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Elastic Thermal Stresses in Delta Wings

Part II

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Symbols listed in Part I of this report are not repeated here.

\( b \) \ldots \ldots \text{half base-length of wing}

\( l \) \ldots \ldots \text{wing chord}

\( l_0 \) \ldots \ldots \text{distance between base and x-axis.}

\( s \) \ldots \ldots \text{exponent in representation of wing cross-section.}

\( \beta \) \ldots \ldots \text{angle at base of triangle.} \quad \beta = \frac{\pi}{2} - \alpha.

\( \mu(\zeta) \) \ldots \ldots \text{complex function replacing } \phi(\zeta) \text{ of Part I}
General Remarks

1.) Introduction. In Part I of this report the basic equations for shallow conical shells have been derived assuming the temperature distribution in the shell to be symmetric with respect to the vertical plane of symmetry of the delta wing. Boundary conditions have been formulated and a perturbation method has been developed consisting of two steps termed first and second approximation, respectively. Temperature distribution has been resolved into two parts: one symmetric with respect to the horizontal plane of symmetry of the delta wing, and the other skew-symmetric with respect to that plane. The general solution and a numerical example have been given for the symmetric part using first approximation only.

In the present Part II of the report the study of thermal stresses in delta wings is continued. Equations of first approximation for the skew-symmetric temperature part are derived and solved. A numerical example is given. Then the equations of second (and final) approximation are obtained and solved for both the symmetric and the skew-symmetric part. The numerical example is continued into the second approximation.

Frequent reference is made in the following to Part I. Equations or figures contained in Part I are quoted by adding a Roman I to the equation number or figure number: Eq.(I-12-1) or Fig.I-3.

1) Parkus (see list of references).
Skew-Symmetric Temperature Distribution

2.) General Equations. The two basic equations (I-7-2) remain, of course, valid in the present case of a temperature distribution skew-symmetric with respect to the x,y-plane, Fig.1. Boundary conditions (I-8-3), however, have to be changed. Since upper and lower shell now experience the same deflection w but opposite tangential displacements u and v, Fig.2, the conditions

\[
\begin{align*}
\bar{q}_n &= q_n + \frac{\partial m_s}{\partial s} = \pm \frac{\partial W}{\partial n} n_n \quad (2-1) \\
\frac{\partial w}{\partial x} &= 0 \\ 
q_x &= -\frac{\partial W}{\partial n} n_x \quad n_{xy} = 0 \quad (2-2)
\end{align*}
\]

have to be satisfied along base (upper sign) and legs (lower sign) of the triangle, cf. Fig.1. Boundary conditions (I-8-4) along the y-axis (line of symmetry) remain unchanged:

\[
\begin{align*}
\frac{\partial v}{\partial x} &= 0 \\ 
q_y &= -\frac{\partial W}{\partial n} n_y \\
\frac{\partial v}{\partial y} &= 0 \\
\frac{\partial w}{\partial y} &= 0 \\
\varphi &= 0 \\
\frac{\partial \varphi}{\partial y} &= 0
\end{align*}
\]

Equations (2-1) have now to be expressed in terms of the deflection \( w \) and the stress function \( \varphi \). Eqs.(I-7-5) and (I-7-6) render

\[
\begin{align*}
m_{\varphi} &= -N \left[ v^2 w - (1-v) \frac{\partial^2 w}{\partial r^2} + \varphi_1 \right] \\
m_{r\varphi} &= -(1-v)N \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial w}{\partial \varphi} \right) \quad (2-3)
\end{align*}
\]
Furthermore

\[
q_\varphi = - \frac{N}{r} \frac{\partial}{\partial \varphi} (v^2 w + \tau_1) \tag{2-4}
\]

Base and leg of the triangle are given by \( \varphi = 0 \) and \( \varphi = \beta = \frac{\pi}{2} - \alpha \), respectively. Hence, the first two of conditions (2-1) take on the form

\[
\begin{align*}
\nabla^2 w - (1-v) \frac{\partial^2 w}{\partial r^2} + \tau_1 &= 0 \\
\frac{\partial}{\partial \varphi} (v^2 w + \tau_1) + (1-v) r \frac{\partial^2 w}{\partial r^2} (\frac{1}{r} \frac{\partial w}{\partial \varphi}) - \frac{1}{N} \frac{\partial w}{\partial \varphi} \frac{\partial^2 F}{\partial r^2} &= 0
\end{align*}
\]

along \( \varphi = 0 \) and \( \varphi = \beta \).

From eqs. (I-5-3) and (I-3-7) one finds using polar coordinates and denoting the radial displacement by \( u \) and the change of the angle \( \varphi \) by \( \varepsilon \)

\[
\begin{align*}
\frac{\partial u}{\partial r} &= \frac{1}{En} \left[ \nabla^2 F - (1+v) \frac{\partial^2 F}{\partial r^2} + (1-v) D \tau_0 \right] \\
\frac{\partial u}{\partial \varphi} + \frac{\partial}{\partial \varphi} \frac{c}{r} w &= \frac{1}{En} \left[ (1+v) \frac{\partial^2 F}{\partial r^2} - \nabla^2 F + (1-v) D \tau_0 \right] \\
\frac{\partial u}{\partial \varphi} + r^2 \frac{\partial \varepsilon}{\partial r} &= 2 \frac{1}{En} \frac{\partial}{\partial \varphi} \left[ \frac{F}{r} - \frac{\partial F}{\partial r} \right]
\end{align*}
\]

Along \( \varphi = 0 \) and \( \varphi = \beta \) we have \( u = 0 \) and \( \varepsilon = 0 \) for all values of \( r \). Hence, from the first equation

\[
\nabla^2 F - (1+v) \frac{\partial^2 F}{\partial r^2} + (1-v) D \tau_0 = 0 \tag{2-5b}
\]

\[1^{st}\] Melan-Parkus, p.59
Differentiating the first equation with respect to $\varphi$ and the third with respect to $r$, and subtracting, there follows

$$\frac{\partial}{\partial \varphi} \left[ \nabla^2 F + (1 + \nu) r \frac{\partial^2 F}{\partial r^2} + (1 - \nu) D \varphi_0 \right] = 0 \quad (2-5c)$$

along $\varphi = 0$ and $\varphi = \beta$.

The four equations (2-5a) to (2-5c) replace the boundary conditions (2-1).

3. Perturbation procedure. Expanding $w$ and $F$ in series in powers of $c$, Eqs. (I-9-2), and retaining only the first two terms one obtains eqs. (I-9-3) and (I-9-4):

$$\begin{align*}
\nabla^2 \nabla^2 w_1 &= \frac{p}{r} - \nabla^2 \varphi_1 \\
\nabla^2 \nabla^2 F_1 &= - (1 - \nu) D \nabla^2 \varphi_0 \\
\nabla^2 \nabla^2 w_2 &= - \frac{1}{r} \frac{\partial^2 F_1}{\partial r^2} \\
\nabla^2 \nabla^2 F_2 &= \frac{E h}{r} \frac{\partial^2 w_1}{\partial r^2}
\end{align*} \quad (3-1)$$

Performing the same expansion in the boundary conditions (2-5a) to (2-5c) and using (I-9-1) one finds

$$\begin{align*}
\nabla^2 w_1 - (1 - \nu) \frac{\partial^2 w_1}{\partial r^2} + \tau_1 &= 0 \\
\frac{\partial}{\partial \varphi} \left[ \nabla^2 w_1 + (1 - \nu) r \frac{\partial^2 w_1}{\partial r^2} + \tau_1 \right] &= 0 \quad (3-3)
\end{align*}$$
\[ \begin{align*} 
\nu^2 F_1 - (1+\nu) \frac{\partial^2 F_1}{\partial r^2} + (1-\nu)D \tau_o &= 0 \\
\frac{\partial}{\partial \varphi} \left( \nu^2 F_1 + (1+\nu)r \frac{\partial^2 F_1}{\partial r^2} + (1-\nu)D \tau_o \right) &= 0 \\
\nu^2 w_2 - (1-\nu) \frac{\partial^2 w_2}{\partial r^2} &= 0 \\
\frac{\partial}{\partial \varphi} \left( \nu^2 w_2 + (1-\nu)r \frac{\partial^2 w_2}{\partial r^2} \right) &= \frac{1}{N} \frac{\partial f}{\partial \varphi} \frac{\partial^2 F_1}{\partial r^2} \\
\nu^2 F_2 - (1+\nu) \frac{\partial^2 F_2}{\partial r^2} &= 0 \\
\frac{\partial}{\partial \varphi} \left( \nu^2 F_2 + (1+\nu)r \frac{\partial^2 F_2}{\partial r^2} \right) &= 0 
\end{align*} \]

The four pairs of boundary conditions (3-3) to (3-6) are valid along base and legs of the triangle, i.e. for \( \varphi = 0 \) and \( \varphi = \beta \). The remaining four conditions (2-2), valid along the line of symmetry \( x = 0 \), take on the form (I-9-7) and (I-9-8) and, as in the case of the symmetric temperature distribution, can be taken care of automatically by extending the validity of the preceding equations to the entire triangular region of the wing. For the second approximation, however, an additional external load

\[ q = -21 \frac{df}{dx} n_x \]

has to be placed along the y-axis, cf. eq. (I-9-9).
4.) First approximation. The solutions of eqs. (3-1) are split up into particular solutions \( w_p^1, F_p^1 \) of the complete equations and solutions \( w_h^1, F_h^1 \) of the homogeneous equations. If, as in Part I of this report, attention is restricted to temperature changes only then \( p = 0 \) in eq. (3-1) and the particular solutions may be obtained from eqs. (I-10-2):

\[
\begin{align*}
\nabla^2 w_p^1 &= -\tau_1 \\
\nabla^2 F_p^1 &= -(1-v)D \tau_0
\end{align*}
\]

(4-1)

No boundary conditions need be taken into consideration.

Eqs. (4-1) can be solved by the same method as used in sec. 10 of Part I. Another possibility consists in employing the method of finite differences.

The particular solutions thus obtained will, in general, not satisfy the boundary conditions (3-3) and (3-4). Substituting \( w_1 = w_p^1 + w_h^1 \) into (3-3) and making use of eqs. (4-1) one obtains

\[
\begin{align*}
\nabla^2 w_h^1 - (1-v) \frac{\partial^2 w_h^1}{\partial r^2} &= (1-v) \frac{\partial^2 w_p^1}{\partial r^2} \\
\frac{\partial}{\partial \varphi} \left[ \frac{1}{r} \nabla^2 w_h^1 + (1-v) \frac{\partial^2 w_h^1}{\partial r^2} (\frac{w_h^1}{r}) \right] &= - (1-v) \frac{\partial^3 w_p^1}{\partial \varphi \partial r^2} (\frac{w_h^1}{r})
\end{align*}
\]

(4-2)

The same functional representation is assumed for \( w_h^1 \) as given by eq. (I-11-3)

\[
2w_h^1 = \bar{z}\mu(z) + z\bar{\mu}(z) + \bar{\chi}(z) + \bar{\chi}(z)
\]

(4-3)
where \( \mu(z) \) and \( \chi(z) \) are analytic functions\(^1\) of the complex variable
\[
z = x + iy = re^{i\varphi} - a, \quad a = 1 \tan \alpha + il_c
\] (4-4)

A bar denotes the conjugate complex quantity. Using
\[
\frac{\partial}{\partial x} = e^{i\varphi} \frac{\partial}{\partial z} + e^{-i\varphi} \frac{\partial}{\partial \bar{z}}, \quad \frac{1}{r} \frac{\partial}{\partial \varphi} = i(e^{i\varphi} \frac{\partial}{\partial z} - e^{-i\varphi} \frac{\partial}{\partial \bar{z}})
\]
and writing
\[
\chi'(z) = \psi(z)
\] (4-5)

one finds
\[
\nabla^2 w_h = 2 \left[ \mu'(z) + \bar{\mu}'(z) \right]
\] (4-6)
\[
\frac{1}{r} \frac{\partial}{\partial \varphi} \nabla^2 w_h = 2i \left[ \mu''(z) e^{i\varphi} - \bar{\mu}''(z) e^{-i\varphi} \right]
\]
\[
\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial w_h}{\partial \varphi} \right) = \frac{i}{2} \left[ \bar{z} \mu''(z) + \psi'(z) \right] e^{2i\varphi} - \frac{i}{2} \left[ z \mu''(z) + \bar{\psi}'(z) \right] e^{-2i\varphi}
\] (4-7)
\[
\frac{\partial^2 w_h}{\partial r^2} = \frac{1}{r} \left[ \bar{z} \mu''(z) + \psi'(z) \right] e^{2i\varphi} + \frac{1}{2} \left[ z \mu''(z) + \bar{\psi}'(z) \right] e^{-2i\varphi} + \mu'(z) + \bar{\mu}'(z)
\] (4-8)

---

\(^1\) The letter \( \mu \) is now used instead of \( \varphi \) to avoid confusion with the polar angle.
In order to simplify the boundary conditions the second
of eqs. (4-2) is multiplied by \( dr = e^{-i\varphi}dz = e^{i\varphi}d\bar{z} \), and
integrated:

\[
2i \left[ \mu'(z) - \overline{\mu'(z)} \right] + \frac{1}{2} (1-v) \left[ \overline{z\mu''(z)} + \psi'(z) \right] e^{2i\varphi} -
\]

\[
- \left[ \overline{z\mu''(z)} + \overline{\psi'(z)} \right] e^{-2i\varphi} \right] = - (1-v) \left[ \frac{\delta^2}{\delta \varphi \delta r} \left( \frac{w_1^P}{r} \right) + C_1(\varphi) \right]
\]

\( C_1 \) is real. The first of eqs. (4-2) assumes the form

\[
(1+v) \left[ \mu'(z) + \overline{\mu'(z)} \right] - \frac{1-v}{2} \left[ \overline{z\mu''(z)} + \psi'(z) \right] e^{2i\varphi} +
\]

\[
+ \left[ \overline{z\mu''(z)} + \overline{\psi'(z)} \right] e^{-2i\varphi} \right] = (1-v) \frac{\delta^2 w_1^P}{\delta r^2}
\]

Multiplying this equation by \( i \) and adding it to equation
above renders

\[
(3+v)\mu'(z) - (1-v)\overline{\mu'(z)} - (1-v) \left[ z\mu''(z) + \overline{\psi'(z)} \right] e^{-2i\varphi}
\]

\[
= (1-v) \left[ \frac{\delta^2 w_1^P}{\delta r^2} + i \frac{\delta^2}{\delta r \delta \varphi} \left( \frac{w_1^P}{r} \right) + i C_1(\varphi) \right]
\]

This equation is now once more integrated with respect to \( r \).

One has with eq. (4-4)

\[
\int z\mu''(z)e^{-2i\varphi}dr = \int (\overline{\mu''(z)} - ae^{-2i\varphi})\mu''(z)dr =
\]

\[
e^{i\varphi} \int z\mu''(z)d\overline{z} + (ae^{i\varphi} - ae^{-i\varphi}) \int \mu''(z)d\overline{z}
\]

\[
e^{i\varphi} \left[ z\mu'(z) - \overline{\mu'(z)} \right] + (ae^{i\varphi} - ae^{-i\varphi}) \mu'(z) = e^{-i\varphi} z\mu'(z) - e^{i\varphi} \mu'(z)
\]
and hence

$$\lambda \mu(z) - z\mu'(z) - \psi(z) = e^{i\varphi} \left[ \frac{\partial w_1}{\partial r} + i \frac{1}{r} \frac{\partial w_1}{\partial \varphi} \right] + i C_1(\varphi)z + C_2(\varphi)$$

where

$$\lambda = \frac{3 \nu}{1 - \nu}$$  \hspace{1cm} (4-9)

The triangular region is simply connected and $C_1$ and $C_2$ may be put equal to zero. Therefore the boundary condition reads finally

$$\lambda \mu(z) - z\mu'(z) - \psi(z) = \left( \frac{\partial w_1}{\partial r} + i \frac{1}{r} \frac{\partial w_1}{\partial \varphi} \right) e^{i\varphi} \hspace{1cm} (4-10)$$

along $\varphi = 0$ and $\varphi = \beta$.

For the stress function $F_1$ both differential equation and boundary conditions are the same as for the deflection $w_1$. Hence, upon putting as in eq. (11-19),

$$2F_1 = \bar{z} \phi(z) + z \phi(\bar{z}) + X(z) + \bar{X}(z)$$  \hspace{1cm} (4-11)

$$\psi(z) = x'(z)$$

one has at once from eqs. (3-4)

$$w \phi(z) - z \phi'(z) - \psi(z) = \left( \frac{\partial P}{\partial r} + i \frac{1}{r} \frac{\partial P}{\partial \varphi} \right) e^{i\varphi} \hspace{1cm} (4-12)$$

where

$$w = \frac{3 - \nu}{1 - \nu} \hspace{1cm} (4-13)$$
5. Example. Consider the particular case where the temperature increases from its initial uniform value zero to a constant value $T_0$ in the upper half of the wing and, at the same time, decreases to the value $-T_0$ in the lower half. Then

$$\tau'_0 = (1+v)aT_0, \quad \tau'_1 = 0$$

and, from eqs. (4-1), taking symmetry with respect to $x = 0$ into account,

$$w_1^p = 0, \quad \Phi_1^p = -\frac{C}{2}(x^2 + y^2) = -\frac{C}{2}z\bar{z} \quad (5-1)$$

where

$$C = \frac{1-v^2}{2} DaT_0 \quad (5-2)$$

Substitution into eq. (4-10) renders $w_1^h = 0$ and hence $w_1 = 0$, while from eq. (4-12) one has

$$\mathcal{X}\Phi(z) - \overline{\Phi'(z)} - \overline{\Psi(z)} = -Cz \quad (5-3)$$

The solution of this equation can easily be given in closed form:

$$\Phi(z) = -\frac{C}{\mathcal{X} - 1} z, \quad \Psi(z) = 0 \quad (5-4)$$

Eq. (5-3) is then satisfied not only on the boundary of the triangular region but everywhere within this region. From eq. (4-11) one finds

$$\Phi_1^h = -\frac{C}{\mathcal{X} - 1} z\bar{z} \quad (5-5)$$
and hence, using eq. (5-1),

\[ F_1 = - \frac{G}{2} \frac{\partial^2 - 1}{\partial y^2} \epsilon_2 \]  

(5-6)

The corresponding state of stress is uniform compression in the upper half and uniform tension in the lower half of the wing. In particular, one finds in the upper shell

\[ n_x = \frac{\partial^2 F_1}{\partial y^2} = -\frac{\epsilon_1}{\epsilon_2} \quad C = n_y, \quad n_{xy} = 0 \]  

(5-7)

In the case of a more complicated temperature distribution no closed solution can, in general, be found and a numerical procedure has to be used. In order to demonstrate this the problem stated above will now be solved numerically.

Eq. (5-3) is transformed from the triangular region into the circular domain by means of the mapping function

\[ z = \omega(\zeta), \text{ cf. eq. (I-11-6)}, \]

\[ \chi \phi(\zeta) - \frac{\omega(\zeta) \phi'(\zeta) - \psi(\zeta)}{\omega'(\zeta)} = CH(\zeta) = -C \omega(\zeta) \]  

(5-8)

valid on the boundary \( \zeta = \delta \) of the unit circle. Using the same expansions as in eqs. (I-11-8a) and (I-11-22) to (I-11-24) and utilizing the Cauchy formula (I-11-12) the following equation is obtained

\[ \sum_{k=0}^{\infty} A_k \zeta^k - \sum_{k=0}^{\infty} \sum_{m=1}^{n+1} m \lambda_m \alpha_{m+k-1} \zeta^k - B_0 = \sum_{k=0}^{\infty} H_k \zeta^k \]

Comparison of coefficients of equal powers of \( \zeta^k \) renders the following set of linear equations for the expansion
coefficients $A_k$ of function $\Phi(\zeta)$

$$c_0A_1 + 2c_1A_2 + \ldots + (n+1)c_nA_{n+1} + B_0 = -H_0c \quad (5-9)$$

and

$$2A_k - \sum_{m=1}^{n} m c_{m+k-1} A_m = H_kc \quad (k = 1, 2, \ldots n) \quad (5-10)$$

A similar set of equations for the coefficients $B_k$ of the function $\Psi(\zeta)$ is obtained by replacing eq.(5-8) by its conjugate complex and performing the same operations as before. One gets

$$-B_k = cH_k + \sum_{m=1}^{n} m \bar{c}_{m-k-1} A_m \quad (k = 1, 2, \ldots 2n-1) \quad (5-11)$$

Coefficients $c_n$ are all real quantities and have already been tabulated in Part I, Table 12, for a set of 10 triangles. Coefficients $H_k$ of the expansion (I-11-22)

$$H(\sigma) = \sum_{-n}^{+n} H_k\sigma^k \quad (5-12)$$

have to be calculated by means of Fourier expansion of $H(\sigma)$ along the unit circle, cf. sec.I-16. In the present example we have

$$H(\sigma) = -\omega(\sigma) = -\text{Re}^i\gamma = -R(\cos\gamma + i\sin\gamma)$$

Real and imaginary part of $H(\sigma)$ as functions of the polar angle $\gamma$ on the circumference of the unit circle can be taken from tables I-1 to I-10, cf. Fig.I-10. As in the example in Part I a triangle with angle $2\alpha = 100^0$ is assumed.
Figs. 3 and 4 show the two functions $-\text{Re}\{H(\theta)\}$ and $-\text{Im}\{H(\theta)\}$. A Fourier analysis renders the corresponding Fourier coefficients which in turn determine the coefficients $H_k$, by means of the equations

$$H_0 = ia_0, \quad H_n = \frac{1}{2}(a_n - b_n), \quad H_{-n} = \frac{1}{2}(a_n + b_n) \tag{5-13}$$

The Fourier coefficients are given on Figs. 3 and 4 for $n = 6$ while the coefficients $H_k$ are contained in Table 1. The approximation curves are also represented (broken lines).

<table>
<thead>
<tr>
<th>$k$</th>
<th>$H_k$</th>
<th>$H_{-k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-0.261</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-10.301</td>
<td>1.801</td>
</tr>
<tr>
<td>2</td>
<td>0.431</td>
<td>-1.941</td>
</tr>
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<td>-0.081</td>
</tr>
<tr>
<td>5</td>
<td>0.021</td>
<td>0.071</td>
</tr>
<tr>
<td>6</td>
<td>0.047671</td>
<td>0.047671</td>
</tr>
</tbody>
</table>

Table 1. Power series coefficients of $H$

Each value in the tables 1 and 2 is to be multiplied by the corresponding factor $\mu$.

The exact value of all quantities $H_{-k}$ is zero. Table 1 reflects the error involved in the approximation of the mapping function.

Eqs. (5-9), (5-10) and (5-11) can now be set up and solved. The results are given in Table 2. One notices that the $A_k$ are not precisely proportional to the $H_k$ and the $B_k$ are not all zero as would be the case with the exact solution. The error is due to the relatively small number $n = 6$ of terms in the series expansion.
Table 2. Power series coefficients of $\Phi$ and $\Psi$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$A_k$</th>
<th>$B_k$</th>
<th>$A_k$</th>
<th>$B_k$</th>
<th>$B_k$</th>
<th>$B_k$</th>
<th>$\mu$</th>
</tr>
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</tr>
</tbody>
</table>

Second Approximation

6. Symmetric temperature distribution. As can be seen from a comparison of eqs. (1-9-4) and the corresponding boundary conditions (1-9-6) with eqs. (1-9-3) and boundary conditions (1-9-5) the steps to be performed in the second approximation are, in principle, the same as in the first approximation. They deviate, however, in two respects. First, the equations for the particular solutions $w_2^0$ and $\psi_2^0$ can no longer be reduced to the simple form (1-10-2). Second, a line load has to be placed along the axis of $y$ in order to have the boundary conditions satisfied there.

The determination of the particular solutions

$$
N_2 \phi_2^0 - \frac{1}{r} \left( \frac{\partial^2 \phi_2^0}{\partial r^2} + \frac{\partial^2 \psi_2^0}{\partial r^2} \right) = 0
$$

(6-1)

may be performed either in the triangular z-plane or in the
circular $\zeta$-plane, where $z = \omega(\zeta)$. Since $w^P_1$ and $P^P_1$ are usually known in the $z$-plane a series expansion similar to (I-10-5) and (I-10-7) may be used for the corresponding parts $w^P_{21}$ and $P^P_{21}$ of $w^P_2$ and $P^P_2$:

\[
\begin{align*}
w^P_{21} &= q_{mn} \varphi^{m+k} \cos \frac{n\pi \varphi}{\alpha} \\
N_{21} &= P_{mn} \varphi^{m+k} \cos \frac{n\pi \varphi}{\alpha}
\end{align*}
\]  
(6-2)

and, in case where $m + k = \frac{n\pi}{\alpha}$,

\[
\begin{align*}
w^P_{21} &= q_{mn} \varphi^{m+k} \ln \alpha \cos(m+k)\varphi \\
P^P_{21} &= P_{mn} \varphi^{m+k} \ln \alpha \cos(m+k)\varphi
\end{align*}
\]  
(6-3)

In contradistinction, $w^h_1$ and $P^h_1$ are known in the $\zeta$-plane. The corresponding parts $w^P_{22}$ and $P^P_{22}$ of $w^P_2$ and $P^P_2$ may therefore more conveniently be represented by series expansions in terms of $\varphi$ and $\zeta$. Using the relations

\[
\begin{align*}
v^2 &= 4 \frac{\delta^2}{\delta z \delta \bar{z}} = \frac{4}{\omega'(\zeta)\omega'(\zeta)} \frac{\delta^2}{\delta \zeta \delta \bar{\zeta}} = \frac{1}{\omega'(\zeta)\omega'(\zeta)} \Delta = \Theta(\varphi, \zeta) \Delta \\
\Delta &= \frac{\delta^2}{\delta \varphi^2} + \frac{1}{\varphi} \frac{\delta}{\delta \varphi} + \frac{1}{\varphi^2} \frac{\delta^2}{\delta \phi^2}, \quad \Theta(\varphi, \zeta) = \frac{1}{\omega'(\zeta)\omega'(\zeta)}
\end{align*}
\]  
(6-4)

and taking eq.(4-8) into account eqs.(6-1) may be written

\[
N\Delta(\Theta w^P_{22}) = -\frac{1}{8\pi} \text{Re} \left[ \frac{\omega'(\zeta)}{\omega'(\zeta)} \tilde{\varphi}'(\zeta) + \Psi^*(\zeta) \right] e^{2i\varphi} + 2\tilde{\varphi}^*(\zeta)
\]  
(6-5)
\[ \Delta(\Theta P_{22}) = \frac{Eh}{\pi^2} \text{Re} \left\{ \left[ \frac{\omega(\zeta)}{\omega'(\zeta)} \right] \frac{\phi^*(\zeta) + \psi^*(\zeta)}{2} e^{2i\varphi} + 2\phi^*(\zeta) \right\} \] (6-6)

where, in accordance with the notation of eqs.(I-11-28),

\[ \phi^*(\zeta) = \frac{\phi'(\zeta)}{\omega'(\zeta)}, \psi^*(\zeta) = \frac{\psi'(\zeta)}{\omega'(\zeta)}, \phi^*(\zeta) = \frac{\mu'(\zeta)}{\omega'(\zeta)}, \psi^*(\zeta) = \frac{\psi'(\zeta)}{\omega'(\zeta)} \]

In order to evaluate the right-hand side of eqs.(6-5) and (6-6) the function \(1/\omega'(\zeta)\) has to be expanded into a series

\[ \frac{1}{\omega'(\zeta)} = \sum a_n \zeta^n = \sum a_n \rho^n e^{in \theta} \] (6-7)

The coefficients \(a_n\) in this expansion may be found by putting \(\rho = 1\) and using eq.(I-13-2) which gives the values of \(\omega'(\zeta)\) along the circumference of the unit circle. From eq.(6-4) one has then

\[ \Theta(\phi, \varphi) = \sum \sum a_m a_n \rho^{m+n} e^{i(n-m)\theta} = \sum \sum \beta_{rs} \rho_r \cos s \theta \] (6-8)

The solution of eqs.(6-5) and (6-6) proceeds now in the following manner. The numerical values of the right-hand sides of these equations are calculated at a sufficient number of points in the \(\zeta\)-plane using a polar coordinate network \(\rho, \theta\). From these values a series expansion is obtained with the aid of some suitable numerical method:

\[ \frac{-1}{\pi^2} \text{Re} \left\{ \left[ \frac{\omega(\zeta)}{\omega'(\zeta)} \right] \phi^*(\zeta) + \psi^*(\zeta) \right\} e^{2i\varphi} + 2\phi^*(\zeta) = \sum \sum a_m \rho^{m-2} \cos n \theta \] (6-9)

\[ \frac{Eh}{\pi^2} \text{Re} \left\{ \left[ \frac{\omega(\zeta)}{\omega'(\zeta)} \right] \mu^*(\zeta) + \psi^*(\zeta) \right\} e^{2i\varphi} + 2\psi^*(\zeta) = \sum \sum a_m \rho^{m-2} \cos n \theta \]
The expansion will contain cosine terms only since all functions are symmetric with respect to the y-axis. A similar expansion - with unknown coefficients - is then assumed for the following quantities:

\[
\begin{align*}
\Theta \Delta w_{22}^P &= \sum_m \sum_n \left[ b_{mn} \rho^m \cos n \phi + \sum_n b_n \rho^n \ln \rho \cos n \phi \right] \\
\Theta \Delta P_{22}^P &= \sum_m \sum_n \left[ b_{mn} \rho^m \cos n \phi + \sum_n b_n \rho^n \ln \rho \cos n \phi \right]
\end{align*}
\]  

(6-10)

where the prime at the summation sign indicates that terms with \( m = n \) have to be omitted. From the definition of the operator \( \Delta \), eq.(6-4), there follows

\[
\Delta(\Theta \Delta w_{22}^P) = \sum_m \sum_n (m^2 - n^2) b_{mn} \rho^{m-2} \cos n \phi + 2 \sum_n b_n \rho^{n-2} \cos n \phi
\]

\[
\Delta(\Theta \Delta P_{22}^P) = \sum_m \sum_n (m^2 - n^2) b_{mn} \rho^{m-2} \cos n \phi + 2 \sum_n b_n \rho^{n-2} \cos n \phi
\]

Comparing coefficients with eq.(6-9) one finds

\[
\begin{align*}
b_{mn} &= \frac{A_{mn}}{m^2 - n^2} \quad & b_n &= \frac{A_{nn}}{2n} \\
B_{mn} &= \frac{a_{mn}}{m^2 - n^2} \quad & B_n &= \frac{a_{nn}}{2n}
\end{align*}
\]  

(6-11)

In order to obtain \( w_{22}^P \) and \( P_{22}^P \) the foregoing steps would have to be repeated. This, however, leads to very time-consuming calculations. We proceed therefore in an approximate fashion by applying the same method used to obtain eq. (6-9), i.e. we evaluate eqs.(6-10) and (6-8) at all points of the coordinate
network and expand:

\[
\Delta w_{22}^p = \sum \sum c_{mn}\rho^{m-2} \cos n \varphi \right)
\Delta F_{22}^p = \sum \sum c_{mn}\rho^{m-2} \cos n \varphi \right)
\]

Then, upon putting

\[
\begin{align*}
w_{22}^p &= \sum \sum w_{mn}\rho^m \cos n \varphi + \sum w_n\rho^n \ln \rho \cos n \varphi \\
F_{22}^p &= \sum \sum F_{mn}\rho^m \cos n \varphi + \sum F_n\rho^n \ln \rho \cos n \varphi
\end{align*}
\]

and substituting into eqs.(6-12) one finds, corresponding to eqs.(6-11),

\[
\begin{align*}
w_{mn} &= \frac{c_{mn}}{m^2-n^2} \\
F_{mn} &= \frac{c_{mn}}{m^2-n^2}
\end{align*}
\]

To \(w_2^p\) a second solution pertaining to the line load (3-7) has to be added. The solution for an infinite plate strip simply supported along its edges and carrying a line load is available in the literature\(^1\) and may be used here. Introducing a coordinate system \(x,y\) as shown in Fig.1 one has

\[
w_2^q = \frac{1}{4\pi^2} \sum_n \frac{a_n}{n}\left(1 + \frac{n|x|}{n}\right)e^{-\frac{n|x|}{1}} \sin \frac{ny}{1} \quad (6-15)
\]

\(^1\) Girkmann, p.170.
where \( y^* = y + 1\). The coefficients \( a_n \) are those of the Fourier expansion of the line load,

\[
q(y^*) = -21 \frac{\partial f}{\partial x} n_{x1} \bigg|_{x=0} = \sum a_n \sin \frac{ny^*}{1}
\]  

(6-16)

For a supersonic wing profile the following approximation may be used

\[
lf = (\beta^s - \varphi^s)y^* = (\beta^s - \varphi^s)r \sin \varphi
\]  

(6-17)

where \( s \) is a suitable chosen positive number. Then

\[
1 \frac{\partial f}{\partial x} = s \varphi^{s-1} \sin^2 \varphi, \quad l \frac{\partial f}{\partial y} = \beta^{s-1}(\varphi + \frac{s}{2} \sin 2\varphi)
\]  

(6-18)

The ratio of the leading-edge slope to the trailing-edge slope, taken along sections \( x = \text{konst} \), is given by \( s \sin 2\beta/2\beta \).

The normal force \( n_{x1} \) at \( x = 0 \) is the sum of the particular part \( n_{x1}^P \) and of the homogeneous part \( n_{x1}^h \). The first is in terms of \( P^P_1 \), eq.(I-10-5), given by

\[
n_{x1}^P = \frac{\partial^2 P_1^P}{\partial y^2} = \frac{\partial^2 P_1^P}{\partial \xi^2} \bigg|_{\xi = 0}
\]  

(6-19)

while for the second one finds from eqs.(I-11-30)

\[
n_{x1}^h = n_{x1}^h \bigg|_{\xi = 0} = \frac{\phi_1^h(\zeta) + \psi_1^h(\zeta)}{\omega(\zeta)} + \frac{1}{\omega'(\zeta)} \left[ \omega(\zeta)\phi_1^h(\zeta) + \psi_1^h(\zeta) \right]
\]  

(6-20)

The coefficients \( a_n \) in the expansion (6-16) may now be calculated by some suitable numerical method.

The homogeneous solutions \( w^h_2 \) and \( y^h_2 \) of the second approximation
are introduced in the same form as in the first approximations, eqs.(I-11-3) and (I-11-19)

\[
\begin{align*}
2w_2^h &= \bar{\omega}_\mu(z) + z\mu(z) + \chi(z) + \bar{\chi}(z) - \\
2F_2^h &= \bar{\omega}_\Phi(z) + z\Phi(z) + X(z) + \bar{X}(z)
\end{align*}
\]

(6-21)

The corresponding boundary conditions, eqs.(I-9-6), are identical with those of the first approximation, eqs.(I-9-5). Hence, eq.(I-11-4) remains unchanged:

\[
\mu(z) + z\mu'(z) + \Psi(z) = -\left(\frac{\partial w_2}{\partial x} + i \frac{\partial w_2}{\partial y}\right) F^{p q} \quad (6-22)
\]

or, after transformation into the \(\zeta\)-plane,

\[
\mu(\zeta) + \frac{\omega(\zeta)}{\omega'(\zeta)} \mu'(\zeta) + \Psi(\zeta) = -\left(\frac{\partial w_2}{\partial x} + i \frac{\partial w_2}{\partial y}\right) F^{p q} = G(\zeta) \quad (6-23)
\]

Similarly, for \(F_2^h\)

\[
\Phi(\zeta) + \frac{\omega(\zeta)}{\omega'(\zeta)} \Phi'(\zeta) + \Psi(\zeta) = -\left(\frac{\partial F_2}{\partial x} + i \frac{\partial F_2}{\partial y}\right) = H(\zeta) \quad (6-24)
\]

We note that

\[
\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} = e^{i\Phi} \left(\frac{\partial}{\partial \zeta} + \frac{i}{\zeta} \frac{\partial}{\partial \varphi}\right) \quad (6-25)
\]

7. **Skew-symmetric temperature distribution.** Eqs.(3-2) are valid in this case. They are identical with those of the symmetric distribution. Moreover, the same line load as
in the symmetric case given by eq. (6-16) has to be placed on the axis of y. Boundary conditions, however, are now in the form of eqs. (3-5) and (3-6).

The particular integrals \( w_2^P + w_2^q \) and \( P_2^p \) are determined in exactly the same way as in the symmetric case, sec. 6.

For the homogeneous solution \( w_2^h \) one has from boundary conditions (3-5) on \( \varphi = 0 \) and \( \varphi = \beta \), with \( w_2^P + w_2^q = w_2^* \),

\[
\nabla^2 w_2^h - (1-v) \frac{\partial^2 w_2^h}{\partial r^2} = (1-v) \frac{\partial^2 w_2^*}{\partial r^2} - \nabla^2 w_2^*
\]

\[
\frac{\partial}{\partial \varphi} \left[ \frac{1}{r} \nabla^2 w_2^h + (1-v) \frac{\partial^2 w_2^h}{\partial r^2} \right]
\]

\[
= \frac{1}{N} \left( \frac{1}{r} \frac{\partial F_1}{\partial \varphi} - \frac{\partial}{\partial \varphi} \left[ \frac{1}{r} \nabla^2 w_2^* + (1-v) \frac{\partial^2 w_2^*}{\partial r^2} \right] \right)
\]

\( w_2^h \) is now split up into two parts,

\[
w_2^h = w_{21}^h + w_{22}^h
\]

where

\[
2w_{21}^h = \overline{z_1}(z) + \overline{z_1}(z) + X_1(z) + \overline{X_1}(z)
\]

\[
2w_{22}^h = \overline{z_2}(z) + \overline{z_2}(z) + X_2(z) + \overline{X_2}(z)
\]

The first part \( w_{21}^h \) is obtained as a biharmonic function satisfying boundary conditions (4-2) or (4-10), with \( w_1^P \) replaced by \( w_2^* \). The second part \( w_{22}^h \) is then subject to the
following boundary conditions

\[
\nabla^2 w_{22} - (1-\nu) \frac{\partial^2 w_{22}}{\partial r^2} = -\nabla^2 w_2
\]

\[
\frac{\partial}{\partial \varphi} \left[ \frac{1}{r} \nabla^2 w_{22} + (1-\nu) \frac{\partial^2 w_{22}}{\partial r^2} \right] = \frac{1}{r} \frac{\partial f}{\partial \varphi} \frac{\partial^2 f}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial \varphi} \left( \frac{1}{r} \nabla^2 w_2 \right)
\]

We proceed now as in sec. 4. The second of eqs (7-5) is multiplied by \( dr \) and integrated. Bearing in mind that \( \partial f/r \partial \varphi \) is independent of \( r \), cf. eq. (6-17), and putting the integration constant equal to zero, we obtain

\[
21 \left[ \mu_2^2(z) - \mu_2^2(z) \right] + \frac{1}{4}(1-\nu) \left\{ \left[ \mu_2^2(z) + \psi_2^2(z) \right] e^{2i\varphi} - \left[ z_2^2(z) + \psi_2^2(z) \right] e^{-2i\varphi} \right\} = \frac{1}{r} \frac{\partial f}{\partial \varphi} \frac{\partial^2 f}{\partial r^2} - \int \frac{1}{r} \frac{\partial}{\partial \varphi} \left( \nabla^2 w_2 \right) dr
\]

This equation is multiplied by \( i \) and then subtracted from the first of eqs (7-5), to yield

\[
(3+\nu) \mu_2^2(z) - (1-\nu) \mu_2^2(z) - (1-\nu) \left[ z_2^2(z) + \psi_2^2(z) \right] e^{-2i\varphi} =
\]

\[
= - \int \left( \frac{\partial}{\partial r} - \frac{1}{r} \frac{\partial}{\partial \varphi} \right) \nabla^2 w_2^* dr - \frac{1}{r} \frac{\partial}{\partial \varphi} \frac{\partial^2 f}{\partial r^2}
\]

\[
= - 2 \int e^{i\varphi} \frac{\partial}{\partial z} \left( \nabla^2 w_2^* \right) dr - \frac{1}{r} \frac{\partial}{\partial \varphi} \frac{\partial^2 f}{\partial r^2}
\]

\[
= - 2 \nabla^2 w_2 - \frac{1}{r} \frac{\partial}{\partial \varphi} \frac{\partial^2 f}{\partial r^2}
\]

Another integration with respect to \( r \) renders
\[ \lambda \psi_2(z) - z\psi_1'(z) - \psi_2(z) = \left[ \frac{1}{2} \left( \frac{\partial \xi}{\partial \phi} \right) \phi_1 + 2 \int \nabla^2 \phi_2 d\tau \right] \frac{i\phi}{1-\nu} \]  

(7-6)

where \( \lambda \) is given by eq.(4-9).

The same procedure may be applied to boundary condition (3-6). Letting

\[ \phi^h_2 = \phi^h_{21} + \phi^h_{22} \]  

(7-7)

and putting

\[ 2\phi^h_{21} = \bar{\phi}_1(z) + \bar{\phi}_1(z) + X_1(z) + X_1(z) \]  

(7-8)

\[ 2\phi^h_{22} = \bar{\phi}_2(z) + \bar{\phi}_2(z) + X_2(z) + X_2(z) \]  

(7-9)

one has boundary condition (4-12), with \( \phi^p_1 \) replaced by \( \phi^p_2 \), valid for \( \phi^h_{21} \) while \( \phi^h_{22} \) has to satisfy the following equation

\[ \lambda \phi_2(z) - z\phi_1'(z) - \psi_2(z) = -2 \left[ \int \nabla^2 \phi_2 d\tau \right] \frac{i\phi}{1-\nu} \]  

(7-10)

where \( \lambda \) is given by eq.(4-13).

8. Example. The example of sec.5 shall now be continued into the second approximation. With \( w_1 = 0 \) one finds at once from the second of eqs.(3-2) together with boundary conditions (3-6)

\[ P_2 = 0 \]  

(8-1)

In order to determine \( w_2 \) one has, however, to perform all the steps outlined in sec.7. First the particular solution \( \phi^p_2 \) must be found. Substituting eq.(5-6) into the first of eqs. (6-1) one readily gets
\[ \lambda \mu_1(z) = z \mu_1'(z) - \psi_1(z) = \frac{\partial w_2^*}{\partial x} + i \frac{\partial w_2^*}{\partial y} = g^{(1)}(\theta) \quad (8-4) \]

where \( B(r) \) is some arbitrary biharmonic function which will be determined later.

Next, the solution \( w_2^0 \) due to the line load (3-7) has to be calculated. Substituting \( n_1 \) from eq.(5-7) into eq.(6-16) and using the first of eqs.(6-18) one obtains

\[ q(y^*) = \frac{2C}{\pi s^2} \frac{\pi^2 s^{-1}}{s^2 \pi^2} \sin^2 \phi \quad (8-3) \]

A value of \( s = 0.08 \) has been chosen in the following. Fig. 5 shows the corresponding cross section of the wing along \( x = \text{const.} \) Two other types with \( s = 0.07 \) and 0.1, respectively, are also shown. The relative maximum thickness of the wing in the three cases is 4.0\%, 4.6\% and 5.6\%, respectively.

In accordance with eq.(6-16) expression (8-3) is now developed into a Fourier sine series. As has been done consequently in this example only the first 6 terms in the series are retained. Table 3 shows the corresponding coefficients \( a_k \), with \( v = 0.3 \) in the expression \((\pi + 1)/(\pi - 1) = 2/(1-v)\).

<table>
<thead>
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<th>( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
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<tr>
<td>( a_k/C )</td>
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<td>-0.0868</td>
<td>0.0556</td>
<td>-0.0426</td>
<td>0.0333</td>
<td>-0.0280</td>
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Table 3.

For the calculation of the first part \( w_2^h \) of the homogeneous solution \( w_2^h \) eq.(4-10) is used in the form

\[ \lambda \mu_1(z) - z \mu_1'(z) - \psi_1(z) = \frac{\partial w_2^*}{\partial x} + i \frac{\partial w_2^*}{\partial y} = g^{(1)}(\theta) \quad (8-4) \]
<table>
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<tr>
<th>$\delta \circ$</th>
<th>$\frac{\partial \omega_2}{\partial x}$</th>
<th>$\frac{\partial \omega_2}{\partial y}$</th>
<th>$\frac{\partial \omega_2}{\partial x}$</th>
<th>$\frac{\partial \omega_2}{\partial y}$</th>
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<td>0</td>
<td>1.99</td>
</tr>
</tbody>
</table>

Table 4.

All values are to be multiplied by C/N.
with $w_2 = w_2^p + w_2^q$.

Table 4 contains the values of the derivatives of $Nw_2^{p}/C$ from eq.(8-2), of $Nw_2^{q}/C$ from eq.(6-15) and of $Nw_2^{*}/C$ as functions of the polar angle $\phi$ in the $\zeta$-plane, see Fig.I-6. The as yet undetermined function $B(r)$ in eq.(8-2) has now been chosen in such a way as to make the discontinuity produced by $r^3$ in $\psi = \psi_0$ vanish. One finds $B(r) = -\frac{C}{6N} \frac{2^k + 1}{k(k+1)} br^2$.

The corresponding graph is represented in full lines in Figs.6 and 7.

Real and imaginary part of $G^{(1)}(\kappa)$ in eq.(8-4) are now developed into Fourier series, represented by the broken lines in Figs.6 and 7. The coefficients $G_n$ in the expansion. (I-11-9) follow then from the equations

$$G^{(1)}_o = ia_o, \quad G^{(1)}_n = \frac{i}{2}(a_n - b_n), \quad G^{(1)}_{-n} = \frac{i}{2}(a_n + b_n) \quad (8-5)$$

They are all purely imaginary and are given in Table 5.

<table>
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<th>$k$</th>
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<td>-15.90</td>
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</table>

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G^{(1)}_k$</td>
<td>-0.42</td>
<td>-38.82</td>
<td>4.86</td>
<td>1.02</td>
<td>5.76</td>
<td>1.20</td>
<td>1.92</td>
</tr>
</tbody>
</table>

Table 5.

All values are to be multiplied by $\frac{C}{N}$.
Eqs. (8-4) are now solved by introducing expansions for \(\mu_1\) and \(\psi_1\) of the form (I-11-10) and (I-11-11). The results are presented in Table 6.

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
n & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
a_n^{(1)} & 0 & -9.97 & 1.24 & 0.27 & 1.03 & 0.32 & 0.39 \\
\hline
b_n^{(1)} & -1.86 & -17.96 & 59.60 & 11.30 & 8.40 & 16.88 & 5.73 \\
\hline
\end{array}
\]

(Continued)

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
n & 7 & 8 & 9 & 10 & 11 \\
\hline
b_n^{(1)} & -1.98 & -0.82 & -0.36 & -0.28 & -0.07 \\
\hline
\end{array}
\]

Table 6.

All values are to be multiplied by \(\frac{C}{N}\).

For the calculation of the second part \(w_{22}\) of the homogeneous solution the foregoing procedure has to be applied to eq. (7-6).

Using eq. (6-15) one finds

\[
\nabla^2 w_2 = -\frac{1}{2\pi} \sum \frac{a_n}{n} e^{-\frac{\pi n |x|}{L}} \sin \frac{n\pi y}{L}
\]

This expression is zero along the base \(y = -1\) of the triangle while along the legs one has

\[
\int_0^r \nabla^2 w_2 dr = \frac{1}{2\pi} \sum \frac{a_n}{n^2} \left[ -\frac{\pi n |x|}{L} \left( \cos \frac{n\pi y}{L} - \tan \alpha \sin \frac{n\pi y}{L} \right) e^{-n\pi \tan \alpha} \right]
\]
Eq. (8-2) renders

\[ v^2 w_P = \frac{C}{N} \frac{\alpha + 1}{\alpha - 1} (r - \frac{2}{3} b) \]

whence

\[ \int_0^r v^2 w_P dr = \frac{C}{N} \frac{\alpha + 1}{\alpha - 1} r(\frac{r}{2} - \frac{2}{3} b) \]

In addition, from eq. (6-17),

\[ \frac{1}{r} \frac{\partial f}{\partial \varphi} = (\beta^8 - \varphi^8) \cos \varphi - \varphi^{8-1} \sin \varphi \]

Using eq. (5-6) for \( F_1 \) the right-hand side of eq. (7-6), denoted by \( G^{(2)}(6) \), may now be calculated. Figs. 8 and 9 show the graphs of its real and imaginary parts together with the corresponding Fourier expansions (broken lines). From these coefficients \( G_k^{(2)} \) are found with the aid of eqs. (8-5) and are represented in Table 7.

<table>
<thead>
<tr>
<th>( k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_k^{(2)} )</td>
<td>457.6</td>
<td>-454.3</td>
<td>-23.7</td>
<td>-159.2</td>
<td>65.4</td>
<td>-125.1</td>
<td>17.4</td>
</tr>
</tbody>
</table>

**Table 7**

All values are to be multiplied by \( iC/N \).
The last step consists in the determination of the expansion coefficients of the functions $\psi_2(\zeta)$ and $\phi_2(\zeta)$. Table 8 contains the coefficients.

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_n^{(2)}$</td>
<td>0</td>
<td>-14.5</td>
<td>-4.8</td>
<td>-33.2</td>
<td>11.7</td>
<td>-25.9</td>
<td>3.5</td>
</tr>
<tr>
<td>$b_n^{(2)}$</td>
<td>408.5</td>
<td>409.3</td>
<td>-115.6</td>
<td>69.6</td>
<td>-15.2</td>
<td>23.2</td>
<td>78.3</td>
</tr>
<tr>
<td>n</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b_n^{(2)}$</td>
<td>14.3</td>
<td>-0.4</td>
<td>10.4</td>
<td>1.8</td>
<td>-0.6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 8.
All values are to be multiplied by $iC/N$

References

Acknowledgment.
The author wishes to acknowledge the valuable and untiring assistance of Dipl.Ing.J.Zeman and Dipl.Ing.F.Ziegler in developing equations and performing the extensive numerical work.
Fourier Coefficients:

No[Hz]:

- $A_1 = 0.07, B_1 = 1.08$
- $A_2 = 0.18, B_2 = 0.29$
- $A_3 = 0.20, B_3 = 0.36$
- $A_4 = 0.38, B_4 = 0.76$
- $A_5 = 0.28, B_5 = 0.92$

Fig. 3
Fourier Coefficients:

\[ \text{Im}[H(a)]: \quad a_0 = 0.36 \times \mu \]
\[ a_1 = -10.31 \]
\[ a_2 = 1.72 \]
\[ a_3 = -0.80 \]
\[ a_4 = -1.92 \]
\[ a_5 = 0.60 \]
\[ a_6 = 0.26 \]

\[ \mu = 0.04787 t \]
Fourier Coefficients:

\[
\frac{a_0}{2} = \frac{10.40}{2}
\]

\[
a_1 = 10.40
\]

\[
a_2 = 10.40
\]

\[
a_3 = 10.40
\]

\[
a_4 = 10.40
\]

Fig. 6
Fourier Coefficients

\( a_n = \frac{a_0}{n} \)

\( a_1 = 0.02 \)

\( a_2 = 0.01 \)

\( a_3 = 0.001 \)

\( a_4 = 0.0001 \)

\( a_5 = 0.00001 \)

\( a_6 = 0.000001 \)

\( a_7 = 0.0000001 \)

\( a_8 = 0.00000001 \)

Fig. 7
Fourier Coefficients:

Re $D^m$: $A = \frac{2}{\pi} \frac{\sin \theta}{\sin \theta_N}$

$A = 2.0$  
$A = 2.5$  
$A = 1.0$  
$A = 2.4$  
$A = 0.4$