ecole Polyt. Timisoara 11 (1943-1944) 181-184.

On a FE
(F).
Bull. Sec. sci. Acad. Roum 26 (1943-1944) 79-82.

On a functional property common to circle and logarithmic spirals (R).
Gaz. mat. Buc. 50 (1945) 246-249.

On some extensions of the FE of Cauchy (F).
Bull. Sec. sci. Acad. Roum. 28 (1945) 197-200.

Golab, S. On homogenous functions I. The equation of Euler (F).
C.R. Soc. Sci. Varsovie Cl. III 25 (1932) 105-110.

On a FE in the theory of geometrical objects (G).
Wiadomosci mat. 45 (1939) 97-137.

Grant, I.D. Doubly Homogenous Functional Equations (E).
Amer. math. Monthly 3 (1929) 257-273.

Grévy, A. Study on the FEs (F).
Ann. Ec. Norm. (3) 11 (1894) 249-323.

Gronwall, T.H. On the equations in three variables which can be represented by point nomographs (F).
J. Math. pures appl. (6) 8 (1912) 59-112.

A FE in the Kinetic Theory of Gases (E).
Ann. Math. (2) 17 (1915) 1-4.

Hadamard, J. Two Works on Iteration and Related Questions (E).
Bull. Amer. math. Soc. 50 (1944) 67-75.

Halphen, G H. On certain series for the development of functions of one variable (F).
C.R. Acad. Sci. 93 (1881) 781-783.

On homogenous functions (F).
Rev. Math. spéc. 21 (1911) 130-131.

Hamel, G. A base of all numbers and the non-continuous solutions of the FE $f(x+y) = f(x) + f(y)$ (G).
Math. Ann. 60 (1905) 459-462.

Hantzsche, W., and Wendt, H. On the arithmetic, geometric and harmonic means (G).
Unterrichtsbl. Math. Naturw. 42 (1937) 22-25.

Hartman, P., and Kershner, R. — The Structure of Monotonic Functions (E). Amer. J. Math. 59 (1937) 809-822.

Haruki, H — On a Certain Simultaneous FE Concerning the Elliptic Functions (E). Proc. phys.-math. Soc. Japan (3) 24 (1942) 450-454.

Haupt, O. — On the Theory of the exponential function and the trigonometric functions (G). Sitz. Ber. med. Soz. Erlangen 60 (1928) 155-160.

Hayashi, T. — On a FE Treated by Abel (E). Tokyo sugaku-butsu. (1)8(1[illegible]) 129-134.

On a FE Treated by Abel (E). Z. Math. Phys. 44 (1899) 346-349.

On solution of FEs (E). Tôhoku math. J. 3 (1913) 62-63.

Hecke, E. — A new kind of zeta functions and their relation to the distribution of prime numbers I (G). Math. Z. 1 (1918) 357-376.

A new kind of zeta functions and their relation to the distribution of prime numbers II (G). Math. Z. 6 (1920) 11-51.

Herbrand, J. — Investigation of bounded solutions of certain FEs (F). C.R. Paris 189 (1929) 669-671.

Herschfeld, A. — On Ball's Functional Equations (E). Amer. math. Monthly 38 (1931) 395-396.

Hille, E. — A Pythagorean Functional Equation (E). Ann. Math (2) 24 (1923) 174-180.

A Class of FEs (E). Ann. Math (2) 29 (1928) 215-222.

Hostinsky, B. — (FE related to probability chains) (F) Paris 1939.

Huntington, E.V. — Sets of Independent Postulates for the Arithmetic Mean, the Geometric Mean, the Harmonic Mean and the Root-Mean- Square (E). Trans. Amer. math. Soc. 29 (1927) 1-22.

Hurwitz, W.A. — Note on the Definition of an Abelian Group by Independent Postulates (E). Ann. Math. (2) 8 (1906) 94-96.

Iaggi, E

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Tietzke, H.	On FEs the solutions of which cannot satisfy an algebraic differential equation (Transcendental-transcendent functions) (G). Monatsh. Math. Phys. 16 (1905) 329-364.
Toda, K.	On Systems of Simultaneous Functional Equations (E). J. sci. Hyrosh. University (A) 6 (1936) 69-86.
Toyoda, K.	On Axioms of Linear Functions (E). Proc. Imp. Ac. Japan 17 (1941) 221-227.
Vaidyanathaswamy, R.	The Theory of Multiplicative Arithmetic Functions (E). Bull. Amer. math Soc. 37 (1931) 342.
	The Theory of Multiplicative Arithmetic Functions (E). Trans. Amer. math. Soc. 33 (1931) 579-662.
Veress, P.	On the notion of mean (H). Mat. fiz lapok 43 (1936) 46-60.
Vilner, I.A.	On the nomograms of systems of equations and of analytic functions (F). Akad. Nauk SSSR prikl. Mat. Mech. 4 (1940) 105-116.
van Vleck, E.B.	A FE for the Sine (E). Ann. Math. (2) 11 (1910) 161-165.
	A FE for the Sine (E). Ann. Math. (2) 13 (1913) 154.
van Vleck, E.B., and Doubler, F.H.	A Study of Certain Functional Equations for the ϑ-Functions (E). Trans. Amer. amth. Soc. 17 (1916) 9-49.
Vivanti, G.	A remark on functionals admitting an addition theorem (I). Boll. Unione mat. Ital. 14 (1935) 244-246.
Volpi, R.	Remarks on a purely analytical and elementary theory of trigonometric and hyperbolic functions and their relation with the exponential functions (I). Giorn. Mat. Battaglini 41 (1903) 33-46.

Ward, M., and Fuller, F.B.	The Continuous Iterations of Real Functions (E). Bull. Amer. math. Soc. 42 (1936) 387-396.
Wellstein, J.	Two FEs. (G). Arch. Math. Phys. (3) 16 (1910) 93-100.
Wendt, H.	(See Hantzsche, W.)
Wiener, N.	The Isomorphisms of Complex Algebras (E). Bull. Amer. math. Soc. 27 (1921) 443-445.
Wilson, E.B.	Note on the Function Satisfying the Functional Relation $f(u)\ f(v) = f(u + v)$. (E). Ann. Math. (2) 1 (1899) 47-48.
Wilson, W.H.	On a Certain General Class of FEs (E) Bull. Amer. math. Soc. 23 (1916-1917) 392-393.
	On a Certain General Class of FEs (E). Amer. J. Math. 40 (1918) 263-282.
	On Certain Related FEs (E). Bull. Amer. math. Soc. 26 (1919-1920) 300-312.
	Two General FEs (E). Bull. Amer. math. Soc. 31 (1925) 323-324.
Yosida, K.	On the Groups Embedded in the Metrical Complete Ring (E). Japanese J. Math. 13 (1936) 7-26.

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Part II

A list of publications, with comments, from 1946 on.

Alaci, V.

On a class of FEs (R). *
Analele Acad. R.P. Rom. Sec. Sti. Mat. Fiz. Chim. (A) 3 (1950) 461-477.

Altman, M.

FEs involving a parameter (E)
Proc. amer. math. Soc 11 (1960) 54-61.

An iteration procedure based on Newton's method is established to solve the FE

$$F(x, \mu) = [illegible]$$

where both [illegible] and [illegible] are elements [illegible] Banach space. Convergence and speed of convergence of the iteration process are investigated.

An iterative method of solving FEs (E). *
Bull. Acad. Polon. Sci. Sér. Sci. math. astr. phys. 9 (1961) 57-62.

Iterative methods of higher order (E). *
Bull. Acad. polon. Sci. Sér. Sci. math. astr. phys. 9 (1961) 63-68.

A generalization of a Laguerre method for FEs (E). *
Bull. Acad. polon. Sci. Sér. Sci. math. astr. phys. 9 (1961) 581-586.
*

Anastassiadis, J.

On the solutions of the FE

$$f(x+1) = g(x) f(x) \qquad (F).$$

C.R. Acad. Sci. 252 (1961) 2414-2417.

The above FE is solved under the condition f(1) = 1 and some mild restrictions.

Angheluta, Th.

The FE of bisymmetry (R). *
Studia Univ. Babes-Bolyai mat. 3 (1958) 9-15.

On the FE of Translation (R).
Inst. Politehn. Cluj. Lucrari Sti. (1959) 29-31.

A proof if given for the fact already known that

$$z[z(x,y), t] = z[x, y+t]$$

is solved by

$$z(x,y) = F^{-1}[F(x)+y]$$

Remarks on the FE of Poisson (R).
Inst. Politehn. Cluj. Lucrari Sti. (1959) 33-39.

The author proves that every non-vanishing solution of the

Poisson FE $f(x+y) + f(x-y) = 2 f(x) f(y)$

satisfies the algebraic addition theorem

$$f(x+y)^2 - 2f(x)f(y)f(x+y) + f(x)^2 + f(y)^2 - 1 = 0$$

This is used to prove that every continuous solution of the original FE is analytic.

(Angheluta, Th.) C "E with three unknown f s (R). *
L . Stii Inst. politeh. Cluj. . 23-30.

Anghelutza, T. (See Angheluta, Th.)

Arrighi, G. On the FE (I) $2\varphi(x)\varphi(y) = \varphi(x+y) + \varphi(x-y)$

Boll. Unione mat. Ital. (3) 4 (1949) 255-257.

The solution of the equation, proved by Picard to be cos λ x and cosh λx , assuming continuity, is given, under the weak condition of right continuity.

Aczél, J. The Notion of Mean Values (E).
Norske Vid. Selsk. forhandlinger 19 (1946) 83-86.

The author defines a normal mean value $M(x_1, x_2, \cdots, x_n)$ by the properties : symmetry in the variables; $M(x, x, \cdots, x) = x$; M is monotone increasing in each variable, M is continuous function of the "vector" $[x_1, x_2, \cdots, x_n]$ finally $M[M(x_{11}, x_{12}, \cdots, x_{1n}), \cdots, M(x_{n1}, \cdots x_{nn})]$ is symmetric in all its n^2 variables ("bisymmetry") under these conditions the Kolmogoroff Nagumo theorem is proved :

$$M(x_1, \cdots, x_n) = F^{-1}\left[\frac{1}{n}\{F(x_1) + \cdots + F(x_n)\}\right]$$

where F is continuous and monotone.

On Mean Values and Operations Defined for two Variables (E).
Norske Vid. Selsk. forhandlinger 20 (1947) 37-40.

The validity of the Kolmogoroff-Nagumo theorem (see Aczél : The Notion of Mean Values) is proven, for the case of two variables, replacing the monotonity condition by the weaker condition

$x < m(x, y) < y$ for $x < y$

An analogue of the Kolmogoroff-Nagumo theorem is also given.

(Aczél, J.)

On a FE (F).
Publ. Inst. Math. Acad. Serbe Sci. 2 (1948) 257-262.

The "generalized addition theorem"
is treated. $f(ax+by+c) = \Phi(f(x), f(y))$

On Mean Values (E).
Bull. Amer. math. Soc. 54 (194[illegible]) [illegible]-400.

Let $M_1(x_1),\ M_2(x_1, x_2),\ \cdots,\ M_n(x_1, \cdots, x_n)$

be a sequence of mean value functions of 1,2, ; n variables; a necessary and sufficient condition is searched for under which a continuous and increasing function $f(x)$ exists, so that
$M_n(x_1, \cdots, x_n) = f^{-1}\left[\frac{f(x_1) + \cdots + f(x_n)}{n}\right]$
f being independent of n; the author proves that the bisymmetric condition: $M\left[M(x_{11}, \cdots, x_{1n}), \cdots, M(x_{n1}, \cdots, x_{nn})\right]$

invariant under the exchange $x_{ik} \to x_{ki}$
is necessary and sufficient.

On a class of FEs (G).
Comment. math. Helv. 21 (1948) 247-252.

The only continuous solution of
$f[\alpha_1 x_1 + \alpha_2 x_2] = \beta_1 f(x_1) + \beta_2 f(x_2) + \rho_1(x_1) + \rho_2(x_2) + \rho_0$
with
$\rho_1(0) = \rho_2(0) = 0$
is
$f(x) = ax + b$

On operations defined for real functions (F).
Bull. Soc. math. France 76 (1949) 59-64.

Let $f(x, y)$ be such that $a \leq f(x, y) \leq b$
provided $a \leq x, y \leq b$
Then an operation $x \circ y = f(x, y)$ is defined
; it is shown that
$x \circ y = f(x, y) = \varphi^{-1}[\varphi(x) + \varphi(y)]$
if and only the operation is monotone, continuous and associative.

Aczél, J., Kalmar, L. and Mikusinski, J. G.

On the translation equation (F).
Studia math. 12 (1951) 112-116.

The FE $f[f(x, u), v] = f(x, u+v)$
is dealt with. This is one of the most important FEs, since its

solution enables us to write iterated functions of arbitrary index, using the using the notation $f_n(x) = f(x, n)$

the FE expresses the relation

$$f_n[f_m(x)] = f_{n+m}(x)$$

where $f_n(x)$ is defined by $f_n(x) = f[f_{n-1}(x)]$; $f_0(x) = x$

The authors prove the existence of solutions under various assumtions. Especially, under some monotony assumptions,

$$f(x, u) = \omega^{-1}[\omega(x) + u]$$

using this formula, the generalized ν-th iterated function of f is given for arbitrary real ν by

$$f_\nu(x) = \omega^{-1}[\omega(x) + \nu]$$

Aczél, J.

FEs in applied mathematics (H). *
M. Tud. Akad. III Osztály közleményei 1 (1951) 131-142.

On FEs in several variables I. Elementary solution methods for FEs in several variables (H).
Matlapok 2 (1951) 99-117.

The continuous, increasing solutions of the "mean function" equation

$$m[m(x,y), m(\xi,\eta)] = m[m(x,\xi), m(y,\eta)]$$

are

$$m(x,y) = f^{-1}[p_1 f(x) + p_2 f(y) + p]$$

while the continuous and increasing solutions of

$$F[F(x,y), z] = F[x, F(y,z)]$$

are

$$F(x,y) = f^{-1}[f(x) + f(y)]$$

Some FEs in connection with the theory of continuous groups (H). *
Az Első Magyar Matematikai Kongresszus közleményei 1950 (Budapest 1952) 565-569.

On Composed Poisson-Distributions III (E).
Acta math. Acad. Sci. Hung. 3 (1952) 219-224.

A FE is set up for the probability distribution of the event that exactly k events occur in the time interval $[t_1, t_2]$ the assumption is made that the number of events in two non-overlapping time intervals are independent. The solution is constructed by induction, it contains the distributions of exponential decay and the Poisson distribution as special cases.

Reduction of FEs of several variables to the solution of partial differential euqaton

(Aczél, J.)

Reduction of FEs of several variables to the solution of partial differential equations. Application to nomography (H). *
M. Tud. Akad. Alk. Mat. Int. Közlemények 1 (1952) 311-333.

On the theory of means (H).
Acta Univ. Debrecen 1 (1954) 117-135.

A review article on results in this field since 1930 ; a new result is the most general strictly monotone and twice differentiable solution $M(x,y) = f^{-1}[pf(x)+qf(y)]$
of the quasidistributivity equation
$M[M(x,y),z] = M[M(x,z), M(y,z)]$

On the theory of means (G)
Acta Univ. Debrecen 1 (1954) [illegible], 18

On the theory of means (KU)
Colloquium math. 4 (1956-1957) 33-35

Translation into Russian of the review article, see above.

Outlines of a general treatment of some FEs (G).
Publ. math. Debrecen 3 (1953-1954) 119-132.

The classes considered contain certain well-known and important FEs as special cases ; thus the "addition theorem"
$f(x+y) = F[f(x), f(y)]$
the "generalized Jensen equation "
$2f(x+y) = f(2x)+f(2y)$
and so on. In the majority of cases existence and uniqueness of the solution is proved. Thus, the addition theorem has a strictly increasing and continuous solution if and only f the "addition function" F (x,y) is strictly increasing in both variables, and the associative law
$(x \circ y) \circ z = x \circ (y \circ z)$
holds, where $x \circ y = F(x,y)$

A Solution of Some Problems of K. Borsuk and L. Jánossy (E).
Acta phys. Acad. Sci. Hung. 4 (1955) 351-362.

Associative equations of the type
$F[F(x,y),z] = F(x, F(y,z))$
are treated in connection with L. Jánossy's work on an axiomatic foundation of probability theory.

(Aczél, J.)

Algebraical remarks on the Fréchet solution of the Kolmogoroff equation (F).
Publ. math. Debrecen 4 (1955-1956) 33-42.

The general solution of the FE $P(s,t)\,P(t,u) = P(s,u)$
$s \leqq t \leqq u$
is given under more general conditions than the solution given earlier by Fréchet. This equation plays an important role in polarity theory.

On addition and subtraction functions (G).
Publ. math. Debrecen 4 (1956) 325-333.

Addition theorems :
$f(x+y) = F[f(x), f(y)]$
and subtraction theorems
$f(x-y) = G[f(x), f(y)]$
are investigated. The main results : the addition theorem has a non-constant continuous solution if and only there exists an open interval on the real axis which is a group under the operation
$x \circ y = F(x,y)$
Furthermore : for any solution $f(x)$ $f(cx)$
is also a solution. The subtraction theorem has a continuous solution (which is then strictly increasing) if and only if there exists an open interval of the real axis on which the operation
$u \square v = G(u,v)$
is continuous, transitive, involutory, and if there exists a right hand unit e such that $u \square e = u$.

Some general methods in the theory of FEs in one variable,. New applications of F Es (Ru).
Uspechi mat. nauk. 11 (1956) 3 (69) 3-39.

Several classes of FEs are examined in view of possible applications. These vary as widely as scalar and vectorial multiplication of vectors, the Poisson distribution, and non-euclidean distance. In particular, the results lead to a characterization of the distance function
$$d(X,Y) = c \cdot \arccos \frac{x_1 y_1 + x_2 y_2 + x_3 y_3}{\sqrt{(x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2)}}$$
in elliptic geometry and
$$d(X,Y) = c \cdot \operatorname{arch} \frac{x_1 y_1 - x_2 y_2 - x_3 y_3}{\sqrt{(x_1^2 - x_2^2 - x_3^2)(y_1^2 - y_2^2 - y_3^2)}}$$
in hyperbolic geometry.

Miscellaneous on FEs (G).
Math. Nachr. 19 (1958) 87-99.

Several FEs and systems of FEs are treated, mostly addition theorems.

Aczél, J., and Kiesewetter, H.

On the reduction of degree in a class of FEs (G). *
Publ. math. Debrecen 5 (1957-1958) 348-363.

Aczél, J.

Some general methods in the theory of FEs and some recent applications I. (H). *
M.Tud. Akad. III Osztály közleményei 9 (1959) 375-422.

On the differentiability of the integrable solutions of certain FEs (G).
Ann. Univ. Sci. Budapest 3 (1960).

It is shown that the integrable solutions of the FEs

$$f(x)+\sum_{k=1}^{n} c_k(x)g_k\{f[a_k(x)+b_k y]\} = A(x,y)$$

and

$$f(x)f(y)+\sum_{k=1}^{n} c_k(x)g_k\{f[a_k(x)+b_k\cdot y]\} = A(x,y)$$

are differentiable, provided the g_k are continuous, the a_k and c_k differentiable and $A(x,y)$ integrable in y and

$\frac{\partial A}{\partial x}$ continuous in both variables.

Aczél, J., Hosszu, M. and Straus, E. G.

FEs for Products and Compositions of Functions (E). *
Pacific J. Math. 10 (1960).

Aczél, J.

Some general methods in the theory of FEs, and some recent applications. *
M. Tud. Akad. III Osztály közleményei 10 (1960) 9-32.

Aczél, J., Golab, J., Kuczma, M. and Siwek, E.

The cross ratio as solution of a FE (G). *
Ann. polon. math. 9 (1960) 183-187.

Aczél, J.

(Lectures on FEs and their applications).
Vorlesungen über Funktionalgleichungen und ihre Anwendungen (G).
Birkhäuser, Basel (1961) 324 p.

This is the first book ever to be published on FEs ; a book by Picard, published in 1928, Leçons sur quelques équations fonctionnelles, is a treatise on a small, but well-chosen number of FEs, Ghermanescu's book on FEs, published in 1960, is a treatise on a special class of FEs ; in fact, it deals with one, very general equation which includes difference equations.
Aczél's book does not propose to give a general theory of FEs, since such a theory does not exist. The book consists of two parts : the first part

deals with functions of one variable, the second with functions of several variables; within the first part, two cases are distinguished according to whether the variables appear under the function sign only, or also outside the function sign. One of the most interesting properties of FEs, namely that one equation may determine several unknown functions, is given one chapter by itself. Among the solution methods, reduction to differential and integral equations is treated; numerous applications to the theory of means and many other topics are given.
The [illegible] weakness of the book [illegible] fact that only such equations are [illegible]ted where- roughly speak[illegible]- the number of variables is higher than the number of variables in the unknown function. This excludes the first FE ever treated, and the most important of all, the Abel- Schröder equation. The reason for this [illegible]mission is that this type of equation is much more difficult to treat and requires different methods (For details on this problem, see the Introduction).

Aczél, J.

Several new results in the theory of FEs (H). *
Acta Univ. Debrecen 7 (1960).

Aczél, J.
Ghermanescu, M. and
Hosszu, M.

On cyclic equations (E).
Magyar Tud. Akad. Mat. Kutató Int. Közl. 5 (1960) 215-221.

Let $F(x_1, \cdots, x_p)$ be a function of p variables and let $x_1, x_2, \cdots, x_n$ be a set of variables with $n \geq p$. Let us denote by Ω the operation which substitutes for each variable the next one in the cyclical arrangement $\cdots x_1, x_2, \cdots, x_n, x_1, \cdots$. The FE investigated and solved (without restriction) thus has the form

$$\left[I + \Omega + \Omega^2 + \cdots + \Omega^{n-1} \right] F(x_1, \cdots, x_p) = 0$$

Aczél, J.

Miscellaneous on FE II. (G). *
Math. Nachr. 23 , 39-50 (1961).

Bognár, Z. and
Targonski, Gy. I.

On the determination of conjugate harmonic functions (G).
Public. Math. 3 (1954) 215-216.

Let be $f(x+iy) = u(x,y) + i\,v(x,y)$

Real and analytic part of an analytic function, u and v form a pair of conjugate harmonic functions ; they are connected by the classical formula

$$v(x,y) = \int -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy$$

from the Cauchy Riemann formulae. The paper furnishes an alternative method for the determination of v from u

$$v(x,y) = \operatorname{Im} U(x+iy, 0)$$

where U is the complex extension of the (real) harmonic fucntion u,

provided $f(x)$ is real on the real axis, apart from a possible imaginary constant ; also, quite generally, the FE

$$U(x+iy,0)+iV(x+iy,0)=U(0,y-ix)+iV(0,y-ix)=u(x,y)+iv(x,y)$$

is necessary and sufficient for the Cauchy-Riemann equations to hold.

Bajraktarevic, M.

On the solutions of a FE (Sc).
Hrvatsko Prirodoslovno Drustvo. Glasnik Mat. Fiz.Astr. (2) 8 (1953) 297-300.

The FE $f(x)f(x+1)(x+a_1)\cdots(x+a_n)=1$

is solved under more general conditions then before.

On certain iterated sequences (F).
Naucno Drustvo N.R. Bosne-Hercegovine dj. 4 odjeljenje priv.-tehn. nauka 1 (1953) 1-33.

The FE $f[g(2x)]=g(x)$

is examined, where g is known and f unknown. Conditions are stated under which a unique solution exists.

On certain iterated sequences II (F).
C.R. Paris 236 (1953) 988-989.

The FE $\psi(x+1)=f[\psi(x)]$

(ψ unknown) is treated by relating it to the FE

dealt with in the preceding publication.

On certain solutions of two FEs (Sc). *
Bull. Soc. Math. Phys. Serbie 6 (1954) 172-184.

On a FE.
Glasnik Mat. Fiz. Astr. Drustro. Mat. Fiz. Hrvatska Ser. II 12 (1957) 201-205.

It is shown that the FE
$Q(z)=f\{z, Q[g_1(z)], \cdots, Q[g_q(z)]\}$
has always a solution, under rather general conditions.

(Bajraktarevic, M. Monotone solution of a FE (F).
Acad. Serbe Sci. publ. Inst. Mat. 11 (1957) 43-52.

The FE mentioned in the title is
$F(z) = \varepsilon_0 f[z, F[g(d_0, z)]]$
ε_0 is constant; $f(z,t)$ and $g(d_0, z)$
are known functions. Existence and uniqueness of a strictly monotone solution is given under appropriate conditions.

(... mean value FE (F).
Glasnik mat.-fiz. astr. Drustvo Mat. Fiz. Hrvatske (2) 13 (1958) 243-248.

The existence of certain solutions of the above FE is shown ; these solutions are explicitely given ; the solutions are shown to be invariant under a certain transformation ; finally, a class of functions is given in which the FE is completely solved.

On a solution of the FE $\varphi(x) + \varphi[f(x)] = F(x)$

Glasn. mat. fiz. astr. 13 (1958) 11-13.

Results of Kuczma on the above FE are extended.

Baker, I. N. Solutions of the FE $f(x)^2 - f(x-) = h(x)$ (E).
Canad. Math. Bull. 3 (1960) 115-120.

Solutions are given, under restrictions, for the FE in the title and for the FE $f(x)^2 - f(2x) = g(x)$
which can be reduced to the former.

Bellman, R. A Note on Scalar Functions of Matrices (E).
Amer. Math. Monthly 59 (1952) 391.

Let φ be a function of the n^2 variables a_{ij} $i,j = 1, \dots, n$
arranged as a matrix A, and
$\varphi(AB) = \varphi(BA)$
for every pair A,B ; then $\varphi(A)$ is a polynomial function of the coefficients of λ of the equation

$$\det(A - \lambda I) = 0$$

Berg van den, J. On the FE $\varphi(\alpha x) - \beta\varphi(x) = F(x)$ I, II (G).

Nieuw. Arch. Wisk. (3) 3 (1955) 113-123.

A treatment of the above FE which breaks the unfortunate habit of many authors of looking for strictly increasing solutions only. First bounded solutions are investigated; later, some unbounded solutions are also considered.

Blum, J.R., Norris, M.J., and Wing, G.M. Asymptotic behaviour of solutions of a FE (E). *
Proc. amer. math. soc. 12 (1961) 463-467.

Boas, R.P. Functions which are odd about several points (E).
Niew. Arch. Wisk. (3) 1 (1953) 27-32.

The condition that $f(t)$ is odd about the point $t=x$ is expressed by the Jensen FE $f(x+t)+f(x-t) = 2f(x)$

This FE is treated with respect to the nature of the set of t and x values on which it holds.

Functions which are odd about several points. Addendum (E).
Nieuw Arch. Wisk. 5 (1957) 25.

The author points out the a lemma in his paper with the above title was already found by Surstin in 1915. Some misprints are corrected.

Boswell, R.D. Continuous Solutions of Two Functional Equations (E). *
Amer. math. Monthly 65 (1958) 476.

On Two FEs (E).
Amer. math. Monthly 66 (1959) 716.

The only continuous solution of $f(x+y)=f(x)+f(y)+a(1-A^x)(1-A^y)$ is $f(x)=kx-a(1-A^x)$, a being real and $A>0$; the only continuous solution of $f(x+y) = A^x f(y) + A^y f(x)$ is $f(x) = kx A^x$.

Carstoiu, I.

On some FEs and the symbolic calculus (F).
C.R.Acad.Sci. Paris 224 (1947) 1199-1200.

Five FEs (among them a difference equation) are solved using the Laplace transform. The method is restricted by the fact that existence of the first derivative had to be assumed.

Chaundy, T.W., and Mc Leod, J.B.

On a FE (E).
Quart.J.math.Oxford S.(2) 9 (1958) 202-206.

The FE $f(x) + u f(vx) = U(u,v) f[V(u,v)x]$

is investigated. x, u, and v are variables, f, u, and v are unknown functions, f is assumed to be continuous. The FE arises in a problem concerning statistical thermodynamics of mixtures.

Choczowski, B.

On continuous solutions of some FEs of the n-th order (E).
Ann.Polon.math. 11 1 2(1961) 123-132.

Continuous solutions of the following functional equations are investigated.

(1) $\varphi(x) = H[x, \varphi[f_1(x)], \ldots \varphi[f_n(x)]]$

(2) $\varphi[f_{n+1}(x)] = g[x, \varphi(x), \varphi[f_1(x)], \ldots, \varphi[f_n(x)]]$

(3) $F[x, \varphi(x), \varphi[f_1(x)], \ldots, \varphi[f_n(x)]] = 0$

Climescu, A.C.

On the FE of associativity (F).
Bull.Ecole Polyt. Jassy 1 (1946) 211-224.

Introducing the generalized "multiplication" (group operation)

$$x \circ y = f(x, y)$$

the condition of associativity is expressed by the FE

$$f[x, f(x, z)] = f[f(x, y), z]$$

if u and are defined and single valued on the range of x and y

$$u^{-1}[f\{u(x), u(y)\}]$$

is also a solution. Several special cases are treated and applications given.

Daroczy, J.

Remarks on FEs (H). *
2 Congr.math.hongr.Budapest (1960) 11.

Daróczy, Z.

Necessary and sufficient conditions for the existence of non-constant solutions of functional equations (G).
Acta Sci. Math. 22 (1961) 31-41.

The author starts from the following result of J. Bernstein: the FE $f\left(\frac{x+y}{2}\right) = pf(x) + qf(y)$; $p+q=1$ has non-constant solutions only if $p \neq \frac{1}{2}$. As a generalization, the author investigates the FE
$f(ax+by+c) = pf(x) + qf(y) + r$
and finds necessary and sufficient conditions on the coefficients for the the existence of non-constant solutions.

Dias Tavares, A.

A Theorem on Real Functions of a Real Variable (Po). *
Revista científica 1 no. 1 (1956) 9-11.

Djokovic, D.

(See Doković, D.)

Doković, D.

(See also Mitrinovic, D.S.)

Doković, D.

On some cyclical functional equations which reduce to the equation of Cauchy (E). *
Publik. elektrotech. Fak. Univ. Beogradu Ser. Math Fiz. 1 (1961) 61-64, 21-26.

Dwinas, S.

A Deduction of the Laplace-Gauss Law of Errors (S).
Rev. mat. hisp. amer. (4) 3 (1948) 12-18.

The Gauss density function is shown to be the only density function satisfying $f(x)f(y) = f\left(\sqrt{x^2+y^2}\right)$

also, $\frac{a}{2} e^{-a|x|}$
is the only density satisfying
$$f(x)f(y) = f(|x|+|y|)$$

Elyash, E.S., and Levine, N.

A Note on the Function Az + b (E).
Amer. math. Monthly 66 (1959) 803.

Let $m(z,w) = pz + qw$
be with $p \geq 0$, $q \geq 0$, $p+q=1$

the weighted arithmetic mean of z and w. The following theorem is proven : the only convex regular function satisfying the FE
$f[m(z,w)] = m[f(z), f(w)]$
is the linear function. A more general case includes the dependence of p and q on z and w.

Erdös, J.

A Remark on the Paper "On some Functional Equations" by S. Kurepa (E).
Glasnik mat.-fiz.astr. Društvo Mat. Fiz. Hrvatske 14 (1959) 3-5.

It is shown that every continuous solution of the FE
$$f(x+y, z) + f(x, y) = f(y, z) + f(x, y+z)$$
is of the form
$$f(x, y) = g(x+y) - g(x) - g(y)$$
on the other hand, this is not the general solution of the FE.

Erdös, J., and Golomb, M.

Functions which are Symmetric about Several Points (E).
Nieuw. Arch. Wisk. (3) 3 (1955) 13-19.

The "oddness" FE $f(x+t) + f(x-t) = 2f(x)$

treated earlier by R. P. Boas, is further investigated ; the generalization
$$\sum_{k=1}^{n} a_k f(z + c_k u) = f(z) \quad c_k \neq 0, \ \sum a_k = 1$$
is treated.

Fenyő, I.

On a solution method for certain FEs (G).
Acta math. Acad. Sci. Hung. 7 (1956) 383-396.

The theory of distributions is used to transform certain FEs into distribution equations and to solve them.

Gáspár, J.

A new definition of determinants (G).
Publ. math. Debrecen 3 (1953) 257-260.

A scalar function of a matrix is shown to be the determinant under some commutativity and homogeneity conditions all of which, except one, are very mild.

Gair, A.

On the analytical solutions of certain FEs (R). *
Bul. sti. teh. inst. politeh. Timisoara 5 (1960) 123-127.

Ghermanescu, M.

(See also Aczél, J.)

Ghermanescu, M.

Functional characterization of the trigonometric functions (F).
Bul. Inst. Politeh. Iasi 4 (1949) 362-368.

The addition theorems for the sine and the cosine are investigated assuming measurability of the solutions.

(Ghermanescu, M.) Measurable solutions of certain linear FEs in several variables (F). I, II.
Bull. sci. Ecole Polyt. Timisoara 13 (1948) 18-37, 128-140.

Measurable solutions, especially polynomial solutions, are sought for a number of FEs, most of them difference equations.

Linear FEs (R). *
Acad. R. P. Rom. Bul. sti. mat.-fiz. 3 (1951) 245-259.

On the FE $\sum_{i=0}^{p} A_i f(x+\omega_i) = 0$ (R).

Com. Acad. R.P. Romane 3 (1953) 187-192.

This FE is treated under various assumptions. A characteristic result: $(2\pi i)^{-1} \oint e^{xz} \psi(z) / \vartheta(z)\, dz \quad |z| = 1$ is a solution, where $\vartheta(z) = \sum_{i=0}^{p} A_i e^{\omega_i z}$ and $\psi(z)$ [illegible], but otherwise arbitrary.

On the FE $\sum_{i=0}^{p} A_i f(x+\omega_i) = 0$ II. (R).

Com. Acad. R.P. Romane [illegible] (1957) 499-5[illegible]

Additional solutions are constructed (see the previous paper with the same title) [illegible] of residues.

On the FE $\sum_{i=0}^{p} A_i(x) f(x+\omega_i) = 0$ (R).

Com. Acad. R.P. Romane [illegible] (1958) 341-[illegible].

The FE is a generalization of the one discussed in the previous paper the P_i being polynomials.

A system of FEs (R).
Acad. R.P. Rom. Bul. sti. mat.-fiz. 6 (1958) [illegible]-582.

The FE $f(\vartheta_n(x)) + \sum_{k=1}^{n-1} \varphi_k(x) f(\vartheta_{n-k}(x)) = 0$

is treated; here the functions φ_k are known, so are the $\vartheta_n(x)$

ϑ_n is the n-th iterated function of a known ϑ_n. Special attention is paid to the case where $\vartheta(x)$ is a translation

(Ghermanescu , M.) On FEs in two variables (R).
Acad. R.P. Rom. Bul. sti. mat. fiz. 7 (1955) 963-975

Sixteen results are given concerning a number of FEs, all in two variables. The results are quite general, since only measurability of the solutions is required.

FEs with n- periodic functional argument I (F).
C.R.Acad. Sci. Paris 243 (1956) 1593-1595.

Theorems on the existence of solutions for the FE
$$\sum_{n=0} a_n f[\vartheta_n(x)] = g(x)$$
are given ; here, f is unknown, ϑ is known and ϑ_n denotes its n-th iterate ; $g(x)$ is known and might also be identically zero. Finally, $\vartheta_N(x) \equiv x$ is assumed.

FEs with n-periodic functional argument II (F).
C.R. Acad. Sci. Paris 244 (1957) 543-544.

A continuation of the first part. Existence theorems are given.

A class of linear GFEs (F).
C.R. Acad. Sci. Paris 245 (1957) 274-276.

The Fe $f(P) + \sum_{k=1}^{n} a_k(P) f[\vartheta_k(P)]$
is considered ; P is a point in a multidimensional space ; $\vartheta_k(P)$ as usual the kth iterate of ϑ ; the coefficients satisfy
$$a_k[\vartheta(P)] \equiv a_k(P)$$
, i.e. they are invariants under the substitution ϑ ,or generalized periodical functions in the sense of Rausenberger-. The characteristic equation
$$\lambda(P)^n + \sum_{k=1} a_k(P)\lambda(P)^{n-k} = 0$$
is then defined ; each solution
$$\lambda_i(P) \quad i=1,\cdots,h$$
is also an invariant under the substitution ϑ . These solutions are used to construct the general solution of the FE.

Linear FEs with n-periodic functional argument (R).
Acad. R.P. Romîne Bul. Sti. Sect. Sti. Mat.Fiz. 9 (1957) 43-78.

The FE $\sum_{k=0}^{n} a_k f[\vartheta_k(P)] = 0$
is studied ; the a_k are constants n-periodicity of ϑ is defined by
$$\vartheta_n(P) = \vartheta(P)$$
The inhomegenous case $\sum_{k=0}^{n} a_k f[\vartheta_k(P)] = g(P)$

is also treated. P is a point in a multidimensional space.

(Ghermanescu, M.) Doubly automorphe functions (R)
Acad. R.P. Romîne Bul. Sti. Sec. Sti. Mat. Fiz. 9 (1957) 253-260.

These functions are defined by $f[\vartheta_1(P)] = \lambda_1(P) f(P)$
$f[\vartheta_2(P)] = \lambda_2(P) f(P)$
where $\vartheta_1, \vartheta_2, \lambda_1, \lambda_2$ must satisfy various conditions.

On the FE of Cauchy (F).
Bull. Math. Soc. Sci. Math. Phys. R.P. Roumaine (N.S.)
1 (49) (1957) 33-46.

New methods are shown to solve the Cauchy equation
$$f(x+y) = f(x) + f(y)$$
assuming that the solutions are continuous, resp. measurable. Some related FEs are also treated, among them $\varphi(\alpha x) = \varphi(x)$,
$f(\alpha x + \beta y) - a f(x) - b f(y) = u(x,y)$
etc.

A class of linear FEs (R).
Acad. R.P. Romîne Stud. Cerc. Mat. 9 (1958) 113-126.

The FE $f(P) + \sum_{k=1}^{n} \alpha_k(P) f[\vartheta_k(P)] = 0$
is investigated; α_k, f are known functions, the ϑ_k are the iterates of the unknown function ϑ. P is a point in multidimensional space. The α_k are assumed to be automorphic invariants with respect to ϑ:
$$\alpha_k(\vartheta(P)) = \alpha_k(P)$$

On the functional definition of the trigonometric functions (F).
Publ. math. Debrecen 5 (1957-1958) 93-96.

A simple method is given to solve the FEs
$$f(x) + f(y) = f[xy - \sqrt{(1-x^2)(1-y^2)}]$$
$$f(x) + f(y) = f[xy + \sqrt{(x^2-1)(y^2-1)}]$$
continuous monotone solutions are
$k \cdot \arccos x$ resp. $k \cdot \operatorname{arch} x$.

(Ghermanescu, M.)

On a class of linear FEs (R).*
Studii si cerc. mat. 9 (1958) 113-126.

(Functional Equations)
Ecuatii functionale (R).
Bucuresti, Ed. Acad. Republ. popul. romîne (1960) 521 p.

The book deals mainly with the results of Romanian authors. It investigates various cases of the FE

(*) $\Phi\{P, f(P), f(\vartheta(P)), \dots, f(\vartheta_n(P))\} = 0$

where P is a point in multidimensional space, Φ and ϑ known functions (ϑ may depend on a parameter) and ϑ_n denotes the n-th iterate of ϑ. Φ may be linear, or non-linear. For the case $\vartheta(P) = P - P_0$ we obtain the difference equations, and two chapters of the book are accordingly dedicated to the general theory of difference equations. The "n-periodic" case $\vartheta_n P = P$ is treated, and so are many of the topics which come under the heading of the FE (*).

Linear FEs with orthomorphic functional arguments (F). *
C.R. Acad. Sci. 254 (1962) 402-403.

The FE $\sum_{k=1}^{n} C_k(P) f[\vartheta_k(P)] = 0$

is investigated for the case that the coefficients are automorph (substitutional invariants) with respect to some ϑ : $C_k(\vartheta(P)) = C_k(P)$; P is a point in multidimensional space, ϑ_n denotes the n-th iterate of ϑ.

Gołab, J.

(See also Aczél, F.)

Gołab, S.

On the distributive law of real numbers (G).
Studia math. 16 (1956) 353-358.

The equation $g[f(x,y), z] = f[g(x,z), g(y,z)]$

becomes $(x+y)z = xz + yz$

if $f(x,y) = x+y$; $g(x,y) = xy$

Simple conditions on the functions f and g are given under which an automorphism of the operations $x \oplus y = f(x,y)$ and $x \odot y = g(x,y)$ is established to ordinary addition and multiplication on the field of real numbers.

(Golab, S.) On the FE $f(X)f(Y) = f(X \cdot Y)$ (F).

Colloquium math. 1 (1948) 265.
(See also the next [illegible])

On the equation $f(X)f(Y) = f(XY)$

(F).
Ann. polon. math. 6 (1959) 1-13.

X and Y are 2 x 2 matrices of complex numbers. If the above equations is satisfied for every X,Y, then $f = \Phi(\det X)$
Where Φ is a complex-valued function with the property
$\Phi(xy) = \Phi(x)\Phi(y)$
, i.e. under certain restrictions essentially a power of x.

Golab, S. and Schinzel, A. On the FE $f[x + y f(x)] = f(x)f(y)$ (F).

Publ. math. Debrecen [illegible] (1959) 113-125.

Every continuous solution of this FE consists of pieces of the form $1 + mx$, m real.

Golomb, M. (See Erdős, P.)

Guinand, A.P. The FEs of the [illegible] associativity (F).
C.R. Paris, [illegible]

The equations $f[f(x,y,z),u,v] = f[x,y,f(z,u,v)]$
$f[f(x,y,z),u,v] = f[x,f(y,z,u),v]$
$f[f(x,y,z),u,v] = f[x,y,f(z,u,v)]$
are solved.

Hajek, O. On the FEs of the trigonometric functions (Ru).*
Thecosl. math. J. 3 ([illegible]) (1953) 432-434

Halperin, I. Non Measurable Sets and the equation $f(x+y) = f(x)+f(y)$

(E).
Proc. Amer. math. Soc. [illegible] (1951) 221-224

Some very refined set-theoretical investigations in connection with the above FE.

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Haupt, O. On a uniqueness theorem for certain FEs (G). *
J. reine angew. Math. 186 (1944-1949) 58-64

Hopf, E. On the FEs of the trigonometric and hyperbolic functions (G)
Sitz.-Ber. math. nat. Abt. bayer. Akad. Wiss. (1945-1946) 167-173.

The classical results on the FE $f(x+y) = f(x) + f(y)$

are treated for the case where both sides of the FE are reduced mod 1 ; the classical result of Hamel (accordingly) to which the graph of f is either [illegible] analogue on the cylinder (y reduces mod 1). The FE of the exponential and of the trigonometric functions are treated in the same way.

Horvath, J. Note on a problem of L. Fejér (F).
Bull. Ecole Polyt. Jassy 3 (1948) 164-166.

Let $\mu(x_1, x_2)$ be a mean value function. Conditions on are given so that the FE

$$f\left(\frac{x_1 + x_2}{2}\right) = \mu\left[f(x_1), f(x_2)\right]$$

has a continuous solution.

Hosszu, M. (See also Aczél, J.)

Hosszu, M. On the FE of bisymmetry (H). *
M. Tud. Akad. Alk. Mat. Int. Közleményei [illegible] (1952) 335-342

A Generalization of the FE of Bisymmetry (E).
Studia math. 14 (1953) 100-106

The "generalized bisymmetry equation"

$$F\left[G(x,y), H(u,v)\right] = f\left[g(x,u), h(y,v)\right]$$

is treated.

On the FE of distributivity
Acta math. Acad. Sci. Hung. 4 (1953) 157-167.

The strictly monotone and twice differentiable solutions of the FE

$$F\left[F(x,y), z\right] = F\left[F(x,z), F(y,z)\right]$$

are determined.

(Hosszu, M.)

On the FE of Transitivity (E).
Acta sci. math. Szeged 15 (1953-1954) 203-208.

The transitivity condition
$(x \circ t) \circ (y \circ t) = x \circ y$
of operations between real numbers can be written, using the notation

$$x \circ y = F(x, y)$$

as FE

$$F[F(x,t), \quad t] = F(x, y)$$

This FE is solved under various conditions imposed.

On the FE of Autodistributivity (E).
Publ. math. Debrecen 3 (1953-1954) 83-86.

The monotone and once differentiable solutions kfad
of $M[M(x,y), z] = M[M(x,z), M(y,z)]$
$M[z, M(x,y)] = M[M(z,x), M(z,y)]$
can be expressed in the form

$$M(x,y) = f^{-1}[p f(x) + q f(y)]$$

with

$$p + q = 1$$

Some FEs related with the associative law (E)
Publ. Math. Debrecen 3 (1953) 200-224.

"Associative type" relations and the corresponding FEs are investigated. A typical result : the most general strictly increasing solution of

$$x \circ (y \circ z) = z \circ (y \circ x)$$

is

$$x \circ y = f^{-1}[\alpha^2 f(x) + \alpha f(y) + \beta]$$

Remark on a paper by H. Wundt : "On a FE in the Theory of heat conduction " It is shown (cf. Wundt, H.) that the only differentiable solution of the FE

$$f\left(\frac{x - y}{\log x - \log y}\right) = \frac{f(x) + f(y)}{2}$$

is constant.

Generalization of some FEs with several variables (H).
Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 6 (1956) 439-449.

This is a resumé of five papers by the author published between 1953 and 1956.

(Hosszu, M.)

Unsymmetric means (H).
Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 7 (1957) 207-208.

A continuous unsymmetric, quasi-linear interior mean is a continuous function of the form

$$M(x,y) = f^{-1}[pf(x) + qf(y)]$$

$$p>0,\ q>0 \qquad p+q=1$$

A number of conditions in connection with theorems on such means is modified, with reference to previous work by other authors.

Unsymmetric means (Ru).
Colloquium math. 5 (1957-1958) 32-42.

A Russian version of the preceding paper.

A Generalization of the FE of Distributivity (E).
Acta sci. math. Szeged. 20 (1959) 67-80.

Introducing the "addition" $x \hat{+} y = F(x,y)$ and the multiplication $x \hat{\cdot} y = H(x,y)$, the distributive condition $(x \hat{+} y) \hat{\cdot} z = x \hat{\cdot} z \hat{+} y \hat{\cdot} z$ is expressed by the FE

$$H[F(x,y),z] = F[H(x,z), H(y,z)]$$

This type of FE is discussed in detail with some applications.

Generalizations of the FE of distributivity (H).
Nehézipari Müszaki Egyetem közleményei 3 (1959) 151-166.

A Hungarian version of the preceding paper.

Nonsymmetric Means (E). *
Publ. math. Debrecen 6 (1959) 1-9.

On the FE of translation (H). *
in "2ème Congr. math. hongr. Budapest 1960 II ".

Ianescu, D.V.

On a FE (F).
Mathematica Cluj 1 (24) (1959) 11-26.

The FE
$$\begin{vmatrix} f(x) & f(x+h) \\ f(x+h) & f(x+2h) \end{vmatrix} = 0$$

and its analogue for determinants of higher order is treated.

James-Levy, J. On the problem of general anamorphosis (Ru).
Dokl. Akad. Nauk SSSR (NS) 113 (1957) 258-260.

If the functional relation

$$z = F(x, y)$$

can be written in the form

$$\begin{vmatrix} g_1(x) & f_1(x) & 1 \\ g_2(y) & f_2(y) & 1 \\ g_3(z) & f_3(z) & 1 \end{vmatrix} = 0$$

a nomogram can be constructed to "solve" the equation, i.e. find the value of one variable if the two others are given. The functions $f_1, f_2, f_3, g_1, g_2, g_3$ determine the scales of the nomogram.

Jankó, B. On the method analogous to that of Tchebitcheff and to that of the tangent hyperboles for the approximate solution of non-linear FEs (R). *
Stud. Cer. Mat. Cluj. 11 (1960) 299-305

On a new generalization of the method of the tangent hyperboles for the solution of non-linear functional equations defined in Banach spaces (R). *
Stud. Cer. Math. Cluj. 11 (1960) 307-317, also published in 2.congress math. hongr. Budapest (1900) V.

Jewett, J.W. (See Seebeck, L.L.)

Kalmár, L. (See Aczél, J.)

Kestelman, H. On the FE $f(x+y) = f(x) + f(y)$ (E).
Fundamenta math. 34 (1947) 144-147.

A simple proof is given of the classical result of Ostrowski that a real solution of this FE is linear, provided it is bounded on a set of positive measure.

Kiesewetter, H. (See also Aczél, J.)

Kiesewetter, H.

Structure of linear FEs in connection with the Abel theorem (G).
Z. reine angew. Math. 206 (1961) 113-171.

Linear FEs of the form

$$(1)\quad \sum_{i=1}^{s} a_i f(x_i) + \sum_{k=1}^{p} f[\varphi_k(x_1, \cdots x_s)] \cdot a_{sik} = \text{Const.}$$

are investigated. A special case is

$$(2)\quad \sum_{i=1}^{p+1} f(x_i) = \sum_{k=1}^{r} f[\psi_k(x_1, \cdots, x_{p+1})]$$

which for p=1 becomes the "addition theorem" (in a sense different from the generally used notion)

$$(3)\quad f(x) + f(y) = f[\psi(x, y)]$$

under some restrictions it was shown already by Abel that (3) has a non-trivial solution if and only if the operation

$$(4)\quad x \circ y = \psi(x, y)$$

is associative. The existence of solutions of (3) is thus linked to the algebraic properties generated by $\psi(x, y)$

For the more general case (2) the notion of group had to be generalized for commutative and associative algebraic structures where p simultaneous operations between $p+1$ elements exist.
The paper is essentially devoted to the associative and the cyclical properties of the "argument function" ψ ; the results are described in terms of geometrical models as spherical trigonometry. A bibliography of 35 relevant papers resp. books is given.

Kordylewski, J. and Kuczma, M

On some linear FEs (E). *
Ann. polon. math. 9 (1960) 119-136

On the FE

$$F(x, \varphi(x), \varphi[f_1(x)], \cdots, \varphi[f_n(x)]) = 0 \qquad (E).$$

Ann. Polon. Math. 8 (1960) 55-60.

It is shown that, under not too severe conditions, this FE has an infinity of continuous solutions.

(Kordylewski, J. and Kuczma, M.)

On some linear FEs (E).
Ann. Polon. Math. 9 (1960-1961) 115-126.

The FE $F[f(x)] - T(x)F(x) = g(x)$

for the unknown function $F(x)$ is investigated in this interval $[a,b]$ where a and b are consecutive roots of the equation $f(x) = x$. The results are applied to the FE

$$\sum_{k=0}^{m} A_k F[f_k(x)] = g(x)$$

Kordylewski, J.

On the FE $F[x, \varphi(x), \varphi[f(x)], \ldots \varphi[f^{n}(x)]] = 0$

Ann. polon. math. 2 (1961) 285-294

Existence theorems for the above FE are given.

Kordylewski, J., and Kuczma, M.

On the continuous dependence of some FEs on given functions I. (E).
Ann. polon. math. 10 (1961) 91-98

In the FE

$$\varphi[f(x)] = g(x) + F(x)$$

the dependence of the solution φ on the given functions f and F is investigated.

On the continuous dependence of solutions of some FEs on given functions II. (E). *
Ann. polon. math. 10 (1961) 167-171

Kordylewski, J.

Continuous solutions of the FE

$$\varphi[f(x)] = F[x, \varphi(x)]$$

with the function $f(x)$ decreasing (E).
Ann. polon. math. 11 (1961) 115-122.

Sufficient conditions are given for the existence of a solution.

Kordylewski, J. and Kuczma, M.

On some linear FEs II. (E). *
Ann. polon. math. 11 (1961) 203-207.

Kuczma, M. (See also Aczél, J., Kordylewski, J.)

Kuczma, M. On convex solutions of the FE

$$g[\alpha(x)] - g(x) = \varphi(x) \qquad (E).$$

Publ. Math. Debrecen 6 (1959) 40-47.

Conditions are found under which the above FE possesses at most one convex solution which takes a prescribed value at a given point a. The conditions under which the theorem is proved are the following :
$\alpha(x)$ is continuous, concave and

$$\alpha(x) \geqq x \quad \text{in} \quad [a, \infty] \quad \text{. Moreover,}$$

$$\lim_{n \to \infty} [\varphi(\alpha_n(a)) - \varphi(\alpha_{n-1}(a))] = 0$$

α_n denoting the n-th iterate of α. Under these conditions at most one convex solution exists. In order to have one solution it is necessary that

$$\lim_{x \to \infty} \frac{\alpha(x)}{x} = 1$$

The problem may be considered as a generalization of the FE

$$g(x+1) - g(x) = \log x \quad \text{with} \quad g(0) = 0 \quad \text{of}$$

which the function $\log \Gamma(x)$ is the only convex solution.

On the FE

$$\varphi(x) + \varphi[f(x)] = F(x)$$

Ann. polon. math. 6 (1959) 281-287.

If $f(x)$ and $F(x)$ are continous in the closed interval $[a,b]$, and $f(x)$ is strictly increasing, there exists an infinite number of solutions continuous in the open interval (a,b) while not more than one solution is continuous at a.

(Kuczma, M.)

Remarks on some theorems of J. Anastassiadis (F).
Bull. Sci. Math. (2) 84 (1960) 98-102.

Monotonic solutions of the FE

$$g[\alpha(x)] - g(x) = \varphi(x)$$

are sought which take a prescribed value at a given point. The result is related to the Beta function.

General solution of a FE (E).
Ann. Polon. Math. 8 (1960) 201-207.

This FE is

$$g(x, \varphi(x)) = \varphi[f(x)]$$

On continuous solutions of a FE (E).
Ann. Polon. math. 8 (1960) 209-214.

The FE $g(x, \varphi(x)) = \varphi[f(x)]$

is solved for $\varphi(x)$ under special conditions.

On the form of solutions for some FEs (E). *
Ann. Polon. math. 9 (1960) 55-63.

Remarks on some FEs (E). *
Ann. Polon. math. 9 (1960) 277-284.

Kuczma, M. and Vopenka, P.

On the FE

$$\lambda[f(x)]\,\lambda(x) + A(x)\lambda(x) + B(x) = 0 \qquad (E).$$

Ann. Univ. Sci. Budapest Rolando Eötvös Sect. Math. 3-4 (1960-1961) 123-133.

Conditions are given under which a continuous solution exists.

(Kuczma, M.) A uniqueness theorem for a linear FE (E).
Glasn. mat.-fiz. astr. 16 (1961) 177-181.

It is shown that under suitable restrictions, for the FE

$$\varphi(x) = g(x)\,\varphi[f(x)] + \psi(x)$$

there exists a unique solution which - and the first derivatives of which - take [illegible] ribed values at a given p[illegible]

On the form of solutions of some FEs (E).
Ann. polon. math. [illegible] (19[illegible]) [illegible]

The solution

$$\frac{1}{2}F(x) - \frac{1}{2}\sum_{k=1}^{\infty}(-1)^k\left[F\{f^k(x)\} - F\{f^k(x)\}\right] = g(x)$$

is given for the FE

$$g(x) + g[f(x)] = \varphi(x)$$

The generalization

$$g[f(x)] = g(x, \varphi(x))$$

is considered

General solution of the FE

$$\varphi[f(x)] = g(x, \varphi(x)) \qquad (E).$$

Ann. polon. math. 9 (1961) 2[illegible]-2[illegible]

On monotonic solutions of a FE I. (E).

The FE $\varphi_2(x) = g[x, \varphi(x)]$

is investigated, $\varphi_2(x)$ being the second iterate of $\varphi(x)$.
It is shown that under suitable conditions infinitely many strictly increasing and continuous solutions exist in an interval.

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(Kuczma, M.) On monotonic solutions of a FE II. (E). *
Ann. polon. math. 10 (1961) 161-1[illegible].

On some FEs containing iterations of the unknown functions (E). *
Ann. polon. math. 11 (1961) 1-5

Kurepa, S. On [illegible] FEs (E).
Gla[illegible] mat.-fiz. astr. Drustvo [illegible] fiz. Hrvatske (2) 11 (1956) 3-5.

The solutions of the following FEs are given

$$f(x_1+x_2, x_3) + f(x_1, x_2) = f(x_1, x_3) + f(x_1, x_2+x_3);$$

$$f(x_1+x_2, x_3, x_4) + f(x_1, x_2, x_3+x_4) = f(x_2, x_3, x_4) + f(x_1, x_2+x_3, x_4) + f(x_1, x_2, x_3);$$

$$f(x_1+x_2, x_3, x_4, x_5) + f(x_1, x_2, x_3+x_4, x_5) + f(x_1, x_2+x_3, x_4, x_5) + f(x_1, x_2, x_3, x_4+x_5)$$

It turns out that the solutions are appropriate combinations of arbitrary functions; those are differentiable if the unknown function is assumed to be differentiable.

On some FEs in Banach space (P). *
Stud. math. 19 (1960) 149-158.

On the FE

$$f(x+y) = f(x)f(y) - g(x)g(y) \qquad (E).$$

Glasn. mat.fiz.astr. Jugosl. 15 (1960) 31-48.

The explicit solution of the above FE is given in terms of exponential functions.

(Kurepa, S.) A cosine FE in Hilbert spaces. (E).
Canad. J. math. 12 (1960) 45-50.

Kuwagaki, A. On the FE

$$f(x+y) = R[f(x), f(y)] \qquad (F).$$

Mém. Coll. Sci. Univ. Kyoto (A. Math.) 25 (1951) 139-144.
R is assumed to be rational, thus the FE is a "rational addition theorem". Conditions are given under which continuous solutions $f(x)$ exist.

On the rational FE of the unknown function in two variables (F).
Mém. Coll. Sci. Univ. Kyoto A. Math. 27 (1952-1953) 145-151.

A special case of the problem of the previous paper is dealt with

$$f(x+y, u+v) = R\{f(x,u), f(x,v), f(y,u), f(y,v)\}$$

where R is rational.

On the analytic function of two complex variables satisfying associativity (F).
Mém. Coll. Sci. Univ. Kyoto A. Math. 27 (1952-1953) 225-234.

The above FE is treated under the assumption that for some complex c,

$$f(c,c) = c$$

is valid and that f can be expanded into a power series.

On functions of two variables satisfying an algebraical addition theorem (F).
Mém. Coll. Sci. Kyoto A. Math 27 (1952-1953) 139-143.

The "implicite addition theorem" in two variables"

$$P\{f(x+y, u+v), f(x,u), f(x,v), f(y,u), f(y,v)\} = 0$$

is treated, P being a polynomial and f assumed to be analytic in both variables.

Lambek, J. (See Moser, L.)

Levine, N. (See Elyash, E. S.)

Mc Leod, J. B. (See Chaundy, T. W.)

Martis-Biddau, S. On the characterization of a class of functions (I). *
Collectanea math. 1 no. 1 (1948) 67-84.

Meynieux, R. On a theorem about the analyticity of the solution of a FE (F).
C.R. Acad. Sci. 254 (1962) 3301-3303.

Now results on the piece-wise analyticity of the solutions of the FE

$$F[f(u), g(v), h(u+v)] = 0$$

are given.

On analyticity of the continuous solutions of a FE (F).
C.R. Acad. Sci. 254 (1962) 4413-4414.
(See the preceding paper.)

Results of a similar type are given for the FE

$$\varphi[f(u), g(v)] = h(u+v)$$

Mikusiński, J. G. (See also Aczél, J.)

Mukusiński, J. G. On some FEs (F). *
Annales Soc. Pol. Math. 21 (1948) 346.

Mitrinović, D. S. On a process furnishing FEs the continuous and differentiable solutions of which can be determined (F).
Publ. Fac. Electrotechn. Univ. Belgrade, ser. math. fiz. No. 5 (1956) 1-8.

The FE

$$[f(x) + A g(x)][f(y) + B g(y)] = f(x) + f(y)$$

is solved, besides this the paper deals with some differential-functional equations.

Mitrinović, D. S. and Doković, D. On certain FEs the general solutions of which can be determined (E).*

Publ. Fac. Electrotechn. Univ. Belgrade, ser. math. fiz. no. 61-64 (1961) 1-11.

Mitrinović, D.S., and Presić, S.B.

On a cyclic, non-linear FE (F).
C.R.A. Acad. Sci. 254 (1962) 611-613.

The FE

$$F(x_1, \cdots, x_{2n}) = [f(x_1, x_2) + \cdots + f(x_{2k-1}, x_{2k})] \times$$
$$\times [f(x_{2k+1}, x_{2k+2}) + \cdots + f(x_{2n-1}, x_{2n})]$$

where F is known solution of a cyclical FE is given in terms of arbitrary functions.

Morgantini, E.

On equations in six variables which can be represented by a point nomograph (I).*
R.C. Sem. mat. Univ. Padova 17 (1948) 115-138.

Mycielski, J., and Paszkowski, S.

On a problem of probability calculus (F).
Studia math. 15 (1956) 188-200

The motion of a molecule on a straight line is considered by a method involving FEs.

Milkman, J.

Note on the FE

$$f(xy) = f(x) + f(y), \; f(x^n) = n f(x) \; (E).$$

Proc. Amer. math. Soc. 1 (1950) 505-508.

Solution of those FEs are given under assumptions on the set of which the functions are defined.

The Logarithmic function is unique (E).
Math. Mag. 24 (1950) 11-14.

The FE $f(x) + g(y) = h(xy)$

is treated by reducing it to F(x) + F(y) = F(xy).

Moser, L., and Lambek, J. On monotone multiplicative functions (E).
Proc. Amer. math. Soc. 4 (1953) 544-545.

It is shown that

$$f(m \cdot n) = f(m) f(n); \quad (m,n) = 1; f(n) \neq 0$$

and

$$f(m) \geqq f(n) \quad \text{for} \quad m \geq n$$

implies

$$f(n) = n^k$$

k being constant. The analogous case for continuous argument is well-known ; the interest of the paper lies in the fact that f is a number theoretical function , i.e. defined for positive integer arguments.

Maier, W. On the cyclical structure of certain FEs (G).
Z. reine angew. Math. 206 (1961) 172-174.

FEs with cyclical relations between the variables are studied.

Mitrinović, D. S., and Djoković, D. On an extended class of FEs (F).
C.R. Acad. Sci. 252 (1961) 1718.

An operation consisting of substitutions and summations is defined ; the corresponding FE is solved in terms of the same operation.

Norris, M. J. (See Blum, J.R.)

Pessides, N. On the FEs of Poincaré type (F). *
Composition Math. 10 (1952) 168-212.

Paszkowski, S. (See Mycielski, J.)

Pentikäinen, T. On continuous systems of functions with an algebraic addition theorem (G).
Ann. Acad. sci. fenn. ser. math.-phys. No. 38 (1947) 1-49.

Continuous functions f_1, f_2 on the real interval $[0,T]$ are investigated under the assumption that

$$f_1(u+v),\ f_2(u+v)$$

are algebraic functions of

$$f_1(u),\ f_1(v),\ f_2(u),\ f_2(v)$$

$[0,T]$, can be divided into a finite number of sub-intervals so that the f_i are analytic in each of them.

Pfanzagl, J. — Axiomatic foundations of a general theory of measurement (G). Schriftenreihe des Stat. Inst. der Univ. Wien. N.F. 1 (1959).

This book is related to the theory of FEs through the theory of means, which plays a central part in it.

Praporgescu, N. — On singular FEs (R). Stud. Cerc. mat. Acad. R.P. Romîne 12 (1961) 187-195.

FEs in the theory of Stochastic processes are investigated.

Presić, S.B. — (See also Mitrinović, S.B.)

Presić, S. — On the FE of translation (F). * Publ. Fac. Electrotechn. Univ. Belgrade. ser. math. fiz. No. 44-48 (1960) 15-16.

On the FE $f(x) = f[g(x)]$

Publ. Fac. Electrotechn. Univ. Belgrade. ser. math. fiz. No. 61-64 (1961) 29-31.

The general solution of the above equation is given as well as examples.

Radström, H. — Some elementary FEs and Hilbert's Fifth problem (Sw). Nordisk. mat. Tidsskr. 3 (1955) 129-147.

In the context given in the title, the FE

$$F(x \cdot y) = F(x) + F(y)$$

is dealt with, here

$$x \cdot y = g(x, y)$$

is some operation between e g real numbers

Radó, F. Condition of linear dependence for three functions (R).
Acad. R. P. Romîne Fil. Cluj. Stud. Cerc. Sti. Ser. I. 6(1955) 51-63.

It [illegible] that the condition [illegible] is

$$\begin{vmatrix} F(x) & F(x+h) & F(x+2h) \\ G(x) & G(x+h) & G(x+2h) \\ H(x) & H(x+h) & H(x+2h) \end{vmatrix}$$

for every [illegible] and h in $(-\infty, \infty)$

F, G, H

are defined in this interval.
Subtracting appropriate columns, dividing by h^3 and going to the limit, the condition leads to the vanishing of the Wronskian

FEs in connection with nomography (R).
Acad. R. P. Romîne Fil. Cluj. Stud. Cerc. Mat. 9 (1958) 249-319

This dissertation is a general study on the role played by FEs in the theory of nomography, [illegible] with an introduction to nomography and leading to the most [illegible] results of the author and other research workers in the field.

FEs which characterize nomograms with three straight scales (F)
Mathematica Cluj [illegible] (1959) [illegible]

Conditions are given under which the nomographical representation mentioned in the title is possible.

FEs in connection with nomography (R).
(See the preceding paper)
Studii cerc. mat. Cluj 9 (1958) 249-319.

Read, A. H. The solution of a FE (E).
Proc. Roy. Soc. Ed. A 63 (1952) 336-345.

Redheffer, R.M. — Novel uses of FEs (E).
J. rat. Mech. Anal. 3 (1954) 271-279.

Some new results are gained, other known results confirmed, dealing with problems in electromagnetic theory ; the main interest lies in the method . instead of starting from the Maxwell equations, the author derives his results from FEs, such as

$$f(x,y)+f(y,x)=f(x+y,0)$$

for the problem of energy transfer [illegible] sets in a multimode cavity.

On solutions of Riccati's equation as functions of the initial values (E).
J. rat. Mech. Anal. 3 (1954) 735-743.

Denote by $\varphi(x,t)$ that solution of the Riccati equation which vanishes for $x=t$; for, more generally, which equals $f(t)$ for $x=t$, where f is given) Introducing two auxiliary functions, the author shows that $f(x,t)$ satisfies a number of FEs.

Rényi, A. — Application of integral equations to the solutions of FEs (H). *
Mat. lapok. 6 (1955) 262-268.

Robinson, R.M. — A curious trigonometric identity (E).
Amer. Math. Monthly 64 (1957) 83-85.

$$f(z)=Az,\quad f(z)=A\sin bz$$

and $f(z)=\sinh bz$ with constant A and constant and real b are the only regular functions satisfying the FE

$$|f(x+iy)|=|f(x)+f(iy)|$$

Rosenbaum, R.A. and Segal, S.L. — A FE characterizing the Sine (E).
Math. Gaz. 44 (1960) 97-105.

Investigating the FE

$$f(x+y)f(x-y)=f^2(x)-f^2(y),$$

the authors derive, under some general and sophisticated conditions,

that $k_1 x$ and $k_2 \sinh k_3 x$ are the only solutions for complex x, and $c_1 x$, $c_2 \sin c_3 x$, $c_4 \sinh c_5 x$ are the only solutions for real x

Sakovich, G.N. — Solution of a FE of several variables (Ru). * Ukrain. mat. Zh. 13 (1961) 172-189.

The FE

$$n\,\mathcal{H}(\bar{r}) = \quad (C_n \bar{r})$$

is investigated; here $\bar{r}$ is a real vector and C_n is a given sequence of non-singular matrices.

San Juan, R. — An application of Diophantine approximation to the FE

$$f(x_1 + x_2) = f(x_1) + f(x_2) \qquad (S).$$

Publ. Ist. mat. Univ. Nac. Litora. 6 (1946) 221-223.

It is shown that every finite solution of this equation is either linear or its graph is everywhere dense in the plane

Segal, S.L. — (See Rosenbaum, R.A.)

Siwek, E. — (See Aczél, J.)

Schinzel, A. — (See Gołąb, A.)

Seebeck, L.L., and Jeweit, J.W. — A development of logarithms using the function concept. (E). * Amer. math. Monthly 64 (1957) 647-66[illegible].

Sharkovskij, A.N. — On the solution of a class of FEs (Ru). Ukrain. mat. Zh. 13 (1961) 86-94.

The FE

$$\Phi\left(x, f(x), f[\varphi(x)]\right) = 0$$

is treated.

Stamate, I

A class of mean formulas (R).
Com. Acad. R.P Romîne 8 (1958) 19-22.

A number of means value theorems is geometrically interpreted in terms of tangents to - parametrically given - curves.

On a property of the parabole and the solution of a FE (R). *
Lucr stiint Inst Polit. Cluj, (1959) 101-106.

Remarks in connection with FEs (R). *
Lucr stiint Inst. Polit. Cluj (1959) 107-110.

On the FE

$$f(x+y) = f(x) + f(y) + f(x)f(y)$$

(R).

Lucr. st. int. inst. Polit. Cluj. (1959) 111-118.

The solution for the above addition theorem is given.

Contributions to the integration of a FE (R). *
Lucr stii. Inst. politeh Cluj (1960) 47-51.

On a class of FEs (R). *
Gaz. mat.-fiz. (A) 11 (1960) 587-598.

Straus, E. G.

(See Aczél, J.)

Szekeres, G.

Regular iteration of real and complex functions (E).
Acta. math. 100 (1959) 203-258

The paper is devoted to the solution of the **Schröder equation**

$$f[g(x)] = \lambda f(x)$$

under more general conditions than those given by earlier authors. The result is relevant to the question of non-integral iteration indices, since it provides a solution of the translation equation

$$F\{F(x,\nu),\mu\} = F(x,\nu+\mu)$$

since

$$F(x,\nu) = f^{-1}[\lambda^{\nu} f(x)] \quad (\lambda \neq 0, \neq 1)$$

is such a solution ; thus

$$g_{\nu}(x) = f^{-1}[\lambda^{\nu} f(x)]$$

can be considered as the ν - th iterate of $g(x)$, where ν can be arbitrary real, or complex, under suitable conditions

Targonski, Cy.I. (See also Bognár, Z.)

Targonski, Cy.I. The representation of functions by means of chain series (G). Public. Math. 2 (1955) [illegible]

The problem is solved under some restriction, to find the representation

$$f(x) = x + g(x) + g_2(x) + \cdots$$

where $g_n(x)$ is the n - th iterate of the unknown function g, which is shown to be the solution of the FE

$$f[g(x)] = f(x) - x \quad \text{i.e.} \quad g(x) = f^{-1}[f(x) - x]$$

convergence and uniqueness is proved. Asymptotic expansions like

$$\log(x+1) \approx \frac{1}{2}\left[x - 1 + e^{-2x}(x+1)\right]$$

$$\arcsin x \approx x(1+\cos x) - \sin x\sqrt{1-x^2}$$

result for small x.

Thielman, H.P. On generalized means (E). *
Proc. Iowa Acad. Sci. 55 (1948) 241-247.

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(Thielman, H. P.)

On generalized Cauchy FEs (E).
Amer. math. Monthly 56 (1949) 452–457.

$$F(x+y+nxy) = G(x)\,H(y)$$

is investigated under the condition

$$n > 0, \quad x > \frac{1}{n}, \quad y > \frac{1}{n}.$$

The general solution turns out to be

$$F(x) = a(1+nx)^k, \quad G(x) = a(1+nx)^k,$$
$$H(x) = b(1+nx)^k,$$

a, b, k, being constants.

On a pair of FEs (E).
Amer. math. Monthly 57 (1950) 544–547.

The FEs

$$F(xy) = p(y)\,G(x) + q(x)\,H(y)$$
$$f(xy) = p(x)\,q(y)$$

are solved by first reducing them to the pair

$$f(xy) = g(x)^{p(y)}\,h(y)^{q(x)}$$
$$f(xy) = p(x)\,q(y)$$

and solving this latter pair.

A note on a FE (E).
Amer. J. Math. 73 (1951) 482–484.

Sufficient conditions are given under which an "operation between real numbers" $x \circ y = g(x,y)$ can be written in the form

$$x \circ y = f^{-1}[f(x) + f(y)]$$

where g is continuous and strictly monotone. In particular, if $x \circ x$ is a polynomial of degree higher than 1, then

$$f(x) = k \log(ax+b) \quad \text{or} \quad f(x) = k \arccos(ax+b)$$

Van den Berg, J. (See ..., van den, J.)

Vaughan, H. E. Characterization of the Sine and Cosine (E).
Amer. math. Monthly 62 (1955) 707-713.

The well-known FE

$$g(x+y) = g(x)g(y) + f(x)f(y)$$

is solved.

Vietoris, L. Characterization of the sine and related functions by means of FEs (G).
J. reine angew. Math. 186 (1944-1949) 1-15

Vilner, I. A. Analytical functions of a complex variable of the first nomographic class and their nomograms (E).
Doklady Akad. Nauk USSR 53 (1946) 189-190.

Let be $F(w, z) = 0$, F analytic

$w = p_1 + i p_2$, $z = a + i b$.

F belongs to the first nomographic class if the equation can be written in the form of two real equations of the form

$$f(p_i)X(a) + g(p_i)Y(b) + h(p_i) = 0$$

$i = 1, 2$.

In this case, nomographs exist with straight scales in a and b. All functions in the first nomographic class are determined in terms of elementary functions and elliptic integrals.

Vincze, E. On the characterization of associative functions of several variables (G).
Public. Math. 6 (1959) 241-253.

The theorem that

$$x \vee y = F(x, y) = \varphi^{-1}[\varphi(x) + \varphi(y)]$$

for any continuous, strictly monotone associative operation is generalized to simultaneous operations on n variables ; two different types of formulae arise according to whether n is odd or even.

(Vincze, E.) A generalization of the FE of Abel-Poisson (H).
Mat. Lapok 12 (1961) 18-31.

The generalization

$$F(x+y) + G(x-y) = \mathcal{H}(x)\,\mathcal{K}(y)$$

of the d'Alembert-Poisson FE

$$C(x+y) + C(x-y) = 2\,C(x)\,C(y)$$

is solved in the most general form (complex solution).

Vopenka, P. (See Kuczma, M.)

Wilner, J.A. (See Vilner, I.A.)

Wing, G.M. (See Blum, J.R.)

Wundt, H. On a FE in the theory of heat conduction (G).
Z. angew. Math. Phys. 5 (1954) 172-175.

The FE

$$f\left(\frac{x-y}{\log x - \log y}\right) = \frac{f(x) + f(y)}{2}$$

is treated in detail. The most general differentiable solution is given as

$$f(x) = A \int_1^x \frac{\log t + t^{-1} - 1}{t - \log t - 1}\, dt + B$$

Later, M. Hosszu showed that $f(x)$ is a solution only if $f(x)$ reduces to a constant. (See under Hosszu, M.)

Young, G. S.

The linear FE (E).
Amer. math. Monthly 65 (1958) 37-53.

This is a concise proof that every bounded solution of the FE

$$f(x+y) = f(x) + f(y)$$

is of the form

$$f(x) = ax.$$