TRANSLATION

CALCULATING A THREE-DIMENSIONAL LAMINAR BOUNDARY LAYER ON SPREADING LINES

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Calculating a Three-Dimensional Laminary Boundary Layer on Spreading Lines

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For the case of a flow in the environs of spreading lines of a cone under an angle of attack, of an infinite cylinder with slipping and forward critical point, equations of a three-dimensional compressible boundary layer have been transformed into a system of ordinary differential equations. Introduced is a solving method based on the use of integral ratios and a special form of approximating functions. Numerical solutions have been obtained in a wide range of parameter changes and formulas are given for the calculation of heat exchange, friction and boundary layer characteristics. Calculation results at partial parameter values are in satisfactory agreement with the numerical calculations of other authors.

1. Equations of laminar three-dimensional boundary layer in a compressible gas at stable flow along a curvilinear surface has the form of

\[
\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = \frac{1}{h_1} \frac{\partial \rho u}{\partial x} + \frac{1}{h_2} \frac{\partial \rho v}{\partial y} + \frac{1}{h_3} \frac{\partial \rho w}{\partial z} = 0
\]

Continuum equations...

\[
h_1 \frac{\partial \rho u}{\partial x} + h_2 \frac{\partial \rho v}{\partial y} + h_3 \frac{\partial \rho w}{\partial z} = 0
\]

Energy equation

\[
\frac{c_p}{h_1} \frac{\partial T}{\partial x} + \frac{c_p}{h_2} \frac{\partial T}{\partial y} + \frac{c_p}{h_3} \frac{\partial T}{\partial z} = \frac{\partial}{\partial y} \left( \lambda \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial y} \left[ \frac{\rho}{\rho} \left( v^2 - \frac{6}{2} \right) \frac{u^2 + v^2}{2} \right]
\]

Here - curvilinear orthogonal coordinates on the surface; \( h_1, h_2 \) - Lame coefficients; \( y \) - coordinate normal to the surface; \( u, v, w \) - projections of velocity vector.
on the coordinate axes $x, y, z$ \( \text{fig. 1} \); $p$-static pressure; $\rho$-density; $c_p$-specific heat at constant pressure; $\lambda$-coefficient of heat conduction; $\mu$-viscosity coefficient; $T_0$-braking temperature. Prandtl number $P = \mu c_p / \lambda$.

2. If any one line $z = \text{const}$ appears to be a geodetic line and the line of flow of an ideal fluid on the surface, then from condition

\[
\frac{\partial h}{\partial n} + \frac{\partial g}{\partial n} = 0
\]

and from equation (1.2) follows a trivial solution $w = 0$, $p / \rho = 0$.

In this way, everywhere along these lines, which we will call spreading lines, the components of the full velocity vector lie in one plane (but $\partial w / \partial z \neq 0$) just as at a two-dimensional flow. Branches of the flow outside of the boundary layer and within it diverge in both directions from the spreading line, and the boundary layer in the vicinity of these lines are calculated independently from the development of the boundary layer over the entire surface.

3. In many instances similarity transforms are possible, with the aid of which the system of boundary layer equations can be transformed into a system of ordinary differential equations. We shall discuss the most important case of similarity, when a supersonic flow is directed around bodies.

a) Conical flow. Directing the line $z = \text{const}$ along the formers of the cone, and lines $x = \text{const}$ orthogonal to them we will obtain an expression for the Lame coefficient

\[
h_1 = 1, \quad h_2 = \varphi(z) x
\]

for the round cone with angle of semiopening $\Theta_k$

\[
h_1 = R = \sin \Theta_k \quad (3.1)
\]

The spreading line correspond to the former of the cone; on it are maintained conditions:

\[
\frac{\partial p}{\partial n} = \frac{\partial g}{\partial n} = 0, \quad w = 0, \quad \frac{\partial w}{\partial z} = b
\]

Introducing variables

\[
T = \sqrt{\frac{u}{u^*} - \frac{1}{2} \frac{1}{c^*} \frac{d}{r^*} dy}, \quad f = \frac{u}{u^*}, \quad \varepsilon = \frac{v}{u^*}, \quad S = \frac{r^*}{r_m^*} - 1
\]

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we obtain from (1.1) - (1.4) a system of ordinary differential equations

\[ \frac{1}{3} (l'' + (l + K)') f' = 0 \quad \left( f = \frac{\varphi}{\varphi_0} \right) \]  
\[ \frac{1}{3} (l'' + (l + K)') g' - \frac{2}{3} l = K (g')^3 = - \frac{p_1}{p} (K + \frac{2}{3}) \]  
\[ \frac{1}{3} \left( \frac{1}{p} S' \right)' + (l - K) S' + \frac{n_1}{2 p} \left[ \frac{i}{p} (p - 1) (f')^3 \right] = 0 \]

The parameter K here characterizes the influence of three-dimensionality, index 1 corresponds to conditions outside of the layer and \( \bar{w} \) to the condition on the wall.

The ratio

\[ \frac{p_1}{p} = \frac{f}{l_1} = \omega [1 - (f')^2] + 3w (1 - f) (1 + \omega) \]  

where

\[ \omega = \frac{w_0}{2} \frac{n - 1}{2} \left( \frac{\varphi}{\varphi_0} \right)^2 \]

Boundary conditions:

\[ f' = f = g' = g = 0, \quad S = S, \quad \text{ps} \eta = 0 \]
\[ f' \rightarrow 1, \quad g' \rightarrow 1, \quad S \rightarrow 0 \quad \text{ps} \eta \rightarrow \infty \]  

b) Flow in the vicinity of a spreading line of a cylinder with slipping) Assuming \( \alpha = \) angle of slipping (angle of sweepback for a delta wing). Directing lines \( z = \text{const} \) along the formers of the cylinder, we obtain \( h_1 = h_2 = 1 \). We will designate the rate of slipping \( u_1 = a \) and will consider the case of flow \( v_1 = h_2 \) in the vicinity of the spreading line \( (v_1 \leq a, q_1 = \text{const}, T_1 \sim \text{constant}) \).

Having designated

\[ \eta = \sqrt{\frac{n + 1}{3} \frac{w_1}{w_0} \int_{g_0}^{g_1} d g} \]
\[ f' = \frac{a}{w_1}, \quad q' = \frac{a}{n}, \quad S = \frac{n_1}{w_0} - 1, \quad \beta = \frac{2a}{n + 1} \]

we obtain

\[ (l'' + q') + \beta = 0 \]  
\[ (l'' + q') + \beta = 2 \left[ (g')^3 - \frac{p_1}{p} \right] \]  
\[ \left( \frac{1}{p} S' \right)' + q + \beta = \frac{n}{2 p} \left[ \frac{i}{p} (p - 1) (f')^3 \right] = 0 \]
at boundary conditions (3.10).

The system (3.14) - (3.16) for the case $\beta = 1$ can be obtained directly from (3.12) - (3.14) during transition to variables

$$f' = f \sqrt{\frac{\partial}{\partial x}}, \quad g' = g \sqrt{\frac{\partial}{\partial y}}, \quad \eta' = \sqrt{\frac{\partial}{\partial \eta}}$$ (3.14a)

and with the value $K$ tending toward infinity.

For the case $\omega \rightarrow 0$ equations (3.12) - (3.14) coincide with equations of the boundary layer at a transverse flow around a cylinder.

In case of greater $\omega$ values the transformation of similarity is possible, if $P = 1, \mu \sim 1/T$ with the use of Stewartson variables [h]

c) Three dimensional flow in the vicinity of forward critical point. We shall discuss the development of a boundary layer from the critical point along the line of spreading. We will plot a system of coordinates so that $h_1 = h_2 = 1$, which is absolutely true, if the investigated section of the surface can be turned into a plane.

If outside of the boundary layer $\omega \approx 0$, then the transformation of the similarity is possible at a condition $u_1 = ax^m, v_1 = bx^{m-1}$ and by using variables

$$\eta = \sqrt{\frac{m+1}{2}} \int \frac{p}{u_1} dy, \quad f' = \frac{u}{u_1}, \quad g' = \frac{v}{v_1} \quad (3.15)$$

In this case we obtain

$$(f')' = \left[f + (2 - \beta) \frac{b}{a} g\right] f' = \beta \left[(f')^2 - \frac{\partial f}{\partial y}\right] \quad (3.16a)$$

$$\left(\frac{\partial f}{\partial y}\right)' = \left[f + (2 - \beta) \frac{b}{a} g\right] f' - 2(\beta - 1) f' g' - (2 - \beta) \frac{b}{a} (g')^2 - \left[(2 - \beta) \frac{b}{a} + 2(5 - 1)\right]$$

$$\left(\frac{\partial g}{\partial y}\right)' + \left[f + (2 - \beta) \frac{b}{a} g\right] g' + \beta = 0 \quad (3.16b)$$

Boundary conditions are determined (3.10). When $\beta \rightarrow 0$ the equations can be transformed into form (3.12) - (3.14) with the use of substitutions

$$\frac{b}{a} = 1 + \frac{3}{2} K, \quad \frac{b}{a} g = f + \frac{3}{2} K \frac{g}{a} \quad (3.19)$$

necessary for coordinating the selected coordinate systems on the surface. The case $b/a = 1$ corresponds to an axially-symmetrical flow, and equations (3.16) and (3.17) have a trivial solution $f = g$. The case $b/a = 0$ correspond to a plain flow, and
equations (3.16) - (3.18) convert into (3.12) - (3.14). When \( \beta \to 2 \) the influence of three-dimensionality disappears and the boundary layer develops so as in plain flow.

For approximated solutions it is convenient to use integral ratios, obtainable during the integration of the system (1.1)-(1.4) according to \( y \) from value \( y = 0 \) to the value corresponding to the boundary of the layer. The obtained equations have the form of

\[
\frac{\partial}{\partial x} \left( h_1 \rho_1 u_1 \Delta \theta_{xx} \right) + \frac{\partial}{\partial y} \left( h_1 \rho_1 u_1 \Delta \theta_{yy} \right) = \frac{\partial h_2}{\partial x} \left( \Delta \theta_{xx} + \Delta \theta_{yy} \right) - h_2 \Delta \theta_{yx} \quad (4.1)
\]

\[
\frac{\partial}{\partial x} \left( h_1 \rho_1 u_1 \Delta \theta_{xx} \right) + \frac{\partial}{\partial y} \left( h_1 \rho_1 u_1 \Delta \theta_{yy} \right) = \frac{\partial h_2}{\partial x} \left( \Delta \theta_{xx} + \Delta \theta_{yy} \right) - h_2 \Delta \theta_{yx} \quad (4.2)
\]

\[
\frac{\partial}{\partial x} \left( h_1 \rho_1 u_1 c_p \left( T_e - T_w \right) \Delta \theta_{xx} \right) + \frac{\partial}{\partial y} \left( h_1 \rho_1 u_1 c_p \left( T_e - T_w \right) \Delta \theta_{yy} \right) = h_2 \Delta \theta_{yx} \quad (4.3)
\]

Here \( q_w \) - specific thermal flow, \( \gamma_{wx} \) and \( \gamma_{wy} \) - stresses of component friction forces along the axes \( x \) and \( y \)

\[
\begin{align*}
\Delta \theta_{xx} &= \int_{\theta_{1x}}^{\theta_{2x}} \frac{q_w}{\rho_1 u_1} \left( 1 - \frac{u}{u_i} \right) dy, & \theta_{yx} &= \int_{\theta_{1y}}^{\theta_{2y}} \frac{q_w}{\rho_1 u_1} \left( 1 - \frac{u}{u_i} \right) dy \\
\Delta \theta_{yy} &= \int_{\theta_{1y}}^{\theta_{2y}} \frac{q_w}{\rho_1 u_1} \left( 1 - \frac{u}{u_i} \right) dy, & \theta_{yx} &= \int_{\theta_{1y}}^{\theta_{2y}} \frac{q_w}{\rho_1 u_1} \left( 1 - \frac{u}{u_i} \right) dy \\
\Delta \theta_{xx} &= \int_{\theta_{1x}}^{\theta_{2x}} \frac{\Delta \theta_{xx}}{\Delta \theta_{yy}} dy, & \theta_{yx} &= \int_{\theta_{1y}}^{\theta_{2y}} \frac{\Delta \theta_{xx}}{\Delta \theta_{yy}} dy \\
\Delta \theta_{yy} &= \int_{\theta_{1y}}^{\theta_{2y}} \frac{\Delta \theta_{yy}}{\Delta \theta_{xx}} dy, & \theta_{yx} &= \int_{\theta_{1y}}^{\theta_{2y}} \frac{\Delta \theta_{yy}}{\Delta \theta_{xx}} dy
\end{align*}
\]

The sought for velocity and temperature distributions will be determined as a function of variable \( Y \), representing the ratio of the variable of Downdytsa \( y^* \) to the parameter \( \beta \), proportional to the thickness of the boundary layer

\[
Y = \frac{Y}{\beta}, \quad y^* = \int_{\theta_{1x}}^{\theta_{2x}} \frac{2}{\rho_1 u_1} dy, \quad \beta = \int_{\theta_{1x}}^{\theta_{2x}} \frac{T_e - T_w}{T_e - T_w} \left( 1 - \frac{T_e - T_w}{T_e - T_w} \right) \frac{\rho}{\rho_1} dy \quad (4.5)
\]

Let us now discuss velocity and temperature distribution (in relative coordinates)

\[
\frac{T_e - T_w}{T_e - T_w} = F'_1(Y), \quad \frac{u}{u_1} = F'_1 + x_1 F'_1(Y), \quad \frac{w}{u_1} = F'_1(Y) + x_2 F'_2(Y)
\]

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The parameters $x_1$ and $x_2$ determining deformation of velocity profiles at a variable pressure from without the layer, should be found from three equations of the system (4.1) - (4.3). Function $F_0'$ corresponds to temperature distribution and velocity in gradientless flow ($x_1=x_2=0$) and can be found from the solution of Blasius equations.

The form of the function $F_1'$ is determined from the accurate solution of system (3.4) - (3.6) at $K = 0$ and $S_w = 0$, $P=1$, when the velocity distribution $w/w_1$ can be presented in form of

$$\frac{w}{w_1} = F_1'(Y) + (1 + \omega)F_1''(Y) \quad (4.6)$$

In this way, the adopted velocity and temperature distribution assures total conformity with accurate solutions at a flow along the lines of spreading of a cone, where deformation of the profile $w/w_1$ appears to be maximum. We shall designate:

$$\int_0^y (1 - F_0') dY = A, \quad \int_0^y F_1'(1 - F_0') dY = B, \quad \int_0^y F_0' dY = D \quad (4.7)$$

where for the given type of function $F_0$ and $F_1'$:

$$A = 2.61, \quad B = 1.163, \quad \bar{E} = 0.632, \quad H = 0.231, \quad G = 0.505 \quad (4.7a)$$

All conditions of layer thickness can be further expressed through

$$\begin{align*}
\phi_{xx} &= \Theta(1 + x_1G - x_1E - x_1H) \\
\phi_{yy} &= \Theta(1 + x_1G - x_1E - x_1H) \\
\phi_{rr} &= \Theta(1 + x_1G) \\
\phi_{zx} &= \Theta(1 + x_1G - x_1E - x_1H) \\
\phi_{xz} &= \Theta(1 + x_1G - x_1E - x_1H) \\
\phi_{yy} &= \Theta(1 + x_1G)
\end{align*} \quad (4.7b)$$

$$\delta_s = \delta_s' + \delta_s'' \quad (4.8)$$

The stresses of friction force components

$$\tau_{xx} = \frac{\rho u_1}{\Theta} (F_{xx} + x_1F_{xw}) \quad (4.9)$$
specific thermal flow

\[ q_w = \frac{\lambda_w (T_w - T_\infty)}{\varepsilon} F_{we}^2 \]  

At given functions \( F_0 \) and \( F_1 \) we have

\text{insert eq.(4.10,a) page 36}

5. Using the method, described in previous paragraph, we will obtain a numerical solution of equations (3.4)-(3.6), (3.12)-(3.14) and (3.16) - (3.19). We will consider the case \( P = 1; \mu \sim 1/T \).

a) Conical flow. The system of integral ratios has the form of:

\[ \eta_{xx} + K \eta_{xx} = \frac{1}{3} \frac{f'_w}{n_{xx}} (F_{xx}^2 + x_1 F_{xx}) \]  
\[ \eta_{xx} + 2K \eta_{xx} + \frac{2}{3} (\eta_w + \eta'_w) + K \eta_w = \frac{1}{3} \frac{s'_w}{n_{xx}} (F_{xx}^2 + x_1 F_{xx}) \]  
\[ \eta_{xx} + K \eta_w = -\frac{1}{3} \frac{s'_w}{n_{xx}} n_{xx} \]  

where \( \eta \) is connected with \( \delta \) by the ratio (3.11).

A comparison of (3.4) and (3.6) at \( P = 1 \) shows, that \( f'_w = S_{1w}/S_w \), and consequently

\[ x_1 = 0, \quad \eta_{xx} = \eta_{xx} = \eta_{xx} \]

In this way, to determine \( x_2 \) and \( \eta_{TT} \) remain equations (6.1) and (6.2).

Comparing (5.1) and (5.2) we obtain

\[ (\frac{5}{3} + K) \eta_{xx} + 2K \eta_{xx} + (\frac{2}{3} + K) \eta_{xx} - K \eta_{xx} = \left(1 + \frac{F_{xx}}{F_w}ight) \left(1 - \frac{\eta_{xx}}{n_{xx}} \right) \]  

Using the ratios (4.8) after transformations we obtain an equation for the determination of \( x_2 \)

\[ ax_2^2 + bx_2 + c(1 + w) = 0 \]  
\[ e = K \left( \frac{F_{xx}}{F_w} + 2T \right) \]  
\[ \tilde{s} = (K + 1) \frac{F_{xx}}{F_w} + (3K + \frac{5}{3}) \tilde{s} \]  
\[ \tilde{c} = -\left(K + \frac{1}{3}\right) \left(1 + A \frac{F_{xx}}{F_w} \right) \]

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It can be seen easily, that at $K = 0$ in conformity with the accurate solution

$$x_1 = (1 + \omega) \left( 5.8a \right)$$

The sought for functions:

$$\frac{x}{x_1} = \frac{T_0 - T_w}{x_1 - T} = F'_e(Y), \quad \frac{\omega}{x_1} = F'_e(Y) \cdot x_1 G'(Y), \quad \eta = \frac{Y}{\eta_w} \quad (5.9)$$

$$\eta_{\eta} = \frac{1}{3} \left[ \frac{F'_{\eta}}{1 + K(1 + x_1 G)} \right]^2 \quad (5.10)$$

The values of velocity and temperature derivatives on the wall, necessary to calculate heat exchange and friction, equal

$$\left( \frac{F'_e}{S} \right) = \frac{F'_{c\omega}}{\eta_w} = \frac{\sqrt{V^2}}{3.47 \left( 1 + K(1 + x_1 G) \right)^{\frac{1}{2}}} \quad (5.11)$$

$$\frac{S}{2} = \left( 1 + 1.29 x_1 \right) \quad (5.12)$$

b) Flow in the vicinity of spreading lines of a cylinder with slipping. Integral equations have the form

$$\eta_{\eta} = \frac{F'_{c\omega}}{\eta_w}, \quad \frac{3a + 1}{2} \quad \eta_{\eta} + \frac{n \eta_{\eta}'}{\eta_w} = \frac{n + 1}{2} \left[ F'_{c\omega} + x_2 G'_{c\omega} \right], \quad \eta_{\eta} = \frac{F'_{c\omega}}{\eta_w} \quad (5.13)$$

A comparison of (3.12) with (3.14) or (5.13) shows that

$$\eta_{\eta} = \eta_{\eta}, \quad \eta_{\eta} = \eta_{\eta} \quad (5.14)$$

After transformation we obtain

$$a_2 x_1^2 + b_1 x_1 + c_1 (1 + \omega) = 0, \quad a_1 = \left[ \frac{F'_{c\omega}}{x_1} \frac{C}{3a + 1} \right] \quad (5.15)$$

$$b_1 = \frac{F'_{c\omega}}{3a + 1} E, \quad c_1 = \frac{2a}{3a + 1} \left[ 1 + A T_{\eta} \right] \quad (5.16)$$

Having determined $x_2$, we obtain

$$\eta_{\eta} = \frac{1}{1 + \eta_{\eta}} \left[ \frac{F'_{c\omega}}{3a + 1} \right] \quad (5.16)$$

Velocity and temperature distributions are determined by (5.9), the derivatives at the wall by formula:

$$f' = \frac{F'_{c\omega}}{F'_{c\omega}} = \frac{\eta_{\eta}}{\eta_{\eta}} = 0.47 (1 + x_1 G)^{\frac{1}{2}}, \quad \frac{S}{2} = \left( 1 + 1.29 x_1 \right) \quad (5.17)$$

e) A three dimensional flow in the vicinity of critical point. Integral ratios have the form of:
\[
\frac{3m+1}{2} \eta \psi + \frac{b}{a} \eta_{\psi} - m \eta_{\psi} = \frac{n+1}{2} \eta \psi \left[ F_{\omega}^{\prime} + x_1 F_{\omega}^{\prime} \right] \quad (5.18)
\]
\[
\frac{3m-1}{2} \eta \psi + 2 \frac{b}{a} \eta_{\psi} + \frac{b}{a} \eta_{\psi} + (m-1) \eta_{\psi} = \frac{n+1}{2} \frac{b}{a} \left[ F_{\omega}^{\prime} + x_2 F_{\omega}^{\prime} \right] \quad (5.19)
\]
\[
\frac{m+1}{2} \eta_{\psi} + \frac{b}{a} \eta_{\psi} = \frac{m+1}{2} \eta \psi \quad (5.20)
\]

The relationship between \( x_1 \) and \( x_2 \) will be obtained from condition at \( \eta = 0 \).

From equations (3.16) and (3.17) we have:
\[
f' = \frac{\eta_{\psi}^{\prime}}{\eta_{\psi}} = - \frac{2m}{m+1} \psi \quad \text{and} \quad \eta_{\psi} = \frac{\eta_{\psi}^{\prime}}{\eta_{\psi}} = - \left( \frac{b}{1+m} + \frac{2(m+1)}{b} \right) \psi
\]

From expressions (5.21) we obtain:
\[
\frac{r_{\psi}}{\psi} = \frac{b/\psi + m - 1}{b}
\]

Comparing (5.18) and (5.20), we obtain:

See page 9a for equations (5.23), (5.24), (5.25), and (5.26).

When \( \theta \to 0 \), \( x_1 \to 0 \) to determine \( x_2 \) it is necessary to make a change \( x_1^o = x/b \); at \( x_1^o \to \infty \). \( \theta \to 2 \), \( x_1 \to 0 \), making a change \( x^o = x_1 / \beta \), we obtain an equation for \( x^o \). The value \( \eta_{TT} \) is determined from (5.20):
\[
\eta_{TT} = \left[ \frac{\nu}{(1+x_0 \beta) + x_1 (2-x \beta)(1+x_0 \beta)} \right]^{1/2}
\]

Velocity and temperature distribution is found by formula (5.9). Values of velocity and temperature distributions (derivatives) at the wall are equal:
\[
\frac{\partial u}{\partial \psi} = \frac{u_{\psi}^{\prime}}{\eta_{\psi}}, \quad f' = F_{\psi}^{\prime} + x_1 F_{\psi}^{\prime}, \quad g' = F_{\psi}^{\prime} + x_2 F_{\psi}^{\prime}
\]

Solutions of boundary layer equations obtained for the case \( \mu \sim T \) and \( \mu = 1 \) and during the use of these solutions to calculate heat exchange and friction it is necessary to introduce corrections. We will designate
\[
N_{\psi} = \frac{\psi}{\psi}, \quad R_{\psi} = \frac{\psi_{\omega} \psi}{\psi}, \quad C_{\psi} = \frac{\nu}{\eta_{TT}}
\]

where \( L \) and \( U \) characteristic length and velocity. Expression for the thermal flow has the form of
\[
\eta_{\psi} = \frac{N_{\psi}}{V_{\psi}} \sqrt{\frac{1}{\rho c_p U} \frac{\psi_{\omega} (T_\infty - T_{\psi})}{\rho}} \eta_{\psi} \quad \text{and} \quad T_\psi = \frac{T_{\infty}}{(1+x_1)(1+\omega_1)}
\]

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\[ a_0 x_1^3 + a_2 x_1 + c = 0 \] (5.23)

\[ c = H (\beta + 1) \beta + K_1 (2 \beta - 2) + (2 - \beta) A K_1 + \] 
\[ + \frac{F_{1w}}{F_{0w}} G (\beta + K_1 (2 \beta - 2) + (2 - \beta) A K_1) \] (5.24)

\[ \ell = \frac{F_{1w}}{F_{0w}} (\beta + K_1 (2 - \beta) \beta + \theta [\frac{4 \beta - 2}{2}] + \beta + K_1 (2 - \beta) \beta) \] (5.25)

\[ c = -\beta^2 \left( 1 + A \frac{T_{0w}}{T_{1w}} \right), \quad \beta = \frac{2m}{m + 1} \] (5.26)
Here \( \tau = \text{coefficient of temperature restoration.} \) Approximately \( \tau \sim p^{n_1} \), where \( n_1 \) changes from \( 1/2 \) to \( 1/3 \); \( k_1 = \text{correction multiple, taking into consideration the}\) variability \( 1 \) and \( k_2 = \text{multiple, considering the difference of} P \text{from one. Using the results of calculating on a flat plate with variable properties, we obtain approximately:} \)

\[
k_1 = \left( \frac{\mu_1 p}{\mu_2 p} \right)^{\frac{1}{3}} \left( \frac{k_1}{\mu_2 p} \right)^{\frac{1}{3}} \left( \frac{p}{T_1} \right)^{\frac{1}{3}} (14)
\]

where \( \mu_1 \) and \( \mu_2 \) have been determined at maximum temperature

\[
T' = \frac{T_2}{T_1} + \frac{1}{16} \left( \frac{T_2}{T_1} - \frac{p}{\mu_2} \right) \quad (6.26)
\]

The value \( k_2 = n^2 \), where \( n_2 \) changes from \( 1/3 \) on the flat plate to \( \approx 0.15 \) in three dimensional flow at greater \( x_2 \). We will assume that \( n \approx 0.4 \).

To calculate friction we have analogously:

\[
\tau_w = C/\varepsilon \sqrt{\frac{\mathcal{R}_w}{U}} \sqrt{\frac{p_2}{L}} k_1 \quad (6.3)
\]

a) Conical flow. In this case we have \( L = x_2, U = u_1 \) and

\[
\frac{N_2}{\sqrt{\mathcal{R}_w}} = \frac{1}{2} C/\varepsilon \sqrt{\frac{p_2}{U}} = \frac{1}{2} \frac{f_w}{f_w} \quad (6.4)
\]

The dependence \( \theta = f_w/(1+K) \) upon \( \omega \) at various \( K \) and \( T_w/T_0 \) is shown in fig. 2 and 3; when \( K = 0 \) the solution coincides with the accurate ; when \( K \to \infty \) to compare points are shown data on the calculation of the job [2].

Using formulas (5.17) it is easy to obtain basic characteristics of the boundary layer.

The angle between lines \( z = \text{const} \) and the lines \( \gamma \) flow of an ideal liquid outside of the layer \( \gamma = w_1/u_1 \) and the lines of flow at the wall \( \gamma = \gamma_w/\gamma_w \); hence

\[
\frac{\gamma_w}{\gamma} = \frac{f_w}{f_w} \quad (6.5)
\]

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This ratio characterizes the effect of secondary flows. As is shown by calculations, $\gamma_{1}/f_1$ rises at an increase in $\omega$ and decreases upon cooling the wall and increase in $K$. The expulsion thickness

$$\delta^* = \frac{\delta^*_{x0} + \delta^*_{x1}}{1 + K}$$  \hspace{1cm} (6.6)

Here the values $\gamma_{x0}$ and $\gamma_{x1}$ can be determined by formulas (4.8) and $x_2$ - from equations (5.5);

$$x_1 = 0, \quad \theta = \frac{p_c}{\rho_0} \eta_{TT} \sqrt{\frac{\gamma_{x1}}{w_1}} \quad (6.6)$$

the value $\eta_{TT}$ is determined by formula (5.10).

b) Flow in the vicinity of spreading lines of a cylinder $L = z$, $U = w_1$ with slipping. In this case

$$\frac{N_w}{\sqrt{\eta_{w}}} = \frac{1}{2} C_{xw} \frac{\sqrt{\eta_{w}}}{w_1} = \sqrt{\frac{a+1}{2}} f_1$$ \hspace{1cm} (6.7)

Next analogous to (6.5)

$$\frac{\tau_w}{\tau} = \frac{\tau_{w*}}{\tau_{w*}}, \quad \tau = \frac{w_1}{w_1}$$ \hspace{1cm} (6.8)

As in the case of flow on a cone, secondary flows rise at an increase in $\omega$ and $\beta$ and decreases upon wall cooling.
Calculation results are shown in fig. 4 and 5. Given there are also data from \([3]\) for \(\beta = 1, \omega = (0-10)\) and \(N_{0}/T_{01} = (0.1)\) and from report \([4]\) for \(\omega = 0\). The expansion thickness \(\delta^* = \delta^*_{x}^{*}\) is determined from equations (4.8), (5.15), (5.16).

c) Three dimensional flow in the vicinity of forward critical point. In this case

\[
L = x, \quad U = u, \quad b = K_1, \quad \frac{N_{0}}{V_{m}} = \sqrt{\frac{m+1}{2} \left(-\frac{S_{v}}{S_{u}}\right)} \quad (6.9)
\]

The results of calculating the dependence \((-S_{v}/S_{u})\) on \(K_1\) and \(\beta\) are shown in fig. 6 and 7. Plotted there are also data of calculations from report \([4]\) for \(N_{0}/T_{1} = 1\) and data of report \([2]\) for \(K_1=0\) and \(K_1=1\). The angle between the lines \(z=\text{const}\) and the direction of flow lines at the wall \(\gamma_{\omega}\) is determined by formula

\[
\frac{\gamma_{\omega}}{\gamma} = \frac{S_{v}}{S_{u}}, \quad \gamma = K_1 z \quad (6.10)
\]

As was shown by calculations, the effect of secondary flows decreases with the increase in \(\beta\) and disappears when \(\beta = 2\); the value \(\gamma_{\omega}/\gamma = 1\) at \(K_1=0\) and \(K_1=1\) and lower than unity, if \(0 \leq K \leq 1\). It should be pointed out, that in this interval there is a zone, in which there is no solution of the system (3.16) - (3.18). Expansion thickness

\[
\delta^* = \frac{\delta_{x_{0}}^{*} + K_1 \delta_{y}^{*}}{1 + K_1} \quad (6.11)
\]

The values \(\delta_{x_{0}}^{*}\) and \(\delta_{y}^{*}\) are determined by formulas (4.8), (5.23), (5.27).

7. With the aid of the proposed method it is easy to obtain a solution of equations

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of a two- and three-dimensional boundary layer in a wide range of parameter changes.

As is evident from comparing with available numerical solutions, the accuracy of the method is perfectly sufficient for practical purposes.

In the report are given graphs for $T_w/T_{01} = 0$ and $T_w/T_{01} = 1$. Values for $0 < T_w/T_{01} < 1$ will be obtained with high degree by linear interpolation.

To calculate heat exchange under flight conditions it is necessary either to compute the distributions of parameters of an ideal liquid on the surface of a body, or use experimental data.

Submitted Sep. 14, 1961

Literature


2. Cohen C., Reshotko E. Similar solutions for the compressible laminar boundary layer with heat transfer and pressure gradient. NACA Report, 1956, 1293.


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