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REVIEW of STRUCTURAL DESIGN TECHNIQUES for BRITTLE COMPONENTS UNDER STATIC LOADS

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RALPH L. BARNETT
Mechanics Research
Armour Research Foundation

ARMOUR RESEARCH FOUNDATION OF ILLINOIS INSTITUTE OF TECHNOLOGY

REVIEW OF STRUCTURAL DESIGN TECHNIQUES FOR BRITTLE COMPONENTS UNDER STATIC LOADS

Ralph L. Barnett

Contract AF33(657)-8339
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Aeronautical Systems Division
Air Force Systems Command
United States Air Force
Wright-Patterson Air Force Base, Ohio

ARMOUR RESEARCH FOUNDATION OF ILLINOIS INSTITUTE OF TECHNOLOGY

PREFACE

The second task of the first phase of Armour Research Foundation program 8259, UTILIZATION OF REFRACTORY NON-METALLIC MATERIALS IN FUTURE AEROSPACE VEHICLES, was a "Literature Review of Design Techniques and Analytical Methods". The second part of this task, a critical literature review, is presented here as Volume II of the Final Report on Phase I - Task 2; "Review of Structural Design Techniques for Brittle Components under Static Loads". Volume I, "Literature on Design Techniques and Analytical Methods for Brittle Materials", was issued during April 1963.

The program is being conducted for the Aeronautical Systems Division, Air Force Systems Command, the United States Air Force under Contract AF33(657)-8339. The report period is February 1963 to May 1963. Mr. R. L. McGuire of the Flight Dynamics Laboratory is Technical Monitor for ASD.

The author acknowledges the assistance of P. C. Hermann who proof-read the draft of this report and supervised the preparation of the final report. Dr. N. A. Weil, Director of Mechanics Research, is overall program manager.

Respectfully submitted,

ARMOUR RESEARCH FOUNDATION OF
ILLINOIS INSTITUTE OF TECHNOLOGY

R. L. Barnett, Task Leader

APPROVED BY:

N. A. Weil,
Director of Mechanics Research
and
Program Manager

REVIEW OF STRUCTURAL DESIGN TECHNIQUES FOR BRITTLE COMPONENTS UNDER STATIC LOADS

by

Ralph L. Barnett

ABSTRACT

A systematic review of structural design techniques for brittle materials is conducted with particular emphasis on statistical fracture theories. The properties of both series and parallel statistical models are discussed, and a critical analysis is given for the experimental data which is available in the literature. The relationship of extreme value statistics to the fracture problem is thoroughly exploited. A survey of brittle test specimens and design rules-of-thumb is included, together with a number of very promising design philosophies.

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REVIEW OF STRUCTURAL DESIGN TECHNIQUES FOR BRITTLE COMPONENTS UNDER STATIC LOADS

I. INTRODUCTION

This report presents a critical and comprehensive review of structural design and analysis techniques for materials which behave in a brittle fashion. In the first volume of this final report, an extensive literature search was conducted which formed the starting point for the present investigation. A systematic study of this literature was conducted and the following specific objectives were pursued in the presentation of this effort:

1. Report the results of significant scientific investigations of brittle behavior from the point of view of the structural designer.
2. Collect significant test results and discuss them in the light of available mathematical models.
3. Study the implications of popular analysis method to the design problem.
4. Develop a series of questions which will aid the selection of suitable methods of analysis.
5. Simplify some of the more compact presentations found in the open literature.
6. Prepare a sufficiently comprehensive report that it may serve as a springboard to further investigations in "brittle" design.

Design philosophies, shortcomings in the current state-of-the-art, and rules-of-thumb will be found in various places in this document. Considerable care was devoted to the documentation of source material and it is hoped that we have been reasonably successful in recording our acknowledgments.

II. PHILOSOPHY OF STRUCTURAL DESIGN

II-A. Deterministic and Probabilistic Approaches to Design

II-A-1. Structural Design Problem

The problem of structural design is the disposal of material in such a way that it will, within some level of probability, equilibrate given systems of applied force under appropriate environmental conditions without exceeding permissible amounts of deflection. Formulated in this way the solution to the design problem is not unique and the various possible designs are called adequate designs to indicate that they merely represent a synthesis which satisfies the functional requirements within the confines of existing limitations. If, in addition, designs are required to be minimum weight or minimum cost, we enter the more specialized fields of minimum weight design or minimum cost design. Because we are at present more concerned with the feasibility of the widespread use of brittle materials in structures, rather than efficiency, we are certainly in the area of adequate design.

The theory of adequate structural design is for the most part concerned with stress or deflection analysis of given structures. This means that in practice it can only be used in design by a process of trial and error, in which the structural layout and sizes are first guessed or very roughly calculated, and are then subjected to as complete analysis as the theory will permit. The results of these calculations are compared to some performance yardstick and on this basis the various parts of the structure are judged to be adequate or inadequate. The design is then modified and the thorough-going analysis repeated as a check. In the case of a ceramic structure, the material would be approximated as linear, elastic, isotropic, and homogeneous and a thorough analysis would result in a description of the stresses, strains, and deflections throughout the structure. Now, two sticky questions remain: what is the performance yardstick, and how do we modify an inadequate design. We shall at present concern ourselves only with the first question.

II-A-2. Deterministic Theory

According to the classical theory, the ultimate strength of a material is determined by the internal stresses at a point, assuming that by a suitable

combination of the three principal stresses or strains a characteristic value may be computed for the material in question. This value is supposed to be definitively decisive in judging whether the ultimate strength has been reached or not. This phenomenological approach attempts to describe the reaction of a solid to an external stress system by predicting the behavior of the solid under a triaxial stress state from experimental data obtained from simple uniaxial tests. The following important phenomenological theories were reviewed in Reference 1 in 1952 as possible candidates for the description of the strength of ceramic materials:

1. Maximum stress theory
2. Maximum strain theory
3. Maximum shear stress theory
4. Maximum strain energy theory
5. Distortion energy theory
6. Internal friction theory
7. Mohr theory
8. Stress invariant theory

For materials which exhibit plastic flow (ductile materials), there is considerable test evidence (biaxial) to show that the distortion energy theory is in good agreement with the test results for defining failure by yielding. Both this theory and the maximum shear stress theory are generally recommended for design when the yield strengths in simple tension and simple compression are equal. For ductile materials which exhibit a considerable difference between their tensile and compressive yield strengths, either the internal friction theory or Mohr's theory give good approximations under biaxial stress conditions [2]. For materials which evidence little or no plastic flow before fracture (brittle materials), experimental measurements give many results which may hardly be brought to agree with any of these phenomenological theories.

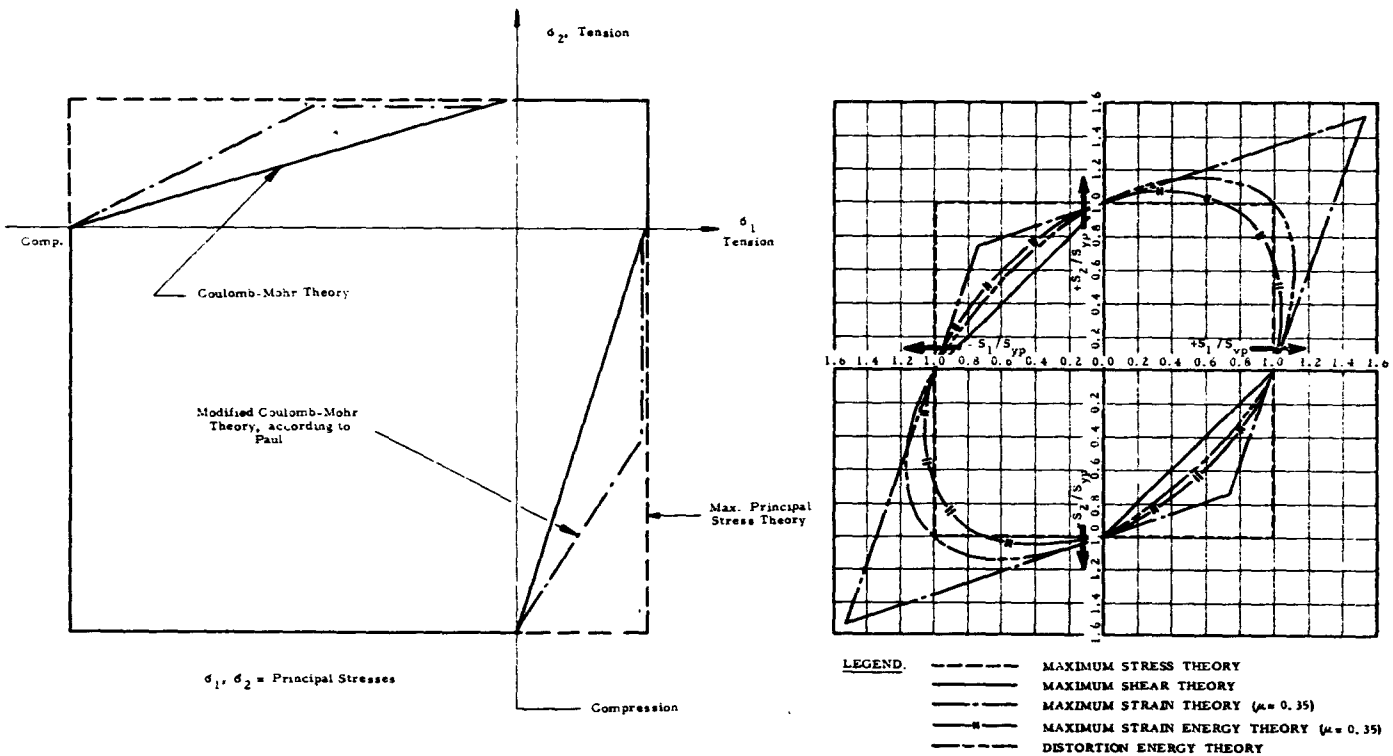
The validity of the hypothesis that failure can be predicted from stress conditions in one single point has been doubted for many years[3]. If the hypothesis is correct and one of the eight failure theories can be used, the designer is in a position to predict the behavior of large structures from simple uniaxial tests. If none of the eight failure theories can be used, a

deterministic approach is still feasible although the question of practicality assumes serious proportions. One can construct a two-dimensional fracture interaction curve similar to those shown in Figure 1, by plotting the data obtained from fracture experiments performed with various ratios of the two principal stresses. In three dimensions most of the corresponding fracture surface can be obtained from experiment with the notable exception of the states of triaxial tension. States of triaxial tension which can be determined from equilibrium conditions alone are currently unknown, and one must resort to the theory of elasticity to determine such states of stress. At best, it is clear that the experimental construction of the fracture surface requires an extensive testing program.

Let us now consider the case where the hypothesis is incorrect, i.e., the strength of a part cannot be predicted from the states of stresses at the various points. Here, not only does the stress state enter as a strength parameter, but perhaps, in addition, the size of the part or its shape (stress gradient) are explicitly involved. Since the range of these additional parameters is essentially unlimited, any direct experimental approach to the development of a general strength criterion would be quite impossible. However, there are two realistic approaches: (1) develop a theory which accounts for the effects of stress state, size, shape, temperature, etc., or (2) conduct service tests on each structural component. We shall comment further on these two approaches after we introduce a further complication.

II-A-3. Statistical Theory

All of the above remarks have been directed toward a deterministic concept of material behavior. We shall now introduce a statistical point of view into the design problem by considering any simple strength test repeated sufficiently often with nominally identical specimens. The resulting ultimate strengths will show a scatter caused by many unknown factors. The designer has never concerned himself with this scatter in ductile design since the range of yield strengths obtained from, say, a thousand tests is so small that one can predict the value of the 1001st test with great confidence. On the other hand, with brittle materials the dispersion of results is so large that even with the thousand test results available, little of firm value can be said regarding the strength of the 1001st test, other than the



a.) Fracture Theories

b.) Yield Theories

Figure 1

TWO DIMENSIONAL REPRESENTATION OF STRENGTH THEORIES

statement that its fracture stress must be non-negative. If, on the other hand, we consider another group of 1000 specimens, we might ask if there is not some relationship between this group and the original. Indeed, the entire foundation of statistics depends on the existence of such a relationship. Since we cannot predict the result of an individual test, we are forced to compromise by predicting only the behavior of a group of tests. One would, therefore, hope that the average stress in the original group of 1000 specimens would be reproduced in subsequent groups of specimens. To be really useful it would be desirable to predict the entire distribution of strengths in subsequent groups of tests. Here, however, we must again accept a compromise in predicting even the behavior of a group since the distribution of strengths can only be estimated when the sample tests are finite in number. Before we deal further with these questions, let us look a little deeper into the nature of the dispersion of fracture strengths.

There is a mechanistic approach to the theory of fracture which analyzes strength properties from the point of view of what makes the material fail. These theories are relatively new, and are concerned with the fact that materials may fracture at stresses 100 to 1000 times below their theoretical breaking strengths. The presence of defects in the material such as cracks, dislocations, and other possible flaws is postulated to account for these disparities. This implies that the strength of a brittle material is determined by a state of stress prevailing within a very small volume whose individual properties determine those external forces at which the ultimate strength is reached. This view is based on a conception set forth by Griffith[4] and later developed by Smekal[5]. Griffith developed his theory in an attempt to explain the extraordinary weakening of materials by surface scratches whose areas were but a minute part of the cross sectional area of the material. He assumed that the scratches or cracks acted as stress concentrators and that, at their extremities, the stress reached approximately the theoretical strength of the material. The calculations of Inglis[6] were used to determine the stresses and strains resulting from typical scratches. The predictions of these calculations were borne out by tests on glass fibers. It should be pointed out that although the mechanistic theories postulate a mechanism of fracture, they still require an

assumption of the criterion for fracture, as do the phenomenological theories. Fracture still is considered to occur at some condition of stress or strain; however, the mechanistic theory has the advantage of providing a description of fracture phenomena.

The Griffith theory raises the question of how many flaws are contained in a specimen. Since Griffith's theory assumes that failure takes place at the most critical flaw in a random distribution of flaws, the large scatter of tensile strengths obtained from brittle materials would indicate that the number of critically oriented cracks cannot be very large. On the other hand, it can be deduced from Smekal's[7] work on glass rods that the total number of flaws is quite large. Smekal found that glass rods broken in tension form a mirror surface which initiates at a critical flaw. Extending from this flaw, the smooth surface gradually gives way to a rough one. He explains this behavior as a slow initial fracture which creates a mirror surface in a plane normal to the applied stress followed by the propagation of lesser flaws due to the transfer of stress which results from the decreased cross section caused by the initial fracture. These secondary flaws eventually run into the initial fracture surface creating an area of increasing roughness. The important implication of these works, from the design point of view, is that the strength of a specimen must decrease with size due to the greater probability of its containing a critical flaw.

The size effects on brittle fracture have been noted for glass by Griffith [4], Weibull [8], and Smekal[9], for gypsum by Duckworth[1] and Patton and Shevlin[10], for porcelain by Weibull[11], Milligan[12], and Bortz[13], for steel at low temperatures by Davidenkov [14], and for aluminum oxide and magnesium oxide by Bortz[15]. These studies, as well as work by others, give credence to the flaw hypothesis and, in addition, point up the inadequacy of the classical theories which predict a unique strength independent of the specimen size.

II-B. Implications of the Statistical Approach to Design

II-B-1. Existence of a Distribution Function

If, in an array of experimentally determined data, we plot the frequency with which each value is observed against the value itself, we have a

means of showing the relative frequency with which we may expect a particular value to occur. Such a plot will usually show the frequency rising to a maximum figure at some point and falling off again as we move from left to right. Extreme values do not occur as often as those near the center. The curve may or may not be symmetrical about the maximum frequency. A mathematical expression of this curve is called the frequency distribution, or the probability density function, and shall be denoted by f(x). Figures 2(a, b, c) are three such frequency curves.

By proper normalizing of the density function, the total area under the curve between its limits can be made to become unity, and by integrating this latter expression, we obtain the cumulative probability function, or, more simply, the distribution function, which we denote by F(x). Thus, a frequency function for a continuous random variable x possesses the following properties:

$$0 \leq f(x) \qquad \text{...frequency function} \tag{1}$$

$$\int_{-\infty}^{\infty} f(x)\, dx = 1 \tag{2}$$

$$\int_{a}^{b} f(x)\, dx = P\left\{a < x < b\right\} \tag{3}$$

$$F(x_o) = P\left\{x \leq x_o\right\} = \int_{-\infty}^{x_o} f(t)\, dt \qquad \text{...monotonically non-decreasing} \tag{4}$$

$$f(x) = F'(x) \tag{5}$$

where $P\left\{x \leq x_o\right\}$ is the probability of choosing at random an individual event having a value of x equal to or less than x_o, and where a and b are any two values of x with $a < b$.

Property (1) is obviously necessary since negative probability has no meaning. Property (2) corresponds to the requirement that the probability of an event that is certain to occur should be equal to unity. We see that by using the distribution function F(x), it becomes possible to estimate the probability that an individual occurrence (x) will be equal to or less than

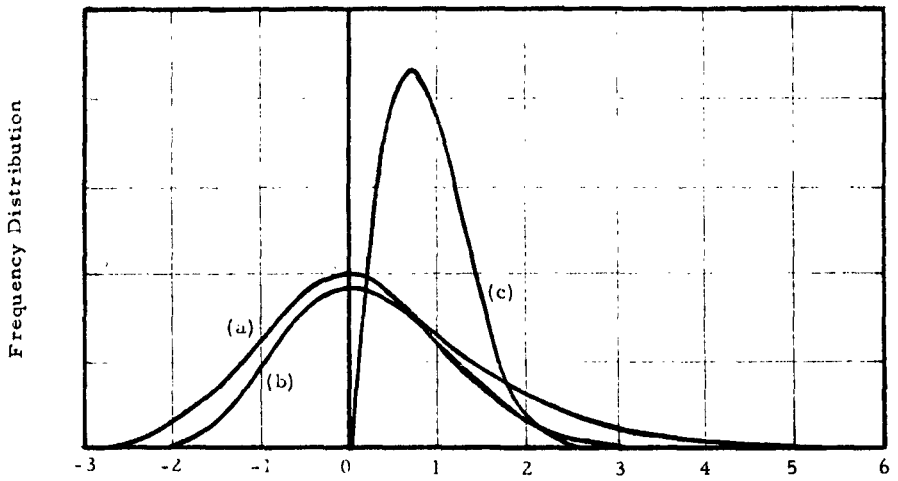


Figure 2

THREE TYPICAL DISTRIBUTION FUNCTIONS
(a) NORMAL; (b) GUMBEL; (c) WEIBULL ($x_u = 0$, $m = 2$)

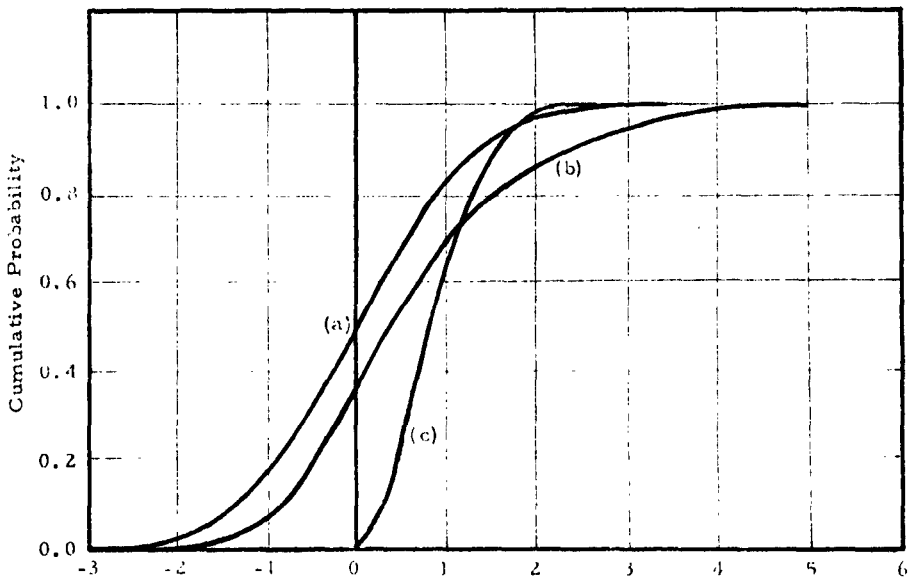


Figure 3

CUMULATIVE PROBABILITY CURVES OF THE THREE DISTRIBUTIONS
(a) NORMAL; (b) GUMBEL; (c) WEIBULL ($x_u = 0$, $m = 2$)

any assigned level of interest. Given this function for the strength of a particular component, a designer could select a design stress corresponding to any desired probability of failure from zero to 100 percent. Figures 3(a, b, c) show the cumulative probability curves for the respective frequency distributions indicated in Figures 2(a, b, c). It will be noted that the cumulative probability curve is, in general, an ogive in form, approaching or passing through zero (absolute impossibility) at one extreme and asymptotically approaching the value of unity (absolute certainty) at the other.

Because the existence of a cumulative distribution function is the most important assumption of any statistical theory, we shall briefly inquire into the meaning of existence. If we consider the diameter of apples and oranges as a statistical variate, we shall find that each of these fruits has a different probability distribution. A fifty-fifty mixture of these fruits has a distribution of diameters and using conventional techniques they can be classified and plotted as a cumulative distribution curve. This curve will of course be different from the curves for oranges alone or apples alone. If another distribution curve were constructed for a 40-60 combination of the fruits, this curve would be different from any of the others. In general, the distribution curves for the oranges alone and the apples alone would suffice to determine the distribution curve of a <u>known</u> combination of the fruits; however, if the proportions are unknown, one cannot find the distribution curve. The situation is even more hopeless when we try to predict the distribution curve of an unknown combination of fruits having available only the distribution curve of an unknown randomly selected combination.

We shall consider a distribution curve to exist for a population when every infinite sample taken from the population has the identical distribution function. From a practical standpoint, we can consider a distribution curve to exist when various large subsets of a large sample all give practically the same distribution curve.

Based on the author's conversations with leading investigators in the field of ceramics, it would appear that many manufacturers of ceramics produce materials which vary from day to day, batch to batch, and some-

times item to item. Sufficiently large variations in the composition and/or treatment of a material precludes the existence of a single material population, and hence, the existence of a distribution function. There are several instances in the literature where there is considerable doubt concerning the validity of the population samples studied, and this doubt demands that attention be directed to this matter for every new material investigated.

II-B-2. General Comments on Distribution Functions

In Figures 2c and 3c we observe that in the Weibull distribution both the variate and the distributions are limited on the left at the value zero. Indeed, there is a form of the Weibull distribution which accounts for a limited variate on the left and right for any real values. If such a distribution were obtained for the strength of a part, the lower limit σ_u would have the significance that parts stressed at or below this level would never fail. If we accept Griffith's flaw hypothesis, the question of whether σ_u equals zero or a finite value is equivalent to questioning the existence of a limiting intensity of a flaw. If such a limit exists, a conservative design procedure might be to assume a uniform saturation of such critical flaws throughout a structure. In trying to detect the presence of a finite σ_u in various test results in the literature, one is concerned with the existence of residual stresses which are known to cause failure without the application of external loading[16], and the fact that many of the lowest strength specimens are ruptured during manufacture or subsequent handling and never get to the testing machine.

Many useful distributions in statistics are based on unlimited variates. Here "common sense" revolts at once and practical people will say: "Statistical variates should conform to physical realities, and infinity transcends reality. Therefore, this assumption does not make sense." This objection is not as serious as it looks, since the denial of the existence of an upper or lower limit is linked to the affirmation that the probability for extreme values differs from unity (or from zero) by an amount which becomes as small as we wish.

To construct a distribution curve for the strength of a given component, we would theoretically require an infinite number of tests of the

component. When we use a finite number of tests we obtain the typical curve shown in Figure 4. This figure, taken from the excellent report by Salmassy[17], indicates that the rareness of extreme events precludes a good definition of the upper and lower limits of the distribution curve. On the other hand, high reliability demands that the designer concern himself with the very low probabilities of failure associated with the lower limit of the distribution curve. The alternative to a very extensive testing program is to find a mathematical description of the distribution function which properly describes the available data, and then use this to determine the stresses associated with low probabilities. Indeed, this is the only possible procedure for finding the zero probability strength, i.e., the stress corresponding to zero probability of fracture

One of the tools for finding the correct description of a probability curve is probability paper. By a suitable transformation of the probability scale in a cumulative probability plot, special graph papers may be constructed so that in many cases the cumulative probability curves will plot as straight lines This has the obvious virtue that extrapolation becomes quite easy. This point is illustrated in Figure 5, where the same distribution curve is plotted using a cartesian scale and using a Weibull scale. We note that the conventional distribution curve cannot be used to predict the zero probability strength σ_u = 10,000 psi. A second feature of probability paper is that it can be used to select the proper distribution function Figures 6a and 6b, taken from Gumbel[18], show how the first Laplacean and the normal distributions, which are indistinguishable when plotted conventionally, are separated on normal probability paper. Probability paper is extremely useful in determining the existence of certain types of distribution functions. The normal probability plot of thirty I.Q. scores in Figure 7 shows three straight lines indicating that the data arise from different normal universes having different averages - the scrambling together of three distinctly different sets of data. If the lines were not approximately parallel as shown, this would indicate a probable difference in the precision (reproducibility) of the three sets of data as well as in the averages. As an additional example, consider the fatigue life of an ST-37 steel which was described by Weibull[19]. The frequency curve for the steel is shown in Figure 8a, and the associated

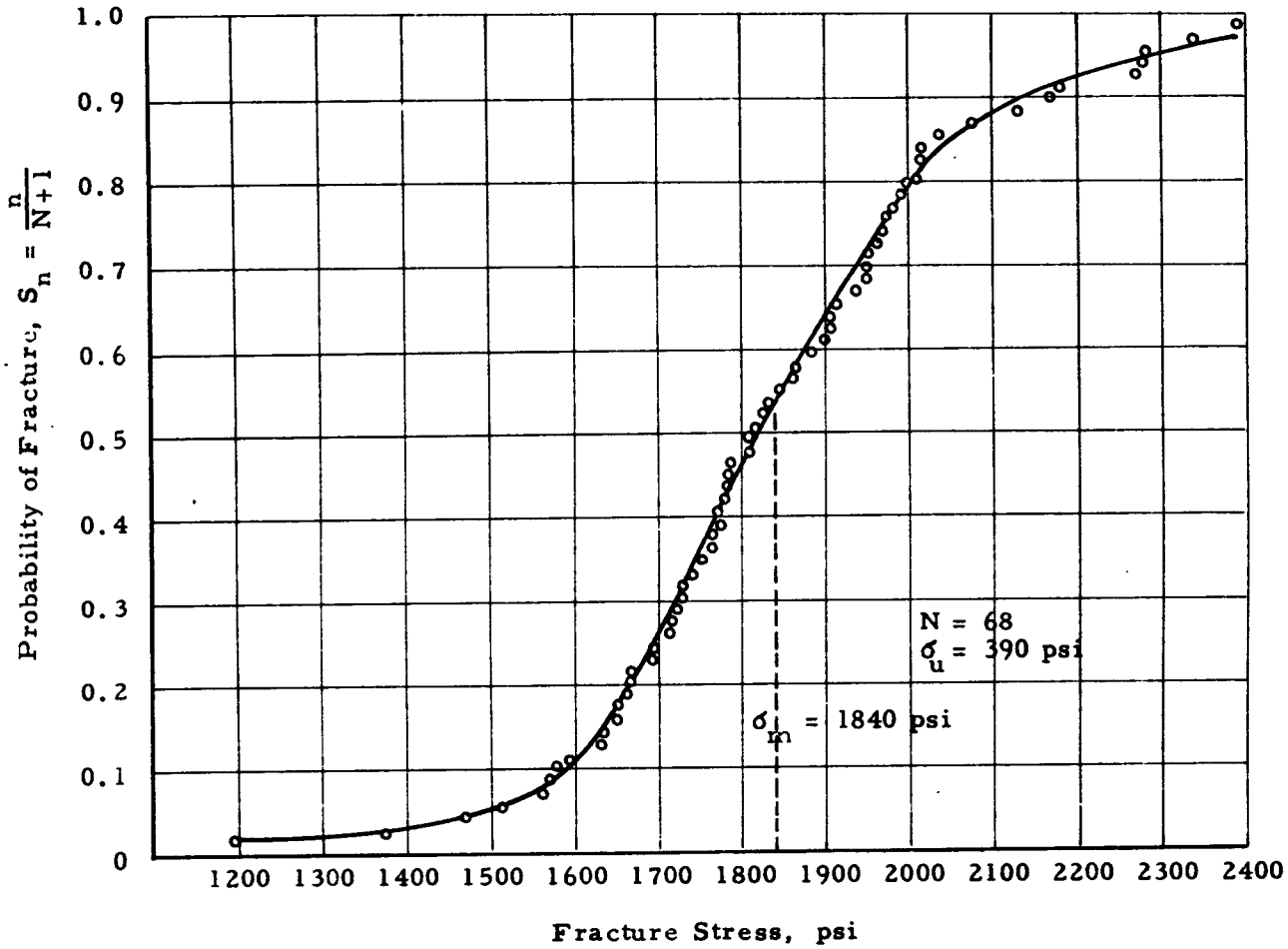
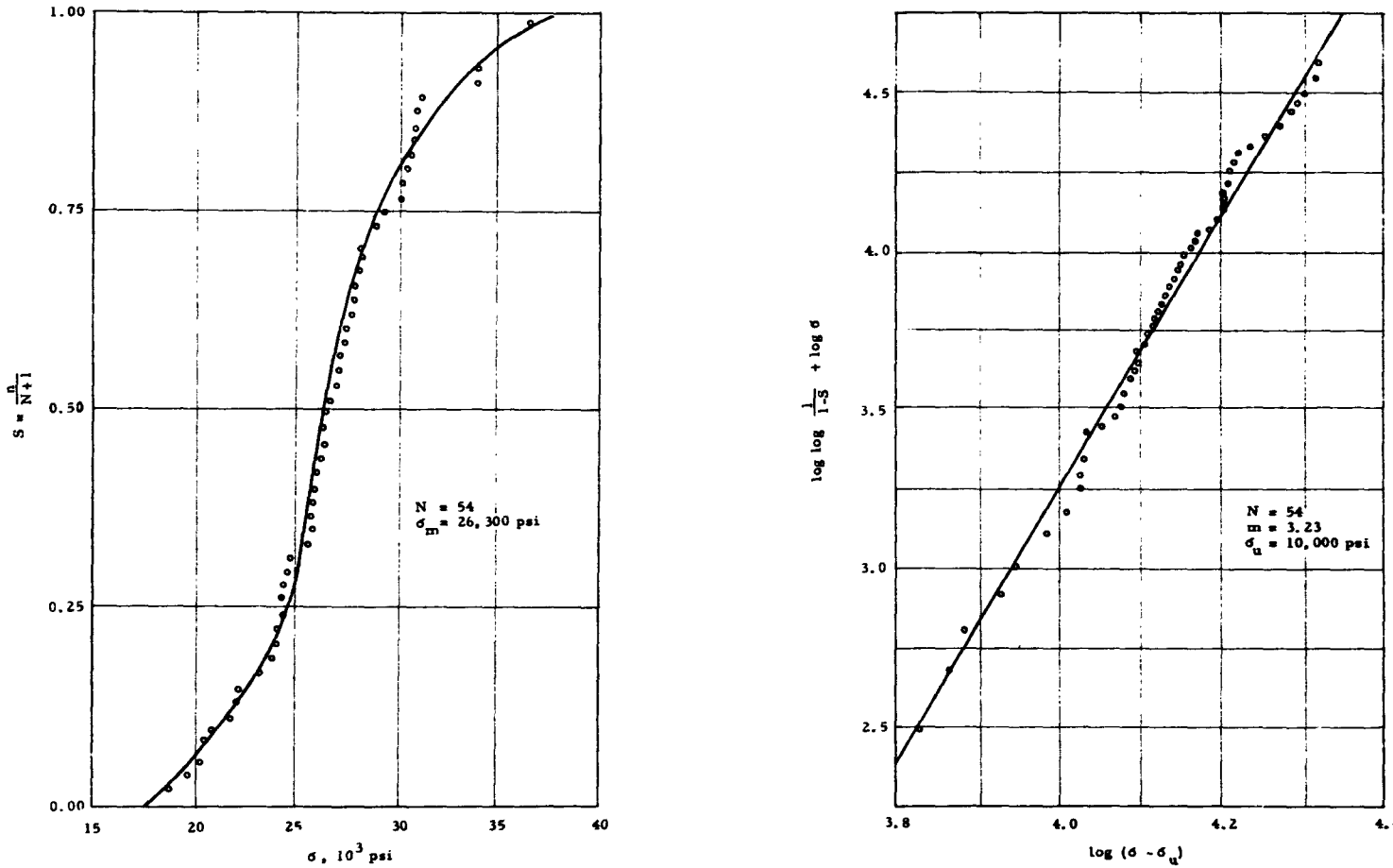


Figure 4

DISTRIBUTION CURVE OF FRACTURE STRESSES FOR TYPICAL BENDING SPECIMENS OF HYDRO-STONE PLASTER (AFTER SALMASSY, REF. 17)

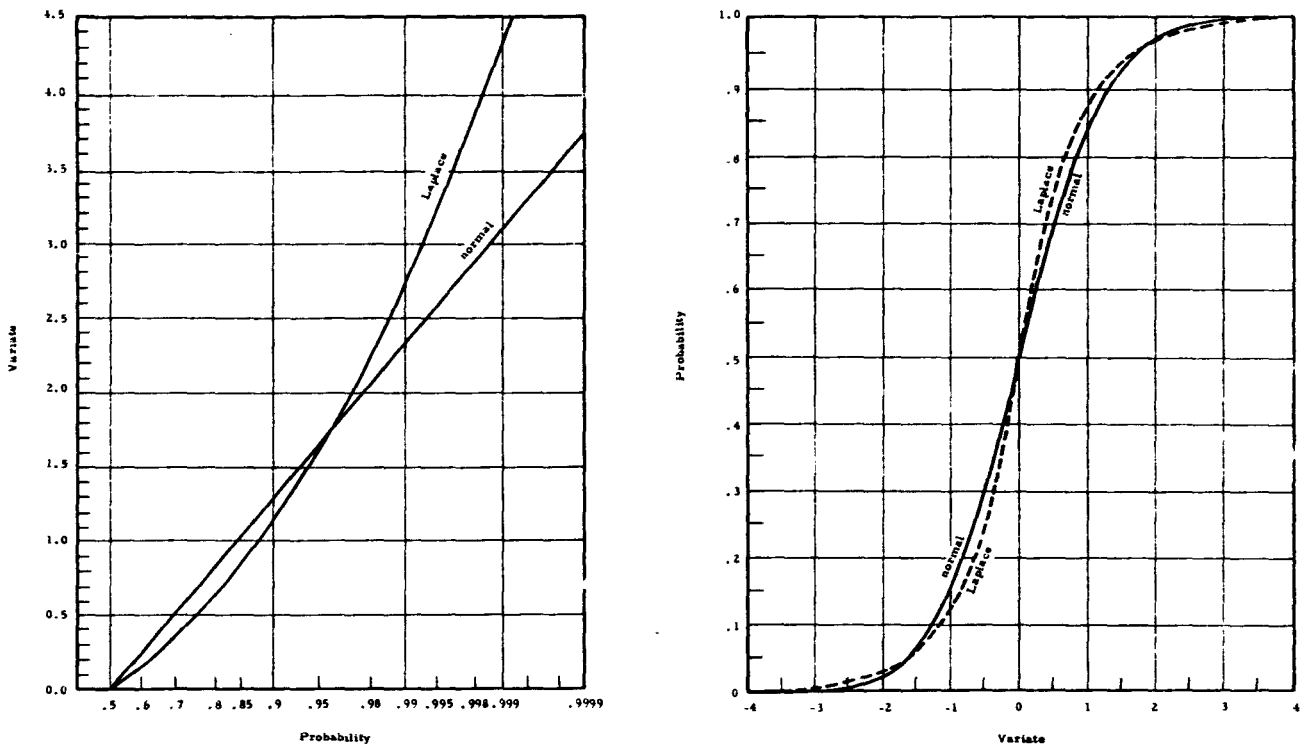


a.) Conventional Plot

b.) Weibull Plot

Figure 5

FRACTURE CURVES FOR FRACTURE STRENGTH OF AS-RECEIVED WESGO AL995 AT 20°C (AFTER BORTZ, REF. 15)



a.) Normal Probability Paper

b.) Cartesian Paper

Figure 6

THE FIRST LAPLACEAN AND THE NORMAL PROBABILITY FUNCTIONS
(AFTER GUMBEL, REF. 18)

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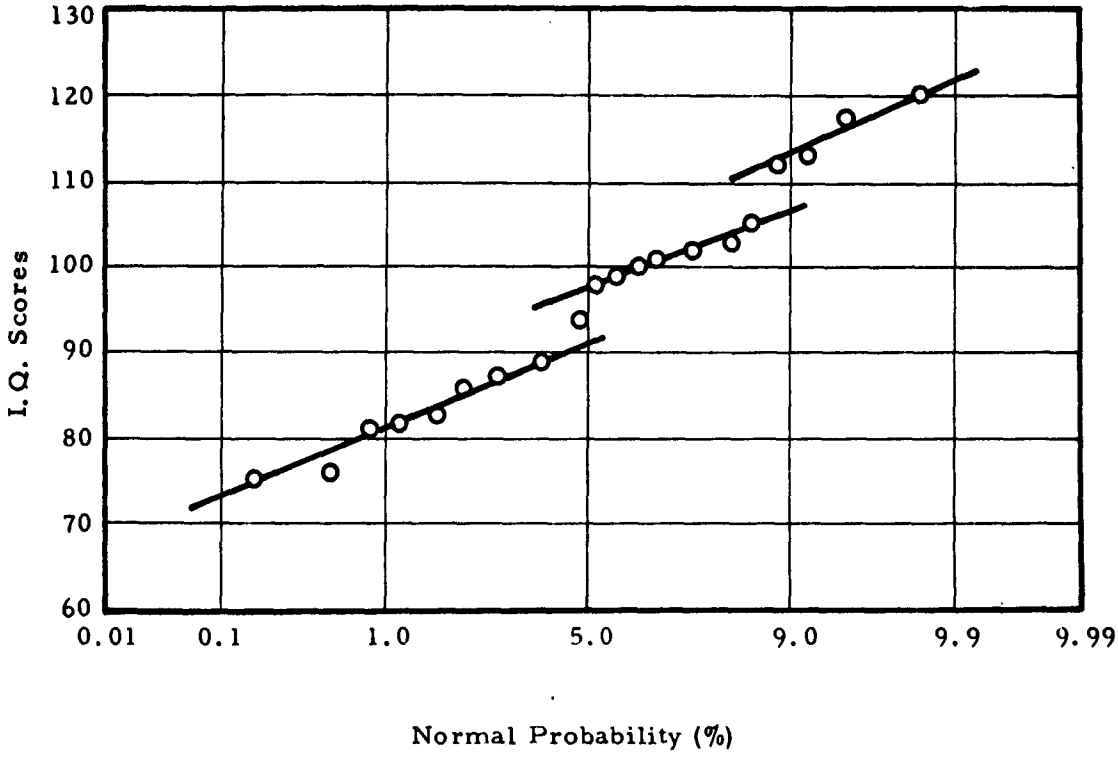
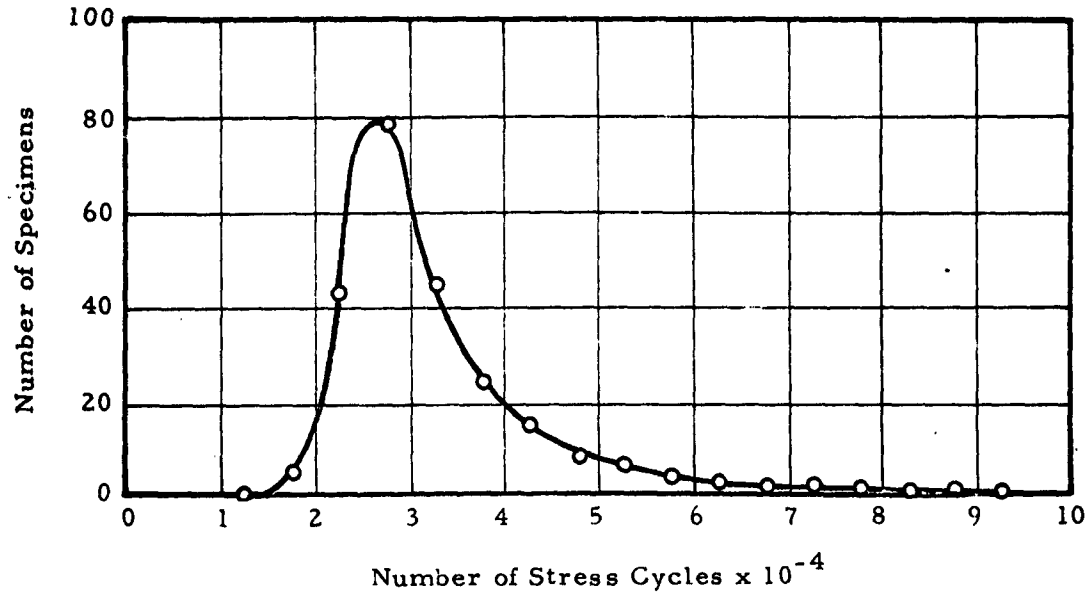
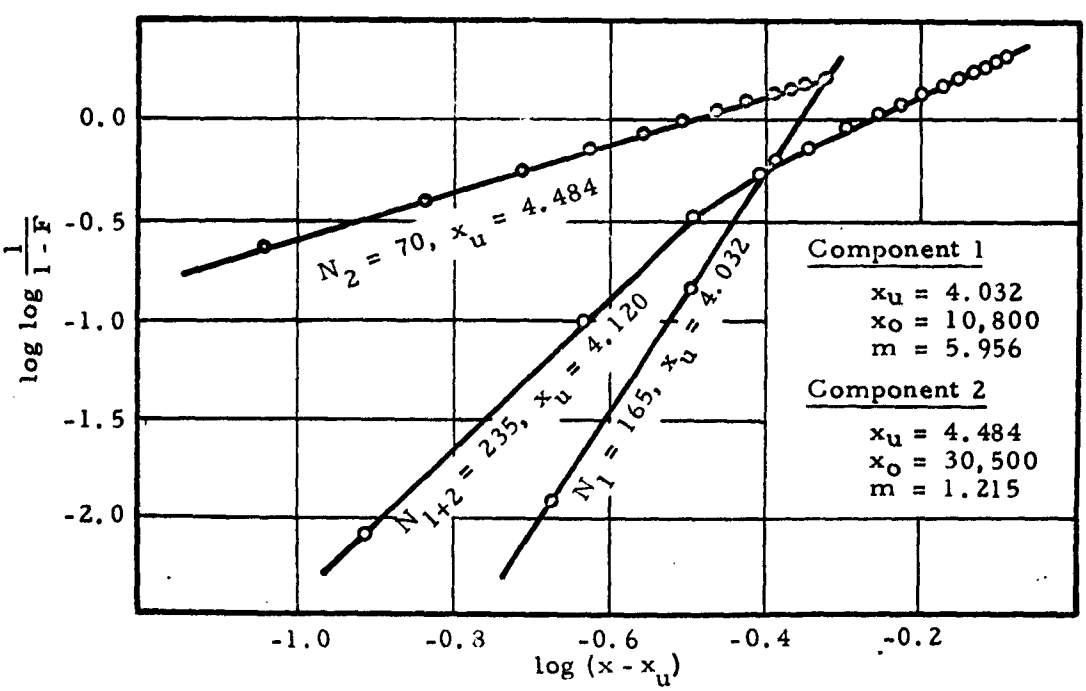


Figure 7

NORMAL PROBABILITY PLOT OF THIRTY I.Q. SCORES
(AFTER LEWIS, REF. 21)



a.) Frequency Curve



b.) Distribution Curve

Figure 8

FATIGUE LIFE OF ST-37 STEEL (AFTER WEIBULL, REF. 19)

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distribution curve plotted on Weibull scales is shown in Figure 8b. The bi-linear curve labeled N_{1+2} represents a mixed universe composed of two populations which each follow a Weibull distribution. No indication of this so-called "complex distribution" is evident from the frequency curve. Kao[20] has presented a graphical technique for unscrambling these complex distributions so the mixed parameters may be estimated.

Probability paper is used extensively in the study of extreme values since it provides a simple graphical method of testing the fit between theory and observations without bothering with troublesome calculations. It may also be used as a criterion for the acceptance or rejection of extreme observations. The paper by Lewis[21] provides a good introduction to the use of probability paper; an extensive treatment can be found in Gumbel[18].

In all of the foregoing discussions, it has been assumed that the distribution curve was derived from a series of tests on bodies of the same size and shape and under exact service conditions, such as temperature, loading, atmosphere, etc. The present knowledge of brittle materials indicates that the mean fracture stress may be affected significantly by such factors. Since the mean stress is a point on the distribution curve, this would indicate that the entire distribution curve may be affected by such factors. Unless the influence of these various factors on the distribution curve can be reasonably predicted, the designer will be required to conduct a series of service tests on each structural component in order to obtain the proper distribution curve needed in the design of the structure.

II-B-3. Statistical Input → Statistical Output

It is the goal of any design procedure to predict the behavior of relatively complex elements from the characteristics of one or just a few simple elements. Specifically, in the case of brittle materials which are statistical in their behavior, we seek a way to derive the probability of survival of a brittle structure under service conditions from data obtained from a simple statistical test. The implication is, that any analysis will at best be able to predict the behavior of a large number of structural elements. The behavior of an individual element still cannot be predicted with any degree of certainty.

Having the survival probability of a structure, the designer is able to predict either:

1. The number of service failures, or
2. The number of prototypes required to produce one prototype which is satisfactory (return period).

In this latter situation, one must have some means of selecting the good parts from the bad, i.e., a service or proof test must be performed on the various prototypes. If the loading conditions are exactly known and the proof test does not damage the one (on the average) good prototype of the group, the resulting structure is 100 percent reliable. It would be reasonable, with this latter procedure, to design parts with a probability of survival of, say, 10 percent, i.e., one good part out of ten. Using the first procedure, one would have to design an aerospace component for an extremely high survival probability of, say, one chance of failure in 100 or in 1000. Referring to Figure 4, the first procedure would then call for a design stress of about 500 psi, whereas in the latter procedure the design stress is 2150 psi.

If we can predict the probability of failure of a structure from the behavior of a simple test, we can relate this probability to any parameter of the structure, such as the working stress, an effective thickness, or perhaps the total weight. This enables the designer to make a rational compromise between the parameter chosen and the survival probability. When the parameter chosen is a statistical variate (perhaps the ultimate load of the part), the relationship between the parameter and the survival probability is precisely a distribution function for the structure.

II-B-4. Series and Parallel Elements

If an element of a structure fails, the entire structure either fails or does not fail. The first type of element, like the link in a chain, limits the strength of the structure to that associated with the failure load of the element. Such elements are called series elements. The second type of element, called a parallel element, may or may not contribute to the strength of the structure; but it does not necessarily limit the strength of the overall structure to that of the element. An example of a parallel element is a strand in a cable.

The treatment of series and parallel elements is entirely different, and for this reason we will study each separately. However, to quote Epstein[22], "In any real problem the situation is probably neither that of elements in series or in parallel, but rather elements distributed in some rather complicated arrangement." In the author's opinion, the fact that the state of the art does not provide a method for dealing with the combined parallel-series problem represents the single greatest shortcoming of statistical strength theory.

III. SERIES MODEL

III-A. Heuristic Approach

III-A-1. Chain Model

Assume that we have a chain made up of n nominally identical links. If we have found, by testing, the probability of failure S at any load x applied to a single link, and if we want to find the probability of failure S_n of the entire chain, we have to base our deductions upon the proposition that the chain as a whole has failed if any one of its parts has failed. Accordingly, the probability of nonfailure of the chain, $(1-S_n)$, is equal to the probability of the simultaneous nonfailure of all the links. Thus we have,

$$(1 - S_n) = (1 - S)^n \qquad S(x) \ldots \text{ distribution function of link} \tag{6}$$

For a very reliable chain we want the probability of failure S_n to be small; consequently, we must use low values of the load x so that the probability of failure S(x) of each link is small. For small S, Eq. 6 becomes

$$(1 - S_n) \doteq 1 - nS \quad \text{or} \quad S \doteq S_n/n \tag{7}$$

Thus, we find the alarming fact that for a large reliable chain the probability of failure of the links must be very, very small. We observe that when $n \longrightarrow \infty$, $S(x) \longrightarrow 0$; hence, in this situation a design is possible only for a distribution function S(x) which is limited on the left, i.e. only if $S(x) = 0$ gives $x = x_u > 0$, where x_u is the zero probability strength below which no failure can occur.

To get a feel for the implications of this model, we will assume that a structure is made up of elements which act like the links of a chain. A low

guess for the ratio between the volume of a typical structure and the gage volume of a typical bending or tension specimen is 1000; hence, let $n = 1000$. Now, even if we are willing to accept nine bad prototypes for every good one, we must still choose a design stress low enough so that only one specimen fails in about every 1000. When we try to read off a probability of only 0.001 from a typical distribution curve such as that shown in Figure 4, we meet with great difficulties. This points out very clearly the importance of having a well-defined lower portion of the distribution curve. Furthermore, we can observe from Figure 4 that for low probabilities a small decrease in the probability of failure corresponds to a large decrease in stress. Thus, we find that the total weight of a structure increases very rapidly for a small increase in reliability. Finally, if we now use a test specimen which is 50 times as large as that assumed in the beginning of this example, we get $n = 20$, which means that an $S_n = 90\%$ requires that $S \doteq 5\%$. At the 5% probability level, the probability curve in Figure 4 has very good definition.

We can summarize the results of this simple chain model as follows:

(1) Reliable design calls for a well defined lower portion of the distribution curve.

(2) Very large structures are not possible unless a zero strength exists.

(3) Bigger specimens are better specimens.

(4) Large reliable structures in brittle materials must work at low tensile stress levels and will consequently be heavy.

(5) A small increase in reliability calls for a large increase in weight.

III-A-2. Design Procedure Based on a Weakest Link Model

If we assume that the various elements of a structure are series elements, we can generalize slightly on the chain model previous discussed, and develop a general design procedure. This procedure will be conservative in the sense that we are assuming the strength of the structure to be limited by either a series or a parallel element. We shall make two additional assumptions. First, we assume that the probability that rupture occurs in a given volume subjected to any uniform stress may be completely

determined by a quantity x which may be calculated from the three principal stresses. Second, we assume that the various distribution functions used in the procedure exist and are obtainable in at least graphical form.

Now, consider a structure subjected to an environmental history of load and temperature.

I. Divide the structure into n convenient volumes $V_1, V_2, \ldots V_n$.
   A. Make as many volumes identical as possible.
   B. Choose statistically homogeneous volumes.
   C. Contain "special conditions" in separate volumes (such as stress concentrations).

II. Determine the critical generalized stress x for each volume and for each environmental condition (loading, temperature, atmosphere, etc.).

III. Taking the critical stress to be uniformly distributed throughout its associated volume, determine the probability curve for each V_i under each environmental condition.
   A. Determine $S_i(x)$ by testing the assumed element, or
   B. Determine $S_i(x)$ using simple tests and theory.

IV. Determine the largest probability of fracture for each volume, $S_1, S_2, \ldots S_n$, using the various x's and probability curves.

V. The probability F that the structure will survive the entire environmental history is given by

$$1 - F = \prod_{i=1}^{n} (1 - S_i) \tag{8}$$

We note that the division into volume elements may be carried out in any manner and that the number of elements n may be finite or infinitely great. We note further that the assumption that x be taken as uniform throughout each volume is conservative for finite volumes and exact when the volumes are taken infinitely small. Finally, we observe that the probability of failure of an element depends only on the stress acting on the element and in no way depends on the conditions which exist in neighboring

volumes. For example, we see that the problem has been formulated in such a way that the probability of fracture of the structure does not depend explicitly on the stress gradient.

III-A-3. Weibull's Theory

111-A-3-a. Development

Using Griffith's flaw theory as a starting point, Weibull[11] reasoned that there would be a distribution of strengths in a given specimen in the sense that a different amount of force will be needed to fracture a specimen at one or another point. This reasoning is, of course, purely conceptual in nature since one cannot actually test the strength of an isolated element without changing the conditions which exist when the element is actually in the body. If one assumes that the flaws are distributed at random with a certain density per unit volume, then the statistical formulation of the problem becomes apparent. If the flaw concept is accepted then the strength of a given specimen is determined by the weakest point in the specimen since it is assumed that a running crack develops here which destroys the entire body.

Beginning his reasoning with the simple chain, Weibull observed that for infinitely small links or volumes, the probability of failure S becomes infinitely small; hence, Eq. 7 can be written exactly as

$$S_n = nS \tag{9}$$

This equation means that the probability of rupture of the entire body is proportional to the volume. Consequently, the distribution function for an infinitely small volume dV may be written

$$S = g(x)\,dV \tag{10}$$

Digression:

Weibull states that, "the only conditions which this formula must fulfill is that g(x) is finite and dV is infinitely small." However, it is quite clear that the product g(x) dV must also be dimensionless since it represents a probability. This may be accomplished by treating g(x) as dimensionless and dividing the product by a unit volume. Then, V may be

interpreted as the number of unit volumes and this interpretation will be used throughout this report.

Remembering that Eq. 8 is valid when n is either finite or infinite, we rewrite this equation as

$$\log(1 - F) = \sum_{i=1}^{n} \log(1 - S_i) \tag{11}$$

When n increased indefinitely, S_i converges to zero, and consequently, $\log(1 - S_i) \rightarrow -S_i$. Then Eq. 11 becomes

$$\log(1 - F) = -\lim_{n \rightarrow \infty} \sum_{i=1}^{n} S_i \tag{12}$$

Now, using Eq. 10, we obtain

$$\log(1 - F) = -\int g(x)\, dV \tag{13}$$

The function g(x), which is essentially the probability distribution of a unit volume, is determined by the distribution constants of the material. Therefore, for a nonhomogeneous solid, g varies from point to point. Furthermore, the argument of g, the generalized stress x, will in general vary from point to point depending as it does on the stress distribution in the solid.

Now, according to Eq. 13, the distribution function becomes

$$F(x) = 1 - e^{-B} \tag{14}$$

where B, the "risk of rupture", is

$$B(x) = \int g(x)\, dV \tag{15}$$

It is now necessary to specify the function g(x). Consider the case of a homogeneous solid subjected to a uniform stress. Here, Eqs. 14 and 15 assume the simple form

$$F(x) = 1 - e^{-V \cdot g(x)} \tag{16}$$

Referring to the properties of distribution functions outlined in Section II-B-1, we can obtain the properties which g(x) must satisfy:

$$F(x) \geq 0 \Rightarrow e^{-V \cdot g(x)} \leq 1 \Rightarrow g(x) \geq 0 \tag{17}$$

$$F'(x) \geq 0 \Rightarrow g'(x) \geq 0 \tag{18}$$

For reasons considered in previous discussions, it is also desirable that the distribution function be limited on the left, i.e., $F(x_u) = 0$, where $x_u \geq 0$. The simplest function satisfying these conditions is the following one suggested by Weibull[11]:

$$\begin{aligned} g(x) &= \left(\frac{x - x_u}{x_o}\right)^m , \quad x \geq x_u \\ &= 0 , \quad x < x_u \end{aligned} \tag{19}$$

where x_u, x_o, and m are constants associated with the material. If V is set equal to unity, we find the cumulative distribution function for a unit volume is given by,

$$F(x) = 1 - e^{-\left(\frac{x - x_u}{x_o}\right)^m} \tag{20}$$

and the associated unit probability density function, by

$$f(x) = \frac{dF}{dx} = \frac{m}{x_o}\left(\frac{x - x_u}{x_o}\right)^{m-1} e^{-\left(\frac{x - x_u}{x_o}\right)^m} \tag{21}$$

The only merit of this distribution function, according to Weibull[19], is to be found in the fact that it is the simplest mathematical expression of the appropriate form. Experience has shown that, in many cases, it fits observations better than other known distribution functions.

III-A-3-b. Properties of the Weibull Distribution

Given the Weibull distribution

$$\begin{aligned} F(x) &= 1 - e^{-B} \qquad x \geq x_u \\ &= 0 \qquad x < x_u \end{aligned} \tag{22}$$

where

$$B(x) = \int \left(\frac{x - x_u}{x_o}\right)^m dV , \tag{23}$$

the mean value x_m of the generalized stress x is found from the first moment of the probability density function about zero; thus,

$$x_m = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^1 x\, dF = x_u + \int_{x_u}^{\infty} e^{-B}\, dx \tag{24}$$

The variance a^2 of the frequency distribution for x is found from the second moment about the mean x_m; therefore,

$$a^2 = \int_{-\infty}^{\infty} (x - x_m)^2 f(x)\, dx = \int_0^1 (x - x_m)^2\, dF = \int_{x_u^2}^{\infty} e^{-B} d(x^2) + x_u^2 - x_m^2 \tag{25}$$

From the definition of g(x) given by Eq. 19, it is clear that the constant x_u is the zero probability strength, i.e., the strength below which the probability of fracture is zero. The constant x_o usually serves merely as a scale parameter; however, there is one situation where a physical meaning can be given to this constant. Let us consider the case of a uniformly stressed body with $x_u = 0$. If m is allowed to approach infinity, the risk of rupture B has a value of zero from zero stress to x_o; here it has the value unity; and from x_o on it has an infinite value. For this case, x_m is

$$x_m = \int_0^{x_o^-} dx + \int_{x_o^+}^{\infty} 0 \cdot dx = x_o \tag{26}$$

where the point x_o has been removed from the integration interval. The associated variance is

$$a^2 = \int_0^{(x_o^-)^2} d(x^2) + \int_{(x_o^+)^2}^{\infty} 0 \cdot d(x^2) - x_o^2 = x_o^2 - x_o^2 = 0 \tag{27}$$

Thus, we find that when m approaches infinity there is only one value of the breaking strength, x_o. This value apparently corresponds to the ultimate strength in classical theory since it satisfies the conditions that rupture occurs as soon as the stress in any point of the body, irrespective of its dimension, has reached a certain determined value. According to this statistical conception, it is precisely $a \neq 0$ that constitutes a criterion of the invalidity of the classical theory of strength.

The constant m, called the flaw density exponent, has been shown by Meyersberg[23] to be an index of the relative number of flaws in the material. The greater the value of m, the greater the number of flaws per unit volume. As the number of flaws per unit volume increases, so does the probability that there will be a flaw of maximum severity in every unit volume of the material. If a unit volume of material is subjected to stress and if fracture is initiated in this volume, it will initiate at the most severe flaw. Then, since the theoretical strength of a material (no flaws) may be considered constant, it can be seen that the fracture stress of a unit volume is a measure of the stress concentration of the most severe flaw in that unit volume. Hence, it can be seen that, as the number of flaws per unit volume increases (m value increases), the narrower the distribution curve becomes, that is, the less scatter in fracture stresses becomes. Usually, the less scatter that a material exhibits, the more homogeneous it is considered. It should be noted that the material constant m might better be conceived as reflecting changes in both the number and severity of flaws since one material might contain flaws of more widely ranging severity than another material in addition to a larger or fewer number of flaws per unit volume. In Table 1 taken from Salmassy[17], typical values for the material constant m are tabulated.

The constants x_u, x_o, and m are assumed to be constants of the material; and consequently, their determination should be possible from any size specimens. This fact can be used for the verification of the Weibull distribution function. Another useful tool for checking the theory is the fact that no observed value of the fracture stress should have a value below x_u.

Table 1

FLAW DENSITY EXPONENT

Material	Material Constant, m	
Glass fibers	1.3	Increasing number of flaws; Increasing homogeneity; Narrowing distribution curve
Nickel-bonded titanium carbide K151A	7	
Hydro-Stone plaster	15	
Steel at temperature of liquid air	24	
Champion's porcelain	35	
Steel	58	
Classically perfect material	∞	

III-A-3-c. Some Important Results

III-A-3-c-i. Zero Probability Strength is Zero ($x_u = 0$)

Take two similar objects of volumes V_1 and V_2. Let x_1 be the generalized stress at some location in V_1 and let x_2 be the corresponding stress in V_2. The relation between these stresses for the same probability of failure is found by equating $F_1(x_1) = F_2(x_2)$ from which it follows that

$$\int_{V_1} \left(\frac{x_1}{x_o}\right)^m dV = \int_{V_2} \left(\frac{x_2}{x_o}\right)^m dV \tag{28}$$

Because the two bodies have similar loadings and geometry,

$$\left(\frac{x_1}{x_2}\right) = \left(\frac{V_2}{V_1}\right)^{1/m} \tag{29}$$

where the x's may be taken as the average fracture stresses. Once more, as in the chain model, we find that the strength x_2 approaches zero as V_2 becomes indefinitely large.

Not only can Equation 29 be used for the determination of m when test results from two different size similar bodies are available; but, when m is known this equation provides an excellent design tool. For example, models of volume V_1 are constructed and tested for their ultimate failure load, P_1. We would then have the distribution curve for the loads P_1. Then, using Eq. 29, the fracture load corresponding to any probability for the volume V_2 would be given by

$$P_2 = P_1 (V_1/V_2)^{\left(\frac{1}{m} - \frac{2}{3}\right)} \tag{30}$$

where P_1 is chosen at the probability level desired in P_2. Note that it is unnecessary to compute or determine the stress distribution in the volume. Also, observe that when m goes to infinity we get the classical result

$$P_2/P_1 = (V_2/V_1)^{2/3} = A_2/A_1 \tag{31}$$

III-A-3-c-ii. Pure Bending-Rectangular Section

The stress in a rectangular beam subjected to terminal couples can be written as

$$\sigma = \sigma_b \frac{y}{h} \tag{32}$$

where σ_b is the maximum fiber stress and h is half of the beam depth. We may then compute the risk of rupture as follows:

$$B_b = \int_V \left(\frac{\sigma - \sigma_u}{\sigma_o}\right)^m dV = \frac{1}{\sigma_o^{\,m}} \int_{h_u}^{h} \left(\sigma_b \frac{y}{h} - \sigma_u\right)^m bLdy$$

$$= \frac{V_b}{2(m+1)} \left(\frac{\sigma_b - \sigma_u}{\sigma_b}\right) \left(\frac{\sigma_b - \sigma_u}{\sigma_o}\right)^m \tag{33}$$

where L is the span length, b is the beam width, $h_u = h \frac{\sigma_u}{\sigma_b}$, and V_b is the beam volume. We note that the argument of B_b is zero for compressive stresses and for tensile stresses which are less than σ_u.

Now, for a tensile specimen of strength σ_t, the risk of rupture becomes

$$B_t = V_t \left(\frac{\sigma_t - \sigma_u}{\sigma_o}\right)^m \tag{34}$$

where V_t is the volume of the specimen. Hence, when the risks of rupture and, therefore, the probabilities of rupture of the bending and tensile specimens are equal, the relationship between σ_b and σ_t becomes

$$\frac{\sigma_t}{\sigma_b} = \left[\frac{1}{2(m+1)} \frac{V_b}{V_t}\right]^{\frac{1}{m}} \left(1 - \frac{\sigma_u}{\sigma_b}\right)^{1+\frac{1}{m}} + \frac{\sigma_u}{\sigma_b} \tag{35}$$

where both stresses are taken at the same probability level, e.g., they may be taken as the mean fracture stresses.

For $\sigma_u = 0$, Eq. 35 reduces to

$$\frac{\sigma_t}{\sigma_b} = \left[\frac{1}{2(m+1)} \cdot \frac{V_b}{V_t}\right]^{1/m} \tag{36}$$

It is worth noting that one could have compared the mean fracture stresses in bending and tension by substituting the respective risks of rupture into the general expression for the mean, Eq. 24. However, the resulting expressions for the two σ_m's are awkward to compare, since the σ_m for the case of pure bending is not a closed form expression. It is interesting that for the case of a classical material ($m = \infty$), Eq. 35 yields

$$\sigma_t = \sigma_b$$

III-A-3-c-iii. Other Specific Results, $x_u = 0$.

By assuming that the probability of fracture in a direction normal to a plane passing through a point depends only on the tensile stresses acting thereon, the total probability of fracture at the point may be found by considering every plane passing through the point [11]. Using this idea, Salmassy [17] developed the following relationships.

Rectangular Bending Member - Circular Torsion Member:

$$\frac{\sigma_{bending}}{\sigma_{torsion}} = \left[\frac{(2m+1)\,(2m+2)\,\beta\, V_{torsion}}{2(m+2)\, V_{bending}} \right]^{1/m} \tag{37}$$

where

$$\beta = \frac{\Gamma(m+1)\;\Gamma\left(\frac{m+1}{2}\right)}{\Gamma(m+\frac{3}{2})\;\Gamma(\frac{m}{2}+1)} \qquad \Gamma \ldots \text{gamma function} \tag{38}$$

Prismatic Tension Member - Circular Torsion Member:

$$\frac{\sigma_{tension}}{\sigma_{torsion}} = \left[\frac{(2m+1)\,\beta\, V_{torsion}}{2\,(m+2)\, V_{tension}} \right]^{1/m} \tag{39}$$

All of the above stresses are the extreme fiber stresses associated with the mean fracture strength of the members.

The usual assumption in the Weibull theory is that every volume of the body contributes to the "risk of rupture"; however, he points out the following: "In fact, it is not unusual that the rupture of a brittle body starts on the surface whose properties may differ from those of the material in the interior of the body, for instance owing to the method of

manufacture. In the extreme case, where all fractures start from the surface and none from the interior of the body, it may be assumed that $g(x) = 0$ for each volume element in the interior, so that the usual volume integral for B needs only to be extended over a surface layer of the small thickness h." Using this assumption, Salmassy works out several important relationships in pages 150 to 155 of Reference 17.

III-A-3-c-iv. Safety Factors for Uniform Tension

The probability of fracture for uniform tension is found from Eqs. 22 and 23 to be

$$F(x) = 1 - e^{-V\left(\frac{x - x_u}{x_o}\right)^m} \tag{40}$$

For this distribution function the mean stress is found from Eq. 24:

$$x_m = x_u + \int_{x_u}^{\infty} e^{-V\left(\frac{x - x_u}{x_o}\right)^m} dx \tag{41}$$

Introducing the change in variable

$$Z = V\left(\frac{x - x_u}{x_o}\right)^m ; \quad dx = x_o V^{-1/m} \frac{1}{m} Z^{\frac{1}{m} - 1} dZ , \tag{42}$$

Eq. 41 becomes

$$x_m = x_u + x_o V^{-1/m} \frac{1}{m} \int_o^{\infty} e^{-Z} Z^{\frac{1}{m} - 1} dZ$$

$$= x_u + x_o V^{-1/m} \Gamma\left(1 + \frac{1}{m}\right) \tag{43}$$

where

$$\Gamma(y) = \int_o^{\infty} e^{-t} t^{y-1} dt; \quad y\Gamma(y) = \Gamma(1 + y) \tag{44}$$

Now, if x_F is the stress associated with a specified probability of fracture F, we may find x_F from Eq. 40, thus

$$x_F = x_u + x_o \left[-\log(1 - F)\right]^{1/m} V^{-1/m} \tag{45}$$

Consequently, the safety factor x_m / x_F becomes

$$\frac{x_m}{x_F} = \frac{x_u + x_o V^{-1/m} \Gamma\left(1 + \frac{1}{m}\right)}{x_u + x_o V^{-1/m} \left[-\log (1 - F)\right]^{1/m}} \tag{46}$$

This safety factor is plotted in Figure 9 for various values of m and $x_u = 0$. We note that the value of the safety factor is very sensitive to changes in the constant m and that this sensitivity increases when greater survival probability is demanded. Furthermore, this effect is most pronounced for small values of m.

III-A-3-d. Estimation of the Weibull Parameters (x_u, x_o, m)

We have indicated in previous discussions that the zero probability strength x_u will probably become the design stress in large structures. For this reason it is important to obtain a good estimate of this parameter. The role of m in extrapolating the strength of a small specimen to a prototype structure is clearly brought out in the discussion attendant to Eq. 30. However, to appreciate the accuracy which is called for in the determination of m, it is convenient to refer to Figure 9. Here, it is apparent that a small error in the determination of m gives rise to a very large error in the safety factor. This is especially true for highly variable materials where m is low. Materials with large dispersions in their fracture strength require great accuracy in the determination of m; but, these are precisely the cases where the estimation of parameters is the most difficult.

When Weibull first proposed his theory of failure, he used a graphical method to determine the parameters of his distribution. This method, which was briefly touched on in Section II-B-2, will be described more fully in the following subsection.

III-A-3-d-i. Graphical Method for Determining x_u, x_o, m

The following method is applicable when data is available from specimens of only one size. It was mentioned previously that the important constant m could be determined from Eq. 29 when data was available from two size specimens.

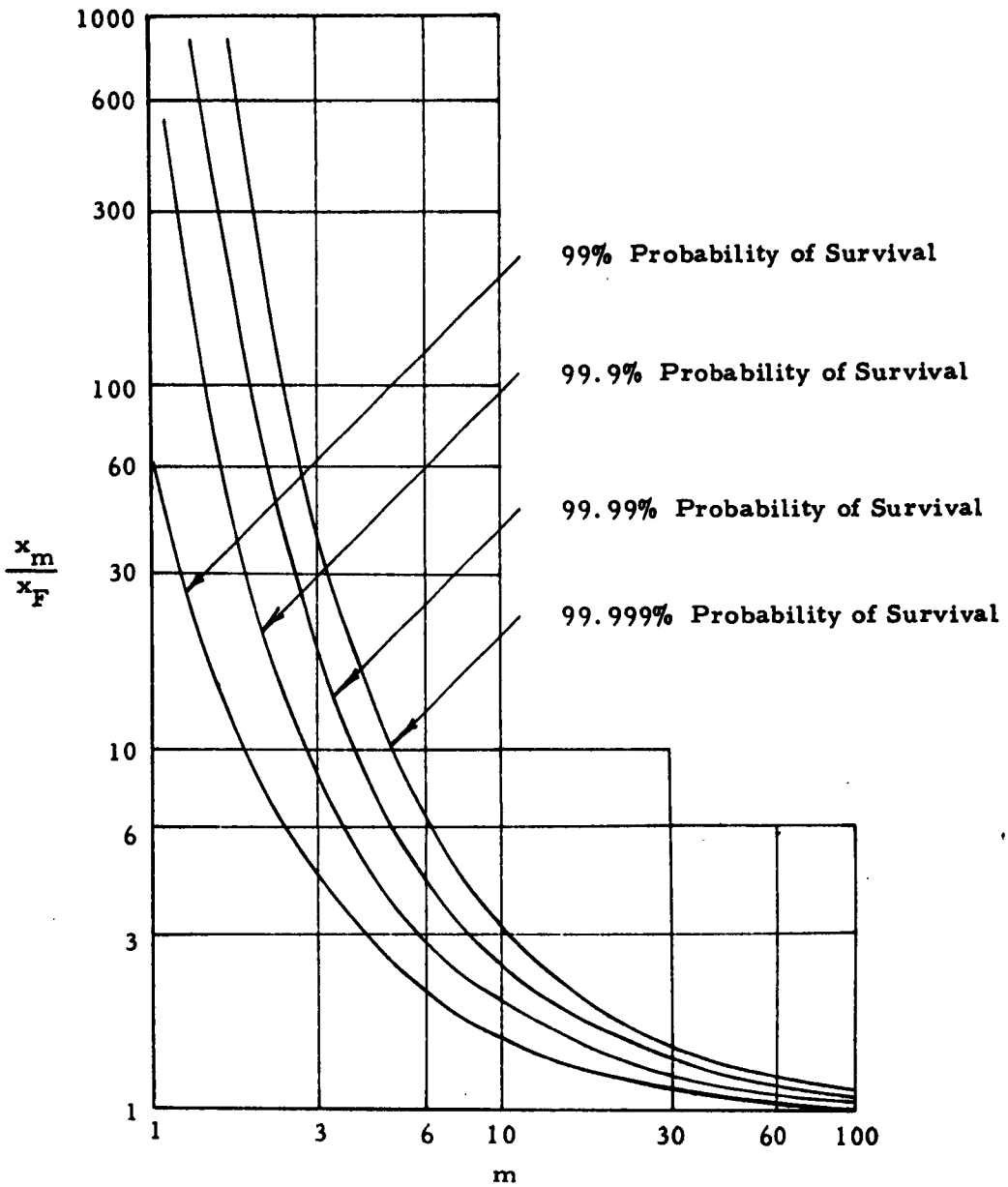

Figure 9

VARIATION OF SAFETY FACTOR WITH THE WEIBULL FLAW DENSITY CONSTANT, m WITH $x_u = 0$ (AFTER ANTHONY AND MISTRETTA, REF. 61)

Weibull's method, when applied to the case of uniformly stressed material, begins by writing Eq. 40 in the form

$$\log \log \frac{1}{1 - F} = m \log (x - x_u) - m \log x_o + \log V \tag{47}$$

It can be seen that a plot of this distribution function will be linear in a system of rectangular coordinates in which $\log \log \frac{1}{1 - F}$ is the ordinate and $\log (x - x_u)$ the abscissa. In addition, m will be the slope of the distribution function in these coordinates. Also, from Eq. 47, a change in the volume of the specimen merely shifts the distribution curve vertically.

The general principle of this method consists in plotting the fracture data in this system of coordinates, drawing the best straight line through the data, and determining the constants m and x_o from the slope and intercepts of the line. As for the determination of the zero probability strength, x_u, Weibull states:

"The application of this method presupposes the knowledge of the constant x_u, which as a rule is not known. In this case, it is advisable to plot the test figures at first in the system of coordinates $\log \log \frac{1}{1 - F}$ and $\log x$. If the figure happens to follow a straight line, this implies obviously that $x_u = 0$. If, on the other hand, the test figures are located on a curved line (concave downwards), it follows that $x_u \neq 0$, and a tentative value is taken for x_u. If this value is too high, the curve will be bent in the opposite direction (concave upwards), and further tentative values must be taken, until the curve approximates a straight line as closely as possible. It will be found that after some experience a few attempts according to this method suffice to bring forth results and that the curvature of the curve is very sensitive to variations of x_u, provided that the test series is passably large, so that the value of x_u may be determined with a degree of accuracy corresponding to the scope of the test series."

A problem arises when plotting the fracture data in the selection of the value of the probability, F, corresponding to a given test value of x. The manner of this selection would seem critical, since it is the variation of probability of fracture, F, with stress, x, that is the object of this analysis. This question is usually discussed under the name Plotting

Positions. An enormous literature is devoted to this question; however, a concise treatment is contained in Reference 24 on extreme values. Weibull determines the probability, F, corresponding to a fracture stress, x, by the formula

$$F_n = \frac{n}{N + 1} \tag{48}$$

Here, N represents the total number of tests in the series, where the N observed fracture stresses are arranged in increasing order from 1 to N. Then, F_n corresponds to the nth observed fracture stress, x_n. If Eq. 48 is substituted into the left-hand side of Eq. 47,

$$\log \log \frac{1}{1 - F} = \log \log \frac{N + 1}{N + 1 - n} \tag{49}$$

The distribution curve corresponding to Eq. 47 is then determined from a plot of

$$\log \log \frac{N + 1}{N + 1 - n} \quad \text{versus} \quad \log (\sigma_n - \sigma_u).$$

The above method would also apply to the case of pure bending. Using the risk of rupture expression given in Eq. 33, the probability of rupture becomes

$$\log \log \frac{N + 1}{N + 1 - n} = (m + 1) \log (\sigma_b - \sigma_u) - m \log \sigma_o$$

$$- \log \sigma_b + \log \frac{V_b}{2 (m + 1)} + \log \log e \tag{50}$$

where we have used the Brigg's logarithms.

It is suggested here for the case where $\sigma_u \neq o$ to plot

$$\left[\log \log \frac{N + 1}{N + 1 - n} + \log \sigma_b\right] \quad \text{versus} \quad \log (\sigma_b - \sigma_u).$$

The value of σ_u is found by trial and error until a straight line plot is obtained. Then, (m + 1) is determined as the slope of this line and σ_o is found as in the case of the tensile specimen above.

The above graphical method corresponds precisely to an earlier discussion of a method referred to as "probability paper." The graphs shown in Figures 5b and 8 b were constructed using the methods outlined in this

section. It should be emphasized that when it is not possible to get a straight line in the Weibull coordinates, the distribution is not of the Weibull type. Furthermore, if two or more straight lines are obtained, this means our population is composed of a mixture of Weibull distributions. A distribution function does not exist in this case in the sense that a multi-valued function is not a function. We can use the composite distribution curve only in the case where the proportions of each population are known - and here the difficulties will be very great.

III-A-3-d-ii. Comments on the Graphical Method and Other Methods

We shall begin by quoting Weibull[25]:

"Up to the past year (1951) the author's usual method (of estimating the parameters) has been to plot the data as shown in the paper (graphical method) and to choose x_u to give the best straight line. In this way it is easy to decide if the distribution is simple or complex, but the procedure is not entirely free of subjectiveness." Indeed, Weibull's comment agrees with many of the experiences of other investigators in the field of statistical fracture strength.

In the discussion of Weibull's paper referred to above, several methods were advanced as alternative methods for estimating the parameters. These methods and several others have been discussed in a very complete manner by Salmassy[17] on pages 126 to 149. (Several typographical errors appear in the formulas.) In particular, six so-called mathematical methods are discussed in addition to a least-squares method and Weibull's standardized-variable method. The methods have all been applied to a series of tests on Hydro-Stone Plaster tension, bending, and torsion specimens of various sizes. The table summarizing their results is reproduced in Table 2.

The results shown in this table are very disconcerting. First, many of the predictions for the zero strength x_u are greater than the observed lowest value. Also, contrary to Weibull's hypothesis, the values of both m and x_u do not appear to be independent of size and stress state. Finally, it appears that the constants also depend on the method of calculation. For

DISTRIBUTION CONSTANTS FOR HYD

For the Distribution

Test[a] Series	Material Constant, m								x_1, Lowest Observed Fracture Stress, psi	
	Mathematical Methods						Graphical Method	Standard Variable Method		
	I	II	III	IV	V[b]	VI				I
Small bend (68)	3.0	1.1	3.0	2.3	2.8	3.1	4.2	3	1195	1210 1
Large bend (99)	7.0	---	4.5	21.0	5.9	6.3	7.7	8	840	170 -
Small tension (82)	4.7	3.1	3.6	4.3	4.0	4.4	3.2	3	705	465
Large tension (36)	5.7	---	3.6	8.1	4.9	5.4	6.5	5	600	165 -
Small torsion (39)	2.3	1.0	3.7	4.9	2.8	2.2	7.0	5	985	925 1
Large torsion (81)	3.0	1.4	3.7	4.6	3.1	2.9	5.0	5	680	595 -

(a) The figures in parentheses indicate the number of specimens in the series.

(b) These values of m were those assumed for the calculation of the constants x_u and x_o by Method

1

Table 2

S FOR HYDRO-STONE PLASTER (AFTER SALMASSY REF. 17)

Distribution Function $F = 1 - e^{-V\left(\frac{x-x_u}{x_o}\right)^m}$

	x_u, Zero Strength, psi									x_o, psi						
	Mathematical Methods							Graphical Method	Least Squares Method	Mathematical Methods						Graphical Method
	I	II	III	IV	V	VI	Avg.			I	II	III	IV	V	VI	
	1210	1485	1225	1295	980	995	1140	1100	980	325	5	180	105	235	725	315
	170	----	650	0	270	235	330	0	235	1505	----	1015	----	1425	1245	1680
	465	720	635	525	380	485	500	700	395	485	250	325	430	490	855	270
	165	----	445	0	190	180	245	0	230	1120	----	870	----	1140	850	1300
	925	1260	615	315	745	950	710	0	610	205	20	480	760	330	670	1100
	595	----	480	345	511	595	505	300	485	1155	1310	1225	1317	1305	600	1340

x_o by Method V; They are the average of the values obtained from the other methods.

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example, the calculated m is found to range from 1 to 21; the zero strength varies from zero to 1485 psi.

In most of the mathematical methods referred to above, one equates the various moments of the Weibull distribution to the corresponding moments of the data. Because the higher moments require much more data for their estimation than the lower ones, it is extremely likely that much of the differences between the various mathematical methods can be attributed to the fact that the small sample sizes used provide widely varying precision in the moment estimates.

The development of any design procedure based on statistical methods depends upon the availability of methods for estimating the various statistical constants. The present state of the art does not appear to provide these methods. Recognizing the shortcomings of the presently available methods, ARF attempted to program the graphical procedure for the digital computer; however, the resulting system of three nonlinear algebraic equations has apparently proved too unwieldy even for the machine[26]. Currently, other systems of equations are being studied at ARF with the objective of resolving this most important problem. It should be mentioned that Weibull has introduced another new method for estimating the parameters of complete or truncated distributions which has not really been studied sufficiently by other people in the field[27].

III-A-3-d-iii. Effective Number of Specimens

The determination of the parameters of a population distribution function requires an infinite amount of data when the statistical variate is continuous. One of the central problems in statistics is the estimation of these parameters from a finite sample size; especially, a small sample size. We are, of course, led naturally to inquire into the smallest sample size possible which is consistent with the functional requirements of our problem. Although there appears to be no easy answer to the question of minimum sample size, a considerable literature has been devoted to its study, particularly, when the population can be treated as normal. The importance of sample size in relation to the "statistical strength problem" is brought out quite dramatically in Figure 10, where the results of fracture

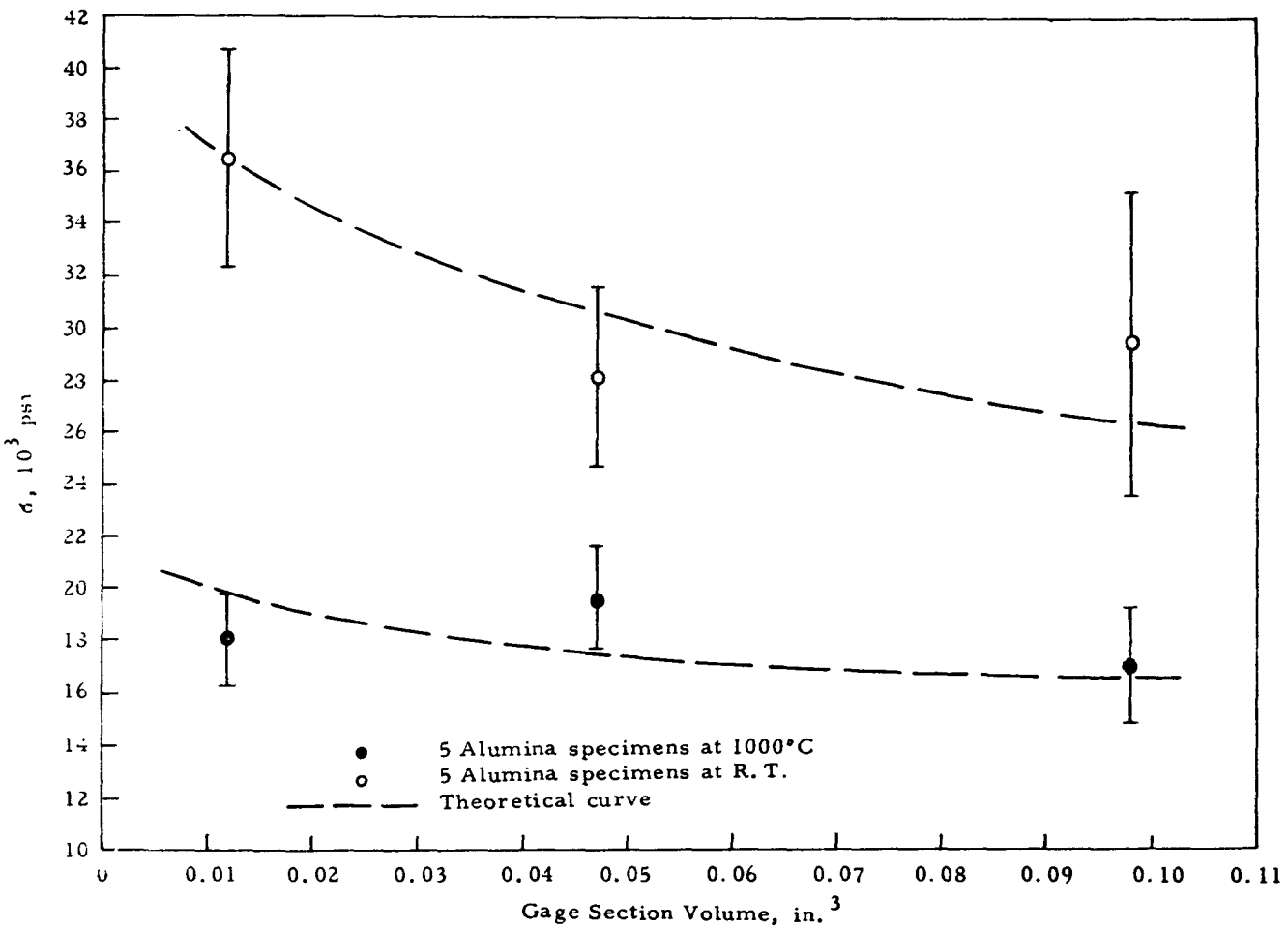

Figure 10

COMPARISON OF WEIBULL'S THEORETICAL VOLUME CURVE BASED ON 1/4 x 1/4 x 3/4 INCH GAGE SECTION FLEXURAL SAMPLES AND ACTUAL FLEXURAL SAMPLE DATA

tests on samples of five specimens are presented. Certainly, the results shown here cannot be explained by the Weibull theory; but more significantly, the results cannot be explained by any weakest link theory for homogeneous solids since they would all predict decreasing strength with increasing size. The random positions assumed by the six points in this figure would hardly surprise any investigator who deals with the measurement of physical quantities. It is obvious, here, that five samples are not enough to reliably determine the mean fracture stress.

In our study of safety factors in Section III-A-3-c-iv, we indicated that the design stress depends very sensitively on the flaw density exponent m. We will now attempt to show how m is related to the actual data of the problem through both the mean and the standard deviation of a sample. For the case of uniform tension, the variance can be found from Eq. 25 when F(x) is given by Eq. 40 and x_m is given by Eq. 43; hence,

$$a^2 = \int_{x_u}^{\infty} (x - x_m)^2 \frac{dF}{dx}\, dx = x_o V m \int_{x_u}^{\infty} \left[\left(\frac{x - x_u}{x_o}\right) - V^{-1/m}\Gamma\left(1 + \frac{1}{m}\right)\right]^2 \cdot$$

$$\left(\frac{x - x_u}{x_o}\right)^{m-1} e^{-V\left(\frac{x - x_u}{x_o}\right)^m} dx \tag{51}$$

Introducing the change of variable

$$z = V\left(\frac{x - x_u}{x_o}\right)^m, \tag{52}$$

the variance becomes

$$a^2 = x_o^2 V^{-2/m} \int_0^{\infty} \left[z^{2/m} e^{-z} - 2\Gamma\left(1 + \frac{1}{m}\right) z^{1/m} e^{-z} + \Gamma^2\left(1 + \frac{1}{m}\right) e^{-z} \right] dz$$

or finally,

$$a^2 = x_o^2 V^{-2/m} \left[\Gamma\left(1 + \frac{2}{m}\right) - \Gamma^2\left(1 + \frac{1}{m}\right)\right] \tag{53}$$

Note that we have established the fact that the standard deviation decreases with increasing volume as $V^{-1/m}$.

When $x_u = 0$, we can eliminate x_o from Eq. 53 by forming the square of the coefficient of variation, i.e., $(a/x_m)^2$.

$$\left(\frac{a}{x_m}\right)^2 = \frac{\Gamma\left(1 + \frac{2}{m}\right)}{\Gamma^2\left(1 + \frac{1}{m}\right)} - 1 \qquad x_u = 0 \tag{54}$$

This formula, originally presented by Rudnick[28], has appeared incorrectly in three publications known to the author. This equation concisely describes the relationship between the sample parameters and the extremely important constant m. We have plotted Eq. 54 in Figure 11. Using this graph, the reader can readily verify that the function is closely approximated by:

$$(a/x_m)^2 = 1/m^2 \tag{55}$$

In a paper by Wallhaus[29], the statement is made that Irwin* used Eq. 53 to show that the variance varies inversely with m. During the preparation of this report a better reference has come to the author's attention, in which Irwin[30] shows that

$$m = \left(1 - \frac{x_u}{x_m}\right) \frac{\sqrt{1.5}}{a/x_m}$$

It turns out, that this expression does not approximate the value of m given by the exact formula of Eq. 54 as well as the simple relationship in Eq. 55.

It is a well-established statistical fact that the higher the moment of the probability density function, the more data are required for its accurate determination. This fact is clearly demonstrated in Table 3, where a tabulation of results of fracture tests on Wesgo Al-995 and Beryllium Oxide is presented[31]. Apparently, for sample sizes of 40 to 60 it is possible to consistently reproduce the mean fracture stress within a small variation. We notice, however, that the variation in the standard deviation, or what

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* Irwin, G. R., notes from a series of lectures presented in the Department of Theoretical and Applied Mechanics, University of Illinois, November, 1961; also private communications with Wallhaus.

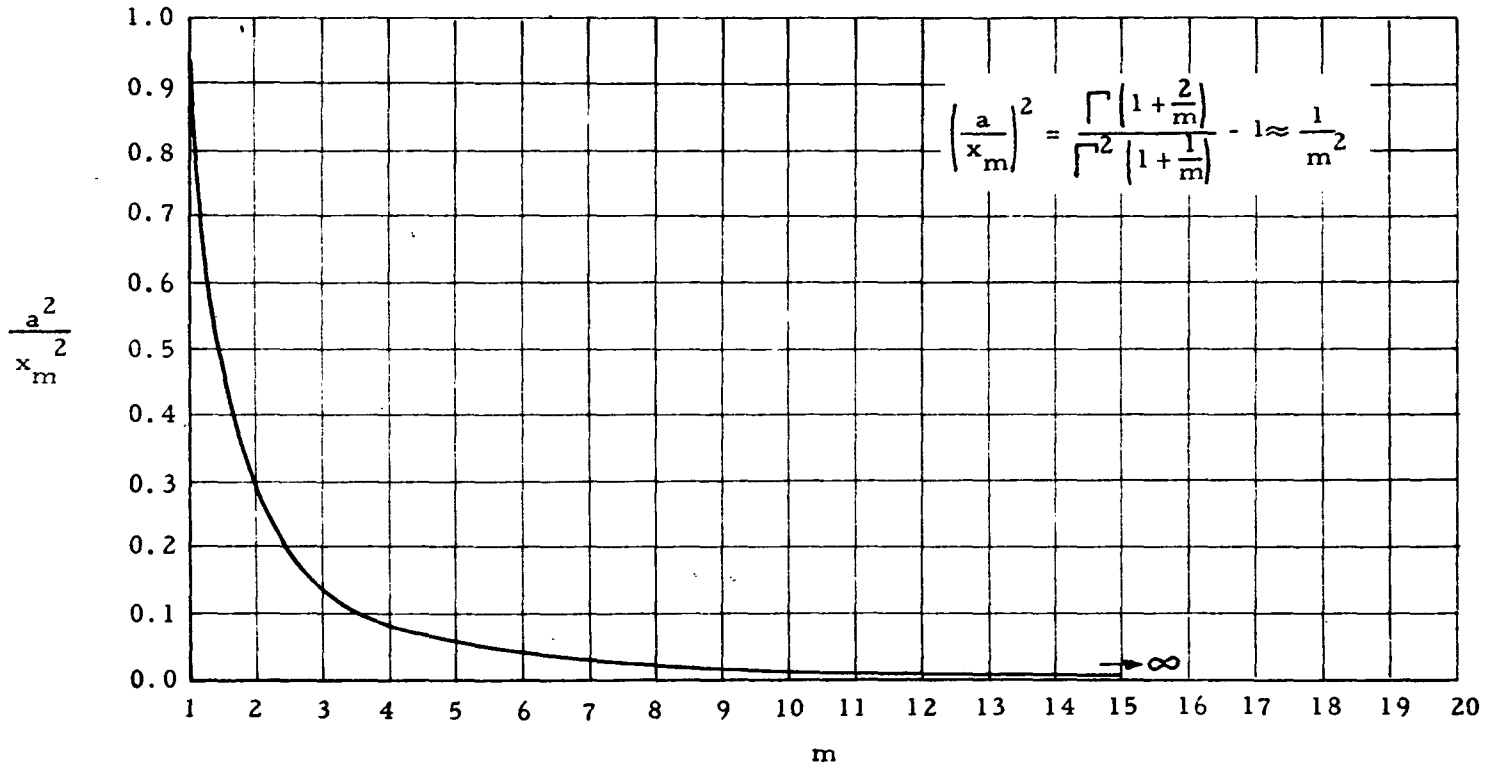

Figure 11

RELATIONSHIP BETWEEN THE FLAW DENSITY CONSTANT m AND THE COEFFICIENT OF VARIATION a/x_m

Table 3

BENDING TESTS ON PRISMATIC SPECIMENS

	(a) Wesgo AL-995 Room Temperature			(b) Beryllium Oxide Room Temperature			(c) Wesgo AL-995 1000° C	
Specimen Type	A	B	C	A	B	C	A	C
Dimensions (in.)	$\frac{3}{16} \times \frac{1}{3} \times 4$	$\frac{1}{4} \times \frac{1}{4} \times 4$	$\frac{1}{3} \times \frac{1}{16} \times 4$	$\frac{3}{16} \times \frac{1}{3} \times 4$	$\frac{1}{4} \times \frac{1}{4} \times 4$	$\frac{1}{3} \times \frac{3}{16} \times 4$	$\frac{3}{16} \times \frac{1}{3} \times 4$	$\frac{1}{3} \times \frac{3}{16} \times 4$
Cross Section (in.)	$\frac{1}{3}$ / $\frac{3}{16}$	$\frac{1}{4}$ / $\frac{1}{4}$	$\frac{3}{16}$ / $\frac{1}{3}$	$\frac{1}{3}$ / $\frac{3}{16}$	$\frac{1}{4}$ / $\frac{1}{4}$	$\frac{3}{16}$ / $\frac{1}{3}$	$\frac{1}{3}$ / $\frac{3}{16}$	$\frac{3}{16}$ / $\frac{1}{3}$
Gage Volume, V, (cu. in.)	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.125
No. of Specimens Tested, N	60	60	60	48	42	50	52	60
Mean Failure Stress, σ_m, (psi)	29,770	31,850	29,750	15,740	15,540	15,650	29,780	29,710
Standard Deviation a, (psi)	2,250	2,210	1,800	2,320	2,540	2,530	2,120	2,000
Coefficient of Variation, v, (%)	7.57	6.94	6.05	14.74	16.34	16.17	7.12	6.73
Mean Stress Gradient at Failure (psi/in.)	178,620	254,800	317,340	94,440	124,320	166,930	178,680	316,910
Lowest Failure Stress, σ_{low}, (psi)	23,540	24,910	25,540	11,830	8,870	10,210	25,060	24,960
Highest Failure Stress, σ_{high}, (psi)	33,800	36,430	33,980	19,900	19,320	21,000	33,700	33,790
	Variation in (a/x_m): 25.1% Variation in x_m: 7.07%			Variation in (a/x_m): 10.86% Variation in x_m: 1.29%			Variation in (a/x_m): 5.80% Variation in x_m: 0.236%	

amounts to the same thing, the variation in the coefficient of variation, can be quite high even for this size sample. In view of Eq. 50, the same error or fluctuation which occurs in the standard deviation also occurs in the constant m. Referring to Fig. 9 we see that this error is magnified when we select the safety factor. We should note that the data reported in Table 3 was obtained by subjecting the specimens to different stress gradients and although this did not effect the mean fracture stress it is possible that it had an effect on the standard deviation.

To clearly point out how devastating the small sample can be, we have included in Table 4 the results of fracture tests on Wesgo AL-995 at 1000°C where only five specimens have been used. Fluctuations of 300% and more are shown in the coefficient of variation.

Up to this point we have considered only the relationship between m and the sample parameters for the case when $x_u = 0$. Following Reference 32 we will now indicate that no essential difference occurs when $x_u \neq 0$. In this latter case it is convenient to introduce the mode of the frequency distribution. Since the mode is the stress associated with the highest point on the frequency curve, it is easily computed by taking the derivative of f(x) and setting it equal to zero:

$$\frac{df(x)}{dx} = \frac{d^2F(x)}{dx^2} = \frac{d^2}{dx^2}\left[1 - e^{-V\left(\frac{x-x_u}{x_o}\right)^m}\right] = 0 \qquad (56)$$

This leads to

$$x_{mode} = x_o\left(1 - \frac{1}{m}\right)^{1/m} V^{-1/m} + x_u \qquad (57)$$

Now, using Eqs. 43 and 53 we obtain,

$$\frac{(x_{mean} - x_{mode})^2}{a^2} = \frac{\left[\Gamma\left(1 + \frac{1}{m}\right) - \left(1 - \frac{1}{m}\right)^{1/m}\right]^2}{\Gamma\left(1 + \frac{2}{m}\right) - \Gamma^2\left(1 + \frac{1}{m}\right)}, \quad x_u \geq 0 \qquad (58)$$

Table 4

FLEXURE STRENGTH OF WESGO AL-995 SPECIMENS AT 1000°C

Atmosphere	Condition	Volume, (in. 3)	No. of Tests	Ave. Failure Stress (10^{+3}psi)	Standard Deviation (10^{+3}psi)	Coefficient of Variation%
Ambient	as-received	0.012	4	22.3	1.37	6.1
Ambient	as-received	0.047	5	23.8	2.19	9.2
Ambient	as-received	0.098	5	19.7	3.60	18.3
Dry argon	as-received	0.012	5	17.6	1.98	11.3
Dry argon	as-received	0.047	5	22.3	3.32	14.9
Dry argon	as-received	0.098	5	17.7	3.82	21.6
Saturated steam	as-received	0.012	5	18.8	2.11	11.2
Saturated steam	as-received	0.047	8	18.2	1.92	10.5
Saturated steam	as-received	0.098	5	17.0	1.68	9.9
Ambient	ground	0.012	5	22.5	1.30	5.8
Ambient	ground	0.047	5	23.5	0.75	3.1
Ambient	ground	0.098	5	23.5	1.71	7.3
Dry argon	ground	0.012	4	22.2	2.23	10.0
Dry argon	ground	0.047	5	19.9	1.65	8.3
Dry argon	ground	0.098	5	19.6	1.92	9.8
Saturated steam	ground	0.012	5	23.5	2.69	11.5
Saturated steam	ground	0.047	5	20.9	0.40	1.9
Saturated steam	ground	0.098	5	20.1	0.42	2.1
Dry argon	annealed	0.012	5	18.0	2.04	11.3
Dry argon	and	0.047	5	19.2	1.07	5.5
Dry argon	unground	0.098	5	17.5	1.78	10.2
Saturated steam	annealed	0.012	5	17.5	1.98	11.3
Saturated steam	and	0.047	5	18.7	1.63	8.7
Saturated steam	unground	0.098	5	16.1	1.63	10.2
Ambient	annealed	0.012	4	18.7	0.40	2.1
Ambient	and	0.047	5	17.2	0.81	4.7
Ambient	ground	0.098	5	16.3	0.90	5.5

The sensitivity of the dependence of m on the data may be inferred from the fact that m varies from 1 to ∞ when $(x_m - x_{mode})^2/a^2$ varies from 1 to zero. The quantity $(x_m - x_{mode})/a$ is sometimes used as a measure of skewness. Since skewness is also defined as the third moment of the frequency distribution about the mean, we can expect that an enormous amount of data will be required to reliably determine m from Eq. 58.

The principal accomplishment of this subsection has been the demonstration that m is inversely proportional to the standard deviation. With this information the question of sample size for a reliable m becomes equivalent to the question of sample size for a reliable determination of the standard deviation. This is a happy situation since a great deal of work has been conducted for this latter problem. Although we will not consider the details of this statistical problem in this report, we shall quote the following theorem which is the heart of the solution for normal distributions:

Theorem: If x is normally distributed with variance σ^2 and s^2 is the sample variance based on a random sample of size n, then ns^2/σ^2 has a χ^2 distribution with n - 1 degrees of freedom. (see any statistics book)

The literature on statistical strength theory does not concern itself to any great extent with the question of sample size; however, there are a few comments that seem pertinent.

1) In his investigation of glass fiber strength, Wallhaus[29] came to the following conclusion: "The use of a large sample size was demonstrated to be essential to avoid distorting the information obtained in an experimental investigation. Significant deviations occured in the Weibull distribution diagrams for sample sizes of from 30 to 50 specimens."

2) As can be judged from Table 2, sample sizes of close to 100 did not lead to consistant results for the determination of the Weibull parameters for Hydro-Stone plaster.

3) In the 1930 paper by Koshal and Turner[33] on the determination of mechanical properties of cotton fibers, a considerable concern was shown for appropriate sample size. We shall quote some of their comments.

"It is seldom indeed that textile tests are made in sufficient numbers to make it possible to ascertain the form of the frequency-distribution of the

results with any degree of accuracy. The curves published for cotton yarns by Andrews and Oxley appear to be the only curves so far published relating to numbers of tests extending to some thousands. Yet if it is desirable to make tests on cotton yarn by the thousand in order to obtain an accurate frequency curve, it is even more necessary for single cotton fibres, as their variability is far greater than that of cotton yarns. Frequency curves for fibre properties have been obtained by various workers; thus, Barratt has given the frequency distributions of the following properties of cotton fibres - length, diameter, breaking load, and extention at break - and has fitted to them curves derived from a formula due to Bateman. But in most cases it is only by courtesy that Barratt's theoretical curves can be said 'to fit' his experimental curves, no doubt because the number of tests employed in the determination of each fibre property was only 100 - a number much too small for the accurate determination of the frequency distribution; moreover, there is no justification for his use of the Poisson formula rather than the normal or other possible type of curve.

..... In a paper dealing with 'Variability as a Problem of Textile Testing,' Pierce gives a table in which he includes two sets of results for the breaking load of cotton hairs. But here again, only 200 tests were made in each case, so that the determination of the type of frequency curve cannot be regarded as at all trustworthy."

In this paper by Koshal and Turner, they consider, among other things, a series of 3000 tests on the breaking strength of yarn. The results of this important work are summarized in Table 5 where we have included the author's comments at the bottom of the table. Note that each of the entries for the population parameters are stated with their attendent probable errors. The probable error is defined as 0.6745 x standard deviation, so that in a repetitive experiment, half the values will lie in the range of the values indicated, e.g., half the mean stresses x_m would lie between (x_m - prob. error) and (x_m + prob. error). The authors conclude for this highly variable material that 1000 tests are almost as significant as the 3000 tests. We will return to this test series in future discussions.

4) As a final comment, the people working at Armour Research Foundation in the area of statistical strength are currently of the opinion

Table 5

FREQUENCY-DISTRIBUTION OF STRENGTH-VALUES OF 3000 FIBRES OF SURAT 1027, A. L. F. (1925-1926)

Strength-class grams	Observed Frequency	Theoretical Frequency		Sets of 1,000 Tests		
	3,000 Tests	3,000 Tests	1,000 Tests	First 1,000	Second 1,000	Third 1,000
0.0 - 0.9	117	127.3	41	38	41	38
1.0 - 1.9	550	531.5	177	182	203	165
2.0 - 2.9	552	596.5	199	184	180	188
3.0 - 3.9	510	522.1	174	172	179	159
4.0 - 4.9	424	407.2	136	156	131	137
5.0 - 5.9	312	295.6	99	105	93	114
6.0 - 6.9	199	202.8	68	65	53	81
7.0 - 7.9	149	132.9	44	46	55	48
8.0 - 8.9	74	82.9	28	20	25	29
9.0 - 9.9	46	50.5	17	14	13	19
10.0 - 10.9	29	29.4	10	7	7	15
11.0 - 11.9	23	16.4	5	7	10	6
12.0 - 12.9	6	8.8	3	2	4	0
13.0 - 13.9	4	4.5	1	1	2	1
14.0 - 14.9	4	2.2	1	1	3	0
15.0 - 15.9	1	1.0	0	0	1	0
Mean	3.91 ±0.029			3.86 ±0.048	3.88 ±0.052	4.04 ±0.050
Distance between mean and mode	1.655 ±0.104			1.699 ±0.172	1.823 ±0.2176	1.6375 ±0.1680
Standard deviation	2.357 ±0.0205			2.272 ±0.0343	2.482 ±0.0374	2.358 ±0.0355
Skewness	0.701 ±0.040			0.748 ±0.070	0.734 ±0.085	0.695 ±0.069

"It is interesting to note what the results would have been if we had tested only 1,000 fibres. For this purpose the 3,000 results have been divided into three successive groups of 1,000 each; column 4 in Table 5 shows the theoretical frequency-distribution of 1,000 results, based on that of the 3,000 results; columns 5, 6 and 7 show the observed frequencies of the three sets of 1,000 results. At the bottom of Table 5 are given the several values of the mean, the distance between the mean and the mode, the standard deviation, and the skewness; from an examination of these values and their associated probable errors, we conclude that the constants are not significantly different for sets of 1,000 tests and 3,000 tests." Koshal and Turner

that 60 to 70 specimens are sufficient for the estimation of the Weibull parameters for ceramic materials.

It is the author's view that the development of a sound structural design theory for brittle materials depends on the availability of reliable estimates of distribution parameters which demands that a rational procedure be put forward for the determination of sample size.

III-A 3-e. Verification of Weibull's Statistical Theory

The verification of Weibull's statistical fracture theory for a given material requires the satisfaction of three seperate questions:

1. Does a distribution function exist for a generalized stress x?
2. Is this distribution function of the Weibull type, i.e., of the form of Eq. 22?
3. Are the distribution constants x_u, x_o, m constants of the material, i.e., are they independent of size and stress state?

Notice, that a material may affirm question (1) and still negate questions (2) and (3). Furthermore, a material may affirm questions (1) and (2), and still negate question (3). Recalling our earlier comments in Chapter II, the non-existence of a distribution function precludes the development of a statistical fracture theory. The Weibull distribution function is simply one of a large number of analytical distribution functions which have been found useful for the description of continuous random variables. In addition to its usefulness in the study of breaking strength, Weibull[19] has shown that it has considerable utility in the study of more diverse phenomena such as (i) the size distribution of fly ash, (ii) the length of Cyrtoideae, (iii) the statures of adult males, born in the British Isles, and (iv) the breadth of beans of Phaseolus Vulgaris. Consequently, the fact that the distribution of strength of a particular size specimen happens to have the form of the Weibull distribution, in no way implies that the distribution of a larger specimen is in any way related to the first distribution. Thus, in addition to having the form of the Weibull distribution, a material must satisfy the hypothesis of the weakest link formulation before the parameters x_u, x_o, m are constants of the material.

It is extremely important to note that the three questions in the preceeding paragraph all relate to some choice of a generalized stress x. Failure to affirm any of these questions may conceivably be traced to the wrong choice of x. It is not uncommon to find generalized stresses, for example, which are quite acceptable for uniaxial and biaxial stress states, but which are not at all suitable for triaxial stresses. For these situations, the Weibull theory is correspondingly limited.

For a homogeneous isotropic material, the generalized stress x will be a function of the principal stresses and perhaps some of their derivatives. As a first order approximation it would seem reasonable to choose x so that it did not depend explicitly on the stress gradient or any higher derivatives of the stress. This corresponds to the assumption that the behavior of an element is independent of the conditions which exist at other elements. There are situations, however, where gradient effects and even shape effects have been observed. Some of these cases are referenced in a survey article by Weibull[34].

When the generalized stress depends explicitly on the principal stresses only, the case of uniaxial stresses allows only one possible choice of generalized stress; however, for polyaxial stresses the possible choices are unlimited. In the little work that has been done in the area of polyaxial stresses, investigators have followed Weibull's procedure of using the normal stress on a plane as the generalized stress. This stress is clearly a function of the three principal stresses and furthermore its use is a simple generalization of the uniaxial case, i.e., when this normal stress is greater than x_u it produces a finite contribution to the probability of fracture, and when it is less than x_u the failure probability is zero.

The use of the Weibull failure theory requires in most cases that the generalized stress x be specified; however, one notable exception exists. For the case of specimens of similar geometry and proportional tractions, and hence stresses, Eq. 24 can be used to predict the size effects for those materials which have zero "zero probability strengths". We note in Eq. 24 that the x's may be taken as any statistical variate such as the external failure load, or one of the stress invariants.

We shall attempt to answer the previous three questions in the light of specific results reported in the literature for various materials.

III-A-3-e-i. Existence of a Distribution Function

In this subsection we shall confine our attention to a few examples in the literature where the distribution function can be shown to be non-existent. It is certainly easier to show this than it is to prove existence.

1. In the report by Salmassy[35] we find a typical description of a situation in which no distribution function can exist.

"Research for the first year of direct Air Force sponsorship .... is covered by AF Technical Report No. 6512, April 1951. This work was limited to K151A, a nickel-bonded titanium carbide product of Kennametal, Incorporated. Although this material exhibited slight plastic flow in compression, bending, and torsion at room temperature, the mode of fracture was that normally found in brittle ceramics. However, research revealed the reproducibility and the homogeneity of K151A to be so poor that any attempt at quantitative correlation of fracture data was futile. Nevertheless, the data of this period, indicated that a flaw-type mechanism offered the most likely basis for developing correlations."

2. The existence of a distribution function is very much in doubt when one obtains the so called "complex" distribution shown in Fig. 8. Weibull has the following remarks concerning such distributions:

"The fundamental question now arises, whether this splitting up is a purely formal operation, or whether it might unveil some hidden real causes. It may be said that any distribution may be represented by a sum of a sufficiently great number of simple distributions, just as any periodical function may be developed in a Fourier series. However, if the number of the components be small and the number of observations sufficiently large, the likelihood of real causes seems to increase. In any case, it is very easy to produce real complex distributions by synthesis."

If the bilinear distribution curve shown in Fig. 8 actually represents a mixture of two distinct components and if these components will not necessarily always be found in the same proportion, a distribution function does not exist.

3. Bending tests were performed on a large number of specimens of commercial refractory porcelain by Salmassy[17]. The data from this test series, for both large and small specimens, are shown in Figs. 12 and 13. Salmassy has a number of important remarks concerning these Weibull plots which we shall quote.

"..... Figs. 12 and 13 show the distribution curves for the specimens plotted logarithmically. As can be seen, both distribution curves are made up of two straight-line portions. If these distribution curves followed Weibull's distribution function all the data should fall on only one line instead of two. Weibull has pointed out that the behavior ..... indicates that the fracture of such a material is complex. This means one of the following:

(1) The material or fabrication of some of the specimens was different from the rest.
(2) All the specimens were not tested in identically the same way.
(3) At a certain stress, some fundamental change takes place in the nature of the fracture of the material.

These results are of special significance, for they point up the possibility of obtaining complex behavior in the practical applications of these statistical principles (Weibull's). If a set of data yields the type of distribution curve shown in Fig. 12, then the designer will know that he may not be obtaining uniform material from his supplier or that his test data have not been obtained in a uniform way. If neither of these are found to be at fault, and the distribution curve is found to be fundamentally complex, then the analysis of the data may be difficult (or perhaps impossible)."

III-A-3-e-ii. Is the Distribution Function of the Weibull Type

It seems appropriate to open this subsection with a quote from Weibull[19] concerning the validity of his distribution function (Eqs. 17 and 18):

"The author has never been of the opinion that this function is always valid. On the contrary, he very much doubts the sense of speaking of the "correct" distribution function, just as there is no meaning in asking for the

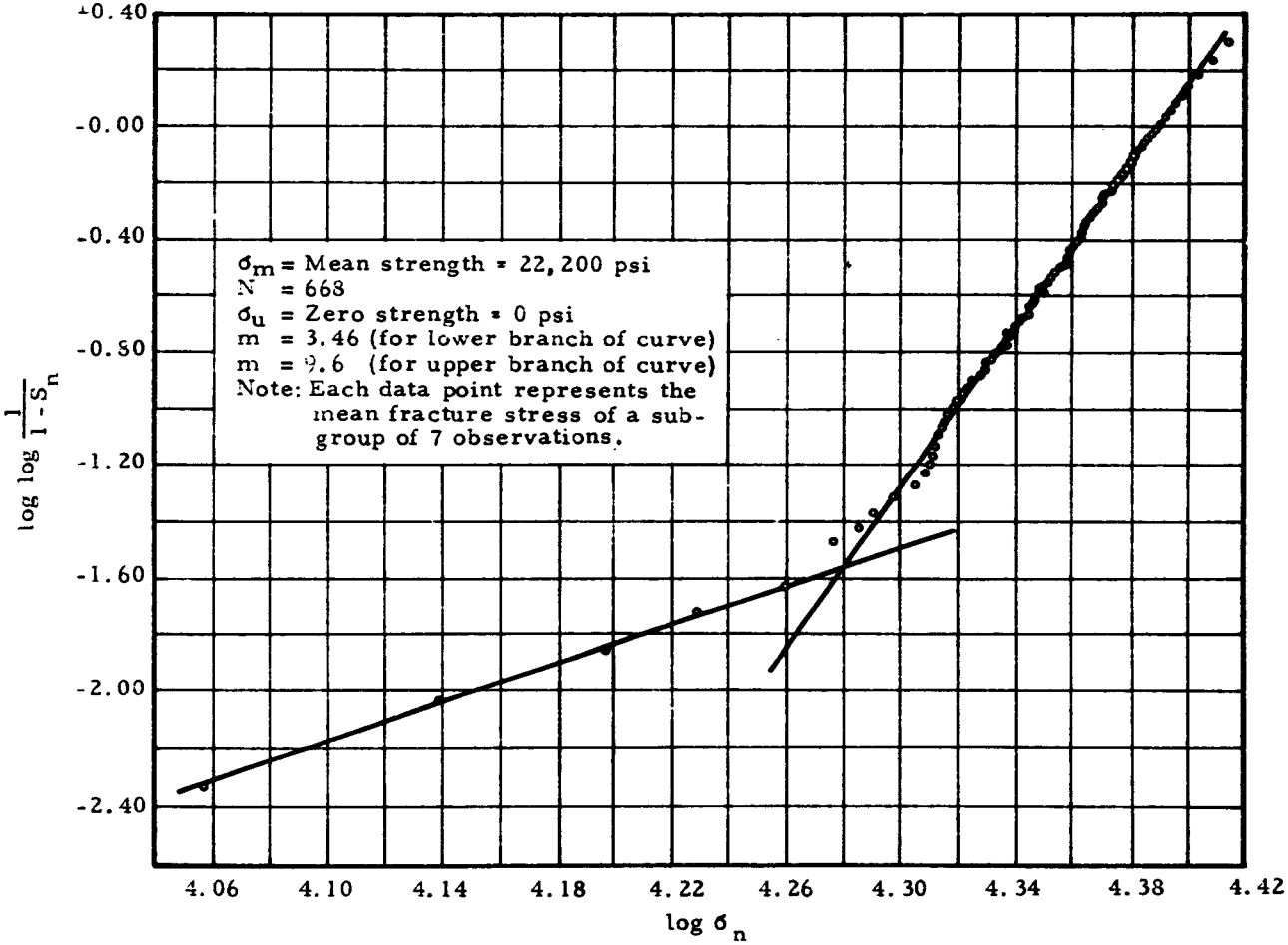

Figure 12

LOGARITHMIC DISTRIBUTION CURVE OF BENDING STRENGTH FOR LARGE SIZE RODS OF A REFRACTORY PORCELAIN (AFTER SALMASSY, REF. 17)

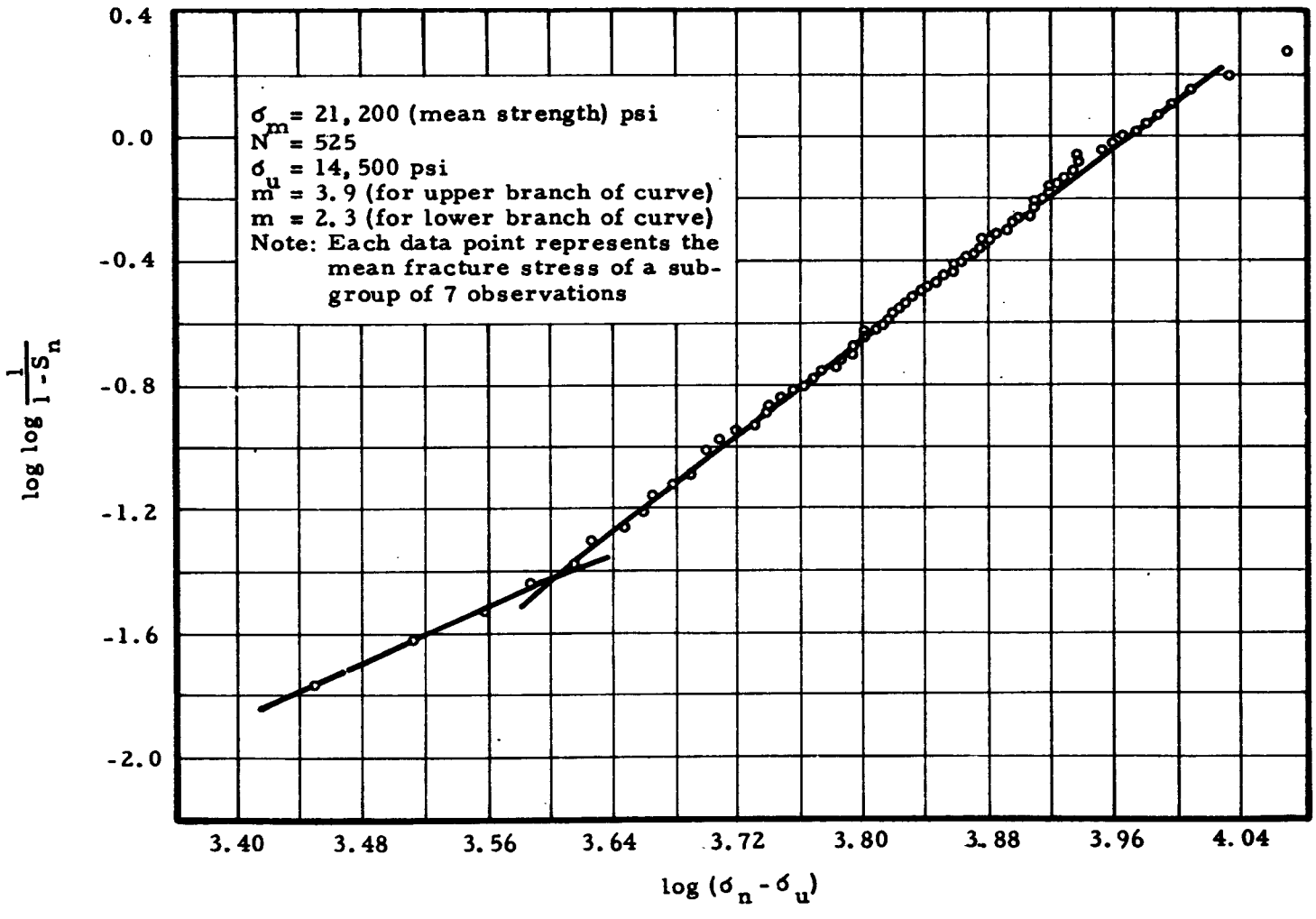

Figure 13

LOGARITHMIC DISTRIBUTION CURVE OF BENDING STRENGTH FOR SMALL SIZE RODS OF A REFRACTORY PORCELAIN (AFTER SALMASSY, REF. 17)

correct strength values of an SAE steel, depending as it does, not only on the material itself, but also upon the manufacturer and many other factors. In most cases, it is hoped that these factors will influence only the parameters. However, accidentally they may even affect the function itself."

The problem of testing the compatibility of a set of observed and theoretical frequencies, is treated extensively in statistics under the subject of "Goodness of Fit".[36] Generally the χ^2 test is used; however, here we will appeal to the simple procedure of plotting the data in Weibull coordinates and determining whether we obtain a straight line.

1. As our first example, we refer to the set of data on the bending strength of Wesgo AL995 which is presented in Fig. 5b. This data was obtained by Bortz[15] at room temperature using a dog bone shaped specimen which will be discussed in Chapter V. Certainly for the specimen used the Weibull distribution appears to represent the data; but, whether the distribution constants associated with this diagram will be useful for other size specimens remains an open question.

2. Let us now return to the series of 3000 tests of Indian Cotton which were studied by Koshal and Turner.[33] These authors fit the data shown in Table 6 with the theoretical frequency curve of Pearson's Type 1, hence,

$$y = 599.3 \left(1 + \frac{x}{18.777}\right)^{0.876716} \left(1 - \frac{x}{29.1947}\right)^{13.631284} \tag{59}$$

where y represents the frequency of any strength x, expressed in grams. The Weibull plot of this data is shown in Fig. 14 from which the Weibull parameters were determined; $x_u = 0.59$ grams, $x_o = 3.73$ grams, and $m = 1.456$. Not only does the Weibull distribution appear to fit the data; but, the following table shows that the fit is even better than that of the Pearson Type 1.

3. In Table 3 of this report, the results of various bending tests are reported for specimens of equal volume but different shapes. Referring specifically to Wesgo AL995 at 1000°C and Beryllium Oxide at room temperature, Weil and Daniel[31] decided to combine the data for the various

Table 6

FIBER STRENGTH OF INDIAN COTTON (AFTER WEIBULL, REF. 19)

Class Identification	Tensile Strength, Grams		
	Weibull Values, n	Observed Values, ñ	Pearson Type 1, n
1	118	117	127
2	646	667	659
3	1232	1219	1255
4	1751	1729	1777
5	2161	2153	2184
6	2461	2465	2480
7	2667	2664	2683
8	2802	2813	2816
9	2886	2887	2899
10	2937	2933	2949
11	2966	2962	2978
12	2982	2985	2994
13	2991	2991	3003
14	2996	2995	3007
15	2999	2999	3009
16	3000	3000	3010

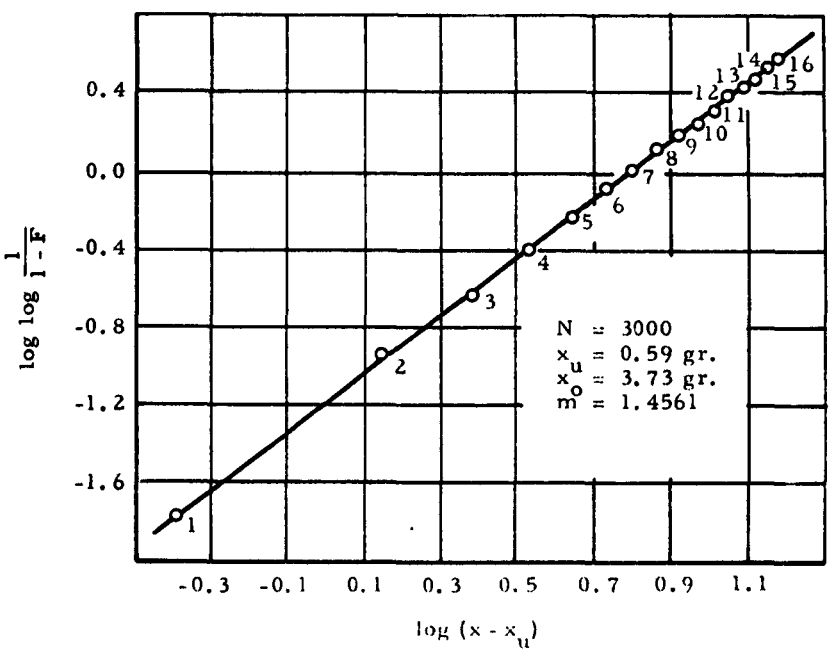

Figure 14

FIBER STRENGTH OF INDIAN COTTON (AFTER WEIBULL, REF. 19)

shapes in view of the fact that their means and standard deviations were so similar. The combined cumulative distribution curves are shown in Figs. 15 and 16 where we observe their excellent linear correlation.

4. As a final example of the applicability of the Weibull distribution we shall consider data obtained from a biaxial stress field. Salmassy[17] developed the distribution curve shown in Fig. 17, using torsion specimens made of Hydro-Stone plaster. Once again, we find a reasonably good linear correlation.

III-A-3-c-iii. Are the Weibull Parameters x_u, x_o, m Constants of the Material

One of the requirements for the successful application of the Weibull theory is that the material to which its applied be of the series or weakest link type. When this assumption holds, it is sometimes possible to characterize a unit volume of material with the Weibull distribution function and then treat the entire structure as if it were a chain with each unit volume as a link. Under these conditions it is possible to determine the statistical parameters of a link, from the known behavior of the entire structure or chain. If we were to apply the Weibull theory to a material which did not behave as a series or weakest link model, every size and stress state would provide a different set of Weibull parameters since the Weibull theory would not properly account for these effects. After presenting his basic theory in Reference 11, Weibull presented another paper, Reference 37, in which he makes the following statement, "At the time when the new theory was set, the experimental data available for its verification were rather scarce, so that the first step required was to provide more complete experimental evidence. From the researches undertaken for this purpose, some unexpected results were obtained, which necessitated an extension and supplementation of the theory." Weibull then proceeded to introduce the concept which he called irregular materials and which we refer to as parallel material. The theory which we refer to as Weibull's theory will not work for materials which do not satisfy the series model.

Perhaps one of the greatest demands placed on any strength theory is to predict the effect of size. If a theory would do nothing else, it would

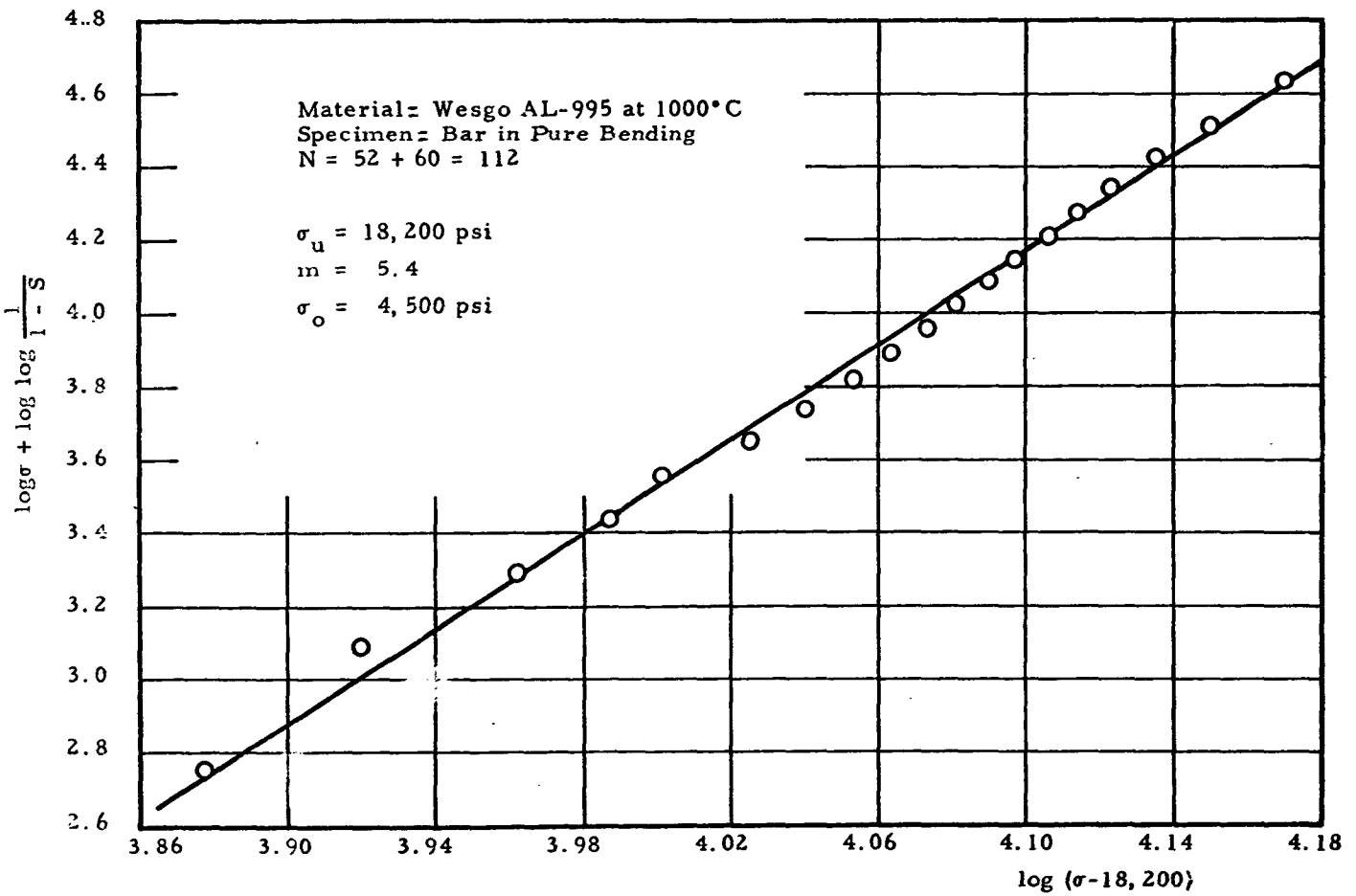

Figure 15

GRAPHICAL DETERMINATION OF MATERIAL PARAMETERS FOR WESGO AL-995 AT 1000°C (AFTER WEIL AND DANIEL, REF. 31)

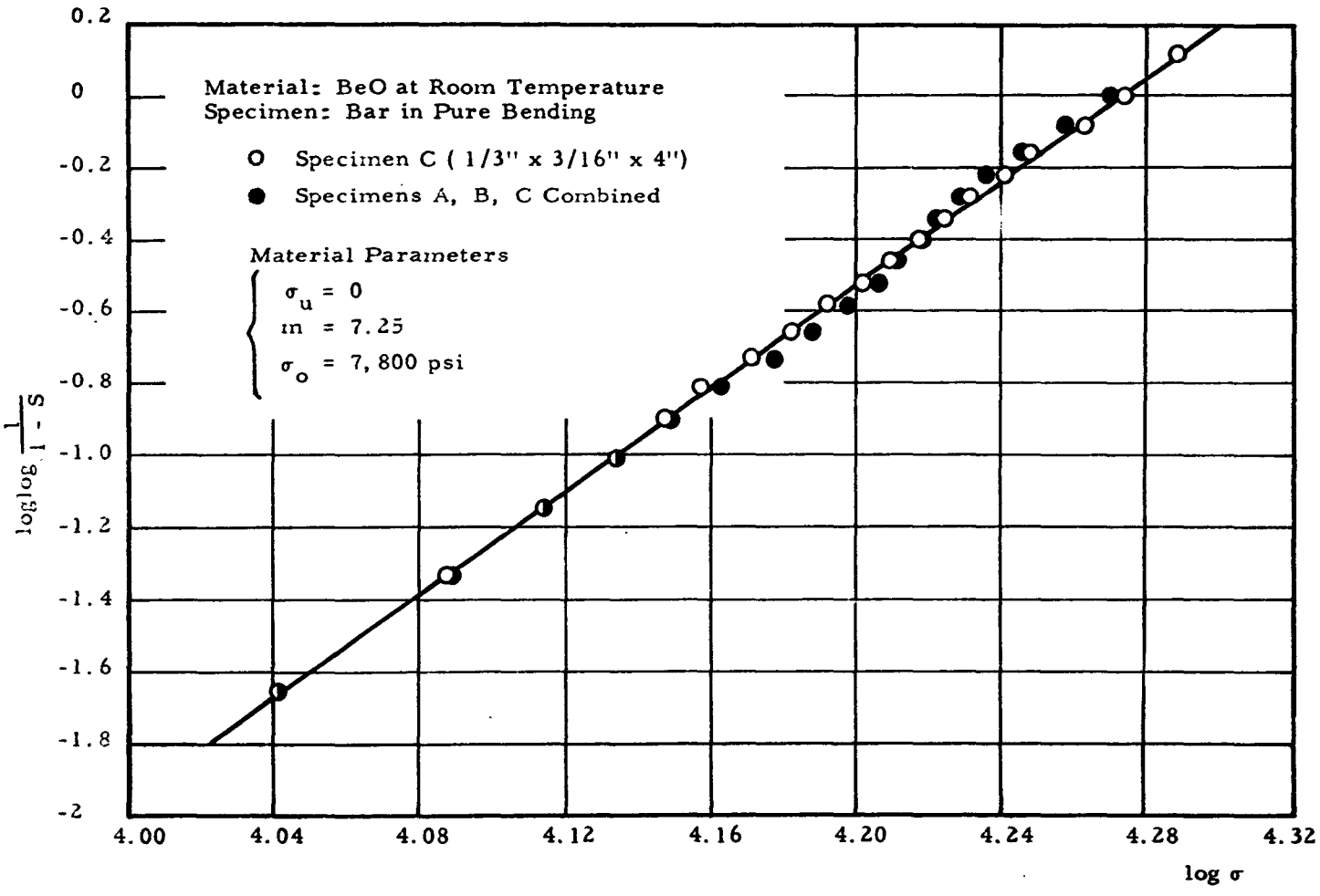

Figure 16

GRAPHICAL DETERMINATION OF MATERIAL PARAMETERS FOR BERYLLIA AT ROOM TEMPERATURE (AFTER WEIL AND DANIEL, REF. 31)

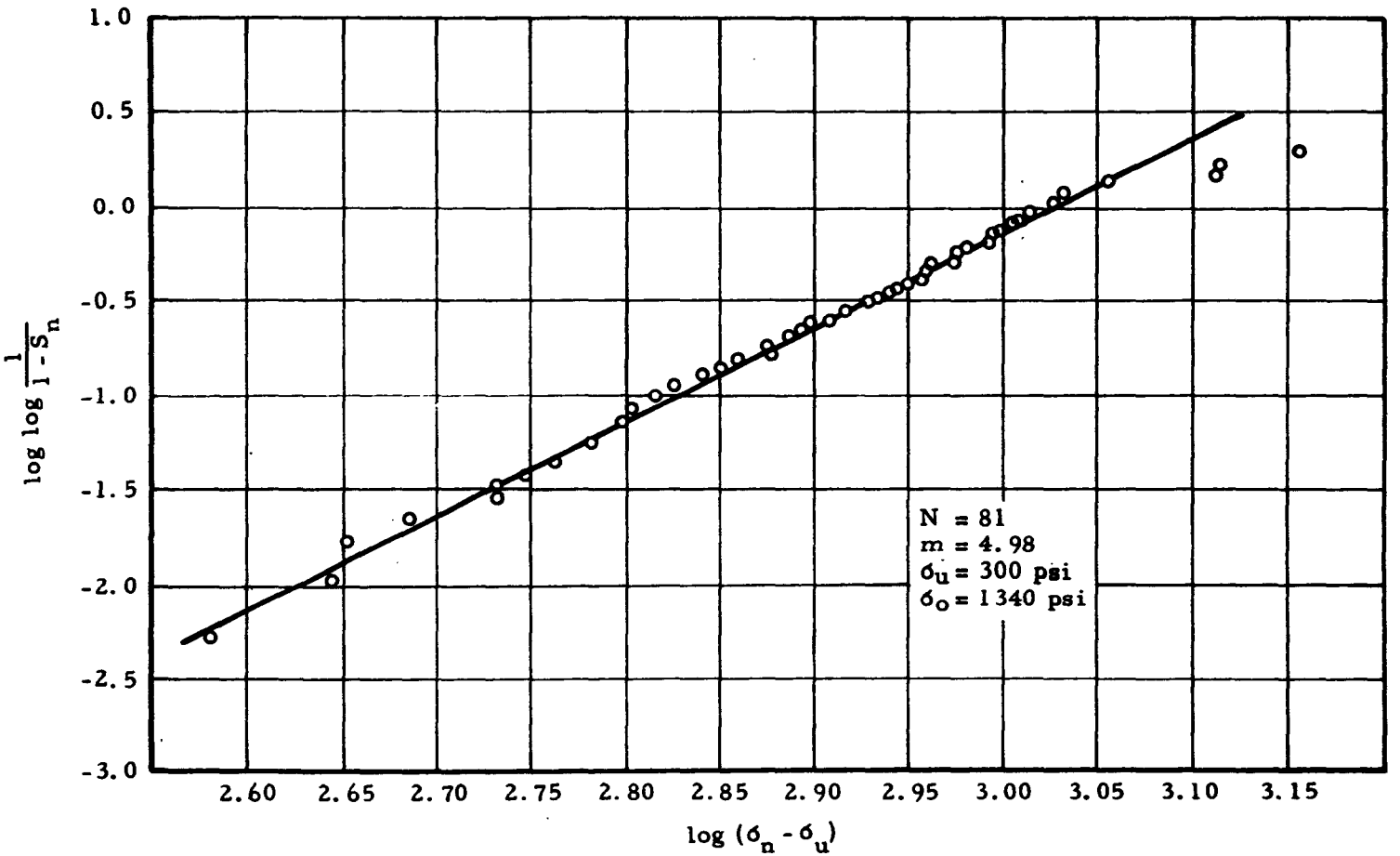

Figure 17

LOGARITHMIC DISTRIBUTION CURVE OF FRACTURE STRESSES FOR NO. 5 SIZE TORSION SPECIMENS OF HYDRO-STONE PLASTER (AFTER SALMASSY, REF. 17)

still allow the designer to predict the behavior of a prototype from that of a model. If m is a constant of a "series material", Eq. 29 enables us to make such a prediction even when we don't know the form of the generalized stress x. We will now consider several investigations which deal with size effect.

1. One of the most widely quoted verifications of Weibull's theory is the work of Davidenkov, Shevandin, and Wittmann[14] on the influence of size on the brittle strength of steel. Using cylindrical bending and tension specimens of brittle phosphorous steel in liquid air, they determined almost identical values of the constant m; namely 23.5 and 25.4. These values of m, calculated from Eq. 29 for the two larger specimens of the three used, were then employed to predict the fracture stress of the smallest specimen. The predictions are found to be off by only 3%. Furthermore, the prediction of the ratio of the strengths in bending and tension as determined by Eq. 36 (which assumes $\sigma_u = 0$) is 1.39 and according to the experiment, 1.40. The average fracture stress is plotted against the volume in Fig. 18 for the bending and tension tests. The standard deviation of the test results is also shown, and as predicted by theory, it is found to decrease with increasing specimen volume. We quote Davidenkov's conclusions:

"The experimental data thus lead to the conclusion that the statistical theory of strength explains satisfactorily and without inner contradictions the influence of size on the brittle strength of steel. From this fact a few important practical conclusions can be drawn.

The shape of the curves in Fig. 18 shows that the brittle strength varies with increase of size more and more slowly, so that it can be assumed that this tends to some definite limit. (The zero strength is not zero as assumed in their calculation of m.) The theoretical meaning of this limit consists in the fact that starting with a sufficiently large specimen, a complete set of all possible non-homogeneities will be present.

The larger the specimen the closer this limit will be approached. It would therefore be advantageous to make tests as far as possible with large specimens under uniform axial stresses."

2. The size of a mild steel specimen has been reported to effect its

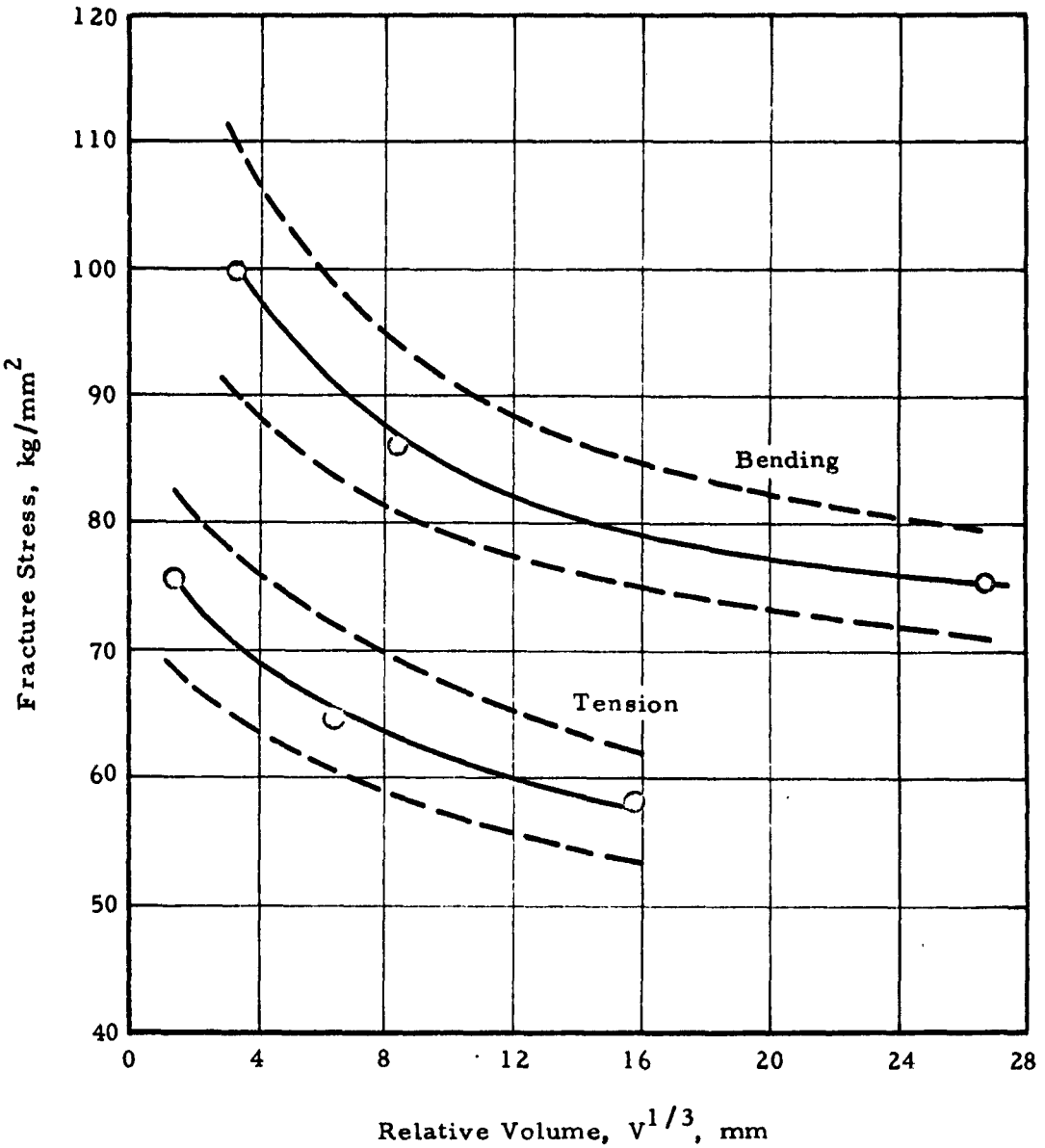

Figure 18

THE EFFECT OF SIZE ON THE MEAN FRACTURE STRESS AND STANDARD DEVIATION IN BENDING AND TENSION OF STEEL TESTED AT THE TEMPERATURE OF LIQUID AIR (AFTER DAVIDENKOV, REF. 14)

room temperature upper yield point strength.[38] The data for these specimens yielded to treatment by Weibull's theory for homogeneous tension members. The data, plotted in Fig. 19, gave a flaw density exponent of m = 58 which indicates a relatively narrow distribution curve. The volume effect is reasonably well predicted over the three decade range considered.

3. Bortz and Weil[39] have recently completed an investigation of the statistical strength properties of Wesgo Al_2O_3, Lucalox Al_2O_3, and ARF MgO. The results of this study are summarized in Table 7 where we observe a great variation of the Weibull parameters from test to test. On the other hand, we also observe an extremely large scatter in the standard deviations which would indicate that the number of specimens are not sufficient for a reliable determination of the Weibull constants.

In each grouping of tests, the medium size specimen appears in the greatest number. Using this fact, the constant m is determined for this specimen size and then used to predict the volume effect of the three specimen group. The results are shown in Figs. 20 to 25. In all cases, of course, the curve passes through the center point. The vertical positions of the points representing the largest and smallest specimens are determined from their associated average breaking strengths. Since the mean strength determinations require less data than the determination of m, the data has been used to its best possible advantage. In spite of this, however, the results are not at all satisfactory from a design point of view. The following comments apply to the various figures:

Figure 20:

In view of the reasonable sample size used, it is surprising that even the weakest link effect is not confirmed. One suspects that each specimen size has been taken from a different population.

Figures 21 and 22:

The weakest link trend is confirmed. Weibull's prediction of decreasing scatter with increasing size in not confirmed; however, the number of specimens used is not satisfactory for a reliable description of the dispersion. Further, the volume is distributed over only one decade and any errors which occur here may be

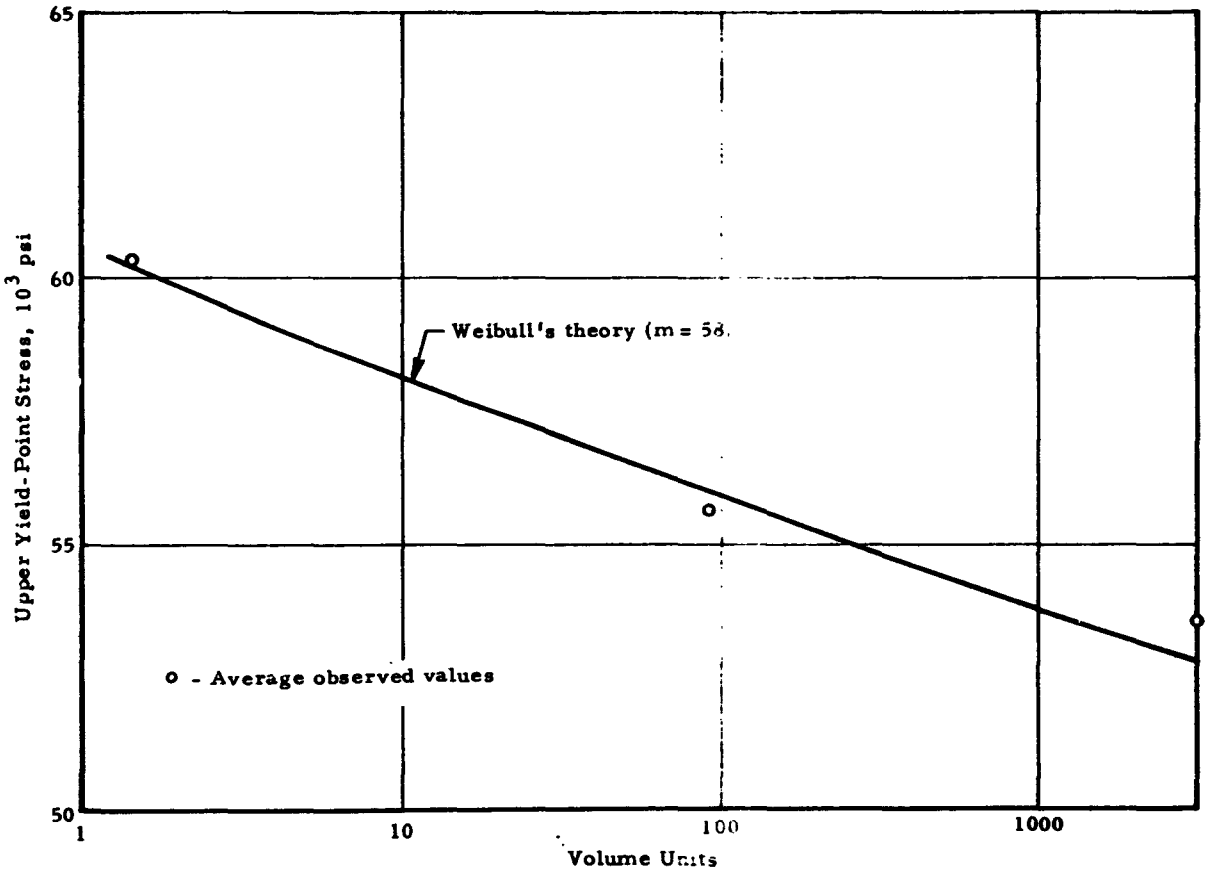

Figure 19

THE EFFECT OF SIZE ON THE UPPER YIELD-POINT STRESS OF A MILD STEEL (AFTER RICHARDS, REF. 38)

Table 7

SUMMARY OF MATERIAL CONSTANTS FROM BEND TESTS OF VARIOUS GAGE VOLUME SPECIMENS TESTED IN AIR

Mat'l.	Temp. °C	Surface Treatment	Heat Treatment	Number of Specimens	Gage Volume in.^{3}	Weibull Parameters: Flow Density Constant, m	Weibull Parameters: Zero Strength x_u, 10^3 psi	Weibull Parameters: Material Constant x_o, 10^3 psi	Statistical Values: Average Strength x_m, 10^3 psi	Statistical Values: Standard Deviation a, 10^3 psi	Statistical Values: Coeff. of Variation %
Wesgo Al_2O_3	20	As received	None	30	0.012	3.70	20.0	1.59	28.8	2.23	7.73
				54	0.047	3.23	10.0	3.24	26.3	1.28	4.87
				25	0.098	11.35	0.0	15.0	27.7	2.54	9.16
		Ground	None	12	0.012	-	-	-	28.9	9.50	33.05
				30	0.047	6.10	10.0	4.37	25.3	3.36	13.30
				19	0.098	-	-	-	23.1	2.03	8.37
		As received	Annealed at 1700°C	5	0.012	-	-	-	34.6	3.50	10.10
				36	0.047	10.00	0.0	18.86	30.8	3.83	12.42
				5	0.098	-	-	-	27.1	2.5	9.22
		Ground	Annealed at 1700°C	5	0.012	-	-	-	25.6	2.81	11.00
				30	0.047	12.50	0.0	18.41	28.6	2.44	8.53
				4	0.098	-	-	-	27.8	2.75	9.88
	1000	As received	None	14	0.012	-	-	-	19.5	2.88	14.75
				20	0.047	3.60	11.0	2.17	19.6	2.66	13.59
				13	0.098	-	-	-	18.1	3.16	17.45
		Ground	None	14	0.012	-	-	-	22.8	2.27	9.97
				15	0.047	-	-	-	21.1	1.97	9.32
				14	0.098	-	-	-	21.1	2.37	11.23
		As received	Annealed at 1700°C	15	0.012	-	-	-	17.8	1.06	5.96
				15	0.047	-	-	-	18.7	1.41	7.52
				15	0.098	-	-	-	16.2	1.55	9.56
Lucalox Al_2O_3	20	Ground	None	24	0.012	1.47	17.5	0.18	27.9	4.91	17.58
				26	0.047	5.05	15.0	4.22	26.5	3.62	13.66
				10	0.098	4.00	10.0	4.57	22.5	1.77	7.87
	1000	Ground	None	22	0.047	7.75	0.0	12.08	22.2	3.21	14.46
				12	0.098	2.72	10.0	1.10	20.3	2.52	12.43
ARF MgO	20	As received	None	19	0.012	6.69	0.0	8.52	20.4	3.08	15.04
				60	0.047	3.25	10.0	0.74	16.6	1.94	11.70
				24	0.098	6.77	0.0	8.10	14.1	2.30	16.32
	1000	As received	None	30	0.012	2.24	10.5	0.26	15.0	1.76	11.74
				49	0.047	5.25	5.0	0.25	14.6	1.38	9.45
				30	0.098	1.13	9.5	0.09	13.1	1.83	13.99

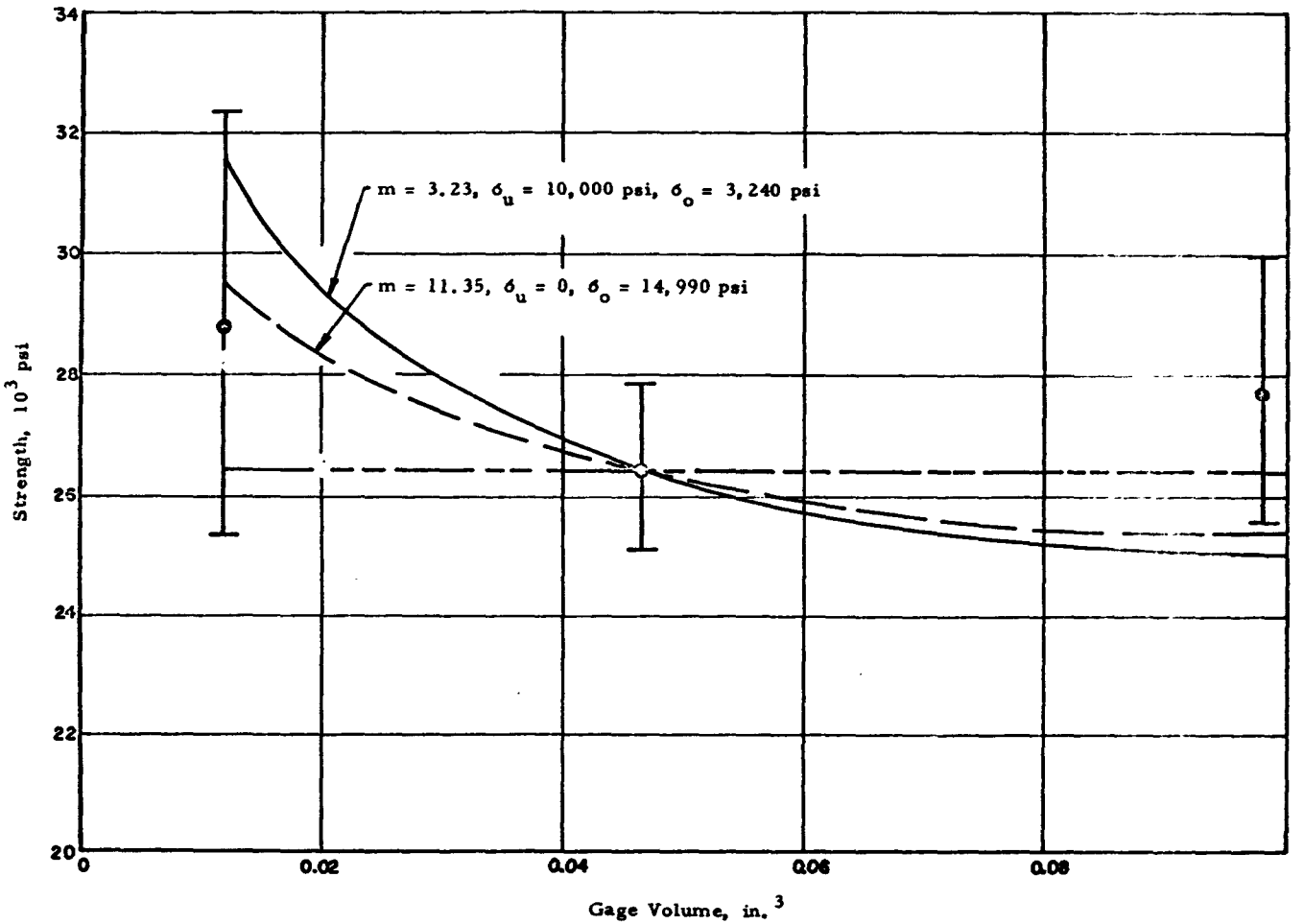

Figure 20

COMPARISON OF THEORY AND EXPERIMENTS FOR SIZE EFFECTS IN AS-RECEIVED WESGO Al_2O_3 TESTED AT 20°C IN AIR

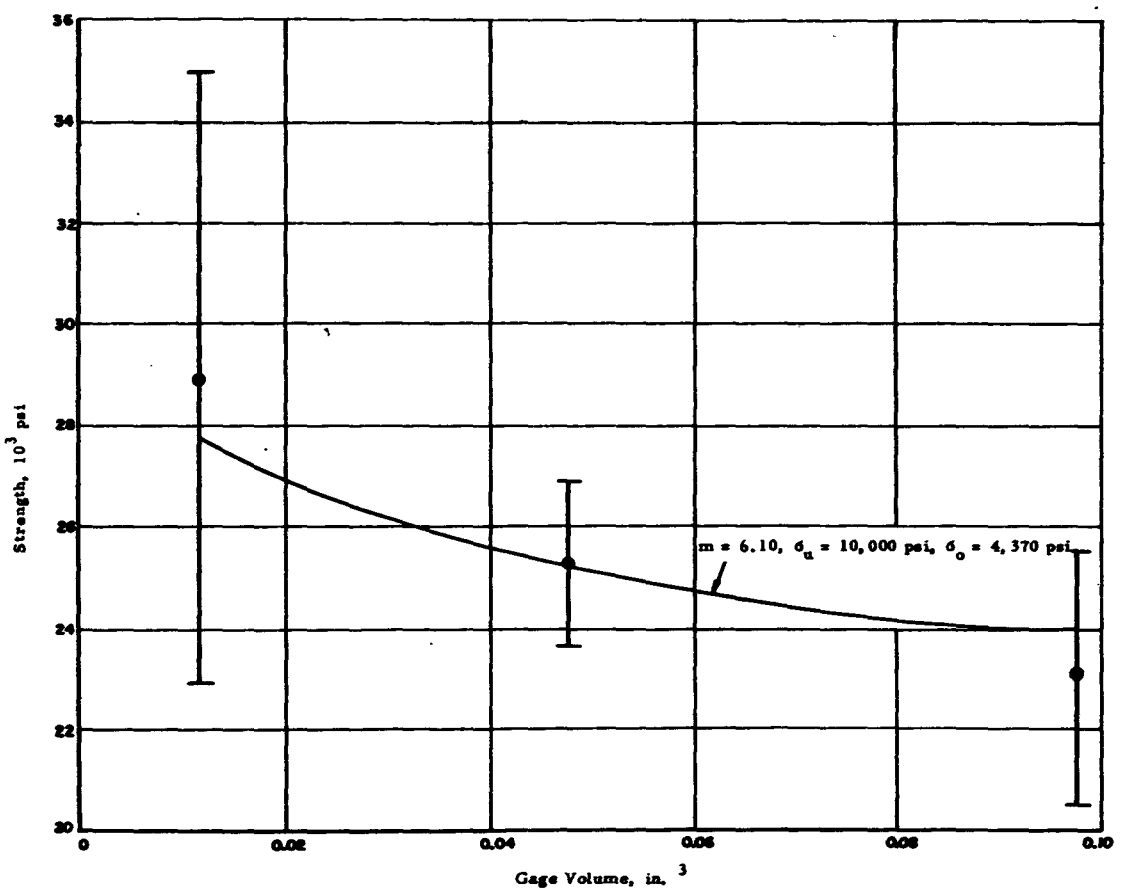

Figure 21

COMPARISON OF THEORY AND EXPERIMENTS FOR SIZE EFFECTS IN GROUND WESGO Al_2O_3 TESTED AT 20°C IN AIR

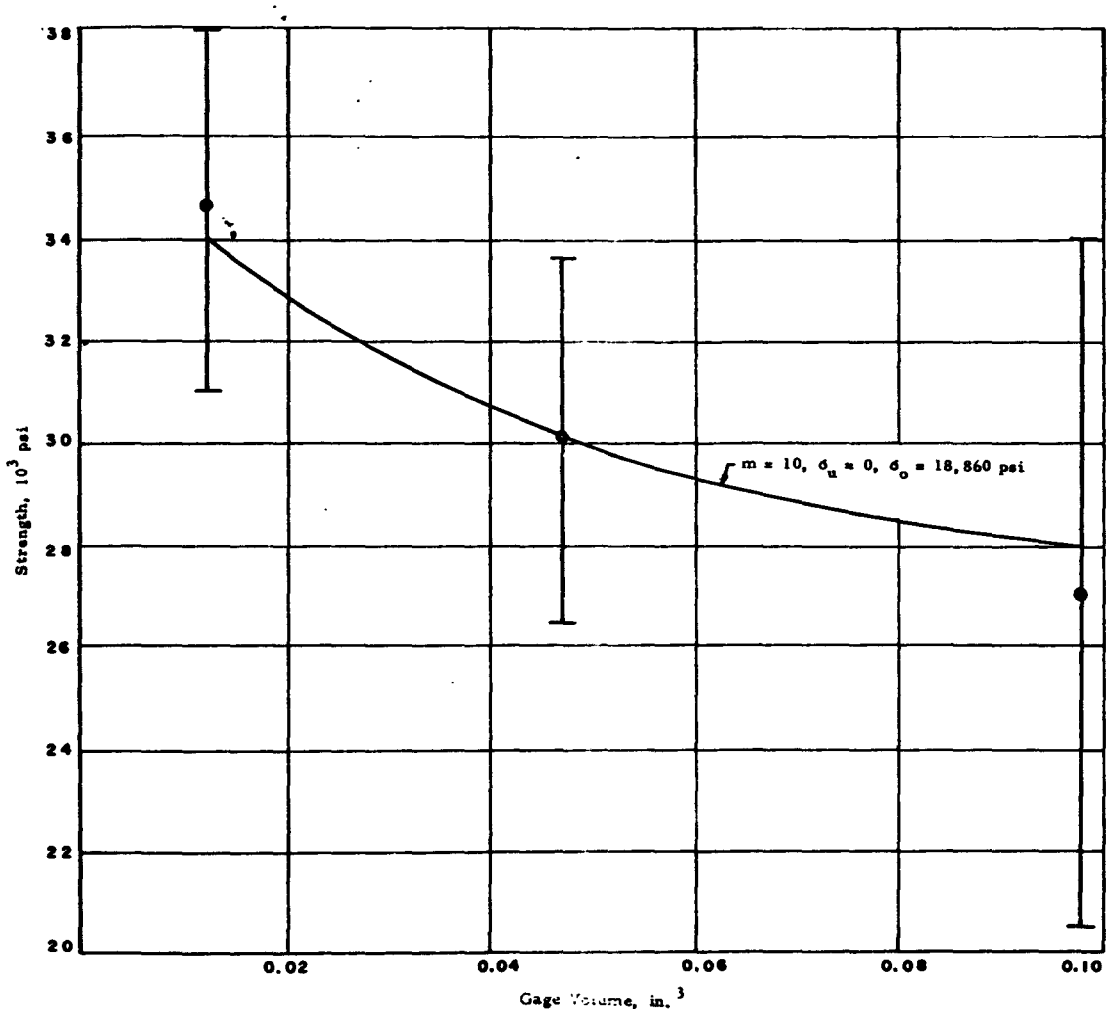

Figure 22

COMPARISON OF THEORY AND EXPERIMENTS FOR SIZE EFFECTS IN ANNEALED WESGO Al_2O_3 TESTED AT 20°C IN AIR

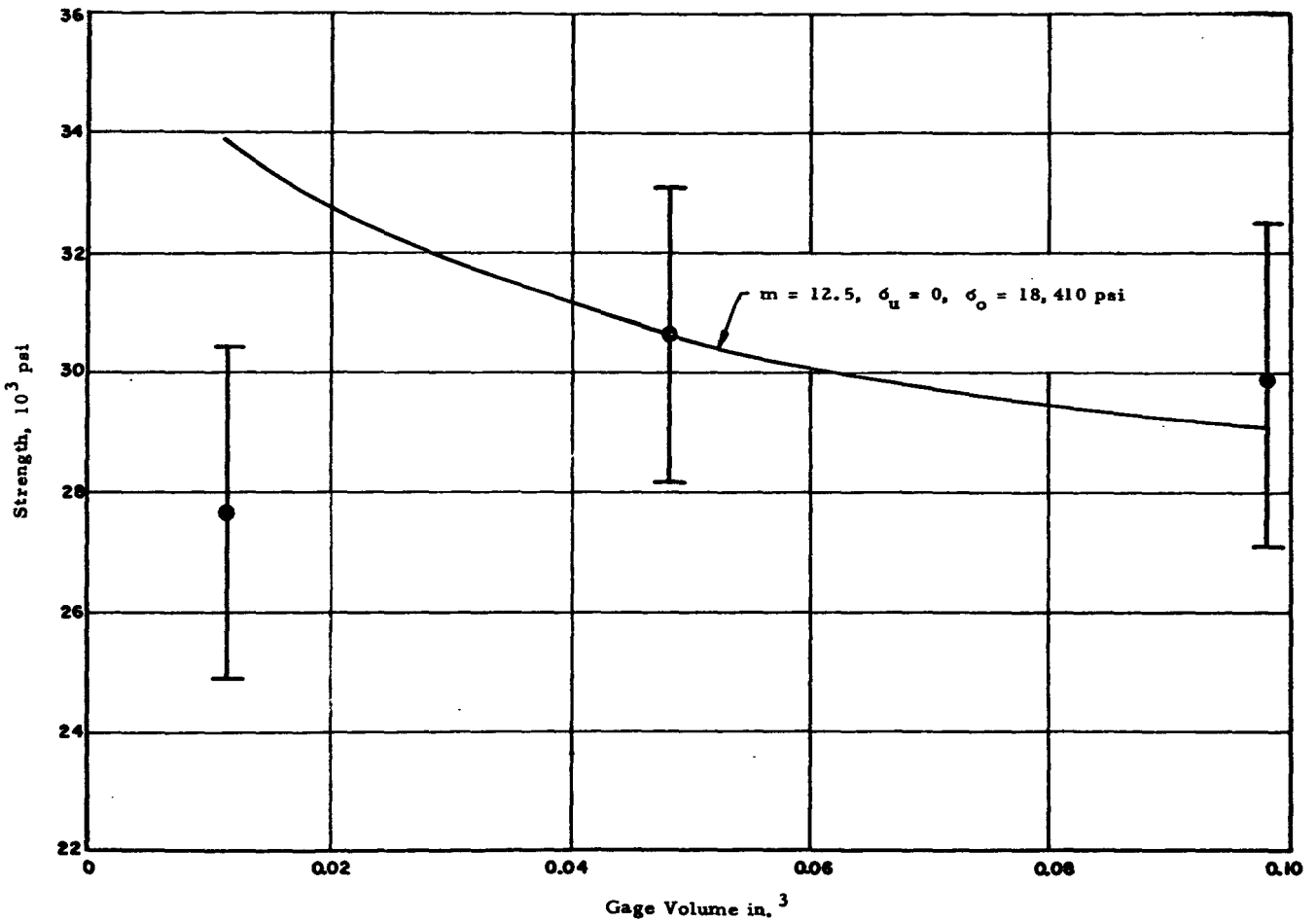

Figure 23

COMPARISON OF THEORY AND EXPERIMENTS FOR SIZE EFFECTS IN GROUND AND ANNEALED WESGO Al_2O_3 TESTED AT 20°C IN AIR

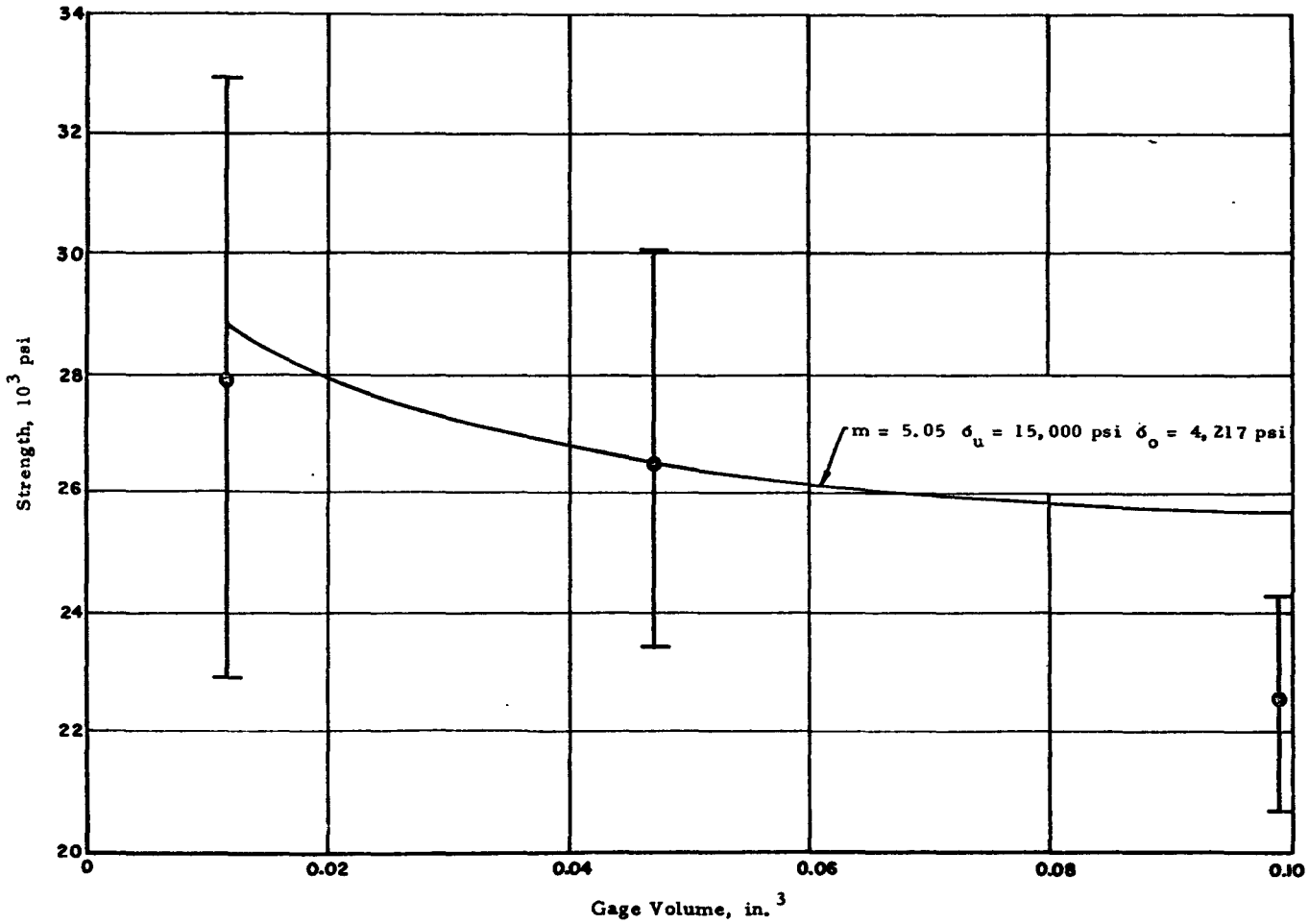

Figure 24

COMPARISON OF THEORY AND EXPERIMENTS FOR SIZE EFFECTS IN LUCALOX Al_2O_3 TESTED AT 20°C IN AIR

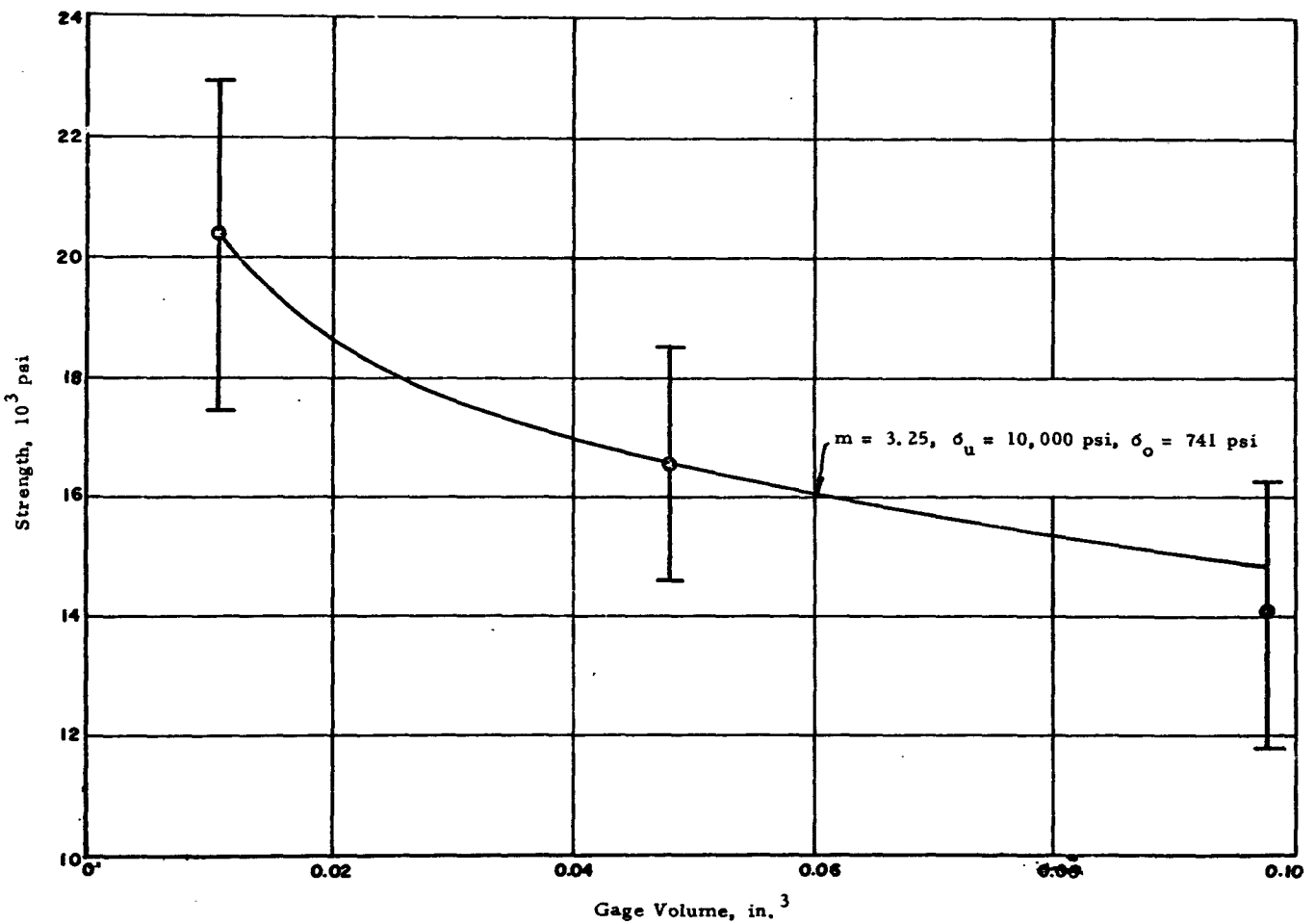

Figure 25

COMPARISON OF THEORY AND EXPERIMENTS FOR SIZE EFFECTS IN ARF MgO TESTED AT 20°C IN AIR

magnified many times when we extrapolate the volume effects over 3 or 4 decades as would be desirable for design purposes.

Figure 23:

Because five specimens are too few to approximate even the mean strength, the results are meaningless.

Figure 24:

Quantitative predictions are most unsatisfactory.

Figure 25:

This is the best of the series considered. We note that it used one of the largest sample sizes in the study. Furthermore, since the preparation of the specimens was undertaken at ARF, one presumes that the control of the manufacturing process was more closely supervised. Still, Weibull's scatter trend is not confirmed and the quantitative predictions in one decade are not close enough for design.

4. In the study by Weil and Daniel[31] previously refered to, one of the specific objectives was to study the adequacy of the Weibull theory for predicting failure under conditions of non-uniform stress distribution. This is accomplished by comparing experimental results for specimens having various stress gradients but the same risk of rupture according to the theory. The results of this investigation are presented in Table 3 where we observe that the mean fracture stress is practically unaffected by the gradients obtained in the different shape, equal volume bending specimens. Since the assumption that the Weibull parameters are constants of the material is utilized in predicting that no gradient effect will be present, this test series confirms the Weibull theory within the range of gradients considered. There is one point which should be raised in regard to the conclusions presented for the analytical portion of this study. The authors point out that in the expression for the risk of rupture for the bending members, the only form in which the specimen dimensions enter is their product, giving rise to the total volume V. They then make the following statement, "An independent variation of length, width, and depth of specimen without a change in volume does not effect the risk of rupture and, therefore, the latter is independent of stress gradient." This conclusion is significant only as a criterion for

experimental verification, since the risk of rupture for a unit volume is assumed not to depend explicitly on the stress gradient. If the authors had started with the assumption that the unit risk of rupture depended explicitly on the stress gradient, and then showed that after integration over the beam volume this dependence disappeared, we would have a very significant conclusion indeed.

5. For the material Hydro-Stone Plaster, Table 2 indicates that the Weibull parameters are not constants of the material.

6. As our final comment, we would like to point out that an almost insignificant amount of data is available for checking the Weibull theory under polyaxial stress states. Such data is badly needed if we are to develop any type of useful design procedure. A small effort devoted to torsion can be found in Salmassy[17]; however, their conclusions are not very decisive. They make the following comments:

a) "The torsion test yields reliable fracture data."

b) "Weibull's statistical theory of strength predicted the effect of size, and the effects observed in the simple stress states of tension, bending, and torsion. Weibull's theory was not adequate, however, for predicting the effects of combined stresses."

c) "The mean fracture stresses in tension, torsion, and bending appear to have the same qualitative relationship for many brittle materials, that is, the fracture stress in bending is higher than that in torsion and is higher in torsion than that in tension. Weibull's theory predicts this effect in these simple stress states, qualitatively. Insufficient data are available to indicate whether Weibull's theory can be used to predict this effect quantitatively."

d) "Under combined stresses, Hydrostone plaster gave an effect which is opposite to that predicted by Weibull's theory."

e) "Although data on cast iron gives a better correlation with Weibull's theory than data on Hydro-Stone plaster, they exhibit a similar deviation from Weibull's theory. The data from these two materials indicate that some stress other than the principal tensile stress may

contribute to the fracture of brittle materials. In essence, this means that Weibull's basic criterion (his choice of a generalized stress) for the fracture of an element of a brittle material may be incorrect. Of particular importance may be the role played by possible localized plastic flow in these supposedly brittle materials."

III-B. Extreme Value Statistics

III-B-1. Introduction and History

Using Griffith's flaw hypothesis as a starting point, we make the assumption that the flaws are distributed at random with a certain density per unit volume. Then, the strength of a specimen subjected to a homogeneous state of stress is determined by the weakest point in the specimen or by the smallest value to be found in a sample of size n where n is the number of flaws. Clearly, n increases as the volume increases and, therefore, the problem of finding out how the strength depends on the volume of a sample is equivalent statistically to studying the distribution of the smallest value as a function of n, the sample size. This statistical problem is treated under the statistical theory of extreme values.[18, 24]

It appears that Peirce[40] was the first investigator to recognize the connection between the strength of a specimen and the distribution of smallest values. His paper in 1926, mentions explicitly that the theoretical work of Tippett[41] is of importance in obtaining quantitative results concerning the distribution of breaking strengths. Tippett's paper in 1925, calculated the numerical values of the probabilities for the largest normal value, for different sample sizes up to one thousand, and the mean range for all normal values from two to one thousand. Tippett's Tables are the fundamental tool for all practical applications of the largest value for a normal distribution.

The fact that most of the studies of extreme values started from this distribution hampered the development, since none of the fundamental theorems of extreme values are related, in a simple way, to the normal distribution. The first paper based on the concept of a type of initial distributions different from the normal one is due to M. Fréchet in 1927. He was the first to obtain an asymptotic distribution of the largest value. Fréchet's paper, published in a remote journal, apparently never gained the recognition it merited. This

was due to the fact that R. A. Fisher and L. H. C. Tippett published in the next year, 1928, the paper that is now referred to in all works on extreme values. They found again, independently of Fréchet, his asymptotic distribution, and constructed two other asymptotes. They especially stressed the extremely slow convergence of the distribution of the largest normal value toward its asymptote, and thus showed the reason for the relative sterility of all previous endeavors.

The theory of extreme values ought to be centered about the exponential distribution because it leads to simple expressions of the most important results. The results obtained from this starting point can easily be generalized. Gumbel started from this basis and derived the asymptotic distribution of the mth extreme value which covered Fisher's distribution as a special case. In 1936, R. Von Mises showed the conditions under which the three asymptotes established by Fisher and Tippett are valid.

Using the theoretical work of the above authors, Gumbel studied many practical applications of extreme value theory; for example, the oldest ages, the distances in time between radioactive emissions, and his famous treatment of floods. Extreme value procedures have been successfully applied to an extensive study of the extreme temperatures and atmospheric pressures in Norway by N. A. Barricelli, and in the United States, to fracture problems by B. Epstein.[22, 42, 43] Epstein's three, almost identical papers, are used estensively in the following discussion of breaking strength.

III-B-2. Asymptotic Theory of Extreme Values

Insofar as applications of the statistical theory to the fracture problem are concerned, one is interested primarily in the distribution of the smallest value in samples of size n, for large values of n. Let F(x) be the probability that the value of the variate X selected randomly from a population is less than a certain x, and let $f(x) = F'(x)$ be the density of probability, henceforth called the initial distribution. Now, the probability that $X \geq x$ is given by $\left[1 - F(x)\right]$. Then the probability that n independent observations of X are all greater than or equal to x is given by $\left[1 - F(x)\right]^n$. This is clearly the probability for x to be the smallest among n independent observations. Then the probability that x is not the smallest value in n independent observations

is

$$1 - \left[1 - F(x)\right]^n$$

But, this is just another way of describing the distribution of smallest values $G_n(x)$, since $G_n(x)$ is the probability of obtaining a "smallest value" less than or equal to x. Thus, the cumulative distribution function of the smallest value in samples of size n is

$$G_n(x) = 1 - \left[1 - F(x)\right]^n \tag{60}$$

The associated probability density function then becomes,

$$g_n(x) = nf(x)\left[1 - F(x)\right]^{n-1} \tag{61}$$

If the initial distribution is known, Eqs. 60 and 61 already enable one to give graphical descriptions of $g_n(x)$ and $G_n(x)$. The means and moments of $g_n(x)$ may be obtained; but, as a rule this leads to integrals that can only be evaluated by numerical methods. The mode of $g_n(x)$, or the <u>most probable value</u> of the smallest value in samples of size n, can be found by finding the maximum of $g_n(x)$, i.e., by solving the equation $g_n'(x) = 0$. If a solution exists it may be written in the form

$$f^2(x_n^*)\,(n-1) = \left[f'(x_n^*)\right]\left[1 - F(x_n^*)\right] \tag{62}$$

If the initial distribution is limited, the distribution of the smallest value decreases monotonically, and no mode in the proper sense exists. The mode of the smallest value in samples of size n have computed by Epstein and his results are presented in Table 8, column (a) for various common probability density functions.

Equations 60, 61, and 62 are adequate if one is merely interested in giving a crude graphical description of the distribution of the smallest value in samples of size n drawn from a parent population f(x). It is necessary to go further, however, if one wishes to study the distribution of the smallest

Table 8

CHARACTERISTICS OF EXTREME VALUE DISTRIBUTIONS

	(a)	(b)	(c)
Probability density function.	x_n^*, Mode of smallest value in samples of size n.	Distribution of smallest values in samples of size n (for n large).	Remarks on general characteristics of the distribution of x, the smallest value in samples of size n (n large).
1. Rectangular distribution $f(x) = 1/(b-a),\ a \leq x \leq b$ $f(x) = 0$, elsewhere	a	$x = a + (b-a)\xi/n$	x is nearly independent of n. $D^2(x) = (b-a)^2/n^2$
2. Cauchy distribution with parameters (μ, λ): $f(x) = \frac{1}{\pi}\frac{\lambda}{\lambda^2+(x-\mu)^2}$	$\mu - \frac{\lambda n}{2\pi}$	$x = \mu - \frac{\lambda n}{\pi \xi}$	Most probable value of x decreases linearly with n. Both mean value and variance are infinite.
3. Laplace's distribution with parameters (μ, λ): $f(x) = \frac{1}{2\lambda}\exp - \frac{\lvert x-\mu \rvert}{\lambda}$	$\mu - \lambda \log \frac{n}{2}$	$x = \mu - \lambda \log \frac{n}{2} + \lambda \log \xi$	Most probable value decreases as a multiple of log n. Variance remains equal to $\lambda^2\pi^2/6$ independently of n.
4. Gauss' distribution with parameters (μ, σ): $f(x) = \frac{1}{(2\pi)^{1/2}\sigma}\exp - \frac{(x-\mu)^3}{2\sigma^2}$	$\mu - \sigma\sqrt{2 \log n} + \sigma\frac{\log \log n + \log 4\pi}{2\sqrt{2 \log n}}$	$x = \mu - \sigma\sqrt{2 \log n} + \sigma\frac{\log \log n + \log 4\pi}{2\sqrt{2 \log n}} + \frac{\sigma}{\sqrt{2 \log n}}\log \xi$	Most probable value decreases as a multiple of $\sqrt{\log n}$. Variance decreases as n increases and is given by $\pi^2\sigma^2/12 \log n$.
5. Weibull distribution with parameters (α, β): $f(x) = \alpha\beta x^{\beta-1}\exp(-\alpha x^\beta),\ x \geq 0,\ \alpha > 0, \beta > 1$ $F(x) = 1 - \exp(-\alpha x^\beta)$	$\frac{1}{\lvert \alpha n \rvert^{1/\beta}}(1-1/\beta)^{1/\beta}$	$x = (\xi/n\alpha)^{1/\beta}$	Most probable value decreases as $n^{-1/\beta}$. Mean value decreases as $n^{-1/\beta}$ and variance as, $n^{-2/\beta}$.

ξ is distributed with probability density function $h(\xi) = e^{-\xi}, \xi \geq 0$.

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values quantitatively, particularly for large values of n. The shortcomings of these equations for large values of n may be circumvented by the introduction of a convenient new variable defined as

$$\xi = nF(x) \tag{63}$$

In terms of this variable $G_n(x)$ becomes,

$$H_n(\xi) = 1 - \left[1 - \frac{\xi}{n}\right]^n \tag{64}$$

The asymptotic cumulative probability function $H(\xi)$ as $n \to \infty$ then becomes

$$H(\xi) = \lim_{n \to \infty} H_n(\xi) = 1 - e^{-\xi} \tag{65}$$

The associated probability density function, $h(\xi)$, can be found from $h(\xi) = H'(\xi)$:

$$h(\xi) = e^{-\xi} \tag{66}$$

Now, using Eqs. 65 and 66, one can completely specify the distribution of x, the smallest value in samples of size n for large values of n. We shall consider several examples of such computations.

Example 1: Rectangular Distribution

The initial distribution is

$$\begin{aligned} f(x) &= \frac{1}{b-a} \qquad a \le x \le b \\ f(x) &= 0 \qquad \text{elsewhere} \end{aligned} \tag{67}$$

Now, F(x) becomes,

$$F(x) = \int_a^x \frac{dx}{b-a} = \frac{x-a}{b-a} \qquad a \le x \le b \tag{68}$$

Using Eq. 60, the cumulative distribution of smallest values becomes

$$G_n(x) = 1 - \left[1 - \frac{x-a}{b-a}\right]^n \qquad a \leq x \leq b \tag{69}$$

and the distribution $g_n(x)$, from Eq. 61, is

$$g_n(x) = \frac{n}{b-a}\left[1 - \frac{x-a}{b-a}\right]^n \qquad a \leq x \leq b \tag{70}$$

The mode or the most probable value of the smallest value in samples of size n, x_n^*, is found by inspection of Eq. 70 since the distribution is limited. Hence,

$$x_n^* = a \tag{71}$$

To get an approximation to the distribution of smallest values in samples of size n for large n, we introduce ξ with Eq. 63; thus,

$$\xi = n\,F(x) = n\left(\frac{x-a}{b-a}\right) \tag{72}$$

or solving for x,

$$x = a + \frac{\xi}{n}(b-a) \tag{73}$$

This expression gives us the distribution of x for large n where ξ is distributed with the probability density function

$$h(\xi) = e^{-\xi} \tag{74}$$

We obtain the typical distribution shown in Fig. 26 by assuming a value of ξ and calculating the strength x from Eq. 73 and its associated frequency $h(\xi)$ from Eq. 74 where we note that x and ξ have the same frequency.

The asymptotic distributions of x for various common initial distributions are shown in Table 8, column (b).

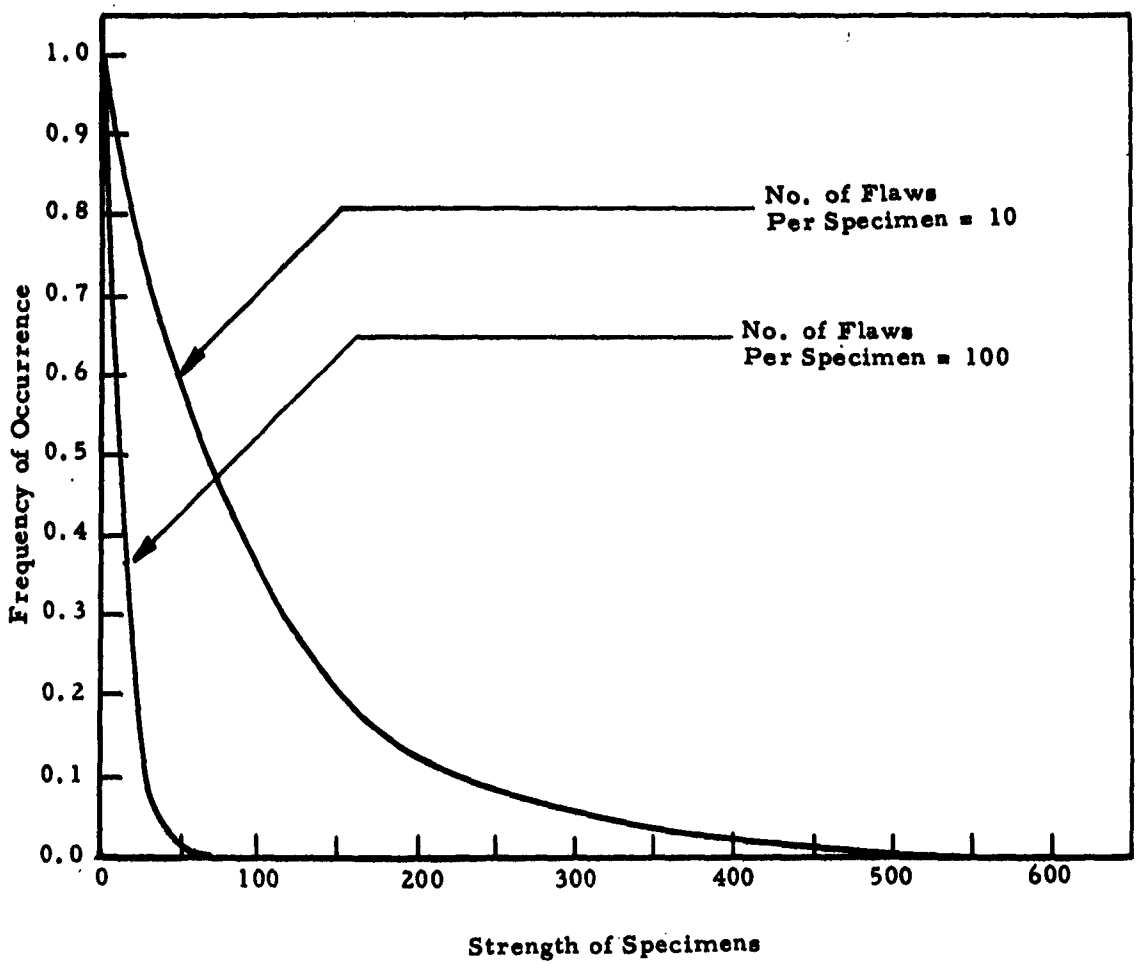


Figure 26

FREQUENCY DISTRIBUTION OF STRENGTHS OF SPECIMENS CONTAINING VARIOUS NUMBER OF FLAWS AND FOR WHICH THE FLAW STRENGTHS ARE RECTANGULARLY DISTRIBUTED WITH a = 0, b = 1000 (AFTER EPSTEIN, REF. 22)

Example 2: Laplace Distribution

When the initial distribution is that of Laplace, the distribution of smallest values in samples of size n (for n large) is given in Table 8, column (b). We note, here, that the frequency of x depends on the frequency of $\log\xi$ rather than ξ, i.e.,

$$x = \mu - \lambda \log\frac{n}{2} + \lambda \log \xi \tag{75}$$

Letting $\eta = \log \xi$ we can find the distribution of $\log \xi$ from the known cumulative distribution function of ξ:

$$H(\xi) = 1 - e^{-\xi} \tag{65}$$

Changing the variable to η ($\xi = e^{\eta}$), the cumulative distribution function becomes

$$J(\eta) = 1 - e^{-e^{\eta}} \tag{76}$$

Differentiating to find the frequency distribution, we have,

$$j(\eta) = e^{\eta} e^{-e^{\eta}} \qquad -\infty \leq \eta \leq \infty \tag{77}$$

which gives the frequency distribution of $\log \xi$. Then the distribtuion of x becomes

$$x = \mu - \lambda \log\frac{n}{2} + \lambda\eta \tag{78}$$

where η is distributed as $e^{\eta} e^{-e^{\eta}}$. To plot the frequency curves shown in Fig. 27 for the Laplace distribution, one assumes a value for η, and calculates x from Eq. 78 and the associated frequency $j(\eta)$ from Eq. 77. We observe that $j(\eta)$ is strongly skewed to the left; thus, the distribution of flaw strengths following the Laplace law, Gaussian law, or more generally one of the type $Ae^{-B|x-\mu|^p}$ for large values of $|x-\mu|$ (where A, B, μ, p are positive constants) will be negatively skewed (long tail to left).

In order to form quantitative judgments about the mean values of x

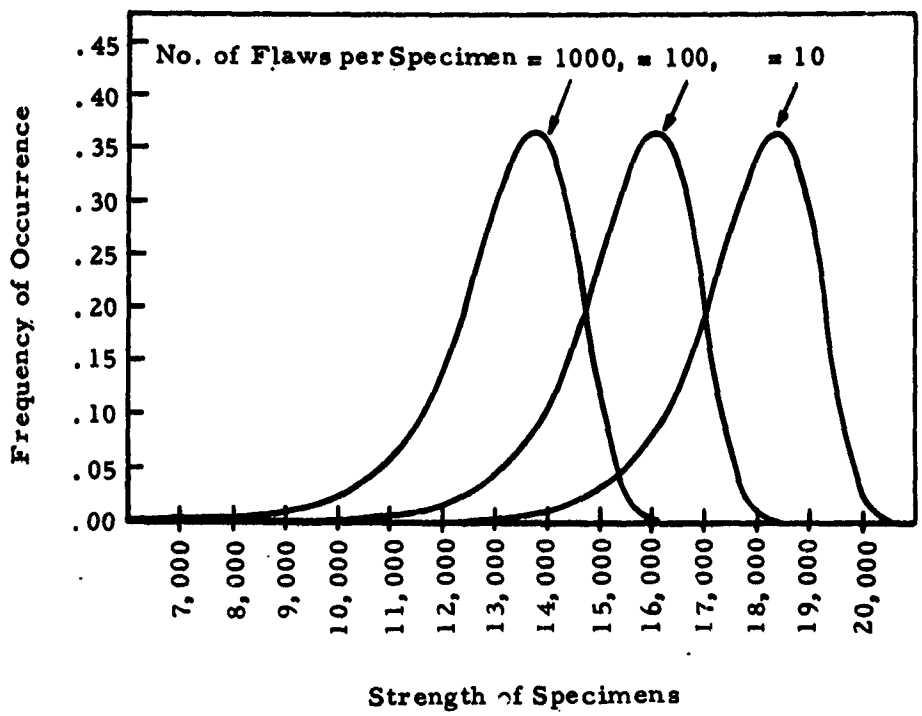


Figure 27

FREQUENCY DISTRIBUTION OF SPECIMENS CONTAINING VARIOUS NUMBERS OF FLAWS AND FOR WHICH THE FLAW STRENGTHS ARE DISTRIBUTED ACCORDING TO THE LAPLACE DISTRIBUTION WITH μ = 20,000, λ = 1000 (AFTER EPSTEIN, REF. 22)

and the dispersion of x as measured by the variance of x, it is useful to calculate the first moment about zero and the second moment about the mean. We recall that

$$\text{mean} \qquad E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx \tag{79}$$

$$\text{variance} \qquad D^2(x) = \int_{-\infty}^{\infty} (x-E)^2 f(x) = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \left[\int_{-\infty}^{\infty} x f(x)\, dx\right]^2 \tag{80}$$

where D is the standard deviation. We shall demonstrate the method of calculation in the following example.

<u>Example 3: $E(\log \xi)$ and $D^2(\log \xi)$</u>

To obtain the mean and variance for the distributions of x tabulated in Table 8, column (b), it is only necessary to consider the random components of x, that is, ξ. Thus, for $\eta = \log \xi$ we obtain,

$$E(\eta) = \int_{-\infty}^{\infty} \eta (e^{\eta} e^{-e^{\eta}})\, d\eta = \int_{0}^{\infty} (\log z)\, e^{-z} dz = -.577 \tag{81}$$

where we have introduced the change in variable $\eta = \log z$. Also,

$$D^2(\eta) = \int_{-\infty}^{\infty} \eta^2 (e^{\eta} e^{-e^{\eta}}) d\eta - E^2(\eta) = \int_{0}^{\infty} (\log z)^2 e^{-z} dz - E^2(\eta)$$
$$= \pi^2/6 \tag{82}$$

Then, it follows for the Laplace case that

$$x_{mean} = \mu - \lambda \log \frac{n}{2} - 0.557\lambda \tag{83}$$

and,

$$D^2(x) = \int_{-\infty}^{\infty} (x - x_{mean})^2 f(x)\, dx = \int_{-\infty}^{\infty} \lambda^2 (\eta - \eta_{mean})^2 dx$$
$$= \frac{\lambda^2 \pi^2}{6} \tag{84}$$

If we compute the quantities $E(\xi)$, $E(1/\xi)$, and $E(\xi^{1/\beta})$, and also the corresponding D^2's, we obtain the results shown in Table 8, column (c).

It is clear how one can interpret Table 8, column (c) physically if one is interested in finding out the consequences of various assumptions about the initial distributions of strengths due to flaws. For instance, a rectangular assumption would imply that no dependence on volume exists; a Chauchy distribution would lead to results which are physically unreasonable. The Laplace, Gaussian, or Weibull assumptions all imply that the strength does decrease with increasing volume. Of these, the Laplace is the only case where the distribution does not become narrower with increasing volume.

III-B-3. The Distribution of Exceedances

In all of the cases discussed in the previous section, the forecast of extreme values is based on the knowledge of the initial distribution. However, there are other methods of forecast which require only the continuity of the initial variate. These methods do not require a knowledge of the initial distribution - they are distribution-free. However, instead of the size of the extremes, they deal only with their frequencies. Such knowledge is sometimes sufficient. If a flood destroys a crop, it does not matter whether the soil is covered with an inch of water or ten feet of water. Similarly, if the strength of a member is exceeded, it may not matter whether it is exceeded by a small amount or a large amount since it fails in either case. Then basically, we want to forecast the number of cases surpassing a given severity within the next N trials. The methods for doing this lead to a forecast by interpolation, based on the essential and reasonable assumptions that the forthcoming trials are taken from the same population as the prior ones and that the observations are independent.

Assume that we have made n observations on any continuous variate x. Starting with the value of the largest observation, we arrange the n observations in decreasing order. The rank m of the largest value is $m = 1$, and for the smallest value $m = n$. Therefore, the mth observation is the mth largest observation. We ask: In how many cases y, will the past m^{th} observation x_m be equaled or exceeded in N future trials? The number of cases y, called the number of exceedances, is a new statistical variate

having a distribution w (n, m, N, y) where the n, m, and N enter as parameters. This distribution, derived by H. A. Thomas, is

$$w(n, m, N, y) = \frac{\binom{n}{m} m \binom{N}{y}}{(N+n)\binom{N+n-1}{m+y-1}} = w(n, n-m+1, N, N-y) \tag{85}$$

or -

$$w(n, m, N, y) = \frac{n!\, N!\, (m+y-1)!\, (N-x+n-m)!}{y!\, (m-1)!\, (n-m)!\, (N-y)!\, (N+n)!} \tag{86}$$

where the original sample size n may be small, or of the same magnitude as the future sample size N. Nothing is assumed known about the distribution of the initial variate x except its continuity. The conditions for the positive intergers m and y, and for the probability w(y) are

$$1 \leq m \leq n$$

$$0 \leq y \leq N \tag{87}$$

$$\sum_{0}^{N} w(y) = 1$$

We observe that although no assumption about symmetry of the initial distribution was made, the distribution in Eq. 85 possesses two symmetries: The probability that the mth largest among n past observations will be exceeded y times in N future trials is equal to the probability that y among N future trials will fall short of the mth smallest among n past observations and to the probability that the past m^{th} value from the bottom will be exceeded N-y times.

As an example of Eq. 85, we consider the 125 annual floods of the Rhine observed in Basel from 1808 to 1932 taken in groups of five. We consider the first group of five as our sample n; the largest flood in this group was 3400 cubic meters per second. In Fig. 28 we show the observed number of exceedances of this flood in each of the 24 subsequent groups of

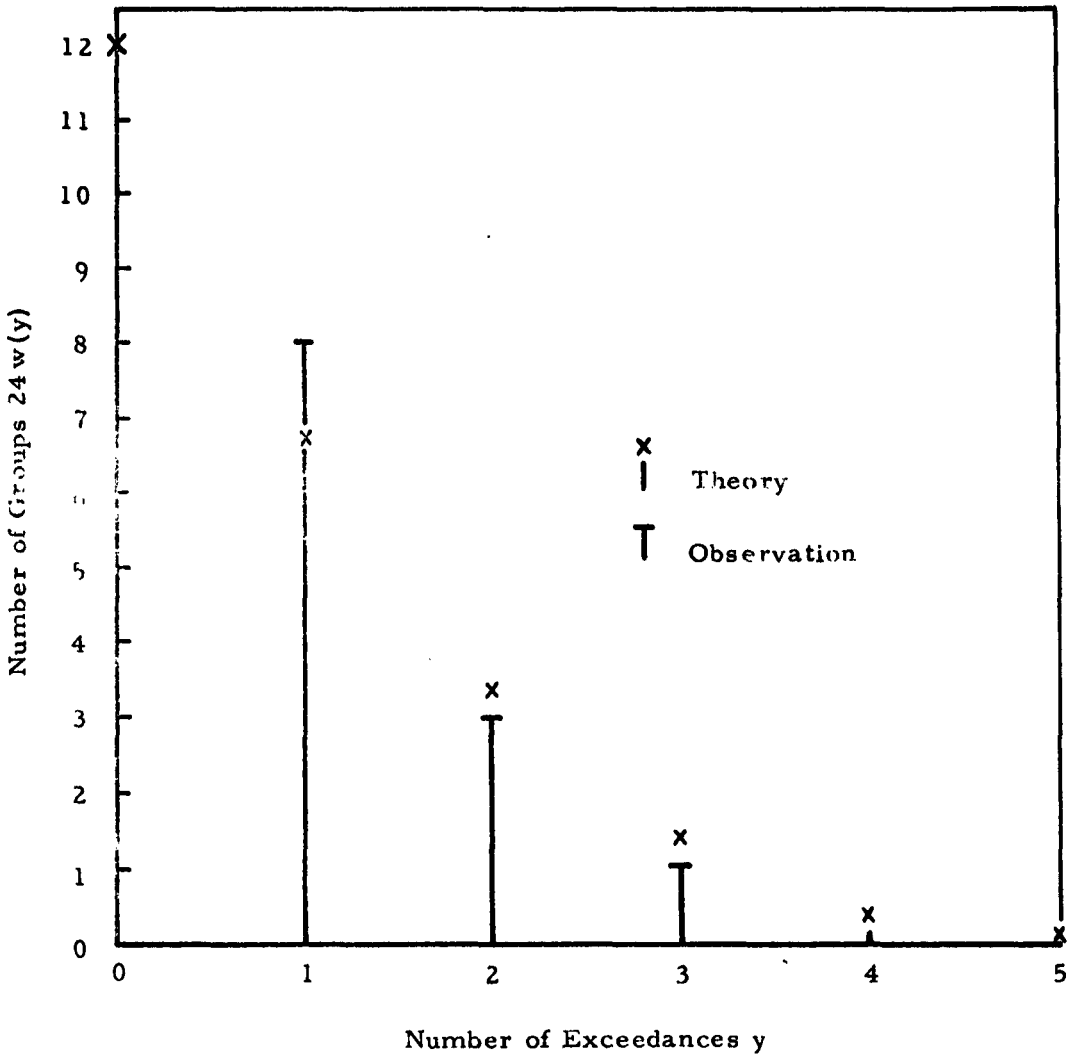


Figure 28

NUMBER OF FIVE-YEAR GROUPS IN WHICH y EXCEEDANCES OVER THE BASE FLOOD HAVE BEEN OBSERVED, RHINE RIVER AT BASEL, 1813-1932 (AFTER GUMBEL, REF. 24)

five years observed, together with the theoretical values obtained from Eq. 85. Note that each future sample of 5 years has a probability of having y exceedances that is given by w(5, m, 5, y). Since we are interested in the exceedances of the largest flood, m = 1. Thus, we find

y	w(5, 1, 5, y)	24w(y)
0	.50000	12
1	.27778	6.66
2	.13889	3.33
3	.05952	1.43
4	.01984	.476
5	.00397	.095

where 24w(y) is the number of groups having values exceeding the original group.

The mean number of exceedances $\bar{y}_m$ over the mth largest value in N future trials is

$$\bar{y}_m = m\frac{N}{n+1} \tag{88}$$

The mean number of exceedances over the smallest value (m = n) is n times the mean number of exceedances over the largest value (m = 1). The variances σ_m^2 and ${}_m\sigma^2$ of the number of exceedances over the mth largest and the mth smallest values are

$$\sigma_m^2 = \frac{m\,(n-m+1)}{(n+1)^2} \cdot \frac{N\,(N+n+1)}{n+2} = {}_m\sigma^2 \tag{89}$$

The variances increase with N and diminish, as usual, with increasing n. The variance is maximum for m = (n+1)/2, i.e., for the median of the original observations. The quotient of the variances of the number of exceedances over the median and over the extremes is

$$\frac{\sigma^2_{(n+1)/2}}{\sigma_1^2} = \frac{(n+1)^2}{4n} = \frac{\sigma_{(n+1)/2}}{\sigma_n^2} \tag{90}$$

Consequently, the variance of the number of exceedances over the median is about n/4 times as large as the variances of the number of exceedances over the extremes. In this sense, the extremes are more reliable than the median, and this quality increases with sample size.

Law of Rare Exceedances:

We now consider briefly the asymptotic behavior of the distribution of Eq. 85. If both n and N are large, we have to distinguish between two cases. In the first, the rank m increases with n such that the quotient m/n remains constant, and the mth values remain near the median. In the second case, m remains constant such that $m \ll n$, and the mth values are extremes.

In the first case, let n = N = 2k - 1, where k is large. Then m = k is the rank of the median of the initial distribution. The use of Stirling's formula leads to the following simple result: The number of exceedances over the median, m = k, of a large sample of size 2k - 1, 2k - 1 future trials, is asymptotically normally distributed with mean and variance equal to k. This distribution may be called the distribution of normal exceedances. It may be stressed that the variance in this case is of the order n/2, i.e., very large.

In the second case, let N and n be large, and m and y be small. From Eq. 85 we obtain for n = N the probability

$$w(n, m, n, y) = \binom{y+m-1}{y}\left(\frac{1}{2}\right)^{m+y} = w(m, y) = w(n, n-m+1, n, n-y) \qquad (91)$$

that the mth largest (or smallest) value will be exceeded y times (or n-y times) in n future trials, independently of n. Since m is small compared to n, the probabilities Eq. 91 may be called the distribution of rare exceedances. For y = 0, we obtain from Eq. 91 the probability

$$w(n, m, n, o) = \left(\frac{1}{2}\right)^{m} = w(n, n-m+1, n, n) \qquad (92)$$

that the largest (or smallest) m^{th} extreme value is never (or always) exceeded. The mean $\bar{y}$ and the variance σ^2 become from Eqs. 88 and 89

$$\bar{y} = m; \sigma^2 = 2m \tag{93}$$

The mean number of exceedances over the mth largest value is the rank m itself. The distribution of rare exceedances is shown in Fig. 29. For m greater than unity, the distribution of rare exceedances has two modes, namely, m-1 and m-2. For m = 1 it has one mode. For m increasing, the probabilities at the mode diminish, the distributions spread and, at the same time, become less asymmetrical. For large values of m, the distribution of rare exceedances, for a standardized variate $z = (y-\bar{y})/\sigma$, converges toward a normal distribution.

The above discussion was taken from the two works of Gumbel[18, 24] where a complete treatment of exceedances may be found. As far as can be determined by this author, these methods have not been applied to the problem of brittle design. In all of the weakest link theories proposed for the fracture problem, an initial assumption is made concerning the initial distribution of flaws or distribution of sizes of cracks. Adequate experimental data of the right type must be available before one can decide in a reasonably rigorous way what the underlying distribution laws are. At the present time this cannot be done. Hence, it seems reasonable to investigate the possibility of using distribution-free methods in our design procedures.

III-C. Other Weakest Link Theories

In the ensuing discussion, we will consider only the situation where we have an isotropic state of stress. The generalizations of any weakest link theory, including those discussed under Extreme Value Statistics, may be applied to any structure and stress state using the notions described in section III-A-2.

III-C-1. Mugele:

In 1951, R. A. Mugele[44] presented the following interesting discussion of Weibull's paper in Reference 19.

"The author's treatment is definitely a contribution to the literature on distribution functions. The range of fields treated in his examples is also impressive.

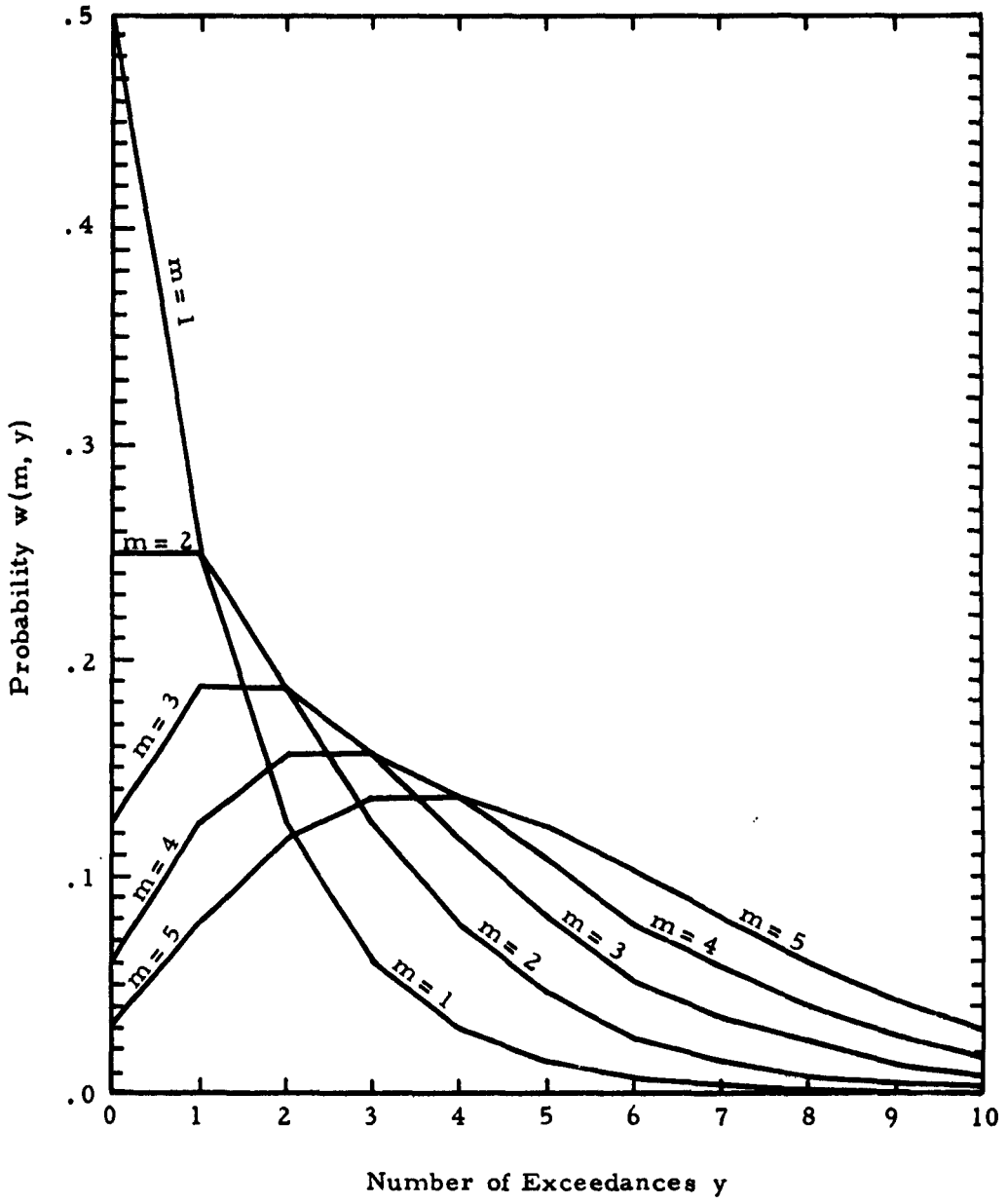


Figure 29
LAW OF RARE EXCEEDANCES (AFTER GUMBEL, REF. 24)

However, the reason for introducing the minimum value x_u, and ignoring the maximum x_m is not entirely clear. Probably it relates to the original applications, ...

Now, for such a case as Fig. 2 (Size distribution of fly-ash) of the paper, one would expect the maximum particle to be more tangible, and also more significant practically, than the minimum.

Incorporation of both a maximum and a minimum value of x will bring Eq. 20 into the form

$$F(x) = 1 - e^{-k\left(\frac{x-x_u}{x_m-x}\right)^m} \tag{94}$$

which will again reduce to Eq. 20 as x_m becomes infinite, and to the Rosin-Rammler[45] type of equation as x_u vanishes.

Of course, one may start with any distribution function where the argument has infinite range, and convert it to one where the range is finite. This has been illustrated in the case of the log-normal distribution by Van Uvan[46] and more recently by Mugele and Evans.[47] The latter reference also gives a critical review of the Rosin-Rammler and other distribution functions." (Unfortunately, the present author has not yet seen this latter reference.)

In 1958, J. A. Kies[48] introduced the distribution given in Eq. 94 without any mention of the above authors. He even specializes the function for glass by assuming $x_u = 0$, as in the Rosin-Rammler case. Apparently, both Mugele and Kies find it objectionable that Weibull's postulate has the feature that it predicts full probability of failure only when the applied stress reaches an infinite value. Accordingly to Eq. 94, a full probability of fracture will be reached when the applied stress is equal to x_m, this value designating the "upper fracture strength" (and not necessarily the theoretical strength) of the material.

Using the techniques described in the section on Extreme Value Statistics, it is straightforward to study the characteristics of this new distribution function. It is obvious from the outset that the gross features

of the theory will not be dissimilar to those of the Weibull function. We hasten to point out, however, that the numerical values of the constants entering Eq. 94 will be different than the corresponding Weibull parameters.

Kies makes the following comments concerning Eq. 94: "For samples of sufficiently small size and high strength the new model has a practical advantage. For sufficiently large size specimens having strengths negligible in comparison with the upper limiting strength the Weibull function is adequate and more convenient." By greater practical advantage, the author explains that the coefficient m in the new function has a greater degree of constancy than in the Weibull function. The experience at ARF indicates that one can obtain more closed form solutions using the Weibull function than that given by Eq. 94. Furthermore, for the structures considered in ceramic, the sizes are usually large and the strengths are about 1/100 of the theoretical strength.

III-C-2. Normal Distribution of Flaws

In 1926, Peirce[40] published a paper on the strength of cotton yarn. In order to develop reliable procedures for the testing of yarn it was essential that one study the effect of certain variables which one might expect on a priori grounds to be capable of greatly influencing the results unless properly controlled. Among the factors considered was the length of the specimen. He observed that if specimens of some fixed length L broke under normally distributed loads, then specimens of length nL, where n is an interger greater than one, will break under loads whose distribution is negatively skewed the larger the value of n. In order to account for this phenomenon, Peirce was led to the formulation of the chain model and to the equivalent statistical problem.

Weibull's work in 1939 used essentially the same ideas as Peirce only he did not, as we have seen, assume a Gaussian distribution of strength for his link or unit volume. Shortly after Weibull's work appeared the Russian physicists, Kontorova[49] and Frenkel and Kontorova[50], published papers which again reiterate the weakest link concept. They consider crystalline specimens with flaws distributed at random throughout the specimens, thus giving a distribution of strengths throughout the specimens. They assume,

as did Peirce, that these strengths are normally distributed. Starting with the distribution of flaw strengths x given by the probability density function,

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma} \tag{95}$$

they found the most probable value of the strength of specimens of volume V large enough to contain many flaws is approximately given by

$$\mu - \sigma\sqrt{2 \log \gamma V - 2 \log 2\sqrt{\pi}} \tag{96}$$

where γ is the average number of flaws per cubic centimeter of the material. We, of course, recognize that the more precise form is given in Table 8, column (a), item 4, i.e.,

$$\mu - \sigma\sqrt{2 \log \gamma V} + \sigma \frac{\log \log \gamma V + \log 4\pi}{2\sqrt{2 \log \gamma V}} \tag{97}$$

Frenkel and Kontorova went to considerable length to emphasize that their approach is to be preferred to that of Weibull because his assumption for a cumulative probability is devoid of physical meaning and because he does not realize that he is dealing with a situation where essentially it is the weakest link which determines the breaking strength of a specimen. Clearly, these criticisms are not justified. The present state of knowledge is not sufficiently precise to say what the initial distribution of flaws should be; consequently, a guess at f(x) is no better than a guess at F(x). There are enough free parameters in both assumptions to give enough freedom to fit existing data reasonably well.

III-C-3. Fisher and Hollomon

In 1947 the two metallurgists, J. C. Fisher and J. H. Hollomon[51], proposed a statistical theory of fracture based on the idea that the material under study is an elastic solid containing many thin disc-like cracks with elliptical cross-sections. The major axes of the ellipses are assumed to be of size x possessing distribution $p(x) = he^{-hx}$, $x \geq 0$. Using the methods of the section on Extreme Value Statistics, this distribution can be handled with great simplicity. As a matter of fact, Epstein[43] treats this case as an

example in distributions of largest values in samples of size n.

In his paper on the strength of glass, Kies[48], compares the Fisher-Hollomon theory with experimental results and concludes that there is little agreement.

III-C-4. Other Theories

There are a number of other investigators who have considered the statistical theories of fracture. Since we have only briefly reviewed their papers in this effort, we shall merely indicate where some of their work may be found. The papers and reports of A. E. Ruark and N. Rosen[52], J. Tucker[53], and J. P. Frankel[54] are referenced in the bibliography.

IV. PARALLEL MODEL

IV-A. Introduction

The parallel model has received only a fraction of the attention which has been devoted to the series or weakest link model. Oddly enough, Peirce [40], who apparently introduced the series model, was also first to consider the parallel model; and furthermore, he did so in the same paper. Considering the behavior of bundles, he deals exhaustively with the underlying physical considerations and derives useful formulae for the strength of large bundles.

The wider significance of the problem was also recognized by Peirce. He points out that a study of the strength properties of certain materials must involve considerations fundamentally similar to those arising in the theory of bundles (called by Peirce "composite specimens"), since each element of the material may be thought of as made up of subelements arranged both in series and parallel along a particular direction of stress. We shall look at a recent extension of Peirce's work in the next section.

In section III-A-3-e-iii, we indicated that Weibull found it necessary to introduce the possibility of a parallel model to explain certain of his test results. He considered the behavior of materials composed of independent parallel elements and called such cases "incoherent irregularity". We record the following comment by H. E. Daniels[55] concerning the validity of Weibull's[35] work.

"Unfortunately there appears to be a flaw in his discussion of probabilities, and formulae are obtained which are open to question. I suggest that his equations (136) and (137) should read

$$dS'_{2/2} = 2\left[S(2\sigma) - S(\sigma)\right] dS(\sigma)$$

and

$$dS''_{2/2} = 2S(\sigma)\, dS(2\sigma)$$

respectively. The final expression for $S_{2/2} = S'_{2/2} + S''_{2/2}$ then agrees with the

present formula (9.1) for $n = 2$."
Since the formulae referred to by Daniels are fundamental to Weibull's subsequent development of the subject, we shall look no further at this effort.

In 1945, Daniels wrote a beautiful paper on the statistical theory of the strength of bundles of threads. He assumes that the breaking strengths x have a continuous probability function F(x) so that

$$\lim_{x \to \infty} x\left[1 - F(x)\right] = 0 \tag{98}$$

His physical model of rupture is as follows: After rupture of one thread the load is supported by the remaining threads until the next one breaks, and so on. Let S be the load and let x_i $(i = 1, 2, 3 \ldots n)$ be the breaking strengths of the individual threads arranged in decreasing order of magnitude, then the bundle breaks if

$$0 < x_i < S/n; \; x_{i-1} < x_i < S/(n-i+1). \tag{99}$$

Let $\dot{S}$ be the largest value of S for which these inequalities are not all true. Then $\dot{S}$ is a random variate representing the breaking load of the bundles of n threads. Finally, let $\dot{x}$ be the maximum of $x\left[1 - F(x)\right]$, then Daniels proves that for large n the value $\dot{S}$ is normally distributed with mean

$$\bar{\dot{S}} = n\dot{x}\left[1 - F(\dot{x})\right] \tag{100}$$

and variance

$$\sigma^2(\dot{S}) = n\,\dot{x}^2 F(\dot{x})\left[1 - F(\dot{x})\right] \tag{101}$$

In other words, Daniels has shown that if a load S on a bundle of n threads is divided so that each thread feels S/n, the bundle will not break under load S if, and only if, there exists an interger $k \leq n$ such that k among the threads have strengths exceeding S/k. It turns out that, whereas the distribution of strengths of single long strands is negatively skewed, the

distribution of strengths of bundles of n threads is asymptotically normal for large n under fairly general assumptions about the distribution of strengths of individual strands.

IV-B. Examples of Strength Computations for Bundles of Threads

1. Using Weibull's distribution function for $x_u = 0$, the mean strength of a bundle of n threads may be found from Eq. 100. Taking the distribution function of one thread as

$$F(x) = 1 - e^{-V\left(\frac{x}{x_o}\right)^m} \tag{102}$$

we obtain,

$$x\left[1 - F(x)\right] = xe^{-V\left(\frac{x}{x_o}\right)^m} \tag{103}$$

where V is the volume of a single fiber. The x which maximizes this quantity is found from,

$$\frac{d}{dx}\left[xe^{-V\left(\frac{x}{x_o}\right)^m}\right] = e^{-V\left(\frac{x}{x_o}\right)^m} - \frac{xVm}{x_o}\left(\frac{x}{x_o}\right)^{m-1} e^{-V\left(\frac{x}{x_o}\right)^m} \tag{104}$$

$$= 0$$

or

$$\dot{x} = x_o\left(\frac{1}{Vm}\right)^{1/m} \tag{105}$$

Then using Eq. 100,

$$\bar{\dot{S}} = n x_o\left(\frac{1}{Vm}\right)^{1/m} e^{-1/m} \tag{106}$$

The variance from Eq. 101 becomes

$$\sigma^2(\dot{S}) = \dot{x}F(\dot{x})\bar{\dot{S}} = x_o\left(\frac{1}{Vm}\right)^{1/m}\left[1-e^{-1/m}\right] n x_o\left(\frac{1}{Vm}\right)^{1/m} e^{-1/m}$$

$$= n x_o^2\left(\frac{1}{Vm}\right)^{2/m}\left(e^{-1/m} - e^{-2/m}\right) \tag{107}$$

2. Following Wallhaus[29], we shall go back to first principles and solve the same problem in essentially the same manner as done by Peirce.

A bundle of fibers is defined to be any number of parallel filaments which are not interwoven. To simplify the analysis the assumtion is made that all fibers are under equal tension during the entire loading cycle to fracture. It is also convenient to make the assumption that all fibers have the same cross sectional area.

The mechanisms which operate to cause failure of the bundle can be best described by considering a constantly increasing load to be applied to a bundle of n fibers. As the load increases, the weakest fiber in the bundle will break causing its load to be equally redistributed among the remaining n-1 fibers in the bundle. This added increment of load may cause another fiber to fail, in which case the progressive fracture of the bundle may continue without any additional increment of the applied load; or the bundle may again reach a state of equillibrium. When the bundle is in a state of equillibrium, the applied load will continue to increase until an unstable state, which leads to progressive failure of the entire bundle, is realized.

Assume that n-r fibers have failed when the constantly increasing load possesses a magnitude P. Let the stress in the bundle corresponding to P be x. Then,

$$P = xA_r \tag{108}$$

where A_r = cross sectional area of the remaining r fibers
if A_n = cross sectional area of the original n fibers

then

$$\frac{r}{n} = \frac{A_r}{A_n} \tag{109}$$

Substituting A_r from Eq. 22 into Eq. 21 yields,

$$P = x\frac{r}{n}A_n \tag{110}$$

Since F(x) is defined as the probability of failure at some stress less than or equal to x, the number of broken fibers at stress x is nF(x) thus,

$$nF(x) = n-r \qquad n \ldots \text{large} \tag{111}$$

Making this substitution in Eq. 110 yields

$$P = xA_n\left[1-F(x)\right] \tag{112}$$

Replacing F(x) in Eq. 112 by the Weibull function given in Eq. 102, the load P is found to be

$$P = xA_n e^{-V\left(\frac{x}{x_o}\right)^m} \tag{113}$$

To find the x which maximizes P we differentiate this expression and set the result equal to zero, thus we obtain

$$x = x_o\left(\frac{1}{Vm}\right)^{1/m} \tag{114}$$

as we previously did in example 1. The maximum load which can be applied to a bundle is then obtained by substituting Eq. 114 into 113.

$$P = x_o A_n\left(\frac{1}{Vm}\right)^{1/m} e^{-1/m} \tag{115}$$

This result is the same as Eq. 106 when n is replaced by the area of n threads. This approach does not enable us to compute the distribution of P, indeed, it does not indicate that there is a distribution of P.

Wallhaus[29] compared the average load given by Eq. 115 for the bundle to the average load obtained from the Weibull theory for a monolithic tension member of the same area and length. He concludes in his Eq. 36 that the monolithic member (series) is stronger than the equivalent bundle (parallel). However, it appears that he misinterpreted the symbol V used in our Eq. 115 as the total volume of the member rather than the volume of one fiber or thread. His conclusion, and consequently, his Fig. 20 are

erroneous.

In closing this chapter, it is well to point out again that most materials are really some combination of the series model and the parallel model. So far, the author has uncovered no treatment in the literature which deals with this combined model. From the designers point of view it might be possible to lower our sights a bit and begin by developing a conservative description rather than a complete description of the series-parallel material.

V. ASPECTS OF THE DESIGN PROBLEM

V-A. Test Specimens

V-A-1. Tension Test

To obtain the stress-strain relationship for a material, it is necessary to have some method of relating the load on the specimen to the stress at some point. In general, the determination of the relationship between load and stress in a solid body requires a knowledge of the material properties which is precisely what we are trying to find. Fortunately, there exists some special combinations of loading and geometry which enable one to relate load to stress by appealing only to equilibrium considerations. Examples of such situations are provided by thin walled cylinders under torsion or internal pressure and odd-shaped solids under hydrostatic pressure and prismatic rods under uniformly applied axial end loads. Certainly, in this group the prismatic rod would seem to be the least involved method of producing tension. However, it is extremely difficult to insure a uniform end pressure and it is here that we must make an appeal to a physical assumption called Saint-Venant's principle. This principle essentially states that all stress distributions on the end of the bar which are statically equivalent to the uniform pressure will produce a uniform stress in the central regions of the bar if the bar is sufficiently long.

Saint Venant's principle enables us to produce any number of varieties of grips and flanged ends and still get σ = Force/Central Area. However, any moment transmitted to the ends of the rod disqualifies the application of the principle since the presence of end moments makes it impossible to have a stress distribution at the ends which is statically equivalent to a uniform pressure. In practice, it is an exacting feat to eliminate the terminal moments. One of the studies in Reference 15 considered the problem of performing tension tests; it was found that with careful alignment of specimens in the grips and the use of crushable shims, the bending strains were still as much as 10 percent of the total strain. Certainly when hundreds of specimens are required for statistical studies, such a test becomes impractical.

Recognizing the difficulties of performing reliable tension tests even at room temperatures, many investigators have proposed other tests for

studying material behavior which make use of the assumption that the stress is a linear function of the strain. Examples of such tests are shown in Fig. 30. We shall briefly comment on the various specimens and techniques.

V-A-2. Diametrical - Compression Test[56]

The diametrical-compression test is used extensively in Europe, South America and Japan for measuring the tensile strength of concrete and is usually referred to as the Brazilian Tension Test. The technique is based on the state of stress developed when a cylindrical specimen is compressed diametrically between two flat platens. This loading produces a biaxial stress distribution within the specimen. Stresses at any point in a cross section can be calculated by elastic theory. Of major interest are the maximum tensile stresses which act across the loaded diameter and which have constant magnitude.

$$\sigma = \frac{2P}{Dt\pi} \tag{116}$$

where P = applied load
D = specimen diameter
t = specimen thickness

Fracture must be initiated by these tensile stresses if the test is to yield useful results.

In addition to tensile stresses, compressive stresses act vertically along the loaded diameter. These stresses vary in magnitude along the loaded diameter from a minimum of $\frac{6P}{Dt\pi}$ at the center to infinitely higher values immediately under the loads. The high shears and compressive stresses under the loads must be reduced if failure is to begin in tension. While various techniques have been tried to eliminate these undesirable modes of failure, it is not always possible to do so and the difficulties are even greater at elevated temperature.

The disk technique is an improvement over the uniaxial test because the need for sample holders and sample alignment for homogeneous materials is eliminated. The sample aligns itself between the platens and the

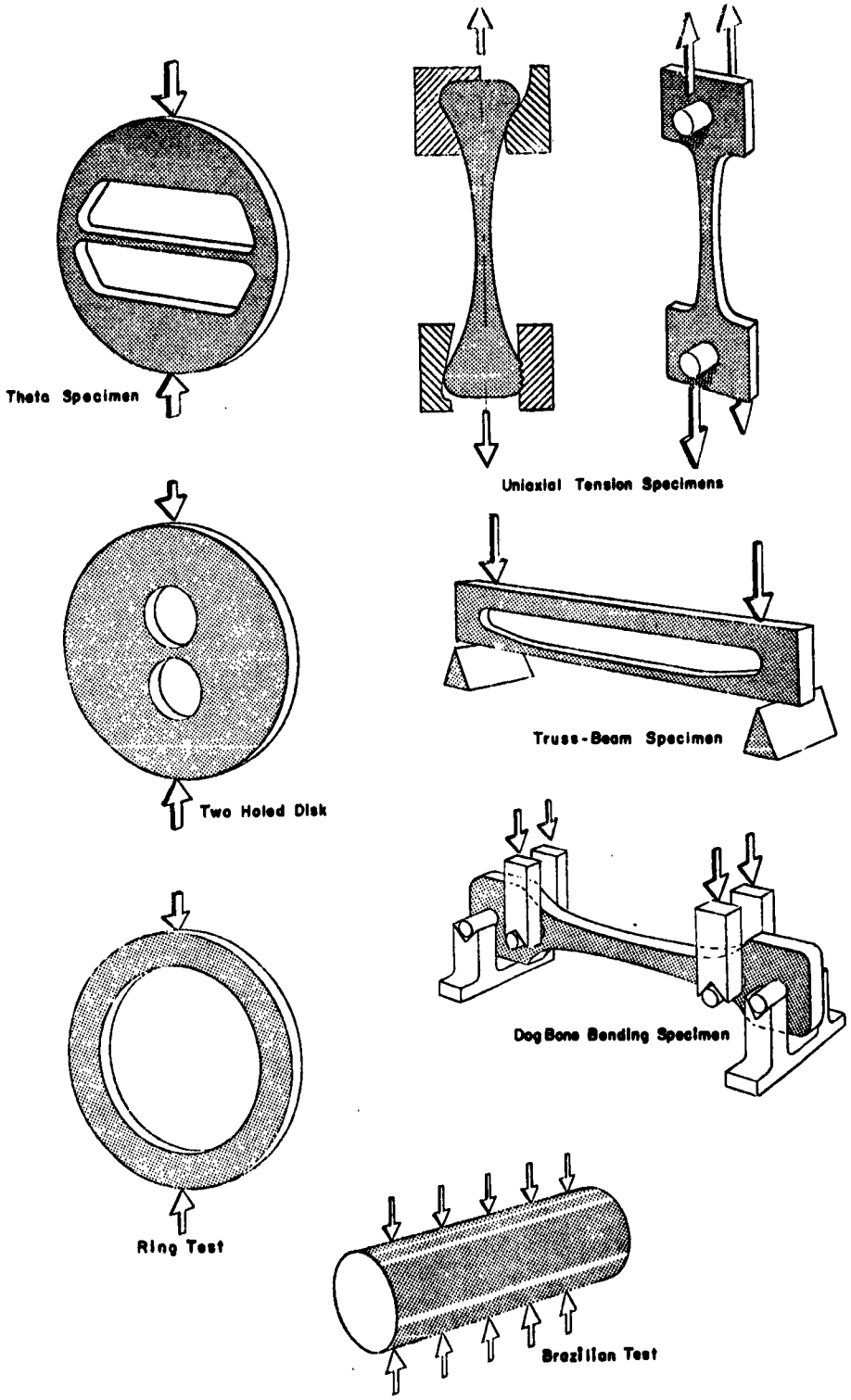


Figure 30
VARIOUS METHODS OF DETERMINING STRENGTH

shapes can be easily tested in a simple furnace without elaborate testing fixtures and equipment temperature protection.

V-A-3. Brittle Ring Test[57]

This is a further development of the disk technique and consists of placing a ring specimen under diametrical-compressive loading. Loading is obtained by compressing the specimen using two flat parallel platens. Maximum tensile stresses are developed on the inner periphery of the ring in the loading plane. Tension, to a lesser degree, is also developed on the transverse diameter of the outer periphery. Compressive stresses are developed on the opposite sides of the neutral axes of these same points; however, these stresses may be neglected since failure will always be caused by the maximum tensile stress on the inner periphery.

The ring specimen has all the advantages of the disk specimen and it will always fail due to maximum tensile stress at lower loads. However, one serious drawback is that it has a variable critical stress field. Experiments with this procedure have produced data which are somewhat higher than that normally obtained from uniaxial tests. Great care should be exercised when using any of the special tests for materials suspected of being influenced by stress gradients.

V-A-4. Theta Specimen[58]

The theta specimen, a further development of the ring test, was developed so that uniaxial tension of a brittle material could be obtained without alignment problems. It also removes the effect due to variable stress fields in determining ultimate tensile strength.

Photoelastic analyses of the theta specimen have shown that the maximum tensile stress occurs in the bar, that the stress distribution in the bar is uniform, and that the fillets at the ends of the bar do not produce any undesirable stress concentrations.

For the normal theta design the primary failure occurs in the bar. The photoelastic pattern is sufficient evidence of this fact and most specimens tested failed with a single break in the bar.

Experimental results of the effect of stress gradients comparing the

ring and theta specimen data have shown that the theta specimen is a better indirect method than the ring for determining the strength of brittle specimens. Its major drawback is that it is very sensitive to dimensional changes.

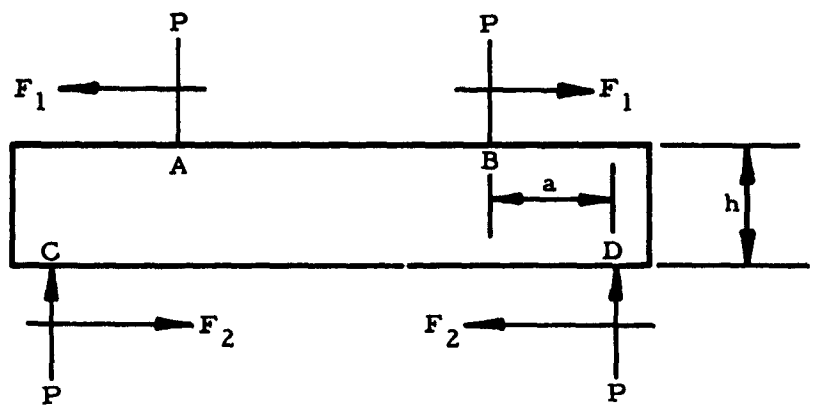
V-A-5. Truss-Beam Specimen

The principle governing the operation of the truss-beam specimen is the same as that which operates in the conventional Queen Post truss. The upper portion of the member acts like a beam and the lower portion acts like a tension tie rod. Using the ordinary truss beam analysis to predict the load in the lower portion of the specimen yielded results which were 1.4 percent higher than found using strain gages. This specimen appears to have the advantage that a large gage section is subjected to uniform stress. The specimen has only recently been developed by Bortz[59] as part of his thesis requirements, and consequently, little actual experience is available for the application of this specimen to ceramics.

V-A-6. Dogbone Specimen

The prismatic bending specimen is perhaps the most attractive specimen next to the prismatic tension specimen. However, like the tension specimen, it has a number of shortcomings. The most important of these is illustrated in Fig. 31. The friction forces shown provide a resisting couple which can increase the apparent strength of the specimen by as much as 30 percent. Clearly this resistance due to friction is proportional to the true fracture loads; consequently, from the statistical point of view we introduce a systematic error when we use this specimen without rollers or other friction reducing methods. Unfortunately, the obvious means of reducing or eliminating the friction do not work at very high temperatures. The dogbone specimen was developed to control the undesirable effects of friction. By placing pins at the neutral axis we essentially reduce the moment arm of the friction forces shown in Fig. 31 from h, the beam depth, to the radius of the pins. To avoid premature fracture due to the stress conditions around the pins, the ends of the specimen have been "beefed up".

On a prismatic bending member subjected to the same loading as shown in Fig. 30 for the dogbone specimen, it is quite common to get failures outside the pure bending region. It has been suggested that specimens which



Load Points : A, B
Support Points: C, D

Figure 31
EFFECT OF FRICTION FORCES ON PURE BENDING

do not fail in the gage length (between the loads) be removed from the sample space. One cannot overestimate the possible danger of such a procedure. If the stronger specimens are excluded from the sample space by such a practice, it will only take a few such exclusions to drastically influence both the mean and the standard deviation of the sample.

V-A-7. Implications to Design:

It is the primary objective of design to generalize the simple to the complex. Normally, the first step is to predict the behavior of a bending member from the known behavior of a tension member. In brittle design the problem is immediately more involved since we must go to a great deal of trouble just to go from an ideal small tension specimen to an ideal large tension specimen. Practically however, we find the situation is still much worse since we have to face enormous problems just to go from one tension specimen to another of the same size. How far are we then from the prediction of bending behavior from results obtained in "simple" tension? It we have so much trouble obtaining a known loading on a specimen under the best laboratory conditions, what shall we do when we try to find stresses under service conditions?

V-B. Loading

In a generalized sense, the normal procedure in "ductile design" is to proportion a structure in such a way that the assumed stress distribution nowhere exceeds the yield stress. When this is done, our plasticity theorems tell us that our design is safe because we have designed in such a way that the final structure has a statically admissible state of stress. The fact that the actual state of stress may be different than the assumed one is of no concern to the designer. On the other hand, in brittle design it is mandatory that the designer know the exact stress conditions throughout the structure. Failure to assess these stresses is certainly one of the major reasons why brittle design is so unsuccessful.

In the case of minimum weight design, an exact knowledge of the loading becomes an even more critical requirement. To demonstrate this characteristic of minimum weight design we shall consider the simple example of the ideal, webless, constant strength beam shown in Fib. 32. We assume

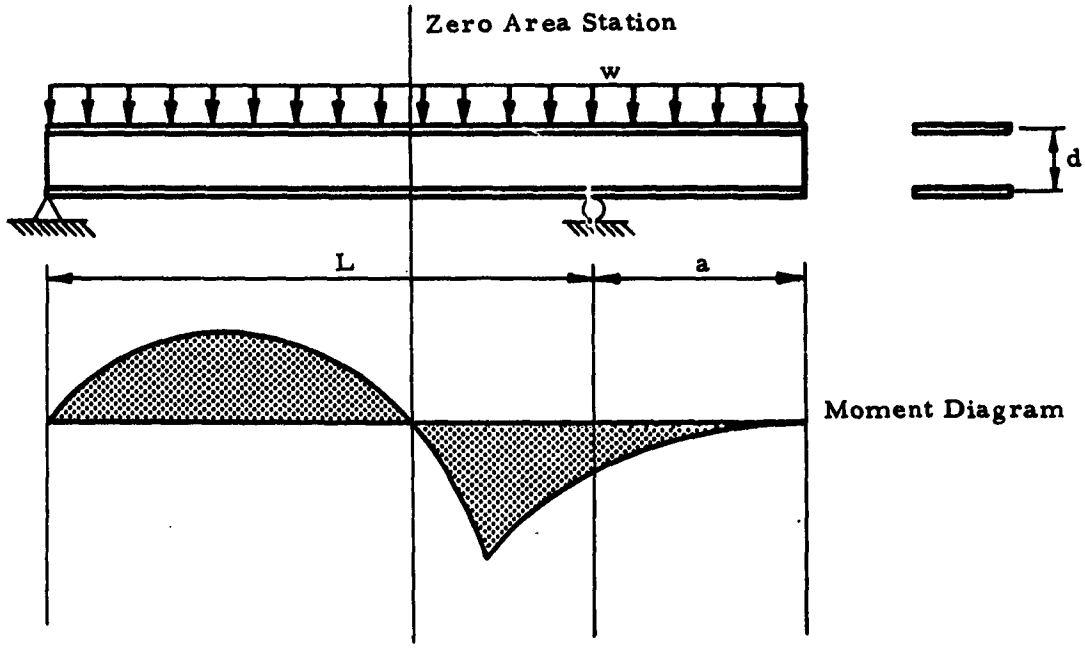


Figure 32

CONSTANT STRENGTH IDEAL I-BEAM

that the flange area of this beam is distributed in such a way that the maximum fiber stress at every station along the beam is equal to a constant, say the yield stress. Clearly, when the depth of such a beam is fixed, the flange area distribution looks like the moment diagram. For the uniform loading considered, it is apparent that a zero area station appears somewhere in the simply supported section. We note that for any other loading condition, large or small, the material in the vicinity of the zero area section is over stressed. Certainly, for the case shown here the entire structure collapses under any other load system. In this case, if another loading is going to act on this beam, we must design for it explicitly. If our overhanging beam was a simple prismatic beam, there would be infinitely many load systems which could be successfully supported.

If a number of different loadings act during the service history of a constant strength beam, the beam can be proportioned by superimposing each expected moment diagram and basing the flange area on the resulting envelop diagram. In this situation every station of the beam will feel the same maximum fiber stress, but, at different times. The probability of survival of such a beam is determined only from the maximum conditions that will occur at every station; hence, the member is analyzed as if it were under a pure moment. Here, the entire bottom flange can be treated using the Weibull theory for a uniform tension member.

V-C. Design Modification

This discussion is centered around Eq. 30, i.e.,

$$P_2 = P_1 \left(V_1 / V_2\right)^{\left(\frac{1}{m} - \frac{2}{3}\right)} \tag{30}$$

When the resistance P_1 is too small, how may the designer modify his design? We observe that for P_2 to be greater than P_1, P_1 must be multiplied by a factor greater than unity, therefore we require that

$$\left(V_1 / V_2\right)^{\left(\frac{1}{m} - \frac{2}{3}\right)} > 1 \qquad m \geq 1 \tag{117}$$

The following actions may be taken by the designer depending upon the value

of m:

m	Modification of V_1 to Increase Capacity
$m < 3/2$	decrease V_1, i.e., make $V_2 < V_1$
$m = 3/2$	nothing can be done
$m > 3/2$	increase V_1, i.e., make $V_2 > V_1$

From the design point of view, the implications of this simple example are both surprising and far reaching. Apparently there are situations where "beefing up" a design weakens it.

Warning:

In many situations where the Weibull distribution function is adequate for analysis, it is entirely unsatisfactory for design. Let us consider, for example, the case where $m < 3/2$. Weibull's theory accurately describes the increase in strength with decreasing volume as long as the size of the volume is reasonable. However, we observe that when $V_2 \to 0$, the tensile capacity becomes infinite, i.e., $P_2 \to \infty$. Clearly, this result is physically untenable and the theory must be modified to handle cases of vanishingly small volumes. As we have seen previously, the distribution function given by Eq. 94 is a possible candidate for such situations.

It is perhaps worthwhile to point out the similiarity of the analysis-design role played by the Weibull distribution function and that of the familiar formula for the maximum bending stress, i.e., $\sigma = Mc/I$. We observe that the cross-sectional area of an ideal bending member is given by the flexure formula as $A = 2M/\sigma d$. When the depth d approaches infinity we observe that the area vanishes. As with the Weibull function, the design problem extends the flexure formula beyond its region of applicability.

In addition to the problem of size effects, the designer must also consider the tricky problem of stress concentrations. Fig. 33 illustrates convincingly the point that strength does not always increase with an increase in dimensions, and that it is possible to increase the strength by reducing the dimensions.

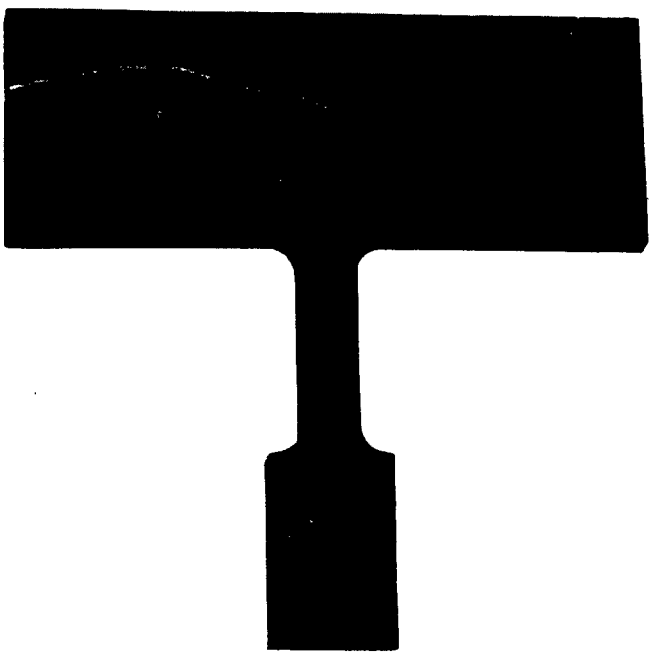


Figure 33

FRACTURE OF A BAKELITE T-BAR IN TENSION ILLUSTRATING THE SIGNIFICANCE OF STRESS CONCENTRATIONS IN BRITTLE MATERIALS UNDER STATIC LOADS; t = 0.272 in., P = 967 lb. (AFTER FROCHT, REF. 60)

V-D. Material Selection for Minimum Weight Design

In this section, we shall consider the minimum weight design of a brittle tension member based on a strength criterion. The Weibull distribution function for a tension member is given by Eq. 40. By rewriting Eq. 40, we can express x as a function of the volume, the material constants, and a specified probability of fracture, F, thus,

$$x = x_u + x_o V^{-1/m} \left[-\log(1-F)\right]^{1/m} \tag{118}$$

where the term in the brackets is always non-negative since the survival probability (1 - F) is always in the closed interval zero to one. Now, the weight of a tension member of cross sectional area A and length L is simply,

$$W = \rho LA = \rho LP/x \tag{119}$$

where the area is expressed as the external load P divided by the design stress x. If we eliminate x from Eqs. 118 and 119 and express the volume as W/ρ, the optimum weight of the tension member can be found from the following algebraic equation,

$$x_u (W/\rho) + x_o \left[-\log(1-F)\right]^{1/m} (W/\rho)^{1-\frac{1}{m}} - (PL) = 0 \tag{120}$$

In general, the roots of this equation must be evaluated numerically for each set of values of the loading index (PL) and the probability of fracture F; consequently, the comparison of materials must also depend on these parameters. For many materials, such as glass and certain ceramics, the zero probability strength, x_u, is zero. Here, Eq. 120 simplifies considerably and an explicit expression can be found for the optimum weight which enables one to delineate a merit index which is independent of the loading index (PL) and the specified fracture probability F. Hence,

$$W = \frac{1}{\left[x_o^{m/(m-1)}/\rho\right]} \frac{(PL)^{m/(m-1)}}{\left[-\log(1-F)\right]^{1/(m-1)}} \tag{121}$$

where the merit index $\left[x_o^{m/(m-1)}/\rho\right]$ is found to depend on the weight density ρ and the two Weibull constants x_o and m. For illustrative purposes, the value of this merit index is tabulated for three materials in Table 9. We note that when $m \to \infty$, the merit index approaches σ_o/ρ, the specific tensile strength of a classical material.

Table 9

MERIT INDICES FOR THREE MATERIALS

Material	m	σ_o	$\sigma_o^{m/(m-1)}/\rho$
Hydro-Stone Plaster	7.7	1,680 psi	5.75×10^4
Porcelain-white glazing	16.2	13,220 psi	29.2×10^4
Beryllium Oxide	7.25	7,800 psi	31.3×10^4

V-E. Comments on Design and "Rule-of-Thumb"

Recognizing that the design of a brittle structure is an art and not a science, the author has searched the literature for various design hints or rule-of-thumb. The search turned out to be most unrewarding with one exception - the work by Bell Aerosystems on the feasibility of designing leading edges using brittle materials. This work has been summarized by Anthony and Mistretta[61] and we shall reproduce most of their section on Design Philosophy:

"The most important ingredient of a design philosophy for brittle materials is believed to be the recognition of the absence of yielding. When the stress at any point in a brittle component, no matter how localized, reaches the limit of material capability, fracture will result. In recognition of this fact, the design philosophy formulated at Bell consists of the following five concepts:

(1) Nonredundant attachments permit the determination of loads at these points, whether the loads are due to externally applied pressures or relative deformations between the leading edge and the wing. If all loads are known, the part can be designed with confidence.

(2) Brittle structures cannot be pulled into position during assembly. Any misfit which introduces high local stress could cause failure. Hence, this situation must be avoided.

(3) Proper account must be taken of all stress risers due to section changes, discontinuities, locally applied loadings, etc. Normal analysis of ductile structures under static loadings neglects stress concentration effects because they are reduced to insignificance by microscopic plastic flow. For brittle materials this relief is essentially absent. Stresses are not relieved by localized straining.

(4) At some point the refractory nonmetallic component must be joined to a metallic structure. Since the nonmetal part is used to withstand extreme temperatures, the joint would be made at a more reasonable temperature, say 2,000° F, where superalloys could be used. Thermal expansion differences could be quite large. If, however, the projections of all mating metallic and nonmetallic surfaces intersect at a single point, there will be no changes in the fit of the two parts resulting from differential thermal expansion. This design rule is based on the fact that while the dimensions of a structural element change with temperature, all angles remain constant.

(5) Because of the brittleness of the material and the lack of yielding capability, the stress concentrations produced by material flaws precipitate material failure. (The term material flaws refers not to defective materials as such, but rather to those macroscopic or submicroscopic items - such as inclusions, voids, density variations, lattice defects, etc. - which cannot be detected by available inspection techniques and are an inherent characteristic of the particular material as produced by present technology.) Since the flaws are a random occurrence, the design allowable strength must be based upon a desired probability of survival under given loading conditions. The flaw concept appears to be most satisfactory in defining the structural performance of brittle materials - particularly, a Weibull statistical distribution function. The effect of the unknown flaws or stress concentrations within the material may be treated on a statistical basis as an effect on the allowable stress - that is, allowable strength values are chosen to ensure a prescribed probability of survival. This procedure requires that

material strength properties be determined statistically by conducting tests of a large number of specimens for each experimental condition. Furthermore, since the presence of other than gross flaws cannot be detected by currently available inspection techniques, it is necessary to relate allowable stresses to a particular process control procedure in order to ensure similarity between the test bar and component material."

The five design concepts are summarized in Table 10.

Table 10
DESIGN WITH BRITTLE MATERIALS

Problem	Solution
1. Attachment loads	Non-redundant attachments
2. Tolerances and fits	Self-aligning and adjustable surfaces
3. Stress concentrations	Refined stress analysis
4. Thermal expansions	Intersection of mating surfaces
5. Material flaws	Statistically determined strength

In addition to the five concepts by Anthony and Mistretta a number of ideas suggested themselves, as this present study progressed, and these thoughts are itemized below:

1. Try to change the weakest link behavior of materials by stratifying or foaming them.

2. Keep things compressive. It may be possible to make the technique of prestressing serve the same role in ceramics which it performs in concrete.

3. Before constructing a prototype in a brittle ceramic, make a prototype "mock up" out of an inexpensive brittle material like plaster.

4. In certain types of attachments it may be possible to use ductile inserts which limit the loads transfered to their yield loads. Here, as long as the transfer forces are bounded, it may not be necessary to know their exact distribution.

VI. SUMMARY

Ideally, a structural designer would like to be able to predict the behavior of a complex prototype structure from a knowledge of the behavior of a unit volume of material. When this cannot be accomplished, it is a reasonable and practicable compromise to be able to predict structural integrity through the use of small physical models or specimens. The common feature of these two approaches is the necessity of having a scaling law which relates the strength of the small specimen to the full size structure.

The most successful approach to the description of strength for materials which behave in a brittle fashion is a statistical approach. There are several important implications which attend the use of statistics in design. First, a statistical input produces a statistical output. Second, the effort and cost of obtaining data and verifying techniques are several orders of magnitude greater than that associated with conventional "ductile" design. Finally, a capability must exist for the consistant production of material from the same universe. Stated in another way, a distribution function must exist for the material. Unfortunately, there is considerable doubt concerning this capability for many of the ceramics and cermets of current interest.

A realistic statistical model for the description of brittle material strength will probably involve both series and parallel elements. However, because the series or weakest link behavior appears to be predominant, most investigators have studied only this model. Within the framework of the series model, we find many statistical theories, including the famous theory of Weibull. It has recently been recognized that the statistical strength problem is a special case of extreme value statistics and the adoption of this new point of view has served to unify the various weakest link theories.

Almost all of the past work on statistical failure theories has been concerned with uniaxial stress problems where the tensile stress is the only reasonable statistical variate. When multi-axial stresses are considered, there is no obvious statistical variate. The discovery of a suitable generalized variate, such as the octahedral stress or perhaps the maximum strain energy, is probably the most significant hurdle to the prediction of strength in multi-axial stress problems.

A small amount of very significant work has been devoted to the study of the parallel model. By considering the behavior of large bundles of filaments in tension, it has been possible to describe the general features of the parallel model under very broad assumptions for the characteristics of the individual filaments. The combination of this work with the series model could lead to a more effective description of brittle behavior. As a short range goal, it should be noted that the structural designer would be happy to compromise for any reasonable and conservative theory which would predict brittle strength.

One of the biggest obstacles to the development of satisfactory design procedures for brittle materials is the dearth of experimental data. Most of the available tests support the weakest link hypothesis from a qualitative standpoint. Attempts to verify specific theories quantatively have in the large been inconclusive. Generally, too few tests were used to properly estimate the distribution parameters. Where great numbers of specimens were used, either the programs considered only one size specimen or a distribution function did not exist for the materials considered. It should be strongly emphasized that the adequacy of a specific theory can only be judged in the light of a specific material and manufacturing process. The validity, of say the Weibull theory, must be ascertained for every material of interest.

There are a number of special problems which manifest themselves in brittle design. Most of these can be attributed to the difficulties involved in predicting or producing stress distributions in brittle components. Others are associated with the statistical nature of the materials. For example, the familiar situation in stress concentration problems of "strengthening by removal of material" can also arise in highly variable materials under homogeneous stress states. Such unusual behavior coupled with the need for high reliability at low weight, demands that radically different design philosophies be developed. One such philosophy is suggested by the statistical concept known as the return period.

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