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THE NORMAL BIVARIATE DENSITY FUNCTION  
AND  
ITS APPLICATIONS TO WEAPON SYSTEMS ANALYSIS,  
A REVIEW

by  
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#### ABSTRACT

The normal bivariate density function is derived from á priori considerations. It is discussed in terms of probability area in a plane, and as a correlation surface. Several numerical methods of solving the normal bivariate distribution double integral are presented, and a curve is included for converting elliptical error distributions to circular probable errors. Regression and correlation coefficients are discussed. Relative to weapons systems analysis, examples are given of uses in studying impact and location errors. Analyses of search and detection for stationary and moving objects are given specific mathematical treatment. An Appendix examines the elliptical properties of normally correlated distributions. The investigation has resulted in a reference paper for the normal bivariate density function.

## INTRODUCTION

A normal bivariate distribution of data has a density function,  $f(x, y)$ , as follows:

$$f(x, y) = \frac{A}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp \left[ \frac{-1}{2(1-\rho^2)} \left[ \frac{(x-\bar{x})^2}{\sigma_x^2} + \frac{(y-\bar{y})^2}{\sigma_y^2} - \frac{2\rho(x-\bar{x})(y-\bar{y})}{\sigma_x\sigma_y} \right] \right] \quad (1)$$

where,  $A$  = area under the curve (unity for a density function),  
 $\sigma_x$  and  $\sigma_y$  are x and y directional standard deviations,  
 $\rho$  = data correlation coefficient, (equals one for perfect correlation).  
 $\bar{x}$  and  $\bar{y}$  are the means of x and y values. They are the centroids of the distribution and are equal to zero when centered at the origin.

Since area under a density function is interpreted as probability, a definite double integral of  $f(x, y)$  for  $A = 1$ , represents the probability of an event occurring within the limits. It is termed the normal bivariate distribution integral.

$f(x, y)$  also represents a normal correlation surface, often called the normal bivariate surface. It permits analysis of x and y relationships and enables value predictions by means of regression equations.

As an analytical tool, the normal bivariate density function finds important military applications in target attack analysis and in problems such as searching for an object.

To investigate the normal bivariate density function and its uses, first it will be derived from a priori considerations. Second, its double integral will be discussed as a probability area in a plane. Third, the role of the function as a correlation surface will be examined. Finally, its

analytical application to weapons systems problems will be discussed. An Appendix presents some elliptical properties of normally correlated distributions.

#### DERIVATION

The equation of the normal probability curve follows from the basic statement of a normal, symmetrical, probability distribution of data (events). Letting the origin of a normal curve be at the mean, the derivative of the function at  $x = 0$  must be zero. Moreover, considering both sides of the  $y$  - axis, the curve gradually approaches the  $x$  - axis so that the derivative approaches zero as  $x$  increases in absolute value without limit. During this process  $y$  approaches zero. All of these properties can be expressed in a simple differential equation as follows:

$$dy = -k xy dx. \quad (2)$$

The negative sign insures that there will be a maximum and not a minimum at  $x = 0$ .  $k$  is a constant. Solving,

$$\int \frac{1}{y} dy = -k \int x dx, \quad (3)$$

$$\log y = -k \frac{x^2}{2} + \log C \quad (4)$$

where  $\log C$  is the integration constant.

Then,  $y = C e^{-k x^2/2}, \quad (5)$

and  $A = C \int_{-\infty}^{\infty} e^{-k x^2/2} dx. \quad (6)$

From definite integral tables, c. f. p. 89 of reference (2),

$$C \int_{-\infty}^{\infty} e^{-k x^2/2} dx = \frac{C}{2\sqrt{k/2}} \Gamma(1/2) = C \sqrt{\frac{2\pi}{k}}, \quad (7)$$

and  $C = \frac{A}{\sqrt{2\pi/k}}.$  (8)

Ignoring tables, and integrating (6) by parts ( $dv = dx$ , and  $u =$  the exponential expression) results in

$$A = C \left[ x e^{-k x^2/2} \right]_{-\infty}^{+\infty} + C k \int_{-\infty}^{+\infty} x^2 e^{-k x^2/2} dx. \quad (9)$$

Since the first expression takes an indeterminate form,  $\infty/\infty$ , new numerators and denominators are obtained by independent derivatives, and the limiting value of the expression then becomes zero. Since, by definition, the  $n$ -th moment of a frequency distribution is defined as

$$n\text{-th moment} = \int_b^a x^n f(x) dx, \quad (10)$$

and since from fundamental principles the standard deviation is the square root of the second moment about the mean, then it follows, considering (6), that the second expression in (9) becomes  $A k \sigma_x^2.$

$$\text{Thus, } A = A k \sigma_x^2, \text{ or } k = 1/\sigma_x^2. \quad (11)$$

From equation (8),

$$C = \frac{A}{\sigma_x \sqrt{2\pi}} \quad (12)$$

The normal curve, then, from (5) and (8) and dropping the A, is

$$f(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-x^2/2\sigma_x^2}. \quad (13)$$

A similar derivation follows for  $f(y)$  from symmetry. Hence,

$$f(y) = \frac{1}{\sigma_y \sqrt{2\pi}} e^{-y^2/2\sigma_y^2} \quad (14)$$

$f(y)$  is also known as a Gaussian error function and specifies the probability of an error of value  $y$  as a function of that value. If the  $x$  and  $y$  coordinates of any point in the distribution are assumed to be independent (a property of two-dimensional Gaussian distribution), and since the probabilities of simultaneous independent events combine as products, it follows that the probability of a point (event) occurring in the rectangular region

$$\begin{aligned} x_1 &\leq x \leq x_2 \\ y_1 &\leq y \leq y_2 \end{aligned}$$

is

$$f(x, y) = \left( \int_{x_1}^{x_2} f(x) dx \right) \left( \int_{y_1}^{y_2} f(y) dy \right) = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x) f(y) dy dx \quad (15)$$

From (13) and (14), . . . ,

$$F(x, y) = \frac{1}{2\pi \sigma_x \sigma_y} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \exp \left[ -\frac{1}{2} \left( \frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} \right) \right] dy dx, \quad (16)$$



which is the definite normal bivariate distribution integral for independent  $x$  and  $y$ .

If  $x$  and  $y$  are not independent variables, a correlation coefficient must be introduced as in equation (1). The development of the correlation coefficient,  $\rho$ , ( $-1 < \rho < 1$ ), for (16), will be left to the reader who is referred, e. g. to pp. 241-43 of Elderton (3), pp. 160-63 of Kenney (9), or pp. 155-57 of Hoel (8). For those who do not consider equation (2) as intuitively obvious, West (19), pp. 135-38, presents a tedious but ingenious derivation of this differential equation.

#### PROBABILITY AREA IN A PLANE

A probability density function is a rate of change of probability with respect to random variable values. A definite integral of the function, therefore, is an area equivalent to the probability that an event (point) occurs between the limits. If the  $x$  and  $y$  values are independent and normally distributed, the occurrence probability of the event ( $x, y$ ) depends only on the standard deviations and prescribed limits of the occurrence.

Equation (16) represents the probability area in a rectangular region. But integrations can be made over elliptical or circular arcs as well as within elliptical or circular "rings," i. e., regions bounded by equi-probability contours. Letting  $J =$  the exponential of equation (1), the locust of  $J = c^2$ , where  $c$  is a constant, is an equi-probability ellipse, so that, on this ellipse,  $f(x, y)$  is constant. Also, the probability that a random point (event) will be in the ellipse defined by  $c$  is

$$p = 1 - e^{-c^2/2}. \quad (17)$$

The probability,  $\acute{p}$  that a point ( $x, y$ ) taken at random will be in the elliptic ring formed by the ellipses with parameters  $c$  and  $c + \Delta c$  is approximately

$$\acute{p} = c e^{-c^2/2}. \quad (18)$$

The Appendix treats further the elliptical properties of normally correlated distributions.

$$\underline{\sigma_y = \sigma_x}$$

If 
$$\sigma_x \equiv \sigma_y = \sigma, \quad (19)$$

and if the area of interest is a circle of radius R centered on the origin, then a transformation to polar coordinates and use of the Jacobian changes equation (16) to

$$F(r, \theta, \sigma) = \frac{1}{2\pi\sigma^2} \int_0^{2\pi} \int_0^R e^{-r^2/2\sigma^2} r dr d\theta. \quad (20)$$

This expression integrates readily by elementary function to

$$F(r, \sigma) = 1 - e^{-R^2/2\sigma^2}, \quad (21)$$

and specifies the probability that a random event, of standard deviation  $\sigma$ , will occur within the area of radius R, the event being, of course, part of the distribution in question.

$$\underline{\sigma_y \neq \sigma_x}.$$

For the common case of  $\sigma_y \neq \sigma_x$ , the double integral does not yield to elementary functions and numerical integration must be used. Harter (7) has applied a reduction technique in preparation for the numerical integration. In particular, the method develops circular probability areas equivalent in value to the actual elliptical error distributions. The probability that a point (x, y),

chosen randomly and independently from a normal bivariate distribution, will lie within a circle with center at the origin and radius  $K \sigma_x$  is

$$P(K, \sigma_x, \sigma_y) = \iint f(x, y) dx dy$$

$$\sqrt{x^2 + y^2} < K \sigma_x. \quad (22)$$

Let  $x/\sigma_x = r \cos \theta$  and  $y/\sigma_y = r \sin \theta$ ,

then

$$P(K, \sigma_x, \sigma_y) = \frac{\sigma_x}{2\pi \sigma_y} \int_0^{2\pi} \int_0^K \exp\left(-\frac{1}{2} r^2 \left[ \cos^2 \theta + \frac{\sigma_x}{\sigma_y} \sin^2 \theta \right]\right) r dr d\theta. \quad (23)$$

Next, let  $\sigma_y/\sigma_x = \alpha$  (the condition of  $\alpha \leq 1$  is maintained by defining the larger of the two standard deviations as  $\sigma_x$ ), and  $\phi = 2\theta$ . Then,

$$P(K, \alpha) = \frac{1}{2\pi \alpha} \int_0^{2\pi} \int_0^K \exp\left(-\frac{r^2}{4\alpha^2} \left[ (1 + \alpha^2) - (1 - \alpha^2) \cos \phi \right]\right) r dr d\phi. \quad (24)$$

Substituting  $r^2/4\alpha^2 = \psi$ ,

$$P(K, \alpha) = \frac{2\alpha}{\pi} \int_0^\pi \int_0^{(K/2\alpha)^2} \exp\left[-\psi \left[ (1 + \alpha^2) - (1 - \alpha^2) \cos \phi \right]\right] d\psi d\phi. \quad (25)$$

Taking one integration yields,

$$P(K, \alpha) = \frac{2\alpha}{\pi} \int_0^{\pi} \frac{1 - \exp\left(-K^2/4\alpha^2 [(1+\alpha^2) - (1-\alpha^2)\cos\phi]\right)}{(1+\alpha^2) - (1-\alpha^2)\cos\phi} d\phi. \quad (26)$$

For the numerical integration, values of  $\cos\phi$  can be read into a computer and values of the exponential function are calculated as needed. A printout of circular error probabilities can be programmed for selected values of  $K$  and  $\alpha$ .

Considering equation (21), it will be noted that for the circular distributions of  $\sigma_y = \sigma_x$ , ( $\alpha = 1$ ), equation (26) becomes

$$P(K, 1) = 1 - e^{-K^2/2}. \quad (27)$$

Weingarten, and DiDonato (18) have simplified the non-circular normal bivariate distribution integral by another approach. A transformation is made to polar coordinates, and reductions made by transformations to a single integral multiplied by a Bessel function. The argument of the function is determined and values taken from standard tables, thereby simplifying the numerical integration.

$$\text{Again let } \alpha = \sigma_y / \sigma_x, \text{ and } K^2 = (x^2 + y^2) / \sigma_x^2.$$

Then in polar coordinates,

$$P(K, \alpha) = \frac{1}{2\pi\alpha} \int_0^{2\pi} \int_0^K \exp\left[-\frac{1}{2}r^2\left(\frac{1+\alpha^2}{2\alpha^2} - \frac{k-\alpha}{2\alpha^2}\cos 2\phi\right)\right] r dr d\phi \quad (28)$$

$$\text{where } 0 \leq \sigma_y / \sigma_x \leq 1. \quad (29)$$

$P(K, \alpha)$  is simply transformed to

$$P(K, \alpha) = \frac{1}{\pi \alpha} \int_0^{K^2/2} \exp(-\beta z) \int_0^\pi \exp(Y z \cos \theta) d\theta dz, \quad (30)$$

where  $Y = (1 - \alpha^2) / 2\alpha^2$ , and

$$\beta = (1 + \alpha^2) / 2\alpha^2.$$

The second integral is found in Bessel function texts, e. g., p. 46 of Gray, Mathews, and MacRobert (5), and may be expressed as

$$\int_0^\pi \exp(Y z \cos \theta) d\theta = \pi I_0(Yz). \quad (31)$$

Where  $I_0(Yz)$  is a first kind, zero order, hyperbolic Bessel function of the argument  $(Yz)$ . Values for ranges of the argument are available in tables.

The computer calculations and printout selection are now principally concerned with solving

$$P(K, \alpha) = I_0(Yz) 1/\alpha \int_0^{K^2/2} \exp(-\beta z) dz, \quad (32)$$

a task easily handled, e. g., by the General Electric 225. Printouts can be programmed for probability radii, such as CEP vs standard deviation functions such as  $\sigma_y / \sigma_x$ ; or, reference curves can be prepared such as, e. g., CEP/ $\sigma_x$  vs  $\sigma_y / \sigma_x$  (cf. Figure 1 and reference (13)).

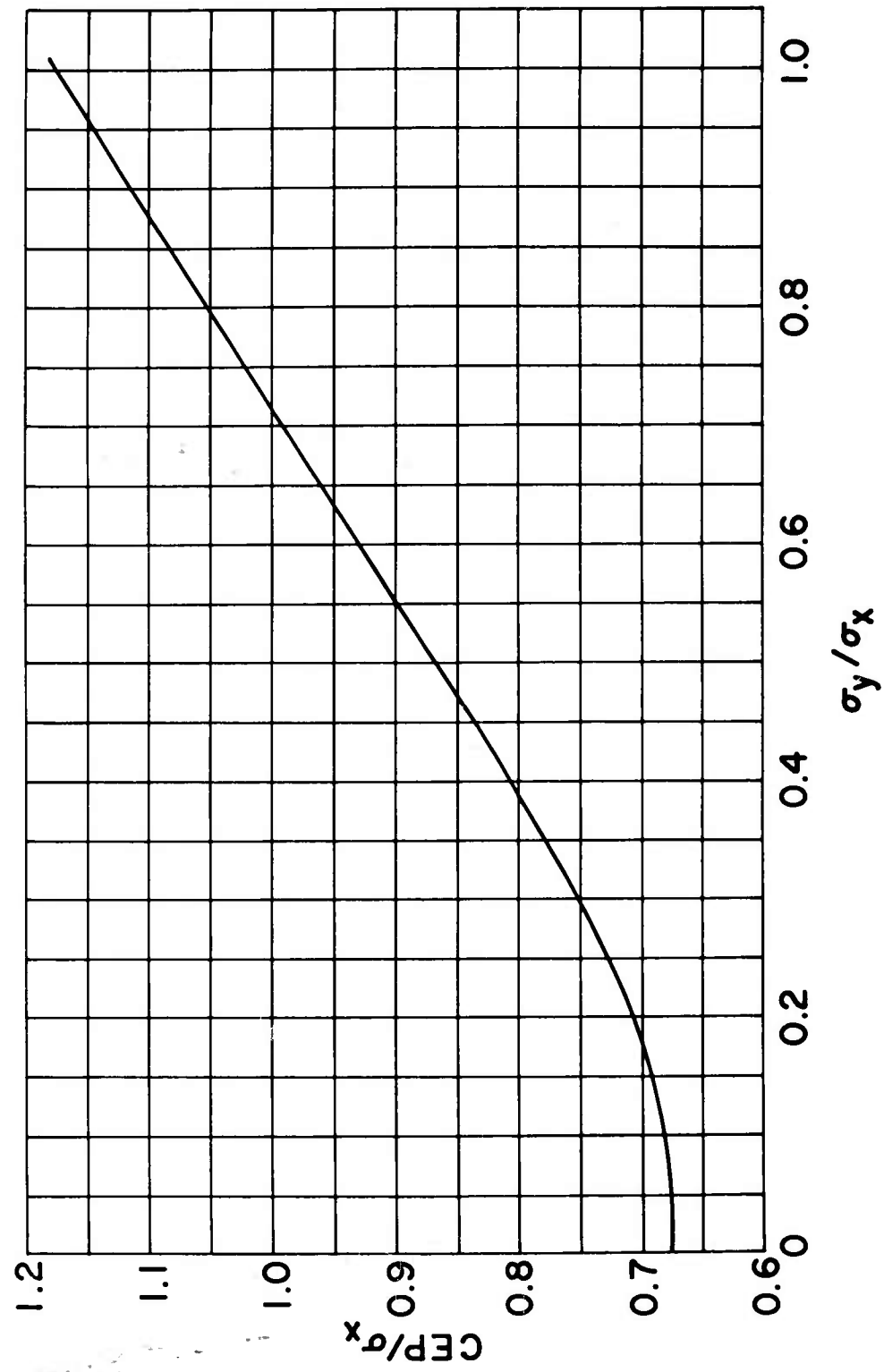


Figure 1. CEP as a Function of  $\sigma_y, \sigma_x$

### Relation to Standard Normal Variables

Two variables  $(x, y)$  are distributed in bivariate normal fashion when each can be expressed linearly as a function of two independent, standard normal variables  $U$  and  $V$ . Then

$$\begin{aligned}x &= a U + b V + d, \text{ and} \\y &= f U + g V + h.\end{aligned}\tag{33}$$

The joint density function of the independent variables  $U$  and  $V$  is now the product of the standard normal densities of  $U$  and  $V$ , or,

$$f_{U, V}(u, v) = \frac{1}{2\pi} \exp \left[ - (u^2 + v^2) / 2 \right]\tag{34}$$

The function

$$\begin{aligned}f_{U, V}(u, v, \rho) &= \frac{1}{2\pi \sqrt{1 - \rho^2}} \int_u^\infty \int_v^\infty \exp \left[ - \frac{1}{2} \left( \frac{1}{1 - \rho^2} \right) \right. \\&\quad \left. (x^2 - 2\rho xy + y^2) \right] dx dy\end{aligned}\tag{35}$$

has been tabulated by Pearson (12), and, with the aid of this table, the integral of the two dimensional normal density functions over a rectangular area may be calculated. Since the function contains three arguments, the table is rather voluminous. A simpler table of a function with two arguments, from which it is comparatively easy to calculate  $F_{U, V}(u, v, \rho)$ , was published by Nicholson (11) in 1943. Also, note reference (20).

It would be instructive for the reader to examine Lindgren's development (10), p. 99, of the bivariate normal density function in terms of the determinant of a second moment matrix. In relation to the constants of equation (33), and

also expressed as variances and covariance, the second moment matrix, M, the starting point, is expressed as

$$M = \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix} = \begin{pmatrix} a^2 + b^2 & af + bg \\ af + bg & f^2 + g^2 \end{pmatrix} \quad (36)$$

### THE CORRELATION SURFACE

In addition to being a probability density in a plane, the normal density function of two variables may be treated as the equation of a surface in three dimensions. Thus,

$$f(x, y) = \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1 - \rho^2}} \exp \left[ -\frac{1}{2(1 - \rho^2)} \left( \frac{x^2}{\sigma_x^2} - \frac{2xy}{\sigma_x \sigma_y} + \frac{y^2}{\sigma_y^2} \right) \right] \quad (37)$$

is the equation of a surface, of which the corresponding volume is uniaxially symmetrical, being centered at the origin. It is called the normal correlation surface and is pictorially presented in a number of texts, including p. 154 of Hoel (8), or p. 586 of Hald (6). All sections of this surface cut by planes parallel to the XZ-plane are normal curves; and all sections parallel to the yz-plane are normal curves. All sections cut parallel to the XY-plane are ellipses.

The eccentricity of the elliptical contours of different normal correlation surfaces varies with the degree of correlation existing in the corresponding data universe. A surface with narrow elliptical contours signifies a universe in which there is high correlation. If the variables are completely independent in the probability sense, and if the variables have been converted to standard normal variables, the contour lines are circles. If the variables have not been converted to standard normal, and  $\rho = 0$ , the contour lines may be



ellipses, but their major and minor axes will coincide with the X and Y axes. As may be intuitively suspected, the elliptical contour lines orthogonally projected to the XY-plane represent areas of probability for the event (x,y).

Since conversion to standard normal variables transforms the contour lines to circles, the conversion provides a basis for expressing elliptical probability areas (elliptical error distributions) as circular probable errors.

#### Regression and Correlation Coefficients

The line of regression which fits the means of a y-array on x (Figure 2 shows such a line) is

$$y = \rho \frac{\sigma_y}{\sigma_x} x. \quad 0 \leq \rho \leq 1. \quad (38)$$

Similarly,

$$x = \rho \frac{\sigma_x}{\sigma_y} y. \quad (39)$$

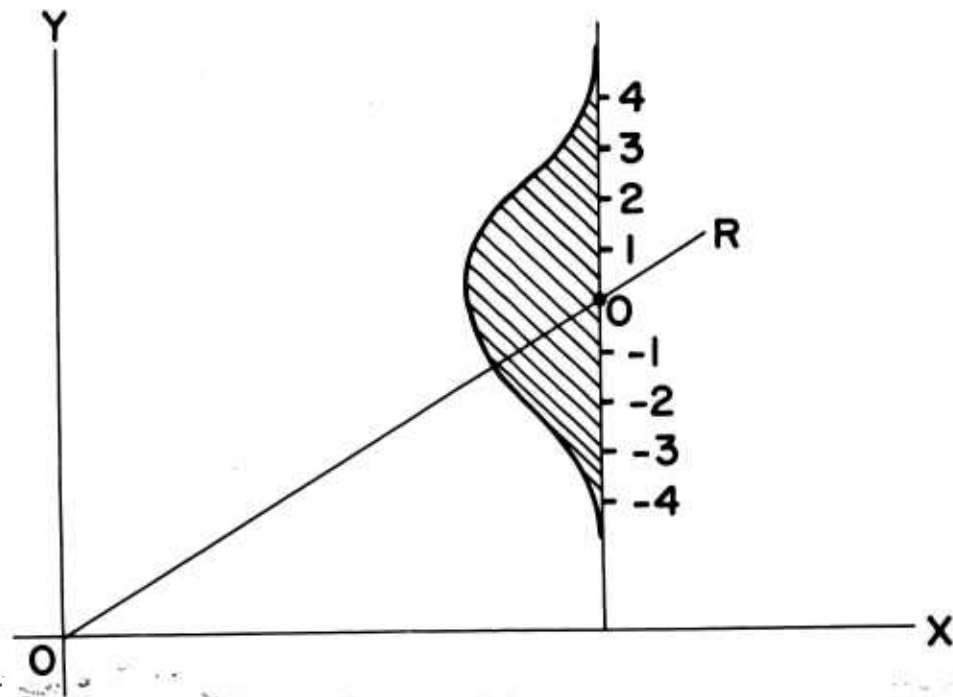


Figure 2. An X-Array of y's and Deviations from a Predicted Value

The expressions  $\rho \sigma_y / \sigma_x$  and  $\rho \sigma_x / \sigma_y$  are called regression coefficients, and  $\rho$ , as defined in the Introduction is the correlation coefficient between  $x$  and  $y$ . It is important to note that although the value of the correlation coefficient is independent of the original units of measurement, the values of the standard deviations and, therefore, of the regression coefficients are not.

The regression coefficient is very useful not only for purposes of statistical measurement but also for providing the best interpretation of the correlation coefficient. If, e.g., the average of a set of variables is regarded as the most probable value, a chosen value of  $x$  need only be multiplied by the value of the regression coefficient in equation (38) to give the most probable associated value of  $y$ .

If the equation of the regression line is written as

$$\frac{y}{\sigma_y} = \rho \frac{x}{\sigma_x}, \quad (40)$$

it is obvious that if the deviations of  $x$  and  $y$  are considered as expressed in standard units by dividing by the value of the standard deviation, the regression coefficient becomes identically the correlation coefficient and is then subject to the same interpretation previously given for the regression coefficient.

It is easily verified that the correlation coefficient is the geometric mean of the two regression coefficients.

A final remark pertinent to correlation - an array of  $y$  values for selected  $x$  implies a conditional distribution; and it is important to remember that the conditional distributions of a joint normal distribution are also normal.

#### APPLICATIONS TO WEAPONS SYSTEMS ANALYSIS

The normal bivariate density function is important to weapons systems analysis whenever an appropriate analytical probabilistic model contains two

variables which are jointly normally distributed. A very important application is the study of impact and location errors. Examples of pertinent areas are:

1. Warhead delivery
2. Gcodetic measurements
3. Mapping measurements
4. Intelligence estimates
5. Artillery attacks
6. Depth charge control
7. Design of warhead attack patterns
8. Determination of geometry and probability density of an expected impact area.
9. Determination of geometry and impact area for a selected probability.
10. Optimization of search and detection programs.
11. Space capsule return, search, detection and recovery; crashed aircraft search plans.
12. Lunar impact and landing analyses.
13. General accuracy analyses, including conversion of elliptical probabilities to circular probabilities. The CEP, e.g. is a convenient input to the single-shot-kill probability.
14. Radar sightings
15. Ephemeris calculations

The section on probability areas in a plane discusses the handling of the normal bivariate distribution integral that would apply to most of the above problem areas. In addition, a typical problem, that is applicable to many of the above areas, is presented and solved by Hald (6), pp. 601-2. The problem is as follows: Given a two-dimensional normal distribution with means  $(0, 0)$ , standard deviations  $\sigma_x = 1.5$  and  $\sigma_y = 1.0$ , and the correlation coefficient  $\rho = 0.75$ . Find the regression lines and the 95% (probability) contour ellipse.

A brief discussion on applications to search and detection analysis will complete the paper.

### Search and Detection for a Stationary Object

Randolph and Harrison (15) have analyzed the effectiveness of a given search force under selective deployment, and, in doing so, have made skillful use of the normal bivariate distribution.

If an object is located in a rectangle bounded by  $x, x + \Delta x$ ;  $y, y + \Delta y$ , the conditional probability that it is detected is defined as  $F[\phi(x, y)] \Delta x \Delta y$ , and the probability that the object is located as well as detected in this rectangle is approximately

$$f_{X, Y}(x, y) F[\phi(x, y)] \Delta x \Delta y, \quad (41)$$

where  $X$  and  $Y$  are mutually independent, normally distributed random variables with zero means. Then, the probability that an object is detected in  $R$ , the domain of  $f_{X, Y}(x, y)$  is

$$P = \iint_R f_{X, Y}(x, y) F[\phi(x, y)] dx dy. \quad (42)$$

The conditional probability of detection is essentially determined by test and experiment, although analysis is usually necessary to predict the effectiveness of new, untried equipment.

If the object is located within a normal probability density, then

$$f_{X, Y}(x, y) = \frac{1}{2\pi \sigma_x \sigma_y} \exp \left[ -1/2 \left( \frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} \right) \right]. \quad (43)$$

and  $XY$  plane projections of the correlation surface contour lines form elliptical boundaries which are of fixed probability density. In the discrete sense, elliptical rings formed by successive contour lines will also have a constant probability density (area). Consequently, an optimum search effort is constant for all points in a selected ring. The probability that the object

is in a selected ring is given by equation (18) where the constant has been defined as the square root of the exponential of equation (1). Also see equation (67).

To find the coverage region for a selected detection probability, let

$$u = \frac{x}{\sigma_x}, \text{ and } v = \frac{y}{\sigma_y}. \quad (44)$$

Then the density of U and V is given in equation (34) and, in the UV plane, R is the region such that

$$R = \left\{ u, v : f_{U,V}(u, v) = d \right\}, \quad (45)$$

d being a constant. But for U and V independent,

$$f_{U,V}(u, v) = f_U(u) \cdot f_V(v). \quad (46)$$

$$\therefore d = \frac{1}{2\pi} \exp\left(-\frac{r_1^2}{2}\right) = \frac{1}{\sqrt{2\pi}} f_{R_1}(r), \quad (47)$$

where  $R_1$  is a random variable of the radius, r is the particular radius of the circular region R, and  $f_{R_1}(r)$  is the normal density at a point  $r_1$ , with  $r^2 = u^2 + v^2$ .

If R is an elliptical area such that

$$R = \left\{ x, y : \frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} = 1 \right\}, \quad (48)$$

then in the UV plane the corresponding area is enclosed by a circle of unit radius, such that

$$R = \left\{ u, v : u^2 + v^2 = 1 \right\} . \quad (49)$$

A circular normal bivariate density function is now implied, with the region, R, the area inside one standard deviation. Hence the area of R is  $\pi$ , and, from equation (47),

$$d = \left( \frac{1}{\sqrt{2\pi}} \right) \left( \frac{e^{-1/2}}{\sqrt{2\pi}} \right) = \frac{0.2427}{\sqrt{2\pi}} = 0.0965 \quad (50)$$

The relationship for P (cf equation (42)) and d is as follows:

$$P = \int_R \int f_{U,V}(u,v) du dv - A(R) d, \quad (51)$$

here  $A(R)$  = area of the R region. Since  $u^2 + v^2 = 1$ ,

$$\int_{U,V} f_{U,V}(u,v) du dv = 0.6827. \quad (52)$$

$$\therefore P = 0.6827 - \pi(0.0965) = 0.38. \quad (53)$$

To interpret, if the search objective is to achieve a 0.38 probability of locating the object, the searching effort need only be conducted within the circle formed by  $u^2 + v^2 = 1$ . Obviously, this is equivalent to a search effort in the XY plane of the ellipse

$$\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} = 1. \quad (54)$$

Thus, the normal bivariate distribution integral can be used in optimizing search strategy (minimum area and, therefore, minimum time) for a selected detection probability.

#### Search Density Function for a Moving Object

The probability density function for the location of a stationary object is fixed with respect to time, once the related parameters are defined; but, for an object capable of movement, the positional distribution changes with time. With the conditions as defined by equations (19), (20), and (21), the probability density function that an object is in the area element  $dA$  at a distance  $R$  from a central point (e.g., the point where the object was previously sighted at time  $t$ ) can be defined as  $f(r) dA$ . The speed of the object can be expressed as  $S$ , but in an unknown direction. However, directions are random inasmuch as any one direction may be considered as probable as any other. Then,

$$f(R) dA = \frac{1}{2\pi \sigma^2} e^{-R^2/2 \sigma^2} dA. \quad (55)$$

Let:

- $L_o$  be the object location at time  $t = 0$ ,
- $L_t$  its location at time  $t$ ,
- $\lambda = \angle OL_t L_o$ ,
- $R_o = \text{distance } OL_o$ ,
- $R_t = \text{distance } OL_t$ .

Figure 3 shows the relationships.

By law of cosines,

$$R_o^2 = R_t^2 + S^2 t^2 - 2R_t St \cos \lambda, \quad (56)$$

since  $St = L_o L_t$ ,

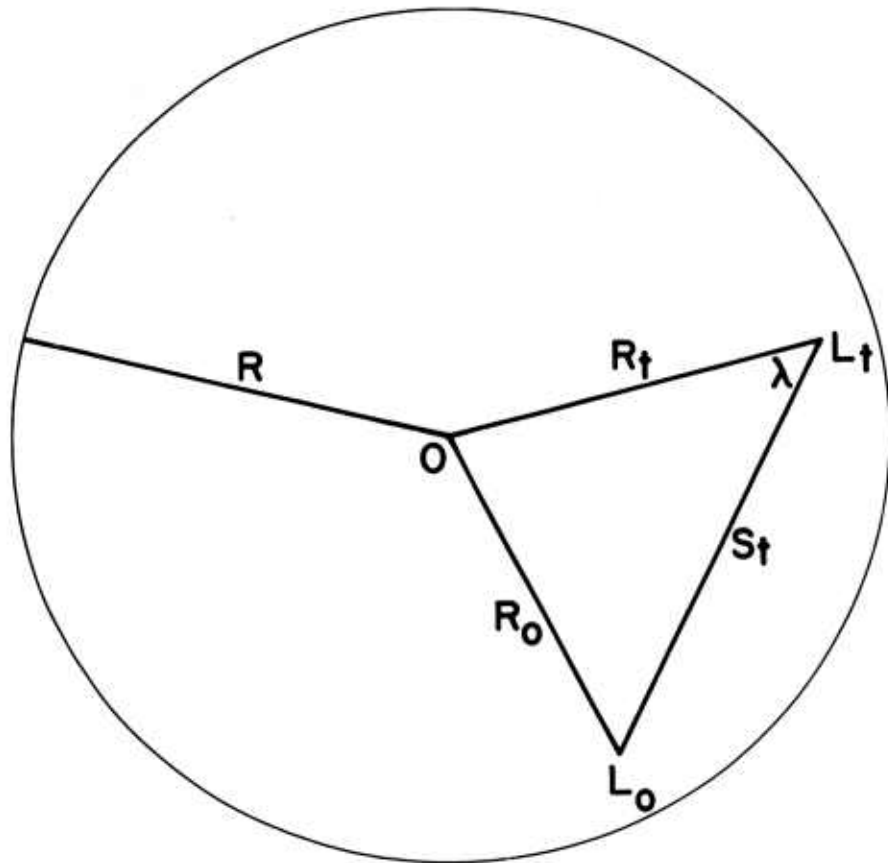


Figure 3. Probabilistic Location Geometry of an Object Moving in a Random (Unknown) Direction



and

$$f(R_o) dA = f(R_t, t) dA = \frac{1}{2\pi \sigma^2} \exp\left(-\frac{R_t^2 + S^2 t^2 - 2R_t St \cos \lambda}{2\sigma^2}\right) dA. \quad (57)$$

Defining the probability of an angle change,  $d\lambda$ , as  $d\lambda/2\pi$ , the probability  $P_t$ , of the object being at  $L_t$  and arriving there from any direction becomes

$$P_t = f(R_t, t) dA d\lambda/2\pi. \quad (58)$$

Also,

$$f(R_t, t) = \frac{1}{2\pi \sigma^2} \exp\left(-\frac{R_t^2 + S^2 t^2}{2\sigma^2}\right) J_0(i R_t St/\sigma^2) \quad (59)$$

where  $i$  is the imaginary term,  $\sqrt{-1}$ , and where

$$J_0(i R_t St/\sigma^2) = I_0(R_t St/\sigma^2), \quad (60)$$

a first kind, zero order Bessel function.

A full study of the search probability density function for a moving object would necessitate a separate detailed publication; but the treatment given here indicates the role of the normal bivariate density function in this type of search. The applications are, of course, in problems of drifting boats, submarine detection and intermittent surveillance, and, in the abstract mathematical sense, a search for optimum values in the regions of certain functions.

### CONCLUSIONS

This investigation has developed a reference paper for the normal bivariate density function. The derivation, properties, solution of the integral, and an indication of the applications should help to clarify the general understanding and role of this important analytical model. New uses will undoubtedly

be found in space systems development in addition to more obvious requirements for multivariate distribution models. In the interests of brevity, examples of correlation uses were not presented. Obviously, however, the applications pertain to any case where two normally distributed, random variables are mutually or singularly dependent.

It is interesting to note that there is sometimes an indirect dependence between mutually independent variables. E.g., increasing missile trajectory energy (lower apogee and attack angle) may increase the downrange standard deviation; but, since trajectory time is less, the crossrange standard deviation (partly a drift error) is decreased.

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## APPENDIX

### Elliptical Properties of Normally Correlated Distributions

The equal-frequency curves obtained by making  $f(x,y)$ , in equation (37), have constant values form an infinite system of homothetic ellipses. Each ellipse is expressed by the following equation:

$$\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - 2\rho \frac{xy}{\sigma_x \sigma_y} = Z_1^2 \quad (61)$$

The area of any of the ellipses is

$$A = \frac{\pi Z_1^2 \sigma_x \sigma_y}{1 - \rho^2} \quad (62)$$

The probability that a random event,  $(x, y)$ , will occur in any ellipse defined by assigning a value to the constant,  $Z_1$ , is expressed as

$$\frac{2\pi\sigma_x\sigma_y}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \int_0^{Z_1} \exp\left[-\frac{1}{2(1-\rho^2)} Z_1^2\right] Z_1 dZ_1, \quad (63)$$

$$\text{or } 1 - \exp\left[-\frac{Z_1^2}{2(1-\rho^2)}\right] \quad (64)$$

Analogous to the concept of Circular Probable Error is the term Elliptical Probable Error. The ellipse representing an EPE is sometimes referred to as the equal frequency ellipse or probable ellipse. It may be defined as that

ellipse of the system having a probability of 0.5 that a random event (x, y) will occur within it. Thus, from equations (63) and (64)

$$e^{-\frac{Z_1^2}{2(1-c^2)}} = 0.5 \quad (65)$$

$$\text{or } Z_1^2 = 1.3863(1-c^2) \quad (66)$$

From (64) it can be confirmed by inspection that the probability that an event (x, y) taken at random will occur in a small ring obtained by assigning values of  $Z_1$  in  $\Delta Z_1$  is as follows:

$$\frac{Z_1}{1-c^2} \exp \left[ -\frac{Z_1^2}{2(1-c^2)} \right] \Delta Z_1 \quad (67)$$

For a constant  $\Delta Z_1$ , the probability must be a maximum when  $Z_1^2 = 1 - c^2$ . Therefore, the ellipse of maximum probability is as follows:

$$\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{2cxy}{\sigma_x \sigma_y} = 1 - c^2 \quad (68)$$

This represents the ellipse along which more events are expected to occur than along any other ellipse of the system.

If now,  $Z_1^2 = 1 - c^2$ , is compared with  $Z_1^2 = 1.3863(1 - c^2)$  it is noted that the equal frequency ellipse, or probable ellipse, is larger than the ellipse of maximum probability.