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Report 1579

AD 405011

DEPARTMENT OF THE NAVY

DAVID TAYLOR MODEL BASIN

HYDROMECHANICS

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THE FRICTIONAL RESISTANCE AND BOUNDARY LAYER OF FLAT PLATES IN NON-NEWTONIAN FLUIDS

by

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RESEARCH AND DEVELOPMENT REPORT

December 1962 Report 1579

(5) 240 800

(6) THE FRICTIONAL RESISTANCE AND BOUNDARY LAYER OF FLAT PLATES IN NON-NEWTONIAN FLUIDS

(7 thru 9) NA

(10) by

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(22) from
Reprint ~~of a paper published in~~ the Journal of Ship Research, Society of Naval Architects and Marine Engineers, ~~Vol.~~ 6, ~~No.~~ 2, Oct~~ober 19~~62.
(Copy not required by DDC

(11) ~~December 19~~62

(12) 13p

(13) NA

(14) ~~Report~~ 1579

(15 thru 19) NA

(20) U

(21) NA

The Frictional Resistance and Boundary Layer of Flat Plates in Non-Newtonian Fluids[1]

By Paul S. Granville[2]

Boundary-layer parameters and frictional resistance formulas for either laminar or turbulent flow are derived for flat plates in power-law non-Newtonian fluids. The results for laminar flow are based on the known velocity profiles for pipe flow, whereas those for turbulent flow are based on the application of similarity laws.

THE possibility of injecting non-Newtonian fluids into the boundary layers of bodies to reduce frictional resistance raises two questions: What is the frictional resistance of bodies in non-Newtonian fluids? Or more fundamentally, what are the characteristics of boundary layers, laminar and turbulent, of non-Newtonian fluids?

As far as can be ascertained attention has been directed mainly to the flow of non-Newtonian fluids in pipes for chemical engineering applications. The work of Metzner and his associates has been particularly noteworthy in this respect. Dodge and Metzner [1][4] have applied similarity laws to the turbulent flow of power-law non-Newtonian fluids in pipes.

The so-called power-law fluids are those whose characteristic stress curves can be fitted by straight lines on log-log plots. In this respect, Newtonian fluids comprise a special case of power-law fluids. The studies in this paper are confined to flat plates in zero pressure gradient as the simplest of bodies with boundary layers. By utilizing the known velocity profiles of the laminar flow of power-law fluids in pipes, local coefficients of frictional resistance for flat plates in laminar flow follow.

The similarity laws are applied to the turbulent boundary layers of power-law fluids on flat plates to ob-

[1] This work was carried out at the David Taylor Model Basin under the Bureau of Ships Fundamental Hydromechanics Research Program.

[2] Physicist, David Taylor Model Basin, Navy Department, Washington, D. C.

[3] The tilde will be used over symbols relating to factors peculiar to power-law fluids.

[4] Numbers in brackets designate References at end of paper.

Nomenclature[3]

A = slope of logarithmic velocity law
a = factor in equation (29)
B_1, B_2 = intercepts of logarithmic velocity law; see equations (22) and (23)
C = constant of integration in equation (47)
C_f = coefficient of frictional resistance
c_1, c_2, c_3, c_4 = linearization constants in equations (60) and (64)
D = frictional resistance or drag
D_1, D_2 = velocity profile constants in equations (36) and (38)
e = base of natural logarithms
F = outer law function, equation (19)
f = friction factor for pipes
f_1 = inner law function, equation (17)
G = subscript for quantities at junction of inner and outer turbulent sublayers
H = shape parameter, $H \equiv \delta^*/\theta$
I_1, I_2 = integrals of outer law velocity profiles, equations (39) and (40)
J_1 = transitional sublayer factor, equation (27)
L = subscript for quantities at junction of laminar and transitional sublayers
ln = natural logarithm to base e
log = common logarithm to base 10
n = flow-behavior index of power-law fluids, equation (2)
0 = subscript for limit of overlapping of inner and outer laws
P_1, P_2 = constants in logarithmic resistance formulas, equations (61) and (66)
R = radius of pipe
$\tilde{R}_D$ = Reynolds number of pipe $\tilde{R}_D \equiv \bar{U}^{\frac{2}{n}-1} 2R/\tilde{\nu}^{\frac{1}{n}}$
$\tilde{R}_x$ = Reynolds number based on length for power law fluids, $\tilde{R}_x \equiv \bar{U}^{\frac{2}{n}-1} x/\tilde{\nu}^{\frac{1}{n}}$
R_θ = Reynolds number based on momentum thickness for power-law fluids, $R_\theta \equiv U^{\frac{2}{n}-1}\theta/\nu^{\frac{1}{n}}$
r = radial distance from center of pipe
T = subscript for quantities at junction of transitional and inner turbulent sublayers
U = free-stream velocity
$\bar{U}$ = average velocity in pipe
u = tangential velocity in boundary layer
u_τ = shear velocity, $u_\tau \equiv (\tau_w/\rho)^{1/2}$
x = distance along boundary layer
y = normal distance from wall
$\tilde{y}^*$ = nondimensional y for power-law fluids $\tilde{y}^* \equiv u_\tau^{\frac{2}{n}-1}/\tilde{\nu}^{\frac{1}{n}}$
α_1 = velocity profile constant, equation (35)
β_1 = velocity profile constant, equation (37)
δ = boundary-layer thickness
δ^* = displacement thickness
$\tilde{\eta}$ = boundary-layer Reynolds number, $\tilde{\eta} \equiv u_\tau^{(2/n)-1}\delta/\tilde{\nu}^{1/n}$
θ = momentum thickness
$\tilde{\mu}$ = viscosity of power-law fluids; see equation (2)
$\tilde{\nu}$ = kinematic viscosity of power-law fluids, $\tilde{\nu} \equiv \tilde{\mu}/\rho$
ρ = density of fluid
σ = local resistance parameter, $\sigma \equiv U/u_\tau$
τ = shearing stress in fluid
τ_w = shearing stress at wall

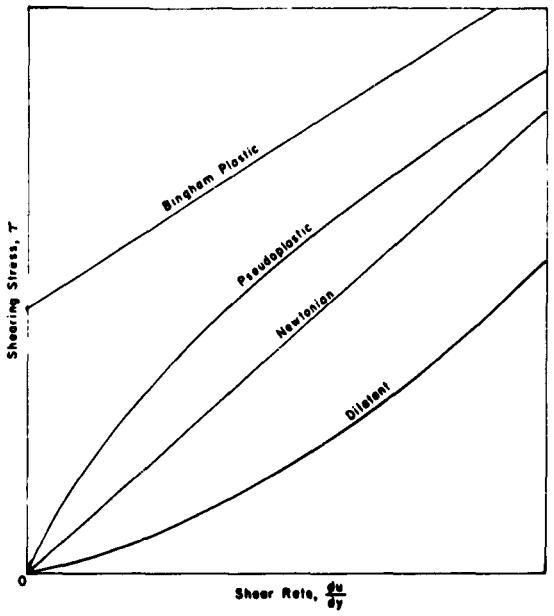

Fig. 1 Shear curves of time-independent non-Newtonian fluids

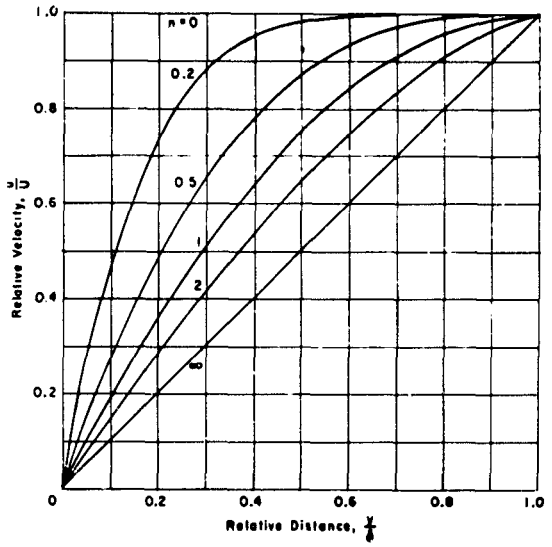

Fig. 2 Laminar velocity profiles for power-law fluids

tain the usual boundary-layer parameters of momentum thickness, displacement thickness, and shape parameter. The overall coefficients of frictional resistance follow in terms of a local-resistance parameter. The subsequent elimination of this parameter provides the more familiar logarithmic resistance-type formulas.

An example is worked out for the frictional resistance of a flat plate in an aqueous solution of a synthetic polymer exhibiting a reduction in resistance for pipe tests in the turbulent regime. Here the values of the pertinent boundary-layer factors are deduced from the pipe data.

Non-Newtonian Fluids

Definition

Non-Newtonian fluids [2] are a class of viscous or real fluids for which the shearing stress τ is no longer a fixed ratio to the rate of shear; this fixed ratio is termed the coefficient of viscosity for Newtonian fluids. For two-dimensional flows the rate of shear is du/dy and then

$$\tau = f\left(\frac{du}{dy}\right) \tag{1}$$

where u is the velocity in the direction of the shearing stress and y is the distance normal to the direction of the shearing stress. Representative curves are shown schematically in Fig. 1.

As stated by Metzner [3], Newtonian fluids are: (*a*) All gases, and (*b*) all liquids or solutions of low molecular weight (i.e., nonpolymeric) materials.

Non-Newtonian fluids are:

1 Solutions or melts of polymeric materials of high molecular weights.

2 Suspensions of solids in liquids, particularly if the solid tends to swell, solvate, or otherwise associate with the liquid phase.

"The distinguishing feature of non-Newtonian systems is seen to be that the colloidal rather than the molecular properties are of significance."

Classification

Non-Newtonian fluids may be divided broadly into three main categories [3]:

1 Fluids with properties independent of the time or duration of shear.

2 Fluids with properties dependent on the time or duration of shear.

3 Viscoelastic fluids which have some of the characteristics of solids, such as elastic recovery from deformations.

Most engineering studies to date have dealt with the first and simplest category, which will also be the one considered in this paper.

The time-independent non-Newtonian fluids may be subdivided into three categories whose characteristic shear curves are shown in Fig. 1:

1 Bingham plastics which require a finite shearing stress to initiate movement. Otherwise, the relationship between shearing stress and shear rate is linear like Newtonian fluids.

2 Pseudoplastic fluids for which the shear curve is nonlinear and curves downward. These include the majority of non-Newtonian fluids.

3 Dilatant fluids for which the shear curve is also nonlinear but curves upward.

Power-Law Fluids

The power-law fluids are those characterized by linear plots in log-log coordinates of the curves of shearing stress versus shear rate, or

$$\tau = \tilde{\mu}\left(\frac{du}{dy}\right)^n \tag{2}$$

These fluids include pseudoplastic fluids $n < 1$, dilatant fluids $n > 1$, and Newtonian fluids $n = 1$.

Term n is the flow-behavior index and $\tilde{\mu}$ is usually termed the viscosity or consistency index (often symbolized as K). Here $\tilde{\mu}$ is called the power-law fluid viscosity, since for Newtonian fluids $n = 1$, μ is the ordinary coefficient of viscosity. However, it should be noted that the dimensions of $\tilde{\mu}$ depend on n for non-Newtonian fluids.

The simple analytic statement of power-law fluids lends itself readily to mathematical analysis, as shown in this paper.

Laminar Boundary Layer

Velocity Profile

In lieu of attempting to solve the equations of motion for the laminar flow of power-law non-Newtonian fluids wherein the difficulties are compounded by the nonlinearity of the shearing-stress terms, a simple expedient is to assume that the known velocity profiles for pipe flow [3] hold sufficiently close for the boundary-layer flow on flat plates. When the boundary-layer thickness δ is substituted for the pipe radius, the velocity profile becomes

$$\frac{u}{U} = 1 - \left(1 - \frac{y}{\delta}\right)^{1+\frac{1}{n}} \tag{3}$$

where u is the velocity in the boundary layer parallel to the plate.

U is the free-stream velocity outside the boundary layer, and y is the distance normal to the plate. Also

$$u = 0 \quad \text{at} \quad y = 0$$

$$u = U \quad \text{at} \quad y = \delta$$

Typical velocity profiles are shown in Fig. 2.

For displacement thickness

$$\delta^* \equiv \int_0^\delta \left(1 - \frac{u}{U}\right) dy,$$

$$\frac{\delta^*}{\delta} = \frac{n}{2n + 1} \tag{4}$$

and momentum thickness

$$\theta \equiv \int_0^\delta \left(1 - \frac{u}{U}\right) \frac{u}{U}\, dy,$$

$$\frac{\theta}{\delta} = \frac{n(n + 1)}{(2n + 1)(3n + 2)} \tag{5}$$

and shape parameter $H \equiv \delta^*/\theta$,

$$H = \frac{3n + 2}{n + 1} \tag{6}$$

The solution here for Newtonian fluids $n = 1$, is

$$H = 2.5 \tag{7}$$

In contrast, the exact Blasius solution for flat plates [4] which is sufficiently close is

$$H = 2.61 \tag{8}$$

The limiting conditions for $n = 0$ are a constant velocity U for

$$y > 0, \ \frac{\delta^*}{\delta} = 0, \ \frac{\theta}{\delta} = 0, \text{ and } H = 2$$

The limiting conditions for $n \to \infty$ are a straight-line velocity profile

$$\frac{u}{U} = \frac{y}{\delta}, \ \frac{\delta^*}{\delta} = \frac{1}{2}, \ \frac{\theta}{\delta} = \frac{1}{6}, \text{ and } H = 3$$

Shearing Stress at Wall

The local skin friction or shearing stress at the wall τ_w is given by

$$\tau_w = \tilde{\mu}\left(\frac{du}{dy}\right)_w^n \tag{9}$$

where

$$\left(\frac{du}{dy}\right)_w = \frac{du}{dy} \quad \text{for } y = 0$$

From velocity profile, equation (3),

$$\tau_w = \tilde{\mu}\left[\left(\frac{n + 1}{n}\right)\left(\frac{U}{\delta}\right)\right]^n \tag{10}$$

or for shearing-stress coefficient,

$$\frac{\tau_w}{\rho U^2} = \nu\left(\frac{n + 1}{n}\right)^n \frac{U^{n-2}}{\delta^n} \tag{11}$$

where $\nu \equiv \tilde{\mu}/\rho$ will be termed the power-law fluid kinematic viscosity.

Substituting θ for δ from equation (5) yields

$$\frac{\tau_w}{\rho U^2} = \left[\frac{(n + 1)^2}{(2n + 1)(3n + 2)}\right]^n \frac{1}{\mathbf{R}_\theta^n} \tag{12}$$

where $\mathbf{R}_\theta$ is the momentum-thickness Reynolds number for power-law fluids,

$$\mathbf{R}_\theta \equiv \frac{U^{\frac{2}{n}-1}\theta}{\nu^{1/n}}$$

For Newtonian fluids $n = 1$, equation (12) becomes

$$\frac{\tau_w}{\rho U^2} = \frac{0.267}{\mathbf{R}_\theta} \tag{13}$$

In contrast, the exact Blasius solution [4] is

$$\frac{\tau_w}{\rho U^2} = \frac{0.220}{\mathbf{R}_\theta} \tag{14}$$

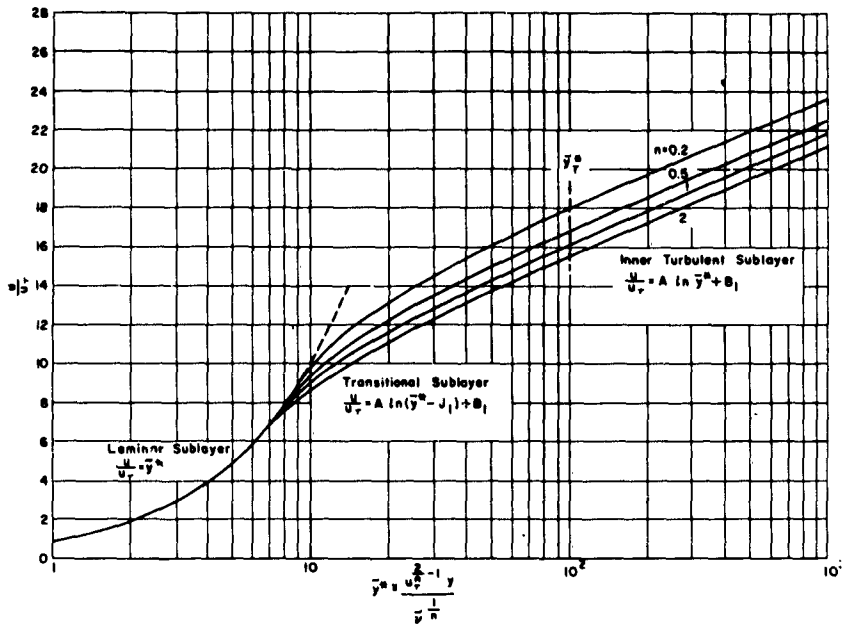

Fig. 3 Typical inner law velocity profiles (based on data of reference [1])

Consequently equation (12) will be arbitrarily altered to agree with the exact solution in the Newtonian case, or

$$\frac{\tau_w}{\rho U^2} = 0.824 \left[\frac{(n+1)^2}{(2n+1)(3n+2)}\right]^n \frac{1}{\mathbf{R}_\theta^n} \tag{15}$$

Turbulent Boundary Layer

Inner Law or Law of the Wall

Similarity laws for the turbulent boundary layer of non-Newtonian fluids on flat plates may be deduced in the same manner as those for pipe flow [1].

Close to the wall the mean velocity u of the turbulent flow of a particular non-Newtonian liquid[5] parallel to the wall is considered to depend on the normal distance y away from the wall, the shearing stress τ_w at the wall, and the density ρ, the power-law viscosity $\tilde{\mu}$, and the flow-behavior index n of the fluid, or

$$u = f(y, \tau_w, \rho, \tilde{\mu}, n) \tag{16}$$

By dimensional analysis the variables can be grouped significantly in the following nondimensional ratios:

$$\frac{u}{u_\tau} = f_1(\tilde{y}^*, n) \tag{17}$$

where $u_\tau \equiv (\tau_w/\rho)^{1/2}$ is friction or shear velocity,

$$\tilde{y}^* \equiv \frac{u_\tau^{(2/n)-1} y}{\tilde{\nu}^{1/n}}$$

is the power-law fluid Reynolds number for the inner law, and $\tilde{\nu} \equiv \tilde{\mu}/\rho$ is the kinematic viscosity for power-law fluids. Fig. 3 shows characteristic velocity profiles for power-law fluids. The inner law, equation (17), reduces to the well-known Newtonian case for $n = 1$.

Outer Law or Velocity-Defect Law

At some distance away from the wall and for the remainder of the bounday-layer thickness, the velocity defect U-u, for Newtonian fluids, has been found experimentally to be independent of viscosity μ and is only a function of τ_w, ρ, and distance δ-y. Hence for power-law fluids the velocity defect may be assumed to be independent of $\tilde{\mu}$ and n. Dodge and Metzner [1] make the velocity defect independent of $\tilde{\mu}$ but dependent on n. This is inconsistent with the concept of the outer law as pointed out to the author by Tulin and Owen Phillips of Hydronautics, Inc.

Then

$$U - u = f(\tau_w, \rho, y, \delta) \tag{18}$$

or by dimensional analysis,

$$\frac{U-u}{u_\tau} = F\left(\frac{y}{\delta}\right) \tag{19}$$

The characteristic velocity profile is shown in Fig. 4.

[5] Since the turbulence level may be affected by other rheological effects such as the development of normal stresses, it is preferable to specify the similarity laws only for each particular liquid.

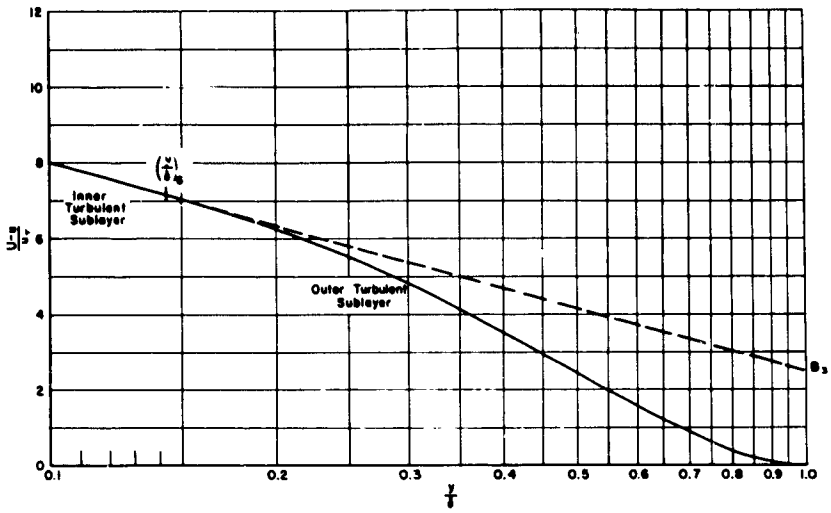

Fig. 4 Outer law velocity profile

Logarithmic Velocity Law

Within the region of the boundary layer where the inner and outer laws overlap, a logarithmic expression results as a consequence of the analytic requirements. This region will also be termed the inner turbulent sublayer.

Equating the derivative of velocity u with distance y of the inner and outer laws, equations (17) and (19), yields

$$\frac{\partial u}{\partial y} = \frac{u_\tau^{2/n}}{\bar{\nu}^{1/n}} \frac{\partial f_1}{\partial \bar{y}^*} = -\frac{u_\tau}{\delta} \frac{dF}{d(y/\delta)} \tag{20}$$

or

$$\bar{y}^* \frac{\partial f_1}{\partial \bar{y}^*} = -\left(\frac{y}{\delta}\right) \frac{dF}{d(y/\delta)} = A \tag{21}$$

Since the left-hand side of equation (21) is only a function of $\bar{y}^*$ and n and the right-hand side of (21) is only a function of y/δ they may be equated to a factor A which is independent of $\bar{y}^*$, y/δ, and n. From the left-hand side of equation (21) there results after integration,

$$f_1 = \frac{u}{u_\tau} = A \ln \bar{y}^* + B_1(n) \tag{22}$$

and from the right-hand side,

$$F = \frac{U - u}{u_\tau} = -A \ln \frac{y}{\delta} + B_3 \tag{23}$$

Factor B_1 is necessarily a function of n from the integration of a partial derivative. Figs. 3 and 4 show the logarithmic velocity profiles.

Since both the inner and outer laws hold in the overlapping region, adding equations (22) and (23) results in

$$\sigma \equiv \frac{U}{u_\tau} = A \ln \eta + B_1(n) + B_3 \tag{24}$$

where

$$\bar{\eta} \equiv \frac{u_\tau^{(2/n)-1}\delta}{\nu^{1/n}}$$

Also

$$\bar{\eta} = \exp\left[\frac{1}{A}(\sigma - B_1 - B_3)\right] \tag{25}$$

Sublayers

Various sublayers may be distinguished in the boundary layer according to the behavior of the velocity profiles, as indicated schematically in Fig. 5. These are:

1 The laminar sublayer next to the wall wherein the turbulent fluctuations are effectively dampened out.

2 The transitional sublayer wherein the shearing stresses are affected by both laminar and turbulent contributions.

3 The inner turbulent sublayer wherein the inner and outer laws overlap.

4 The outer turbulent sublayer where only the outer law prevails.

Velocity Law for the Laminar Sublayer

The inner law, equation (17), holds here however, with no specification as to the exact functional relationship. Within the laminar sub-layer for power-law fluids

$$\tau = \bar{\mu}\left(\frac{du}{dy}\right)^n \tag{2}$$

For the thin laminar sublayer $\tau = \tau_w$ and with boundary condition $u = 0$ at $y = 0$, there results

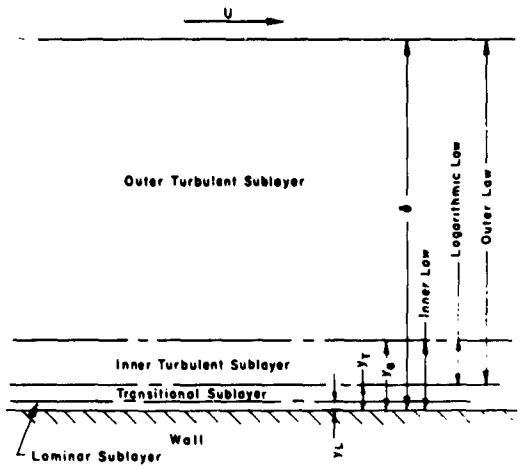

Fig. 5 Sublayers within boundary layer

$$\frac{u}{u_\tau} = \bar{y}^* \tag{26}$$

which also agrees with the inner law.

Velocity Law for the Transitional Sublayer

The transitional sublayer is bounded by the laminar sublayer and the inner turbulent sublayer. For the Newtonian case [5], the velocity profile for the transitional sublayer as originally derived by Squire is

$$\frac{u}{u_\tau} = A \ln (\bar{y}^* - J_1) + B_1 \tag{27}$$

where

$$J_1 = B_1 - A \ln \frac{e}{A} = \bar{y}_L^* - A \tag{28}$$

This relation starts at the outer edge of the laminar sublayer y_L and merges asymptotically with the logarithmic velocity law, equation (22).

It will now be assumed that a similar relationship holds also for power-law fluids, with B_1, J_1, and $\bar{y}_L^*$ being functions of n. Fig. 3 shows typical plots of the velocity profiles for the transitional sublayer.

Velocity Law for the Outer Turbulent Sublayer

For Newtonian fluids Häma [6] fitted a parabola to the nonlogarithmic part of the outer law, or

$$\frac{U - u}{u_\tau} = a\left(1 - \frac{y}{\delta}\right)^2 \tag{29}$$

A similar relation will be assumed for non-Newtonian fluids.

This relation is to merge smoothly with the logarithmic relation equation (23); that is, with equal tangents. Hence, if the point of junction is $(y/\delta)_G$, equating derivatives results in

$$a = \frac{A}{2[1 - (y/\delta)_G](y/\delta)_G} \tag{30}$$

Then equating (23), and (29) produces

$$\frac{1}{(y/\delta)_G} - 2 \ln \frac{1}{(y/\delta)_G} = 2\frac{B_1}{A} + 1 \tag{31}$$

where $(y/\delta)_G$ is to be determined implicitly.

Fig. 4 shows a plot of the parabolic velocity profile.

Boundary-Layer Parameters

The boundary-layer parameters δ^*, θ, and $H \equiv \delta^*/\theta$ are obtained from the similarity laws by appropriately integrating u with respect to y piece-wise across each sublayer. The results are shown Table 1 which is similar to that in reference [5].

Then

$$\frac{\delta^*}{\delta} = \frac{1}{\sigma}\left(D_1 - \frac{\alpha_1}{\bar{\eta}}\right) \tag{32}$$

$$\frac{\theta}{\delta} = \frac{1}{\sigma}\left(D_1 + \frac{\alpha_1}{\bar{\eta}}\right) - \frac{1}{\sigma^2}\left(D_2 + \frac{\beta_1}{\bar{\eta}}\right) \tag{33}$$

and

$$\bar{R}_\theta \equiv \frac{U^{(2-n)/n}\theta}{\nu_{\bar{p}}^{1/n}} \equiv \frac{\theta}{\delta}\bar{\eta}\,\sigma^{(2-n)/n-1}$$

$$= \bar{\eta}\,\sigma^{(2-n)/n-2}\left[\left(D_1 + \frac{\alpha_1}{\bar{\eta}}\right) - \frac{1}{\sigma}\left(D_2 + \frac{\beta_1}{\bar{\eta}}\right)\right] \tag{34}$$

where

$$\alpha_1 = \frac{\bar{y}_L^{*2}}{2} - J_1\left(\frac{u}{u_\tau}\right)_T \tag{35}$$

$$D_1 = \left(\frac{y}{\delta}\right)_G (F_G + A) + I_1 \tag{36}$$

$$\beta_1 = \frac{\bar{y}_L^{*3}}{3} - J_1\left[\left(\frac{u}{u_\tau}\right)_T - A\right]^2 - A(\bar{y}_L^{*2} - A\bar{y}_L^* + A^2) \tag{37}$$

$$D_2 = \left(\frac{y}{\delta}\right)_G [(F_G + A)^2 + A^2] + I_2 \tag{38}$$

$$I_1 \equiv \int_{(y/\delta)_G}^1 F d\left(\frac{y}{\delta}\right) = \frac{a}{3}\left[1 - \left(\frac{y}{\delta}\right)_G\right]^3 \tag{39}$$

and

$$I_2 \equiv \int_{(y/\delta)_G}^1 F^2 d\left(\frac{y}{\delta}\right) = \frac{a^2}{4}\left[1 - \left(\frac{y}{\delta}\right)_G\right]^4 \tag{40}$$

Since $\bar{\eta}$ is given in terms of σ in equation (25), δ^*, θ, and H are then functions of σ (and n) according to the preceding relationships.

Frictional Resistance of Flat Plates

Momentum Equation

Considerations of the momentum changes of any fluid [4], Newtonian or non-Newtonian, flowing past a

flat plate indicate that the von Karman momentum equation is also applicable to non-Newtonian fluids, or for flat plate in zero pressure gradient

$$\frac{d\theta}{dx} = \frac{\tau_w}{\rho U^2} \tag{41}$$

where x is the streamwise distance from the leading edge.

Since the frictional resistance or drag D for a flat plate of unit breadth is

$$D = \int_0^x \tau_w dx \tag{42}$$

the drag coefficient

$$C_f \equiv \frac{D}{\frac{1}{2}\rho U^2 x}$$

$$C_f = 2\frac{\theta}{x} = 2\frac{\mathbf{R}_\theta}{\mathbf{R}_x} \tag{43}$$

where

$$\mathbf{R}_\theta \equiv \frac{U^{(2/n)-1}\theta}{\bar{\nu}^{1/n}}$$

and

$$\mathbf{R}_x = \frac{U^{(2/n)-1}x}{\bar{\nu}^{1/n}}$$

Laminar Flow

Substituting the expression for shearing-stress coefficient for laminar flow, equation (15), into the momentum equation (41) and integrating from $x = 0$ produces

$$\theta = 0.824^{1/n+1}\left[\frac{(n+1)^{2+(1/n)}}{(2n+1)(3n+2)}\right]^{n/(n+1)} x^{1/(n+1)} \tag{44}$$

The drag coefficient for flat plates in the laminar flow of non-Newtonian fluids from equation (43) becomes

$$C_f = 2(0.824)^{1/(n+1)}\left[\frac{(n+1)^{2+(1/n)}}{(2n+1)(3n+2)}\right]^{n/(n+1)} \frac{1}{\mathbf{R}_x^{n/(n+1)}} \tag{45}$$

Turbulent Flow

Since $1/\sigma^2 \equiv \tau_w/\rho U^2$, the momentum equation (47) becomes

$$\bar{\mathbf{R}}_x = \int \sigma^2 d\bar{\mathbf{R}}_\theta \tag{46}$$

or integrating by parts

$$\bar{\mathbf{R}}_x = \sigma^2\bar{\mathbf{R}}_\theta - 2\int \bar{\mathbf{R}}_\theta \sigma d\sigma + C \tag{47}$$

From the statement for $\mathbf{R}_\theta$, equation (34), and $\bar{\eta}$, equation (25) in terms of σ, there results after integration by parts

$$\int \bar{\mathbf{R}}_\theta \sigma d\sigma = A\sigma^{(2/n)-1}\bar{\eta}\left\{D_1\left[1 - \frac{A(2-n)}{n\sigma} + \dots\right] - \frac{D_2}{\sigma}\left[1 - \frac{A}{n}\frac{(2-2n)}{\sigma} + \dots\right]\right\} + \frac{\alpha_1\eta}{2}\sigma^{2/n} - \frac{\beta_1 n}{2-n}\sigma^{(2/n)-1} \tag{48}$$[6]

Then

$$\bar{\mathbf{R}}_x = D_1\sigma^{2/n}\bar{\eta}\left[1 - \left(2A + \frac{D_2}{D_1}\right)\frac{1}{\sigma} + 2A\left(\frac{2-n}{n} + \frac{D_2}{D_1}\right)\frac{1}{\sigma^2} + \dots\right] + (1-n)\alpha_1\sigma^{2/n} + \left(\frac{3n-2}{2-n}\right)\beta_1\sigma^{(2/n)-1} + C \tag{49}$$[6]

The solution of C_f as a function of $\mathbf{R}_x$ is hence given implicitly in terms of σ by equations (34) and (49).

For Newtonian fluids $n = 1$, equation (49) reverts to that given in reference [7].

Evaluation of Constant of Integration

The evaluation of the constant of integration C depends on the starting point of the turbulent flow. If the turbulent flow is assumed to start at the leading edge, it is found that the similarity laws do not hold all the way but begin only at the point where the overlapping of the inner and outer laws starts. The point is given by $y_T = y_G$, where y_T is the limit of the overlapping region nearest the wall and y_G the farthest. Then

$$\bar{\eta}_0 = \frac{\bar{y}_T^*}{(y/\delta)_G} \tag{50}$$

and

$$\sigma_0 = A \ln \bar{\eta}_0 + B_1 + B_3 \tag{51}$$

where the subscript 0 refers to the limit of overlap.

The evaluation of the constant of integration for the case of complete turbulent flow then requires the assumption of the variation of σ^2 with $\mathbf{R}_\theta$ from the leading edge to the limit of overlap. As shown in reference [8] for Newtonian fluids, the curve of σ^2 versus $\mathbf{R}_\theta$ starts at zero where it is tangent to the laminar line and gradually reaches the limit of overlap where it becomes tangent to the curve specified by the similarity laws, equation (34). A like procedure may be applied to non-Newtonian fluids.

The more realistic procedure is to have the initial portion of the flat plate laminar and, hence, with a constant of integration specified for each particular point of transition. The constant of integration is evaluated from the momentum thickness developed by the laminar flow, or

$$\mathbf{R}_{\theta_{lam}} = \mathbf{R}_{\theta_{turb}} \tag{52}$$

at the point of transition. The value of σ at the point of transition is determined implicitly from equation (34). There the value of C, the constant of integration, is ob-

[6] The terms containing β_1 in equations (48) and (49) do not hold for $n = 2$. Instead for $n = 2$, the term is $-\beta_1 \ln \sigma$ for equation (48) and $\beta_1(2 \ln \sigma - 1)$ for equation (49).

Table 1 Velocity Profile Integrals for Pipe Flow

Region	Limits	Velocity Law	$\int \frac{u}{u_\tau} d\tilde{y}^\circ$	$\int \left(\frac{u}{u_\tau}\right)^2 d\tilde{y}^\circ$
Laminar Sublayer	$0 \le \tilde{y}^\circ \le \tilde{y}_L^\circ$	$\frac{u}{u_\tau} = \tilde{y}^\circ$	$\frac{1}{2}\tilde{y}_L^{\circ 2}$	$\frac{1}{3}\tilde{y}_L^{\circ 3}$
Transitional Sublayer	$\tilde{y}_L^\circ \le \tilde{y}^\circ \le \tilde{y}_T^\circ$	$\frac{u}{u_\tau} = A \ln(\tilde{y}^\circ - I_1) + B_1$	$(\tilde{y}_T^\circ - I_1)\left(\frac{u}{u_\tau}\right)_T - A\tilde{y}_T^\circ$	$(\tilde{y}_T^\circ - I_1)\left[\left(\frac{u}{u_\tau}\right)_T - A\right]^2 - A(\tilde{y}_L^\circ - A)^2 + A^2(\tilde{y}_T^\circ - \tilde{y}_L^\circ)$
Inner Turbulent Sublayer	$\tilde{y}_T^\circ \le \tilde{y}^\circ \le \tilde{y}_G^\circ$ $\tilde{y}_G^\circ = \eta\left(\frac{y}{\delta}\right)_G$	$\frac{u}{u_\tau} = A \ln \tilde{y}^\circ + B_1$	$\tilde{y}_G^\circ\left[\left(\frac{u}{u_\tau}\right)_G - A\right] - \tilde{y}_T^\circ\left[\left(\frac{u}{u_\tau}\right)_T - A\right]$	$\tilde{y}_G^\circ\left[\left(\frac{u}{u_\tau}\right)_G - A\right]^2 - \tilde{y}_T^\circ\left[\left(\frac{u}{u_\tau}\right)_T - A\right]^2 + A^2(\tilde{y}_G^\circ - \tilde{y}_T^\circ)$
Outer Turbulent Sublayer	$\left(\frac{y}{\delta}\right)_G \le \frac{y}{\delta} \le 1$	$\frac{u}{u_\tau} = \sigma - F$	$\tilde{\eta}\left\{\left[1 - \left(\frac{y}{\delta}\right)_G\right]\sigma - I_1\right\}$	$\tilde{\eta}\left\{\left[1 - \left(\frac{y}{\delta}\right)_G\right]\sigma^2 - 2\sigma I_1 + I_2\right\}$
Whole Boundary Layer	$0 \le \tilde{y}^\circ \le \tilde{\eta}$		$\tilde{\eta}(\sigma - D_1) + \alpha_1$	$\tilde{\eta}(D_2 - 2\sigma D_1) + \beta_1$

$$I_1 = \int_{\left(\frac{y}{\delta}\right)_G}^{1} F\, d\left(\frac{y}{\delta}\right) \qquad I_2 = \int_{\left(\frac{y}{\delta}\right)_G}^{1} F^2\, d\left(\frac{y}{\delta}\right) \qquad \alpha_1 = \frac{\tilde{y}_L^{\circ 2}}{2} - I_1\left(\frac{u}{u_\tau}\right)_T \qquad D_1 = \left(\frac{y}{\delta}\right)_G (F_G + A) + I_1$$

$$\beta_1 = \frac{\tilde{y}_L^{\circ 3}}{3} - I_1\left[\left(\frac{u}{u_\tau}\right)_T - A\right]^2 - A(\tilde{y}_L^{\circ 2} - A\tilde{y}_L^\circ + A^2) \qquad D_2 = \left(\frac{y}{\delta}\right)_G [(F_G + A)^2 + A^2] + I_2$$

tained from this value of σ and the $\bar{\mathbf{R}}_x$ corresponding to $\bar{\mathbf{R}}_{\theta_{lam}}$ of the laminar flow.

Logarithmic Resistance Formulas

The elimination of parameter σ from equation (49) produces the more familiar logarithmic resistance formulas with C_f as a function of $\bar{\mathbf{R}}_x$. The procedure starts by combining equations (43) and (47) to give, after neglecting the constant of integration,

$$\frac{1}{\sigma^2} = \frac{C_f}{2}\left(1 - \frac{2\int \bar{\mathbf{R}}_\theta \sigma d\sigma}{\sigma^2 \bar{\mathbf{R}}_\theta}\right) \tag{53}$$

The expressions for $\int \bar{\mathbf{R}}_\theta \sigma d\sigma$ and $\bar{\mathbf{R}}_\theta$ from equations (48) and (34) are inserted into equation (53) to give with α_1 and β_1 dropped as negligible

$$\frac{1}{\sigma^2} = \frac{C_f}{2}\left[1 - \frac{2A}{\sigma} + \frac{2A^2(2-n)}{n}\frac{1}{\sigma^2} + \ldots\right] \tag{54}$$

Through reiteration σ is replaced by C_f within the brackets so that

$$\frac{1}{\sigma^2} = \frac{C_f}{2}\left[1 - 2A\left(\frac{C_f}{2}\right)^{1/2} + \frac{4A^2}{n}\left(\frac{C_f}{2}\right) + \ldots\right] \tag{55}$$

and by the binomial expansion

$$\frac{1}{\sigma} = \frac{C_f}{2}\left[1 - A\left(\frac{C_f}{2}\right)^{1/2} + \left(\frac{4-n}{2n}\right)A^2\left(\frac{C_f}{2}\right) + \ldots\right] \tag{56}$$

and by inversion

$$\sigma = \left(\frac{2}{C_f}\right)^{1/2}\left[1 + A\left(\frac{C_f}{2}\right)^{1/2} + \left(\frac{3n-4}{2n}\right)A^2\left(\frac{C_f}{2}\right) + \ldots\right] \tag{57}$$

Now, after substituting for $\bar{\eta}$ from equation (25) and ignoring α_1, β_1, and the constant of integration, $\bar{\mathbf{R}}_x$ in equation (49) is written in logarithmic form as

$$\ln \bar{\mathbf{R}}_x = \frac{\sigma}{A} - \frac{B_1}{A} - \frac{B_3}{A} + \ln D_1 - \frac{1}{n}\ln\frac{1}{\sigma^2} + \ln\left[1 - \left(2A + \frac{D_2}{D_1}\right)\frac{1}{\sigma} + \ldots\right] \tag{58}$$

Substituting the appropriate expressions for $1/\sigma^2$, $1/\sigma$ and σ from equations (55), (56) and (57), and expanding the logarithm as a series results in common logarithms as

$$\log \bar{\mathbf{R}}_x^n C_f = \frac{\sqrt{2}n}{2.3026A}\left(\frac{1}{C_f}\right)^{1/2} - \frac{n}{2.3026\sqrt{2}}\left(\frac{A}{2} + \frac{D_2}{D_1}\right)(C_f)^{1/2} + \frac{n}{2.3026} - \frac{n(B_1 + B_3)}{2.3026A} + \log 2D_1^n \tag{59}$$

wherein terms of higher order than $(C_f)^{1/2}$ have been neglected.

If the term involving $(C_f)^{1/2}$ is linearized with respect to $1/C_f^{1/2}$ or

$$C_f^{1/2} = c_1 + \frac{c_2}{C_f^{1/2}} \tag{60}$$

then equation [65] becomes

$$\boxed{\log \bar{\mathbf{R}}_x^n C_f = \frac{P_1}{C_f^{1/2}} + Q_1} \tag{61}$$

where

$$P_1 = \frac{n}{2.3026}\left[\frac{\sqrt{2}}{A} - \left(\frac{A}{2} + \frac{D_2}{D_1}\right)\frac{c_2}{\sqrt{2}}\right] \tag{62}$$

and

$$Q_1 = \frac{n}{2.3026}\left[1 - \left(\frac{A}{2} + \frac{D_2}{D_1}\right)\frac{c_1}{\sqrt{2}} - \frac{(B_1 + B_3)}{A}\right] + \log 2D_1^n \tag{63}$$

For $n = 1$, Newtonian fluids, equation (61) reduces to the well-known Kármán-Schoenherr formula.

Furthermore, if $\log C_f$ is linearized with respect to $1/C_f^{1/2}$ or

$$\log C_f = c_3 + \frac{c_4}{C_f^{1/2}} \tag{64}$$

then equation (61) becomes

$$\log \bar{\mathbf{R}}_x^n = \frac{P_2}{C_f^{1/2}} + Q_2 \tag{65}$$

or

$$\boxed{C_f = \frac{P_2^2}{(\log \bar{\mathbf{R}}_x^n - Q_2)^2}} \tag{66}$$

where

$$P_2 = P_1 - c_4 \tag{67}$$

and

$$Q_2 = Q_1 - c_3 \tag{68}$$

Equation (66) provides an explicit relation between C_f and $\bar{\mathbf{R}}_x$.

Pipe Flow

Since some of the frictional resistance properties of non-Newtonian liquids may be deduced from pipe flow, it is appropriate to show in detail the application of the similarity laws to pipe flow. The friction factor in pipe flow is stated in terms of average velocity, which provides a measure of flow-carrying capacity of the pipe.

Average Velocity

The average velocity of flow in a pipe $\bar{U}$ is given by the rate of discharge divided by the cross-sectional area or

$$\bar{U} = \frac{2\pi\int_0^R urdr}{\pi R^2} \tag{69}$$

Table 2 Velocity Profile Integrals

Region	Limits	Velocity Law	$\int \frac{u}{u_\tau} \tilde{y}^* \, d\tilde{y}^*$
Laminar Sublayer	$0 \le \tilde{y}^* \le \tilde{y}_L^*$	$\frac{u}{u_\tau} = \tilde{y}^*$	$\frac{1}{3} \tilde{y}_L^{*3}$
Transitional Sublayer	$\tilde{y}_L^* \le \tilde{y}^* \le \tilde{y}_T^*$	$\frac{u}{u_\tau} = A \ln(\tilde{y}^* - J_1) + B_1$	$\frac{1}{2}(\tilde{y}_T^{*2} - J_1^2)\left(\frac{u}{u_\tau}\right)_T + \frac{B_1}{2}(\tilde{y}_T^{*2} - \tilde{y}_L^{*2}) - \frac{A}{4}(\tilde{y}_T^* + J_1)^2 - \frac{A^2}{4}(\tilde{y}_L^* + J_1)$
Inner Turbulent Sublayer	$\tilde{y}_T^* \le \tilde{y}^* \le \tilde{y}_G^*$ $\tilde{y}_G^* = \tilde{\eta}\left(\frac{y}{\delta}\right)_G$	$\frac{u}{u_\tau} = A \ln \tilde{y}^* + B_1$	$\frac{\tilde{y}_G^{*2}}{2}\left(\frac{u}{u_\tau}\right)_G - \frac{\tilde{y}_T^{*2}}{2}\left(\frac{u}{u_\tau}\right)_T - \frac{A}{4}(\tilde{y}_G^{*2} - \tilde{y}_T^{*2})$
Outer Turbulent Sublayer	$\left(\frac{y}{\delta}\right)_G \le \frac{y}{\delta} \le 1$	$\frac{u}{u_\tau} = \sigma - F$	$\tilde{\eta}^2\left\{\frac{\sigma}{2}\left[1 - \left(\frac{y}{\delta}\right)_G^2\right] - I_3\right\}$
Whole Boundary Layer	$0 \le \tilde{y}^* \le \tilde{\eta}$		$\frac{\tilde{\eta}^2}{2}(\sigma - D_3) + \gamma$

$$I_3 = \int_{\left(\frac{y}{\delta}\right)_G}^{1} F\left(\frac{y}{\delta}\right) d\left(\frac{y}{\delta}\right) \qquad D_3 = \left(\frac{y}{\delta}\right)_G^2 \left(F_G + \frac{A}{2}\right) + 2I_3$$

$$\gamma = \frac{\tilde{y}_L^{*3}}{3} - \frac{J_1^2}{2}\left(\frac{u}{u_\tau}\right)_T - \frac{A^2}{4}(\tilde{y}_L^* + J_1) + \frac{B_1}{2}(\tilde{y}_T^{*2} - \tilde{y}_L^{*2}) - \frac{AJ_1}{4}(2\tilde{y}_T^* + J_1)$$

Table 2 Velocity Profile Integrals

Region	Limits	Velocity Law	$\int \frac{u}{u_\tau} \tilde{y}^* \, d\tilde{y}^*$
Laminar Sublayer	$0 \le \tilde{y}^* \le \tilde{y}_L^*$	$\frac{u}{u_\tau} = \tilde{y}^*$	$\frac{1}{3} \tilde{y}_L^{*3}$
Transitional Sublayer	$\tilde{y}_L^* \le \tilde{y}^* \le \tilde{y}_T^*$	$\frac{u}{u_\tau} = A \ln(\tilde{y}^* - J_1) + B_1$	$\frac{1}{2}(\tilde{y}_T^{*2} - J_1^2)\left(\frac{u}{u_\tau}\right)_T + \frac{B_1}{2}(\tilde{y}_T^{*2} - \tilde{y}_L^{*2}) - \frac{A}{4}(\tilde{y}_T^* + J_1)^2 - \frac{A^2}{4}(\tilde{y}_L^* + J_1)$
Inner Turbulent Sublayer	$\tilde{y}_T^* \le \tilde{y}^* \le \tilde{y}_G^*$ $\tilde{y}_G^* = \tilde{\eta}\left(\frac{y}{\delta}\right)_G$	$\frac{u}{u_\tau} = A \ln \tilde{y}^* + B_1$	$\frac{\tilde{y}_G^{*2}}{2}\left(\frac{u}{u_\tau}\right)_G - \frac{\tilde{y}_T^{*2}}{2}\left(\frac{u}{u_\tau}\right)_T - \frac{A}{4}(\tilde{y}_G^{*2} - \tilde{y}_T^{*2})$
Outer Turbulent Sublayer	$\left(\frac{y}{\delta}\right)_G \le \frac{y}{\delta} \le 1$	$\frac{u}{u_\tau} = \sigma - F$	$\tilde{\eta}^2\left\{\frac{\sigma}{2}\left[1 - \left(\frac{y}{\delta}\right)_G^2\right] - I_3\right\}$
Whole Boundary Layer	$0 \le \tilde{y}^* \le \tilde{\eta}$		$\frac{\tilde{\eta}^2}{2}(\sigma - D_3) + \gamma$

$$I_3 = \int_{\left(\frac{y}{\delta}\right)_G}^{1} F\left(\frac{y}{\delta}\right) d\left(\frac{y}{\delta}\right) \qquad D_3 = \left(\frac{y}{\delta}\right)_G^2 \left(F_G + \frac{A}{2}\right) + 2I_3$$

$$\gamma = \frac{\tilde{y}_L^{*3}}{3} - \frac{J_1^2}{2}\left(\frac{u}{u_\tau}\right)_T - \frac{A^2}{4}(\tilde{y}_L^* + J_1) + \frac{B_1}{2}(\tilde{y}_T^{*2} - \tilde{y}_L^{*2}) - \frac{AJ_1}{4}(2\tilde{y}_T^* + J_1)$$

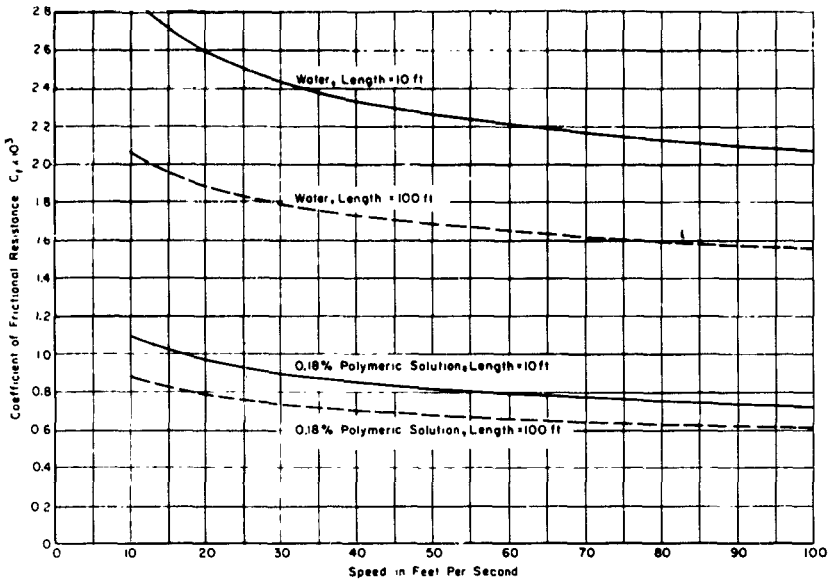

Fig. 6. Comparison of coefficient of frictional resistance for flat plates in turbulent flow

where r is the radial distance from the center of the pipe and R is the radius of the pipe.

In terms of similarity parameters

$$\bar{U} = \frac{2u_\tau}{\bar{\eta}}\left(\int_0^{\bar{\eta}} \frac{u}{u_\tau}\, d\bar{y}^* - \frac{1}{\bar{\eta}}\int_0^{\bar{\eta}} \frac{u}{u_\tau}\, \bar{y}^* d\bar{y}^*\right) \tag{70}$$

where y is the radial distance from the pipe wall towards the center and u_τ, $\bar{\eta}$, and $\bar{y}^*$ have the same definitions as for a flat plate.

Substituting the values of

$$\int_0^{\eta} \frac{u}{u_\tau}\, d\bar{y}^*$$

from Table 1 and

$$\int_0^{\bar{\eta}} \frac{u}{u_\tau}\, \bar{y}^* d\bar{y}^*$$

from Table 2 into equation (70) produces

$$\frac{\bar{U}}{u_\tau} = \sigma - 2D_1 + D_3 + \frac{2\alpha_1}{\bar{\eta}} - \frac{2\gamma}{\bar{\eta}^2} \tag{71}$$

Friction Factor

The friction factor f for pipe flow is defined as

$$f \equiv \frac{\tau_w}{\frac{1}{2}\rho \bar{U}^2} \tag{72}$$

or

$$\left(\frac{2}{f}\right)^{1/2} = \frac{\bar{U}}{u} \tag{73}$$

Then from equations (71) and (24) there results for high values of $\bar{\eta}$

$$\left(\frac{2}{f}\right)^{1/2} = A \ln \bar{\eta} + B_1(n) + B_3 - 2D_1 + D_3 \tag{74}$$

or from the definition of $\bar{\eta}$ another form is

$$\left(\frac{1}{f}\right)^{1/2} = \frac{2.3026A}{\sqrt{2}} \log\left[(f^{1/2})^{(2/n)-1}\, \bar{R}_D\right] - \frac{2.3026A}{\sqrt{2}}\left(\frac{1}{n} + \frac{1}{2}\right)\log 2 + \frac{B_1(n)}{\sqrt{2}} + \frac{B_3 - 2D_1 + D_3}{\sqrt{2}} \tag{75}$$

where

$$\bar{R}_D \equiv \bar{U}^{(2/n)-1}\, 2R/\bar{\nu}^{1/n}$$

Numerical Example

To illustrate the application of the formulas derived from the similarity laws, the frictional resistance of a flat plate in a non-Newtonian liquid is calculated from results in pipes.

The example chosen is a liquid used in fracturing operations for oil wells which shows a lower friction factor than untreated water [9]: 0.18 percent aqueous solution of a synthetic polymer with $n = 0.66$ and $\bar{\mu} = 2.0 \times 10^{-3}$ lb-sec^n/sq ft. A fit of the pipe data gives

$$\frac{1}{f^{1/2}} = 4 \log\left[\left(f^{1/2}\right)^{2/n-1} \bar{R}_D\right] + 17.38 \tag{76}$$

Comparison with equation (75) then yields

$$B_1 + B_3 - 2D_1 + D_3 = 28.00 \tag{77}$$

From Newtonian flow [4]

$$\frac{1}{f^{1/2}} = 4 \log \left(f^{1/2} R_D\right) - 0.40 \tag{78}$$

and $B_1 = 5.1$.

Then

$$B_3 - 2D_1 + D_3 = -3.11 \tag{79}$$

Then for the polymeric solution $B_1 = 31.1$ from equation (77). For both pipe and flat-plate flow [1, 10]

$$2.3026A = 5.66 \tag{80}$$

while $B_1 = 4.9$, reference [10], for flat plates and $B_1 = 5.1$, reference [1], for pipes.

Hence for the polymeric solution B_1 is taken as 30.9 for flat plates.

From values of B_3 and D_2 obtained from reference [7]

$$\log \bar{R}_x{}^n C_f = \frac{0.1693}{C_f^{1/2}} - 3.108 \tag{81}$$

for flat plates in the polymeric solutions.

A comparison of the drag coefficients for a flat plate in water and in the polymeric solution is shown in Fig. 6 for lengths of 10 and 100 ft, respectively. A decided drag reduction is indicated.

Acknowledgment

The author wishes to thank Mr. Raphael D. Cahn for his efforts in checking this paper.

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