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ZERO-CROSSING INTERVALS OF RANDOM PROCESSES

by

A. J. Rainal

THE JOHNS HOPKINS UNIVERSITY
CARLYLE BARTON LABORATORY
BALTIMORE, MD.

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THE JOHNS HOPKINS UNIVERSITY
CARLYLE BARTON LABORATORY
BALTIMORE, MARYLAND

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A. J. Rainal

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TABLE OF CONTENTS

		Page
TABLE OF CONTENTS		
LIST OF ILLUSTRATIONS		
ACKNOWLEDGMENT		
ABSTRACT		
I.	INTRODUCTION	19
II.	THE MEASURING SYSTEM	24
	A. METHOD OF MEASUREMENT . . .	24
	B. SAMPLE SIZE	26
III.	THE SYNTHESIS SYSTEM	30
	A. SHAPING THE POWER SPECTRA .	30
	B. THE "WHITE" NOISE SOURCE . .	31
	C. POWER SPECTRA CONSIDERED . .	39
	D. RANDOM POINT-PROCESS TRANSFORMATIONS	45
IV.	MATHEMATICAL DEDUCTIONS FROM THE PROBABILITY DENSITIES	51
V.	PROPERTIES OF THE PROBABILITY DENSITIES	55
	A. EXPECTATIONS	55
	B. STANDARD DEVIATIONS AND CORRELATION COEFFICIENTS . .	61
VI.	PROBABILITIES AND PROBABILITY DENSITIES DEFINED BY THE ZERO POINTS OF GAUSSIAN PROCESSES . . .	68
	A. POWER SPECTRAL DENSITY $W_1(f)$.	68
	B. POWER SPECTRAL DENSITY $W_2(f)$.	74
	C. POWER SPECTRAL DENSITY $W_3(f)$.	75
	D. POWER SPECTRAL DENSITY $W_4(f)$.	84
VII.	PROBABILITIES AND PROBABILITY DENSITIES DEFINED BY THE ZERO POINTS OF A SINE WAVE PLUS A GAUSSIAN PROCESS .	91

Page

VIII. PROBABILITIES AND PROBABILITY DENSITIES DEFINED BY THE STATIONARY POINTS OF A SINE WAVE PLUS A GAUSSIAN PROCESS 110

IX. STATISTICAL DEPENDENCE OF ZERO-CROSSING INTERVALS 128

X. APPLICATIONS 134

A. THE MOST RANDOM DISTRIBUTION OF SUCCESSIVE ZERO-CROSSING INTERVALS 134

B. TECHNIQUE FOR ESTIMATING THE ASYMPTOTIC EXPONENTIAL BEHAVIOR OF $P_o(\tau)$ 138

XI. CONCLUSIONS 140

XII APPENDIX 142

REFERENCES 145

DISTRIBUTION

LIST OF ILLUSTRATIONS

Figure 1	Digital Measurement of Independent Samples of Zero-Crossing Intervals
Figure 2	Histogram Approximation for $P_o(u)$
Figure 3	Synthesis of Gaussian Noise with Prescribed Power Spectra
Figure 4	Periodic Pulse Sampling of a Noise Voltage y(t)
Figure 5	The Power Spectral Density for a Periodic Pulse Train Having Statistically Independent Pulse Amplitudes, Pulse Duration τ, and Period T_o
Figure 6	The Linear Filter
Figure 7	Randomly Weighted Filter Impulse Responses
Figure 8	The Measured Amplitude Distribution p(v) Compared with Points Representing a Gaussian Distribution
Figure 9	The Synthesis of the System Functions $H_n(j2\pi f)$ for n = 0, 1, 2
Figure 10	The Synthesis of the System Functions $H_n(j2\pi f)$ for n = 3, 4
Figure 11	For the System Function $H_4(j2\pi f)$, the Experimental Values of $\left\|\frac{e_o}{e_1}\right\|$ are Compared with the Gaussian Function $\sqrt{W_4(f)}$
Figure 12	The Linear System
Figure 13	Transformation of Stationary Points to Zero Points
Figure 14	The Synthesis of $D(j2\pi f)$

Figure 15 The Stationary Points of x(t) and the Zero Points of y(t) Occur Simultaneously

Figure 16 Block Diagram of the Synthesis and Measuring Systems.

Figure 17 Point Sets Defined by the Zero-Crossing Points and the Stationary Points of a Gaussian Process

Figure 18 $P_0(u_1)$ is the Probability Density Function for Successive Zero-Crossing Intervals of Gaussian Noise having a Power Spectral Density $W_1(f)$. Theoretical Approximations are Compared with Experimental Results.

Figure 19 $P_1(u_1)$ is the Probability Density Function for the Sum of Two Successive Zero-Crossing Intervals of Gaussian Noise Having a Power Spectral Density $W_1(f)$. $P_0(u_1)*P_0(u_1)$ is Compared with the Experimental Result to Exhibit the Statistical Dependence of Two Successive Intervals.

Figure 20 $p(n, u_1)$ is the Probability that a Given Interval u_1 contains exactly n Zero-Crossing Points. $p(n, u_1)$ and White's $W_+(n, u_1)$ Probabilities are Compared for the Case of Gaussian Noise Having a Power Spectral Density $W_1(f)$ when n = 0, 1

Figure 21 $Z(n, u_1)$ is the Conditional Probability that the nth Zero-Crossing Point from a Given Zero-Crossing Point in du_1 occurs after the Time $du_1 + u_1$. $Z(1, u_1)$ and $Z(2, u_1)$ are Compared for the Case of Gaussian Noise Having a Power Spectral Density $W_1(f)$.

Figure 22	$P_o(u_2)$ is the Probability Density Function for Successive Zero-Crossing Intervals of Gaussian Noise having a Power Spectral Density $W_2(f)$. Theoretical Approximations are Compared with Experimental Results.
Figure 23	$P_1(u_2)$ is the Probability Density Function for the Sum of Two Successive Zero-Crossing Intervals of Gaussian Noise Having a Power Spectral Density $W_2(f)$. $P_o(u_2)*P_o(u_2)$ is Compared with the Experimental Result to Exhibit the Practical Independence of Two Successive Intervals.
Figure 24	$p(n, u_2)$ is the Probability that a Given Interval u_2 Contains Exactly n Zero-Crossing Points. $p(0, u_2)$ and $p(1, u_2)$ are Compared for the Case of Gaussian Noise Having a Power Spectral Density $W_2(f)$
Figure 25	$Z(n, u_2)$ is the Conditional Probability that the nth Zero-Crossing Point from a Given Zero-Crossing Point in du_2 Occurs after the Time du_2+u_2. $Z(1, u_2)$ and $Z(2, u_2)$ are compared for the Case of Gaussian Noise Having a Power Spectral Density $W_2(f)$
Figure 26	$P_o(u_3)$ is the Probability Density Function for Successive Zero-Crossing Intervals of Gaussian Noise Having a Power Spectral Density $W_3(f)$. Theoretical Approximations are Compared with Experimental Results
Figure 27	$P_1(u_3)$ is the Probability Density Function for the Sum of Two Successive Zero-Crossing Intervals of Gaussian Noise Having a Power Spectral Density $W_3(f)$. $P_o(u_3)*P_o(u_3)$ is Compared With the Experimental Result to Exhibit the Statistical Dependence of Two Successive Intervals

Figure 28 $p(n, u_3)$ is the Probability that a Given Interval u_3 Contains Exactly n Zero-Crossing Points. $p(0, u_3)$ and $p(1, u_3)$ are Compared for the Case of Gaussian Noise Having a Power Spectral Density $W_3(f)$

Figure 29 $Z(n, u_3)$ is the Conditional Probability that the nth Zero-Crossing Point From a Given Zero-Crossing Point in du_3 Occurs After the Time du_3+u_3. $Z(1, u_3)$ and $Z(2, u_3)$ are Compared for the Case of Gaussian Noise Having a Power Spectral Density $W_3(f)$

Figure 30 $P_o(u_4)$ is the Probability Density Function for Successive Zero-Crossing Intervals of Gaussian Noise Having a Power Spectral Density $W_4(f)$. Theoretical Approximations are Compared with Experimental Results

Figure 31 $P_1(u_4)$ is the Probability Density Function for the Sum of Two Successive Zero-Crossing Intervals of Gaussian Noise Having a Power Spectral Density $W_4(f)$. $Q(u_4)*Q(u_4)$ is Compared with $U(u_4)-Q(u_4)$ to Exhibit the Statistical Dependence of Two Successive Intervals

Figure 32 $p(n, u_4)$ is the Probability That a Given Interval u_4 Contains Exactly n Zero-Crossing Points. $p(0, u_4)$ and $p(1, u_4)$ are Compared for the Case of Gaussian Noise Having the Power Spectral Density $W_4(f)$

Figure 33 $Z(n, u_4)$ is the Conditional Probability that the nth Zero-Crossing Point From a Given Zero-Crossing Point in du_4 Occurs After the Time du_4+u_4. $Z(1, u_4)$ and $Z(2, u_4)$ are Compared for the Case of Gaussian Noise Having a Power Spectral Density $W_4(f)$

Figure 34 $P_0(u_0, a)$ is the Probability Density Function for Successive Zero Crossing Intervals of a Random Process Consisting of Sine Wave of Frequency $f_0/2$ Plus Gaussian Noise Having the Power Spectral Density $W_0(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a. Theoretical Approximations are Compared with Experimental Results.

Figure 35 $P_1(u_0, a)$ is the Probability Density Function for the Sum of Two Successive Zero-Crossing Intervals of a Random Process Consisting of a Sine Wave of Frequency $f_0/2$ Plus Gaussian Noise Having the Power Spectral Density $W_0(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

Figure 36 $p_a(n, u_0)$ is the Probability that a Given Interval u_0 Contains Exactly n Zero-Crossing Points of a Random Process Consisting of a Sine Wave of Frequency $f_0/2$ Plus Gaussian Noise Having the Power Spectral Density $W_0(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

Figure 37 A Random Process Consisting of a Sine Wave of Frequency $f_0/2$ Plus Gaussian Noise Having the Power Spectral Density $W_0(f)$ Defines $Z_a(n, u_0)$, the Conditional Probability That the nth Zero-Crossing Point From a Given Zero-Crossing Point in du_0 Occurs After the time du_0+u_0. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

Figure 38 $P_o(u_o, a)$ is the Probability Density Function for Successive Zero-Crossing Intervals of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a. Theoretical Approximations are Compared With Experimental Results.

Figure 39 $P_1(u_o, a)$ is the Probability Density Function for the Sum of Two Successive Zero-Crossing Intervals of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

Figure 40 $p_a(n, u_o)$ is the Probability that a Given Interval u_o Contains Exactly n Zero-Crossing Points of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

Figure 41 A Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$ Defines $Z_a(n, u_o)$, the Conditional Probability that the nth Zero-Crossing Point From a Given Zero-Crossing Point in du_o Occurs After the Time du_o+u_o. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

Figure 42 $P_o(u_o, a)$ is the Probability Density Function for Successive Zero-Crossing Intervals of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a. Theoretical Approximations are Compared with Experimental Results.

Figure 43 $P_1(u_o, a)$ is the Probability Density Function for the Sum of Two Successive Zero-Crossing Intervals of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to The Average Noise Power is Denoted by the Parameter a.

Figure 44 $p_a(n, u_o)$ is the Probability that a Given Interval u_o Contains Exactly n Zero-Crossing Points of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

Figure 45 A Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$ Defines $Z_a(n, u_o)$, the Conditional Probability That the nth Zero-Crossing Point from a Given Zero- Crossing Point in du_o Occurs After the Time du_o+u_o. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

Figure 46 $P_o(u_o, a)$ is the Probability Density Function for Successive Zero-Crossing Intervals of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_0(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a. Theoretical Approximations are Compared with Experimental Results.

Figure 47 $P_1(u_o, a)$ is the Probability Density Function for the Sum of Two Successive Zero-Crossing Intervals of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

Figure 48 $p_a(n, u_o)$ is the Probability that a Given Interval u_o Contains Exactly n Zero-Crossing Points of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

Figure 49 A Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$ Defines $Z_a(n, u_o)$, the Conditional Probability that the nth Zero-Crossing Point from a Given Zero-Crossing Point in du_o Occurs After the time du_o+u_o. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

Figure 50 $M_o(u_o, a)$ is the Probability Density Function for Successive Intervals Defined by Adjacent Stationary Points of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a. Theoretical Approximations are Compared with Experimental Results.

Figure 51 $M_1(u_o, a)$ is the Probability Density Function for the Sum of Two Successive Intervals Defined by Adjacent Stationary Points of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

Figure 52 $m_a(n, u_o)$ is the Probability that a Given Interval u_o Contains Exactly n Stationary Points of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

Figure 53 A Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$ Defines $S_a(n, u_o)$, the Conditional Probability that the nth Stationary Point from a Given Stationary Point in du_o occurs after the Time du_o+u_o. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

Figure 54 $M_o(u_o, a)$ is the Probability Density Function for Successive Intervals Defined by Adjacent Stationary Points of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of The Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a. Theoretical Approximations are Compared with Experimental Results.

Figure 55 $M_1(u_o, a)$ is the Probability Density Function for the Sum of Two Successive Intervals Defined by Adjacent Stationary Points of a Random Process Consisting of A Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

Figure 56 $m_a(n, u_o)$ is the Probability That a Given Interval u_o Contains Exactly n Stationary Points of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

Figure 57 A Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$ Defines $S_a(n, u_o)$, the Conditional Probability that the nth Stationary Point from a Given Stationary Point in du_o Occurs after the Time $du_o + u_o$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

Figure 58 $M_o(u_o, a)$ is the Probability Density Function for Successive Intervals Defined by Adjacent Stationary Points of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a. Theoretical Approximations are Compared with Experimental Results.

Figure 59 $M_1(u_o, a)$ is the Probability Density Function for the Sum of Two Successive Intervals Defined by Adjacent Stationary Points of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

Figure 60 $m_a(n, u_o)$ is the Probability that a Given Interval u_o Contains Exactly n Stationary Points of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by The Parameter a.

Figure 61 A Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$ Defines $S_a(n, u_o)$, the Conditional Probability that the nth Stationary Point from a Given Stationary Point in du_o Occurs after the Time du_o+u_o. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

Figure 62 $M_o(u_o, a)$ is the Probability Density Function for Successive Intervals Defined by Adjacent Stationary Points of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a. Theoretical Approximations are Compared with Experimental Results.

Figure 63 $M_1(u_o, a)$ is the Probability Density Function for the Sum of Two Successive Intervals Defined by Adjacent Stationary Points of A Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

Figure 64 $m_a(n, u_o)$ is the Probability that a Given Interval u_o Contains Exactly n Stationary Points of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

Figure 65 A Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$ Defines $S_a(n, u_o)$. The Conditional Probability that the nth Stationary Point from a Given Stationary Point in du_o Occurs After the Time du_o+u_o. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

Figure 66 Each Intensity Pattern Represents an Approximation to the Joint Probability Density of the ith and (i+n)th Zero-Crossing Intervals of a Gaussian Process Having Power Spectral Density $W_3(f)$.

Figure 67 The Truncated Gaussian Probability Density, $P_0(\tau)$, Represents the Most Random Distribution of Successive Zero-Crossing Intervals Having a Given Mean and a Given Standard Deviation.

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ABSTRACT

This report describes a digital system for studying the zero-crossing intervals of random processes. Probabilities and probability densities defined by the zero-crossing points of various Gaussian processes are presented. Probabilities and probability densities defined by the zero-crossing points of a random process consisting of a sine wave plus a Gaussian process are also presented. Finally, probabilities and probability densities defined by the stationary points of a random process consisting of a sine wave plus a Gaussian process are presented. At present none of the probabilities or probability densities can be derived explicitly by analytical methods. The standard deviations associated with the probability densities are also presented. In the case of the Gaussian processes the correlation coefficients for two successive intervals are presented. The first moments associated with the probability densities are compared with the exact theoretical values. All the other experimental results are compared with theoretical approximations. The statistical dependence between the ith zero-crossing interval and the (i+n)th zero-crossing interval is investigated.

I. INTRODUCTION

In many branches of science and technology one encounters the basic density $P_n(\tau)$, the probability density of the interval τ between the mth and (m+n+1)th zero-crossing points of a random process. This density occurs for example in the fields of statistical communication theory, oceanography, statistical mechanics, biophysics, and control engineering. When n = 0 the interval τ is called a zero-crossing interval. A striking experimental fact is that the successive zero-crossing intervals of an audio process contain a great deal of the intelligibility. For the most common random process, a Gaussian process with arbitrary power spectrum, $P_n(\tau)$ cannot be derived explicitly by analytical methods. For a Gaussian process having a certain class of power spectra, S. O. Rice (1) derived two functions which can be used to approximate $P_0(\tau)$ and $P_1(\tau)$ for small τ. More recently J. A. McFadden (2) and M. S. Longuet-Higgins (8) derived approximations for $P_0(\tau)$ which compare favorably with experimental results for a larger class of Gaussian processes having arbitrary power spectra.

Some other important theoretical work concerned with $P_n(\tau)$ has been reported by D. Middleton (3), A. Kohlenberg (4), Kuznetsov, Stratonovich, and Tikhonov (5), V. I. Tikhonov and I. N. Amiantov (6), M. S. Longuet-Higgins (7), C. W. Helstrom (9), H. Steinberg, P. M. Schultheiss, C. A. Wogrin, and F. Zweig (10), D. S. Palmer (11), H. Debart (12), D. Slepian (13), W. M. Brown (14), J. S. Bendat

(15), M. Kac (16), I. Miller and J. E. Freund (17), H. Zuhrt (29), S. Ehrenfeld (30) et al., and L. I. Bialyi (31). Fewer analytical results concerning $P_0(\tau)$, $P_1(\tau)$ exist for such a common non-Gaussian random process as a sine wave signal plus a Gaussian process. In short, one must conclude that mathematical difficulties seriously limit the understanding of this statistical phenomenon. Under such circumstances one is naturally motivated to turn to experiment for further insight.

Most of the experimental work dealing with this problem has been reported by G. M. White (18), Favreau, Low, and Pfeffer (19), C. R. Gates (20), Kjell Blötekjaer (21), and A. I. Velichkin and V. D. Ponomareva (22). In order to explore the problem further an experimental system has been designed at this Laboratory. The system differs considerably from previous experimental systems, and the measurement is digital rather than analog. This approach permits the system to work in conjunction with a digital computer to help analyze the recorded data. Also, the inherent stability of a digital system allows measurements to be made over periods of hours or days if required.

This report presents a description of the experimental system and the following results concerning the zero-crossing intervals of certain Gaussian processes:

(1) $P_0(\tau)$, the probability density function for successive zero-crossing intervals.

(2) $P_1(\tau)$, the probability density function for the sum of two successive zero-crossing intervals.

(3) $p(0, \tau)$, the probability that a given interval τ contains exactly zero zero-crossings.

(4) $p(1, \tau)$, the probability that a given interval τ contains exactly one zero-crossing.

(5) $Z(1, \tau)$, the conditional probability that the first zero-crossing from a given zero-crossing in $d\tau$ occurs after the time $d\tau + \tau$.

(6) $Z(2, \tau)$, the conditional probability that the second zero-crossing from a given zero-crossing in $d\tau$ occurs after the time $d\tau + \tau$.

(7) The expectations and standard deviations associated with $P_0(\tau)$ and $P_1(\tau)$.

(8) The correlation coefficient for two successive zero-crossing intervals.

This report also presents the following results concerning the zero-crossing intervals of a random process consisting of a sine wave signal of frequency $\frac{f_o}{2}$ plus Gaussian noise such that the signal-to-noise power ratio equals a:

(1) $P_o(\tau, a)$, the probability density function for successive zero-crossing intervals.

(2) $P_1(\tau, a)$, the probability density function for the sum of two successive zero-crossing intervals.

(3) $p_a(0, \tau)$, the probability that a given interval τ contains exactly zero zero-crossings.

(4) $p_a(1, \tau)$, the probability that a given interval τ contains exactly one zero-crossing.

(5) $Z_a(1, \tau)$, the conditional probability that the first zero-crossing from a given zero-crossing in $d\tau$ occurs after the time $d\tau + \tau$.

(6) $Z_a(2, \tau)$, the conditional probability that the second zero-crossing from a given zero-crossing in $d\tau$ occurs after the time $d\tau + \tau$.

(7) The expectations and standard deviations associated with $P_o(\tau, a)$ and $P_1(\tau, a)$.

Finally, this report presents the following results concerning the intervals defined by the mathematical stationary points of a random process consisting of a sine wave signal of frequency $\frac{f_o}{2}$ plus Gaussian noise such that the signal-to-noise power ratio equals a:

(1) $M_o(\tau, a)$, the probability density function for successive intervals defined by adjacent stationary points.

(2) $M_1(\tau, a)$, the probability density function for the sum of two successive intervals defined by adjacent stationary points.

(3) $m_a(0, \tau)$, the probability that a given interval τ contains exactly zero stationary points.

(4) $m_a(1, \tau)$, the probability that a given interval τ contains exactly one stationary point.

(5) $S_a(1, \tau)$, the conditional probability that the first stationary point from a given stationary point in $d\tau$ occurs after the time $d\tau + \tau$.

(6) $S_a(2, \tau)$, the conditional probability that the second stationary point from a given stationary point in $d\tau$ occurs after the time $d\tau + \tau$.

(7) The expectations and standard deviations associated with $M_o(\tau, a)$ and $M_1(\tau, a)$.

The above probabilities and probability densities were selected for investigation not only because they are of theoretical interest but also because they are of practical interest, since they are observables which are comparatively simple to measure.

The system measures the basic densities $P_o(\tau)$, $P_1(\tau)$, $P_o(\tau, a)$, $P_1(\tau, a)$, $M_o(\tau, a)$ and $M_1(\tau, a)$. The remaining probabilities are deduced by applying the theory of point-processes. The system can measure these densities at levels other than the zero level.

II. THE MEASURING SYSTEM

A. METHOD OF MEASUREMENT

Instead of measuring successive zero-crossing intervals the system measures samples of zero-crossing intervals which are sufficiently independent. However, the random processes under investigation are metrically transitive (23) or ergodic, and hence the statistics of these samples are equal with probability unity to the statistics of successive zero-crossing intervals.

The method of measuring independent samples of zero-crossing intervals is illustrated in Figure 1. The method consists of opening and closing an electronic gate at the proper times following each initiate command. The gate controls the number of clock pulses counted by an electronic counter. The resulting count in the electronic counter is then a measure of the sampled zero-crossing interval. The count is printed on tape in order to monitor each sampled time interval and to check the average zero-crossing interval. The count is also punched on paper tape which feeds into an IBM-7090 digital computer. The circuit details have been reported in the literature (24).

Selecting a zero-crossing interval that corresponds in time with the instant of an initiate command tends to favor the long intervals. Since in general successive intervals are statistically

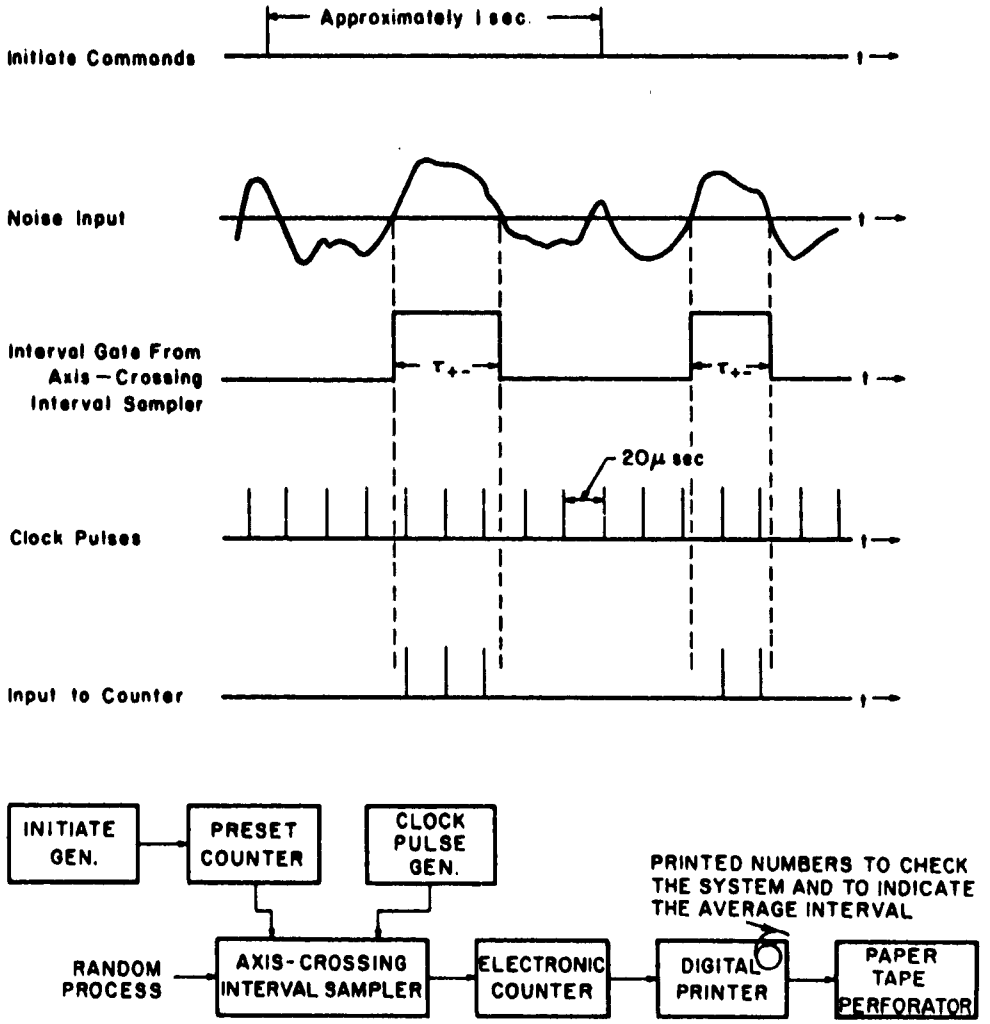


Figure 1 Digital Measurement of Independent Samples of Zero-Crossing Intervals

dependent, this bias may also bias the selection of an interval that occurs after an initiate command. Accordingly, the digital system skips, with equal probability, one or two intervals immediately following the initiate command and then measures the next interval. By skipping these intervals the bias associated with selecting an interval after an initiate command proves to be negligible. The significance of this bias has been discussed recently by McFadden (32). The system has provisions for skipping an arbitrary number of intervals.

B. SAMPLE SIZE

Consider a histogram approximation to a probability density $P_o(u)$ as shown in Figure 2. After measuring n mutually independent samples of time intervals let the number of time intervals in the range u, u + Δu be denoted by S_n and let

$$p = \int_u^{u+\Delta u} P_o(u)\, du \quad . \tag{1}$$

Let the random variable X_k equal one if the result of the kth measurement is in the range u, u + Δu and zero otherwise. Also, let all the X_k have a common probability density with mean p and variance σ^2. Then

$$S_n = \sum_{k=1}^{n} X_k \tag{2}$$

and by using the Tchebycheff inequality

$$\Pr\left\{ |S_n - np| \geq t \right\} \leq \frac{n\sigma^2}{t^2} \quad \text{for all } t > 0 \tag{3}$$

where $\Pr\{E\}$ denotes the probability of the event E. For $t = \epsilon n$

$$\Pr\left\{ \left| \frac{S_n}{n} - p \right| \geq \epsilon \right\} \leq \frac{\sigma^2}{n\epsilon^2} \quad . \tag{4}$$

Accordingly, as $n \to \infty$, $\frac{S_n}{n} \to p$ with probability 1. This is Bernoulli's (1713) celebrated weak law of large numbers which played a central role in the notion of probability. Cantelli's (1917) celebrated strong law of large numbers not only implies the weak law of large numbers but also asserts that with probability one only finitely many of the events

$$\left| \frac{S_n}{n} - p \right| > \epsilon \quad \text{occur as } n \to \infty.$$

In short, the weak law asserts convergence in probability; whereas the strong law asserts convergence with probability one.

In experimental work n is limited by practical considerations. Hence, one needs a quantitative measure of the manner in which $\frac{S_n}{n}$ approaches p with increasing n. In deriving such a measure notice that we are dealing with repeated independent trials having only two possible outcomes. Accordingly, we will adopt a Bernoulli model and consider each trial as a Bernoulli trial. Using the DeMoivre limit Theorem (1718) and letting $q = 1-p$ we have that as $n \to \infty$,

$$\Pr\left\{-x \le \frac{S_n - np}{\sqrt{npq}} \le x\right\} = \Pr\left\{-x\sqrt{\frac{pq}{n}} \le \frac{S_n}{n} - p \le x\sqrt{\frac{pq}{n}}\right\} \to \Phi(x) - \Phi(-x) = 2\,\Phi(x) - 1 \tag{5}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy \quad .$$

Let us require that $\frac{S_n}{n}$ approach p to within a given arbitrary amount during 95% of our statistical experience. That is, let our confidence level be 0.95. Then x = 1.96, and ΔP_o as shown in Figure 2 is given by

$$\Delta P_o \Delta u = \left| \frac{S_n}{n} - p \right| \le 1.96 \sqrt{\frac{pq}{n}} \tag{6}$$

and

$$\Delta P_o \le \frac{1.96}{\Delta u} \sqrt{\frac{pq}{n}} \quad . \tag{7}$$

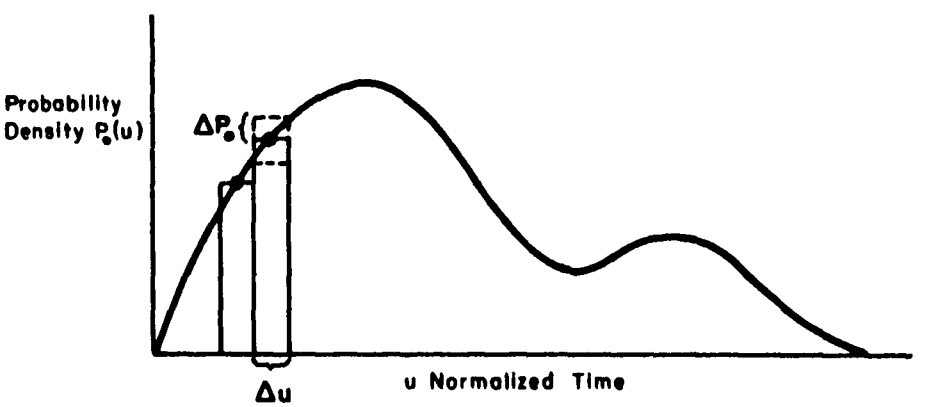


Figure 2 Histogram Approximation for $P_o(u)$.

This is the basic result concerning the sample size. It was used to compute the 95% confidence limits presented with the probability density functions. Using this basic result we choose n to be approximately 40,000.

III. THE SYNTHESIS SYSTEM

A. SHAPING THE POWER SPECTRUM

The method used to synthesize a Gaussian random process with prescribed power spectral density is illustrated in Figure 3. "White" Gaussian noise is applied to a linear network having a system function $H(j2\pi f)$. The output of such a network is a Gaussian random process having a power spectral density proportional to $|H(j2\pi f)|^2$. Operational amplifiers with appropriate feedback networks were used to synthesize $H(j2\pi f)$.

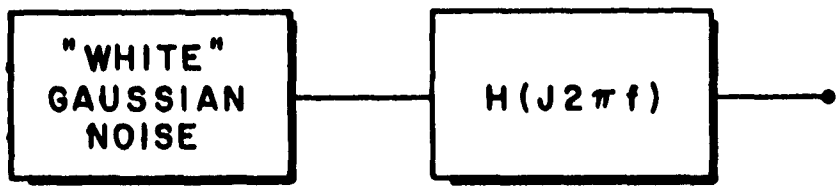


Figure 3 Synthesis of Gaussian Noise with Prescribed Power Spectra.

In order to investigate the zero-crossing intervals of Gaussian processes having certain power spectral densities of interest, the digital system requires a "white" Gaussian noise source having a low frequency cut-off of approximately 1 cps and a high frequency cut-off of at least 8 Kcps. The theory and design of such a noise source has been reported in the literature (25). The theory is described below.

B. THE "WHITE" NOISE SOURCE

Consider a train of periodic pulses $u_n(t-nT_o)$ with fixed duration τ and with amplitudes determined by the corresponding amplitudes of an ergodic random process. This random pulse train may be the result of periodically sampling the amplitude of a noise voltage y(t) as is shown in Figure 4. Generalizing Middleton's (3a) result, the power spectral density of the random pulse train is

$$W(f) = \overline{W_u(f)} \left[\sum_{k=-\infty}^{\infty} K_y(kT_o)\, e^{i\omega k T_o} + \frac{\bar{y}^2}{T_o} \sum_{m=o}^{\infty} \delta\left(f - \frac{m}{T_o}\right) \right] \tag{8}$$

where

$\overline{W_u(f)}$ = average power spectral density of the individual pulses

$K_y(t_1-t_2) = \overline{(y_1-\bar{y}_1)(y_2-\bar{y}_2)}$ = noise autovariance function

$\bar{y}$ = average value of the pulse amplitudes

T_o = sampling period

δ = Dirac delta function.

If T_o is large compared with the noise fluctuation time -- roughly the mean spacing between successive zero-crossings of the noise --, then

$$W(f) = \overline{W_u(f)} \left[(\overline{y^2} - \bar{y}^2) + \frac{\bar{y}^2}{T_o} \sum_{m=o}^{\infty} \delta\left(f - \frac{m}{T_o}\right) \right]. \tag{9}$$

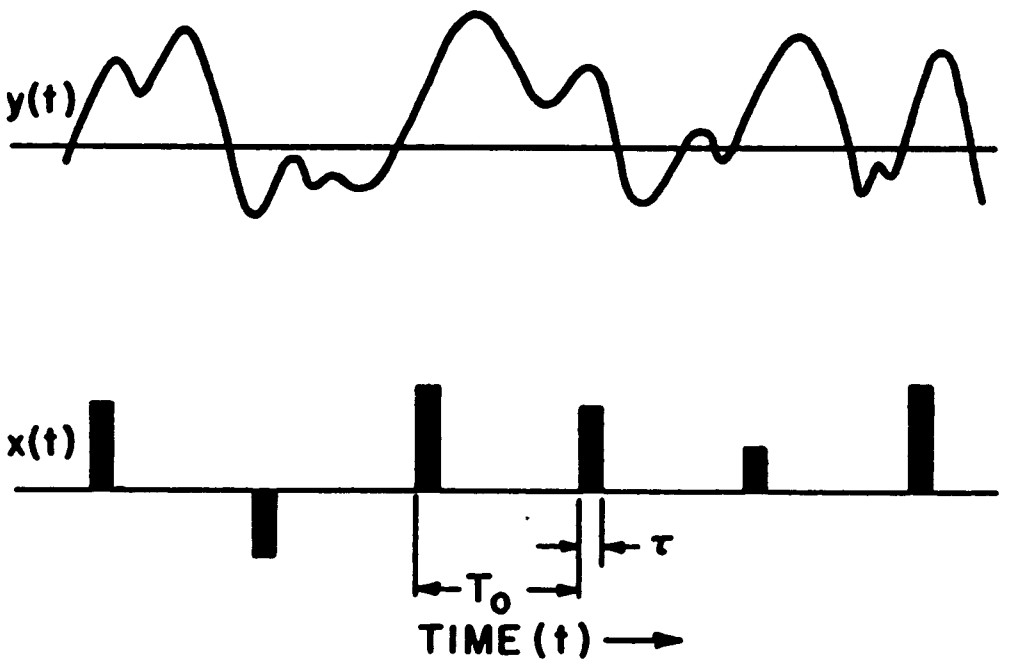


Figure 4 Periodic Pulse Sampling of a Noise Voltage y(t)

The spectrum consists of a continuous part whose distribution is determined solely by the shapes of the individual pulses and whose intensity is proportional to the variance of the pulse amplitudes. In addition there is a discrete spectrum with components at multiples of the sampling rate $1/T_o$, whose intensity is proportional to the square of the average pulse amplitude. A typical W(f) is shown in Figure 5. If $\overline{y} = 0$, then spikes in W(f) vanish. For the white noise source used in the digital system the frequency region from near zero to a little less than $1/T_o$ cps was extracted by a passive linear low-pass filter.

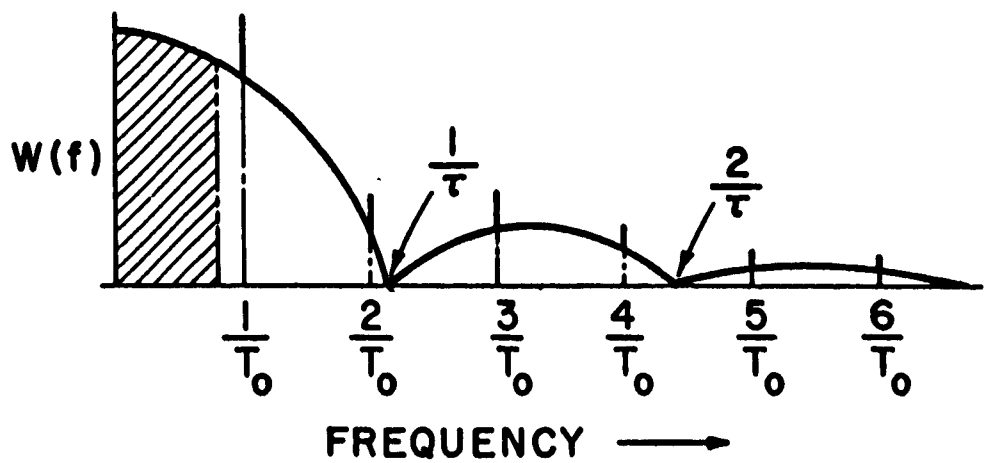
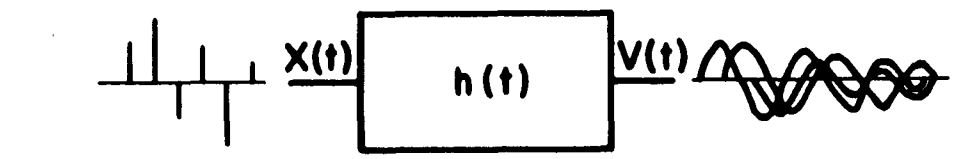


Figure 5 The Power Spectral Density for a Periodic Pulse Train Having Statistically Independent Pulse Amplitudes, Pulse Duration τ, and Period T_o.

Consider the random pulse train x(t), see Figure 4, as the input to a linear filter with impulse response h(t) as is shown in Figure 6. Let x(t) be represented by a weighted sum of narrow pulses:

$$x(t) = \sum_{n=-\infty}^{\infty} y_n u_o(t-nT_o) \quad . \qquad (10)$$



TRAIN OF WEIGHTED NARROW PULSES

SUPERPOSITION OF WEIGHTED IMPULSE RESPONSES

Figure 6 The Linear Filter.

Then, if we assume that the duration of $u_o(t)$ is short compared with the duration of h(t), $u_o(t) \doteq \delta(t)$ and the output V(t) of the filter becomes

$$V(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \sum_{n=-\infty}^{\infty} y_n h(t-nT_o) \quad . \qquad (11)$$

The filter output V(t) consists of the weighted sum of impulse responses whose epochs are at integral multiples of T_o, as shown in Figure 7.

What may be said regarding the amplitude distribution of V(t)? In general very little, for this is related to the more

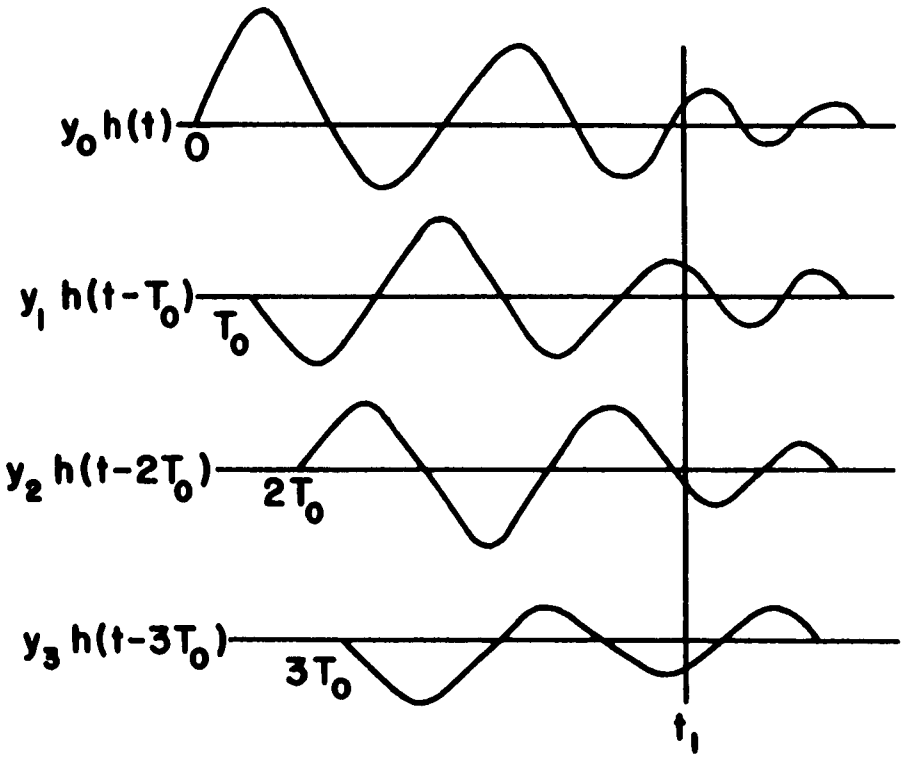


Figure 7 Randomly Weighted Filter Impulse Responses

general problem of determining the output amplitude distribution of a linear system driven by a non-Gaussian random process. However, the form of Equation (11) suggests the application of the central limit theorem of probability. The central limit theorem asserts that under general conditions the sum of a large number of independent random variables approaches a Gaussian distribution. That is to say, if n independent impulse responses produce non-negligible effects on $V(t_1)$, and if n is sufficiently large, then the ensemble of values $V(t_1)$ will be Gaussianly distributed. Since the distribution is tending toward a Gaussian distribution, we can perhaps help to minimize n by providing Gaussianly distributed pulse amplitudes. In any case this is most convenient, for we need only sample the amplitudes of a Gaussian noise voltage whose fluctuation time is approximately one-tenth the sampling period in order to sufficiently approach statistically independent samples. That is, if the sampling rate is 10 Kcps, then we need only sample Gaussian noise having an approximately uniform power spectral density from approximately zero frequency to 100 Kcps.

The amplitude probability density function, p(x), for the random pulse train x(t) whose amplitudes are Gaussianly distributed with zero mean is given by

$$p(x) = \frac{\tau}{T_o} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} + (1 - \frac{\tau}{T_o})\delta(x) \quad . \qquad (12)$$

Also, the characteristic function f(v) of p(x) is given by

$$f(v) = \int_{-\infty}^{\infty} e^{ivx}\, dP(x) = \frac{\tau}{T_o} \exp\left[-\frac{v^2 \sigma^2}{2} \right] + \left(1 - \frac{\tau}{T_o}\right), \tag{13}$$

where P(x) is the probability distribution function of x. Notice that if $\tau = T_o$, then x(t) is a boxcar waveform and is Gaussianly distributed. Of course, the output of the linear filter under these conditions will also be Gaussianly distributed.

In order to apply the central limit theorem to the situation under discussion, the impulse response of the linear filter must have a significant width, or memory, extending out to nT_o. The value of n required will normally be of the order of ten. For our particular case the value of n required to regenerate a Gaussian distribution was experimentally determined to be of the order of 3 or 4. Hence, for a sampling rate of 10 Kcps a low-pass filter having a 5 Kcps cut-off frequency will suffice for the linear filter.

Figure 8 shows the measured amplitude distributions compared with points representing a Gaussian distribution for the case of a sampling rate of 10 Kcps and a low-pass filter having a 5 Kcps cut-off frequency.

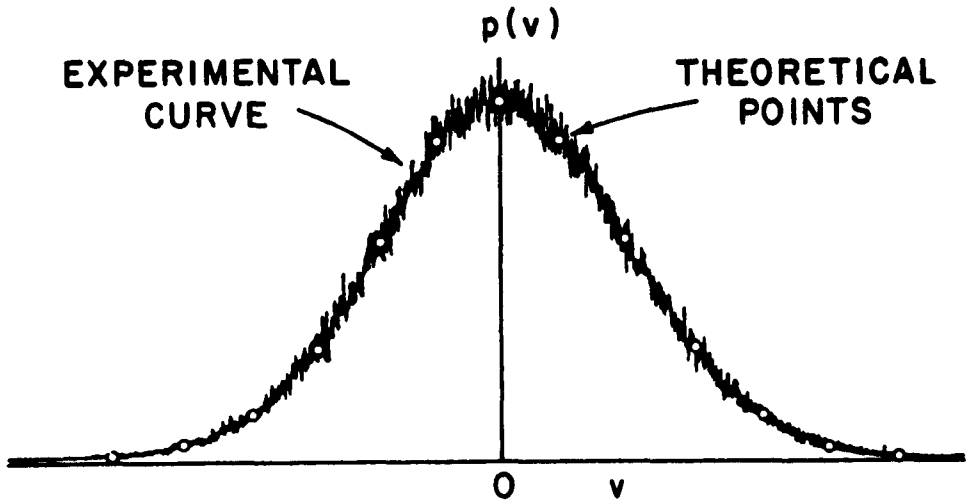


Figure 8 The Measured Amplitude Distribution p(v) Compared with Points Representing a Gaussian Distribution.

C. THE POWER SPECTRA CONSIDERED

The results presented in this report are concerned with the following power spectral densities:

$$W_o(f) = \frac{1}{1+\left(\frac{f}{f_o}\right)^{14}} \quad \text{where } \omega_o = 2\pi f_o \tag{14}$$

$$W_1(f) = \frac{1}{\left[1+\left(\frac{f}{f_1}\right)^2\right]^2} \quad \text{where } \omega_1 = 2\pi f_1 = \frac{1}{\tau_1} \tag{15}$$

$$W_2(f) = \frac{1}{\left[1+\left(\frac{f}{f_2}\right)^2\right]^3} \quad \text{where } \omega_2 = 2\pi f_2 = \frac{1}{\tau_2} \tag{16}$$

$$W_3(f) = \frac{\left(\frac{f}{f_3}\right)^4}{\left[1+\left(\frac{f}{f_3}\right)^2\right]^4} \quad \text{where } \omega_3 = 2\pi f_3 = \frac{1}{\tau_3} \tag{17}$$

$$W_4(f) = e^{-\frac{(f-f_4)^2}{2\sigma^2}} \quad \text{where } \omega_4 = 2\pi f_4 \tag{18}$$

and "Q factor" $= \dfrac{f_4}{2\sigma\sqrt{\ln 4}} = 6.1$.

These particular power spectral densities were selected for investigation because they are representative of the spectra that occur

in many branches of science and technology, and because theoretical approximations for the corresponding $P_o(\tau)$ have been reported (2, 8, 19).

The corresponding system functions $H_n(j2\pi f)$ were synthesized on an analog computer as indicated in Figures 9 and 10. The system function $H_4(j2\pi f)$ corresponding to the Gaussian power spectral density $W_4(f)$ was synthesized by stagger tuning 9 single tuned filters. In this manner the poles of $H_4(j2\pi f)$ were located at the theoretical (33) pole positions corresponding to a 9 th order approximation to the unrealizable Gaussian power spectral density. Figure 11 shows a comparison of the sinusoidal gain of the filter structure as a function of frequency with the theoretical Gaussian function. The comparison is excellent.

The Wiener-Khinchin theorem gives the normalized autocorrelation function for the above power spectra as:

$$\rho_n(\tau) = \frac{\int_o^\infty W_n(f) \cos 2\pi f\tau df}{\int_o^\infty W_n(f)\, df} \qquad n = 0, 1, 2, 3, 4 \tag{19}$$

The results of the integrations yield:

$$\rho_o(\tau) = \mathrm{Sin}\frac{\pi}{14}\left\{ e^{-\omega_o|\tau|} + 2\sum_{n=1}^{3} e^{-\omega_o|\tau|\cos\frac{n\pi}{7}} \cos\left[\frac{n\pi}{7} - \omega_o|\tau|\sin\frac{n\pi}{7}\right]\right\} \tag{20}$$

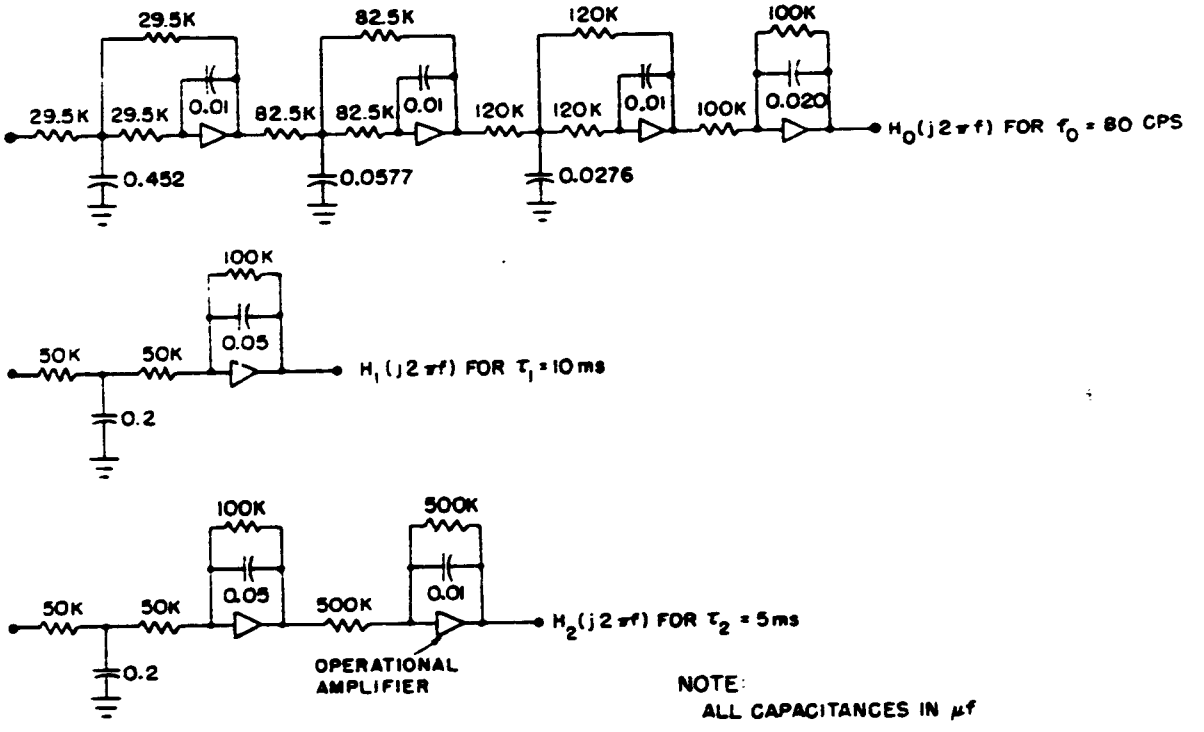


Figure 9 The Synthesis of the System Functions $H_n(j2\pi f)$ for n = 0, 1, 2.

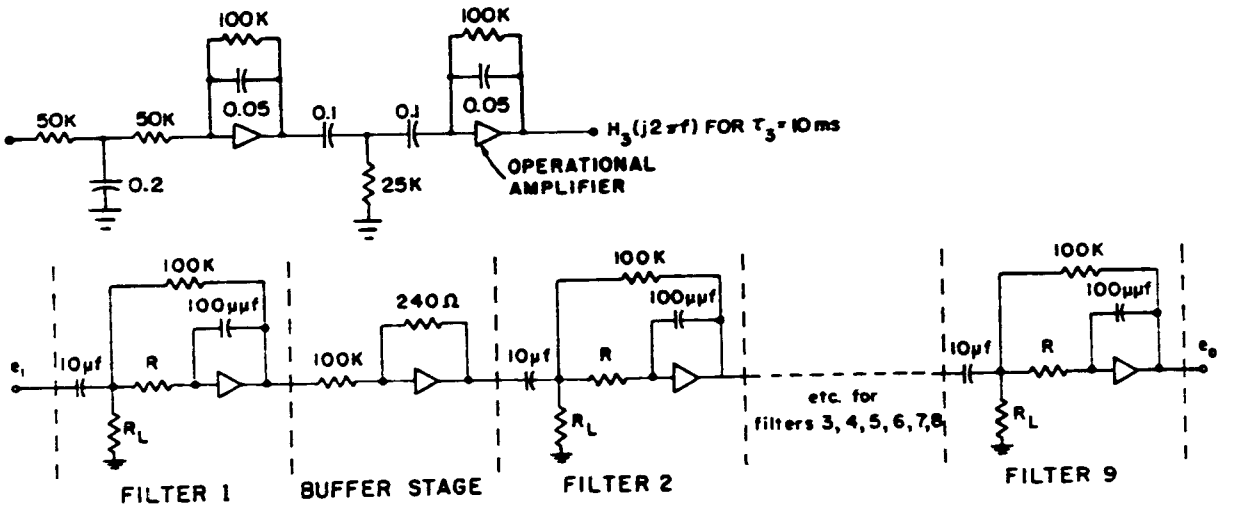


TABLE OF COMPONENT VALUES		
FILTER	R_L	R
1	2800Ω	427KΩ
2	2500	187
3	1850	370
4	1980	205
5	1710	330
6	1740	232
7	1626	301
8	1710	249
9	1630	268

System Function $H_4(j2\pi f)$ For $f_4 = 30.8$ cps

Figure 10 The Synthesis of the System Functions $H_n(j2\pi f)$ for n = 3, 4.

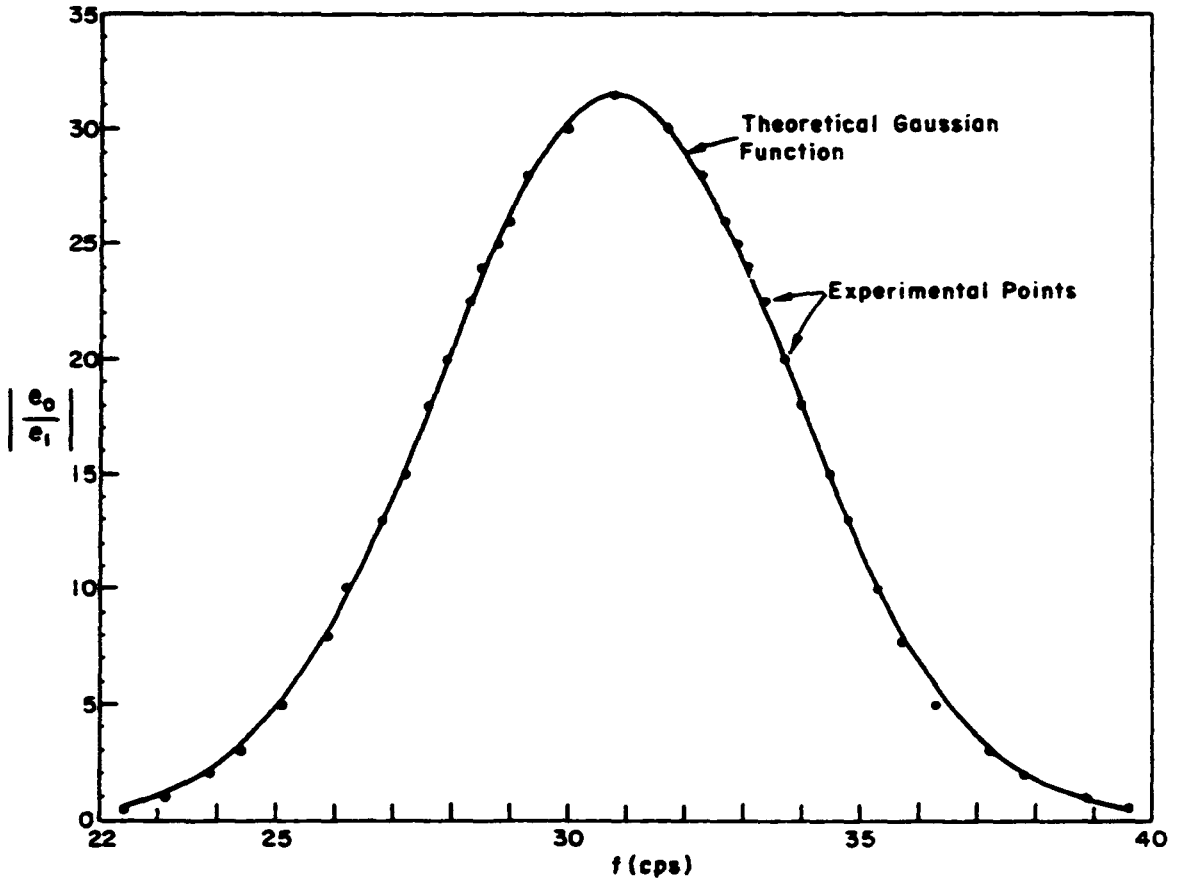


Figure 11 For the System Function $H_4(j2\pi f)$, the Experimental Values of $\left|\frac{e_o}{e_1}\right|$ are Compared with the Gaussian Function $\sqrt{W_4(f)}$.

$$\rho_1(\tau) = (1 + \omega_1|\tau|)e^{-\omega_1|\tau|} \tag{21}$$

$$\rho_2(\tau) = \left[1 + \omega_2|\tau| + \frac{(\omega_2\tau)^2}{3}\right] e^{-\omega_2|\tau|} \tag{22}$$

$$\rho_3(\tau) = \left[1 + \omega_3|\tau| - 2(\omega_3\tau)^2 + \frac{1}{3}(\omega_3|\tau|)^3\right] e^{-\omega_3|\tau|} \tag{23}$$

$$\rho_4(\tau) = \cos \omega_4\tau \; e^{-2(\pi\sigma\tau)^2} \quad . \tag{24}$$

The quantity $\omega_n\tau$ appearing in the normalized autocorrelation functions is defined as normalized time and is denoted by u_n. All probabilities and probability densities reported in this report are plotted with respect to normalized time $u_n = \omega_n\tau$.

The initiate commands discussed in Section II-A above were spaced approximately one second apart. Accordingly, in order to obtain samples of zero-crossing intervals which are sufficiently independent, the time constants and frequencies associated with the power spectral densities were chosen as:

$$\tau_1 = \tau_3 = 10 \text{ m sec} \tag{25}$$

$$\tau_2 = 5 \text{ m sec} \tag{26}$$

$$f_o = 80 \text{ cps} \tag{27}$$

$$f_4 = 30.8 \text{ cps} \quad . \tag{28}$$

Also, for experimental convenience the synthesized power spectral densities had a low frequency cutoff of approximately 1 cps and a high frequency cutoff of 8.5 kc.

D. RANDOM POINT-PROCESS TRANSFORMATIONS

Consider the linear system shown in Figure 12.

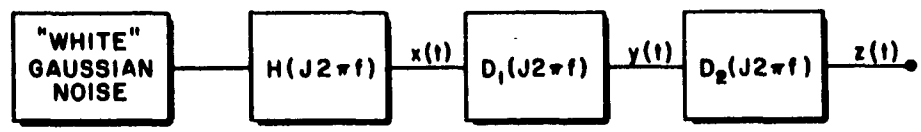


Figure 12 The Linear System.

The power spectral density for the Gaussian random process x(t) is given by

$$W_x(f) = |H(j2\pi f)|^2 \quad . \tag{29}$$

Similarly, the power spectral density for y(t) is given by

$$W_y(f) = |H(j2\pi f)|^2 |D_1(j2\pi f)|^2 = W_x(f) |D_1(j2\pi f)|^2 . \tag{30}$$

If $D_1(j2\pi f)$ is the system function representing an ideal differentiator then the stationary points of x(t) and the zero points of y(t) occur simultaneously.

If y(t) is a Gaussian process having arbitrary power spectral density $W_y(f)$, the average number of zero points per second is given by (27)

$$2\left[\frac{\int_o^\infty f^2 W_y(f)df}{\int_o^\infty W_y(f)df}\right]^{1/2} \quad . \tag{31}$$

Hence, the average number of stationary points per second for a Gaussian process x(t) having power spectral density $W_x(f)$ must be

$$2\left[\frac{\int_o^\infty f^2 W_x(f) 4\pi^2 f^2 df}{\int_o^\infty W_x(f) 4\pi^2 f^2 df}\right]^{1/2} = 2\left[\frac{\int_o^\infty f^4 W_x(f)df}{\int_o^\infty f^2 W_x(f)df}\right]^{1/2} \quad . \tag{32}$$

This last equation agrees with Rice's result (27).

The average number of inflection points per second for a Gaussian process x(t) having power spectral density $W_x(f)$ can be determined in a similar manner. Thus, if also $D_2(j2\pi f) = j2\pi f$, then the average number of inflection points per second for a Gaussian process x(t) having power spectral density $W_x(f)$ must be

$$2\left[\frac{\int_o^\infty f^2 W_x(f)\, 16\pi^4 f^4 df}{\int_o^\infty W_x(f)\, 16\pi^4 f^4 df}\right]^{1/2} = 2\left[\frac{\int_o^\infty f^6 W_x(f)\, df}{\int_o^\infty f^4 W_x(f)\, df}\right]^{1/2} \quad . \tag{33}$$

The same transformation can be extended to higher derivatives.

A method for synthesizing a Gaussian process y(t) having zero points corresponding in time with the stationary points of a

Gaussian process x(t) having a Butterworth power spectral density is shown in Figure 13.

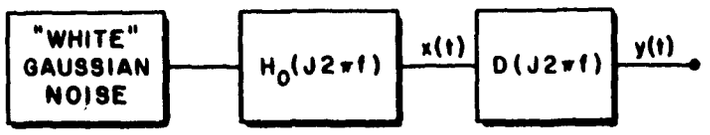


Figure 13 Transformation of Stationary Points to Zero Points.

The circuit having the system function $D(j2\pi f)$ is shown in Figure 14.

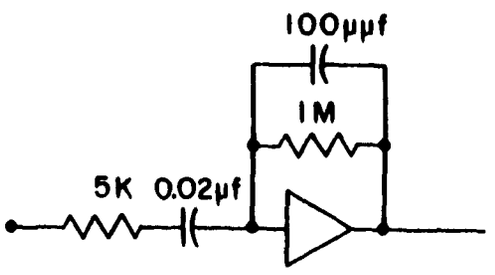


Figure 14 The Synthesis of $D(j2\pi f)$.

The system function $D(j2\pi f)$ is given by

$$D(j2\pi f) = \frac{2j10^{-2}\omega}{\left[1+j\,10^{-4}\omega\right]^2} \quad . \tag{34}$$

Accordingly, $D(j2\pi f)$ has the property of a differentiator since $|D(j2\pi f)|^2$ behaves like f^2 to well within $\pm$ 1 db in the frequency range of interest (0-500 cps).

Taking $D(j2\pi f)$ as $2j10^{-2}\omega$ we find that

$$\sigma_x^2 = \int_o^\infty W_o(f)\,df \;;\quad \sigma_y^2 = \left[4\pi 10^{-2}\right]^2 \int_o^\infty f^2 W_o(f)df \tag{35}$$

and

$$\frac{\sigma_y}{\sigma_x} = 4\pi\,10^{-2}\,f_o\left[\frac{\mathrm{Sin}\frac{\pi}{14}}{\mathrm{Sin}\frac{3\pi}{14}}\right]^{1/2} = 6.007 \tag{36}$$

This final ratio was verified satisfactorily by using an RMS voltmeter.

Oscilloscope photographs illustrating the transformation of stationary points to zero points by the operator $D(j2\pi f)$ are shown in Figure 15. The probability densities associated with the intervals defined by the mathematical stationary points of a random process consisting of a sine wave plus a Gaussian process were determined experimentally by using the operator $D(j2\pi f)$.

Figure 16 shows a complete block diagram of the measuring and synthesis systems.

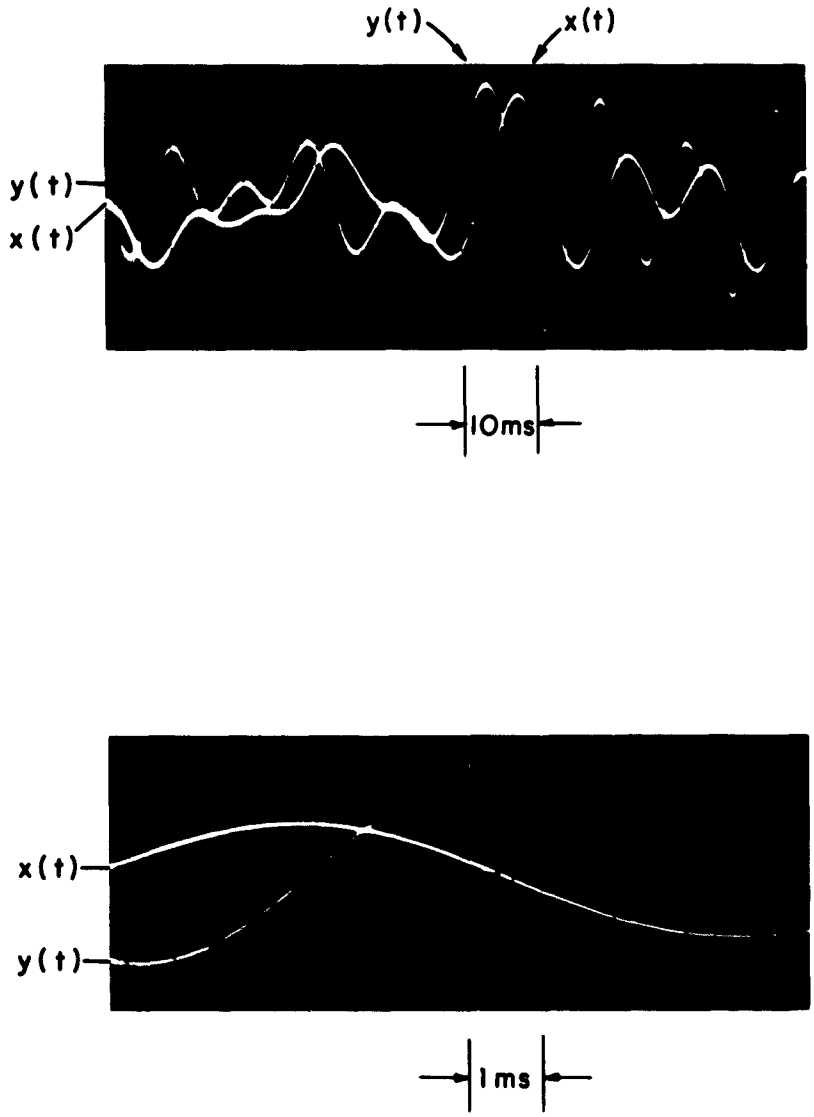


Figure 15: The Stationary Points of x(t) and the Zero Points of y(t) Occur Simultaneously.

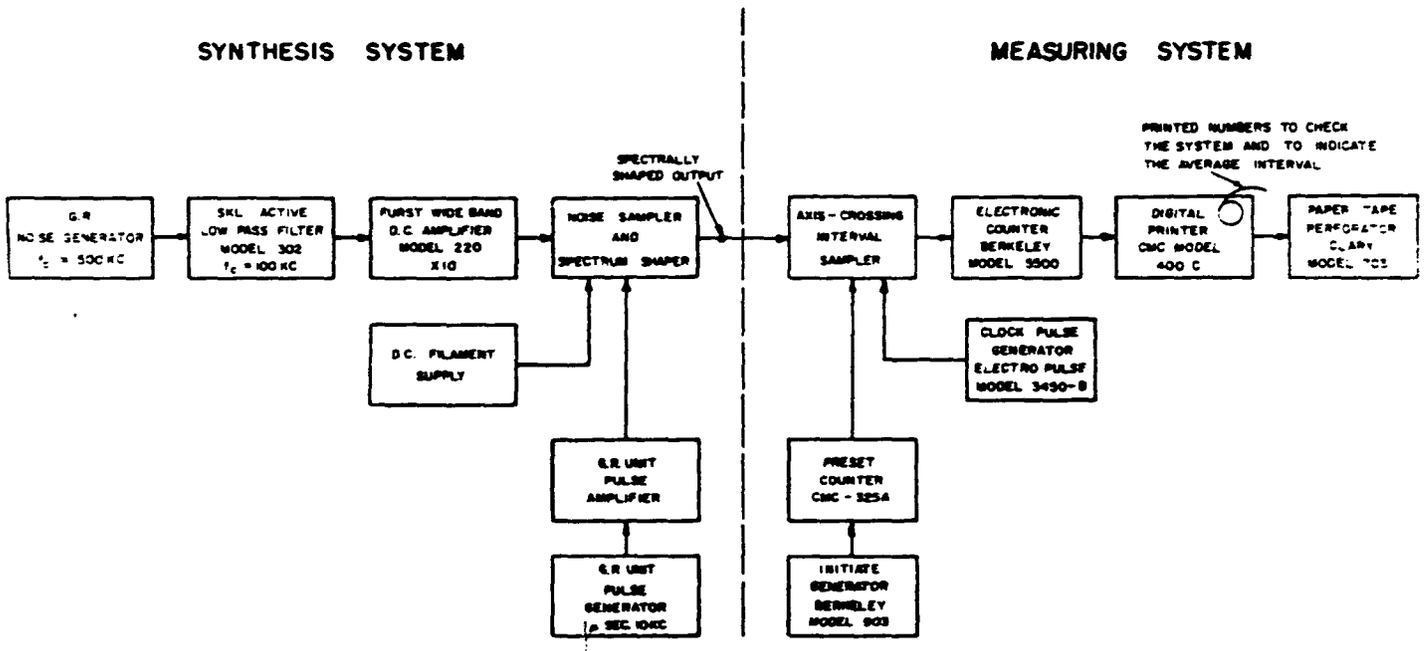


Figure 16 Block Diagram of the Synthesis and Measuring Systems.

IV. MATHEMATICAL DEDUCTIONS FROM THE PROBABILITY DENSITIES

A. Kohlenberg (4), P. I. Kuznetsov, P. L. Stratonovich and V. I. Tikhonov (5), J. A. McFadden (2), and perhaps others, have derived some basic identities relating $p(n, \tau)$, the probability that a given interval $(t, t + \tau)$ contains exactly n points, and $P_n(\tau)$, the probability density of the interval between the mth and (m+n+1)th points. These identities apply to general stationary point-processes. Using McFadden's general result (26) we have for a point set defined by the zeros of a random process:

$$p'(n, \tau) = -\beta \int_{\tau}^{\infty} \left[P_n(\ell) - 2P_{n-1}(\ell) + P_{n-2}(\ell) \right] d\ell \tag{37}$$

where

β = the expected number of zeros per unit time

$P_n(\tau)$ = the probability density function for the sum of n+1 successive zero-crossing intervals

$p'(n, \tau)$ = the derivative with respect to τ of the probability that a given interval τ contains exactly n zeros.

Also, by definition, we have

$$Z(n+1, \tau) = \int_{\tau}^{\infty} P_n(\ell)\, d\ell \tag{38}$$

where

$Z(n+1, \tau)$ = the conditional probability that the (n + 1)th zero-crossing point from a given zero-crossing point in $d\tau$ occurs after the time $d\tau + \tau$.

For n = 0, and the initial condition p(0, 0) = 1, Equation (37) yields

$$p(0,\tau) = \beta \left[\int_0^{\tau} d\tau_1 \int_0^{\tau_1} P_0(\ell)\, d\ell - \tau \right] + 1 . \qquad (39)$$

For n = 1 and the initial condition p(1, 0) = 0, Equation (37) yields

$$p(1,\tau)=\beta \left[\int_0^{\tau} d\tau_1 \int_0^{\tau_1} P_1(\ell) d\ell - 2 \int_0^{\tau} d\tau_1 \int_0^{\tau_1} P_0(\ell) d\ell + \tau \right] . \qquad (40)$$

For n = 0 Equation (38) becomes

$$Z(1,\tau) = \int_{\tau}^{\infty} P_0(\ell)\, d\ell \quad . \qquad (41)$$

For n = 1 Equation (38) becomes

$$Z(2,\tau) = \int_{\tau}^{\infty} P_1(\ell)\, d\ell \quad . \qquad (42)$$

Equations (39), (40), (41), and (42) were used to deduce $p(0, u_n)$, $p(1, u_n)$, $Z(1, u_n)$ and $Z(2, u_n)$ for n = 1, 2, 3 and 4.

Similar expressions hold for the point set defined by the zero points of a random-process consisting of a sine wave signal of frequency $\frac{f_o}{2}$ plus Gaussian noise such that the signal-to-noise power ratio equals a. That is:

$$p_a(0,\tau) = \beta_a \left[\int_0^{\tau} d\tau_1 \int_0^{\tau_1} P_0(\ell, a) d\ell - \tau \right] + 1 \qquad (43)$$

$$p_a(1,\tau) = \beta_a \left[\int_0^{\tau} d\tau_1 \int_0^{\tau_1} P_1(\ell, a) d\ell - 2 \int_0^{\tau} d\tau_1 \int_0^{\tau_1} P_o(\ell, a) d\ell + \tau \right] \tag{44}$$

$$Z_a(1,\tau) = \int_{\tau}^{\infty} P_o(\ell, a)\, d\ell \tag{45}$$

$$Z_a(2,\tau) = \int_{\tau}^{\infty} P_1(\ell, a)\, d\ell \tag{46}$$

where

β_a = the expected number of zeros per unit time when the signal to noise power ratio equals a.

Equations (43), (44), (45) and (46) were used to deduce $p_a(o, u_o)$, $p_a(1, u_o)$, $Z_a(1, u_o)$ and $Z_a(2, u_o)$ for a = 0, .2, 1 and 4.

Similar expressions also hold for the point set defined by the mathematical stationary points of a random process consisting of a sine wave signal of frequency $\frac{f_o}{2}$ plus Gaussian noise such that the signal-to-noise power ratio equals a. That is:

$$m_a(0,\tau) = \gamma_a \left[\int_0^{\tau} d\tau_1 \int_0^{\tau_1} M_o(\ell, a)\, d\ell - \tau \right] + 1 \tag{47}$$

$$m_a(1,\tau) = \gamma_a \left[\int_0^{\tau} d\tau_1 \int_0^{\tau_1} M_1(\ell, a) d\ell - 2 \int_0^{\tau} d\tau_1 \int_0^{\tau_1} M_o(\ell, a) d\ell + \tau \right] \tag{48}$$

$$S_a(1,\tau) = \int_{\tau}^{\infty} M_o(\ell, a)\, d\ell \tag{49}$$

$$S_a(2,\tau) = \int_{\tau}^{\infty} M_1(\ell, a)\, d\ell \tag{50}$$

where

γ_a = the expected number of stationary points per unit time when the signal-to-noise power ratio equals a.

Equations (47), (48), (49), and (50) were used to deduce $m_a(0, u_o)$, $m_a(1, u_o)$, $S_a(1, u_o)$ and $S_a(2, u_o)$ for a = 0, .2, 1 and 4.

V. PROPERTIES OF THE PROBABILITY DENSITIES

A. EXPECTATIONS

Consider the point set shown in Figure 17 and defined by the zero-crossing points of a Gaussian process. Let X, Y be two successive zero-crossing intervals and let Z = X + Y. From symmetry the expectation E(X) equals the expectation E(Y). Hence, E(Z)=2E(X). Accordingly, the expectations, $E_1(u_n)$, associated with the $P_1(u_n)$ densities are related to the expectations, $E_0(u_n)$, associated with the $P_0(u_n)$ densities by:

$$E_1(u_n) = 2E_0(u_n) \qquad \text{for } n = 1, 2, 3, 4 \qquad (51)$$

The exact theoretical expectations, $E_0(u_n)$, associated with the $P_0(u_n)$ densities, or the average values of successive zero-crossing intervals, are given by (27)

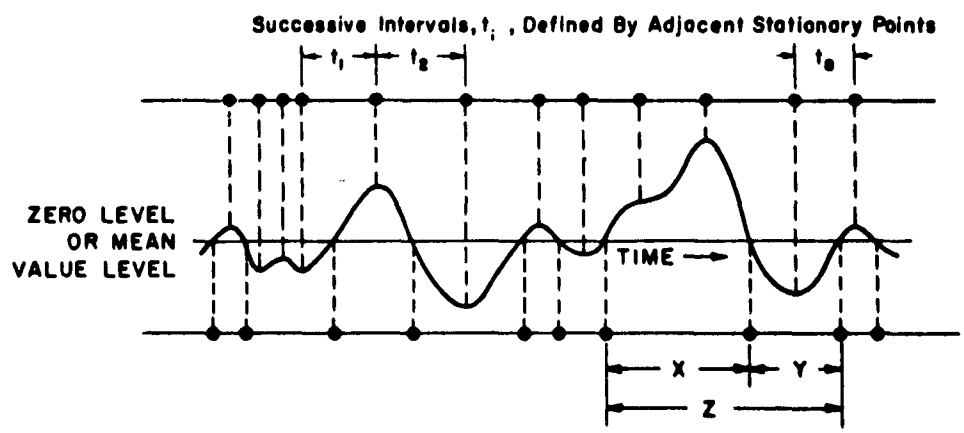


Figure 17 Point Sets Defined by the Zero-Crossing Points and the Stationary Points of a Gaussian Process.

$$E_0(u_n) = \int_0^\infty u_n P_0(u_n) du_n = \frac{1}{2}\omega_n \left[\frac{\int_0^\infty W_n(f)df}{\int_0^\infty f^2 W_n(f)df} \right]^{1/2} = \pi \left[-\rho_n''(0) \right]^{-1/2} \tag{52}$$

for n = 1, 2, 3, 4

$\rho_n''(u_n)$ is the second derivative of the normalized autocorrelation function $\rho_n(u_n)$. The results for n = 1, 2, 3, and 4 are:

$$E_0(u_1) = \pi \tag{53}$$

$$E_0(u_2) = 5.4414 \tag{54}$$

$$E_0(u_3) = 1.4050 \tag{55}$$

$$E_0(u_4) = 3.1341 \tag{56}$$

From Equation (51) the theoretical expectations, $E_1(u_n)$, associated with the $P_1(u_n)$ densities are equal to $2E_0(u_n)$ for n = 1, 2, 3, 4. The expectations $E_0(u_n)$ and $E_1(u_n)$ were also measured. They are compared with the corresponding theoretical values in the tables accompanying Figures 18, 19, 22, 23, 26, 27, 30, and 31. The comparisons are satisfactory in all cases.

Similarly, the expectations, $E_1(u_o, a)$, associated with the $P_1(u_o, a)$ densities are related to the expectations, $E_o(u_o, a)$, associated with the $P_o(u_o, a)$ densities by:

$$E_1(u_o, a) = 2E_o(u_o, a) \qquad \text{for all } a \ . \tag{57}$$

The exact theoretical expectations, $E_o(u_o, a)$, associated with the $P_o(u_o, a)$ densities follow from a general result given by Rice (34) and Middleton (3, 3a):

$$E_o(u_o, a) = \int_o^\infty u_o P_o(u_o, a) du_o = 2\pi\sqrt{b}\left[\sum_{n=o}^{\infty} \frac{(-\frac{1}{2})_n}{(n!)^2}(-ba)^n {}_1F_1(\tfrac{1}{2};n+1;-a)\right]^{-1}$$

for all a (58)

where

$(a)_n = (a)\,(a+1)\,(a+2)\,\ldots\,(a+n-1)$

$(a)_o = 1$

$a = \frac{S}{N} = \frac{\text{average sine wave power}}{\text{average noise power}}$

$b = \frac{\sin\frac{3\pi}{14}}{4\sin\frac{\pi}{14}}$

${}_1F_1$ = the confluent hypergeometric function.

The results for a = 0, .2, 1 and 4 are:

$E_o(u_o, 0) = 5.2587$ (59)

$E_o(u_o, .2) = 5.4065$ (60)

$E_o(u_o, 1) = 5.8286$ (61)

$E_o(u_o, 4) = 6.2506$. (62)

For small values of "a", the random process consisting of a sine wave signal of frequency $\frac{f_o}{2}$ plus Gaussian noise is approximately Gaussian. Accordingly, one could compute easily the approximate theoretical expectations, $E_o^*(u_o, a)$, associated with the $P_o(u_o, a)$ densities by using an equation analagous to Equation (52). That is:

$$E_o^*(u_o, a) = \pi\left[-R''(0, a)\right]^{-1/2} = 2\pi\sqrt{b}\left[\frac{1+a}{1+ab}\right]^{1/2} . \quad (63)$$

$R''(u_o, a)$ is the second derivative of the normalized autocorrelation function, $R(u_o, a)$, defined by the power spectral density $W_o(f)+S\delta(f-\frac{f_o}{2})$. $R(u_o, a)$ is given by:

$$R(u_o, a) = \frac{\rho_o(u_o) + a \cos \frac{u_o}{2}}{1 + a} \tag{64}$$

Equation (63) agrees with Bendat's result (15).

The results for a = 0, .2, 1 and 4 are:

$$E_o^*(u_o, 0) = 5.2587 \tag{65}$$

$$E_o^*(u_o, .2) = 5.3951 \tag{66}$$

$$E_o^*(u_o, 1) = 5.7031 \tag{67}$$

$$E_o^*(u_o, 4) = 6.0306 \quad . \tag{68}$$

From Equation (57) the exact theoretical expectations, $E_1(u_o, a)$, associated with the $P_1(u_o, a)$ densities are equal to $2E_o(u_o, a)$ for all a. The expectations $E_o(u_o, a)$ and $E_1(u_o, a)$ for a = 1, .2, 1, and 4 were also measured. They are compared with the corresponding exact theoretical values in the tables accompanying Figures 34, 35, 38, 39, 42, 43, 46, and 47. The comparisons are satisfactory in all cases.

Finally, the expectations, $\mathcal{E}_1(u_o, a)$, associated with the $M_1(u_o, a)$ densities are related to the expectations, $\mathcal{E}_o(u_o, a)$, associated with the $M_o(u_o, a)$ densities by:

$$\mathcal{E}_1(u_o, a) = 2\mathcal{E}_o(u_o, a) \text{ for all a} \quad . \tag{69}$$

The exact theoretical expectations, $\mathcal{E}_o(u_o, a)$, associated with the $M_o(u_o, a)$ densities follow by applying the transformation associated with the ideal differentiator operator to the general result given by Rice and Middleton:

$$\mathcal{E}_o(u_o, a) = \int_o^{\infty} u_o M_o(u_o, a) du_o = 2\pi \sqrt{\frac{c}{b}} \left[\sum_{n=0}^{\infty} \frac{(-\frac{1}{2})_n}{(n!)^2} (-ca)^n {}_1F_1(\tfrac{1}{2}; n+1; -ba) \right]^{-1}$$

for all a (70)

where

$$c = \frac{\operatorname{Sin} \frac{5\pi}{14}}{16 \operatorname{Sin} \frac{\pi}{14}} \quad .$$

The results for a = 0, .2, 1 and 4 are:

$$\mathcal{E}_o(u_o, 0) = 3.7765 \tag{71}$$

$$\mathcal{E}_o(u_o, .2) = 3.9429 \tag{72}$$

$$\mathcal{E}_o(u_o, 1) = 4.5467 \tag{73}$$

$$\mathcal{E}_o(u_o, 4) = 5.8138 \quad . \tag{74}$$

Again, for small values of "a" the random process consisting of a sine wave signal of frequency $\frac{f_o}{2}$ plus Gaussian noise is approximately Gaussian. Accordingly, one could compute easily the approximate theoretical expectations, $\mathcal{E}_o^*(u_o, a)$, associated with the $M_o(u_o, a)$ densities by using an equation analagous to Equation (52). That is:

$$\mathcal{E}_o^*(u_o, a) = \pi \left[-R_s''(0, a) \right]^{-1/2} = 2\pi \sqrt{\frac{c}{b}} \left[\frac{1 + ab}{1 + ac} \right]^{1/2} \quad . \tag{75}$$

$R_s''(u_o, a)$ is the second derivative of the normalized autocorrelation function, $R_s(u_o, a)$, defined by the power spectral density $\omega^2 W_o(f) + \omega^2 S\delta(f - \frac{f_o}{2})$. This latter power spectral density results by applying the transformation associated with the ideal differentiator operator to the power spectral density

$W_o(f) + S\delta(f - \frac{f_o}{2})$. $R_s(u_o, a)$ and $R(u_o, a)$ are related by:

$$R_s(u_o, a) = \frac{R''(u_o, a)}{R''(0, a)} \text{ for all a} \quad . \tag{76}$$

The results for a = 0, .2, 1, and 4 are:

$$\mathcal{E}_o^*(u_o, 0) = 3.7765 \tag{77}$$

$$\mathcal{E}_o^*(u_o, .2) = 3.9341 \tag{78}$$

$$\mathcal{E}_o^*(u_o, 1) = 4.3994 \tag{79}$$

$$\mathcal{E}_o^*(u_o, 4) = 5.1910 \quad . \tag{80}$$

From Equation (69) the exact theoretical expectations, $\mathcal{E}_1(u_o, a)$, associated with the $M_1(u_o, a)$ densities are equal to $2\mathcal{E}_o(u_o, a)$ for all a. The expectations $\mathcal{E}_o(u_o, a)$ and $\mathcal{E}_1(u_o, a)$ for a = 0, .2, 1, and 4 were also measured. They are compared with the corresponding exact theoretical values and the approximate theoretical values in the tables accompanying figures 50, 51, 54, 55, 58, 59, 62, and 63. The comparisons are satisfactory in all cases.

If one represents the sine wave as the limit of a narrow-band Gaussian process centered at $\frac{f_o}{2}$ as the bandwidth approaches zero, then it becomes clear that the expectations $E_o^*(u_o, a)$ and $\mathcal{E}_o^*(u_o, a)$ are exact for the case of a random process $\xi^*(t, a)$ of the form:

$$\xi^*(t, a) = R(t) \operatorname{Sin}\left[\frac{\omega_o t}{2} + \theta(t)\right] + n(t) \quad . \tag{81}$$

In this last equation the random variable R_t has a Rayleigh probability density function, θ_t is uniformly distributed in the interval $(o, 2\pi)$, and $n(t)$ is Gaussian noise having a power spectral density $W_o(f)$. In other words $E_o^*(u_o, a)$ and $\mathcal{E}_o^*(u_o, a)$ are exact for the Gaussian process, $\xi^*(t, a)$, defined by a very narrow-band Gaussian signal centered at $\frac{f_o}{2}$ plus Gaussian noise having the power spectral density $W_o(f)$. Equation (81) should be compared with random process, $\xi(t, a)$, defined by a sine wave signal plus Gaussian noise which has the form:

$$\xi(t, a) = A_o \operatorname{Sin}\left[\frac{\omega_o t}{2} + \theta(t)\right] + n(t) \text{ where } A_o \text{ is a constant.} \tag{82}$$

B. STANDARD DEVIATIONS AND CORRELATION COEFFICIENTS

Since $Z = X + Y$, the variance of Z is given by:

$$\operatorname{Var}(Z) = \operatorname{Var}(X) + \operatorname{Var}(Y) + 2\operatorname{Cov}(X, Y) \tag{83}$$

where

$$\operatorname{Cov}(X, Y) = E(XY) - E(X)\,E(Y) \quad .$$

The correlation coefficient, $K(X, Y)$, of X, Y is given by

$$K(X, Y) = \frac{\operatorname{Cov}(X, Y)}{\sqrt{\operatorname{Var}(X)\operatorname{Var}(Y)}} \quad . \tag{84}$$

From symmetry, we have

$$\operatorname{Var}(X) = \operatorname{Var}(Y) \quad . \tag{85}$$

Hence,

$$\mathrm{Var}\,(Z) = 2\left[1 + K\right]\mathrm{Var}\,(X) \quad . \tag{86}$$

Let K_n be the correlation coefficient between two successive zero-crossing intervals of a Gaussian process having the power spectral density $W_n(f)$. Then, the variances, $\sigma_1^2(u_n)$, associated with the $P_1(u_n)$ densities are related to the variances, $\sigma_o^2(u_n)$, associated with the $P_o(u_n)$ densities by:

$$\sigma_1^2(u_n) = 2\left[1 + K_n\right]\sigma_o^2(u_n) \text{ for } n = 1, 2, 3, 4 \tag{87}$$

Similarly,

$$\sigma_1^2(u_o, 0) = 2\left[1 + K_o\right]\sigma_o^2(u_o, 0) \tag{88}$$

$$D_1^2(u_o, 0) = 2\left[1 + \mathcal{K}_o\right]D_o^2(u_o, 0) \quad . \tag{89}$$

In the latter two equations $\sigma_1^2(u_o, a)$, $\sigma_o^2(u_o, a)$, $D_1^2(u_o, a)$ and $D_o^2(u_o, a)$ are the variances associated with the respective densities $P_1(u_o, a)$, $P_o(u_o, a)$, $M_1(u_o, a)$ and $M_o(u_o, a)$. Also K_o and $\mathcal{K}_o$ are the correlation coefficients between two successive zero-crossing intervals of the Gaussian processes having the respective power spectral densities $W_o(f)$ and $\omega^2 W_o(f)$.

McFadden (32) recently derived the following equations valid for a general random process:

$$A = \int_0^\infty r(\tau)d\tau = \frac{\sigma_o^2}{E_o(\tau)} \left[\frac{1}{2} + \sum_{n=1}^{\infty} (-1)^n \rho_n \right] \quad (90)$$

$$B = \int_0^\infty \left[U(\tau) - \frac{1}{E_o(\tau)} \right] d\tau = \frac{\sigma_o^2}{E_o^2(\tau)} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \rho_n \right] - \frac{1}{2} \quad (91)$$

where

$r(\tau)$ = the normalized autocorrelation function of the infinitely clipped random process

σ_o^2 = variance of successive zero-crossing intervals

$E_o(\tau) = \frac{1}{\beta}$ = Expectation of successive zero-crossing intervals

ρ_n = correlation coefficient between the ith and (i+n)th zero-crossing intervals

$U(\tau)d\tau$ = the conditional probability that a zero occurs between $t+\tau$ and $t+\tau+d\tau$ given a zero at time t.

If one assumes that successive zero-crossing intervals form a Markov chain in the wide sense, then $\rho_n = \rho_1^n$ and

$$A = \frac{\sigma_o^2}{2E_o(\tau)} \frac{1-\rho_1}{1+\rho_1} > 0 \quad \text{for all } |\rho_1| < 1 \quad (92)$$

$$B = \frac{\sigma_o^2}{2E_o^2(\tau)} \left(\frac{1+\rho_1}{1-\rho_1} \right) - \frac{1}{2} > -\frac{1}{2} \quad . \quad (93)$$

Equations (92) and (93) were used by McFadden (2) to compute σ_o and ρ_1 for various Gaussian processes. We used Equations (92) and (93) to compute σ_o and ρ_1 for most of the Gaussian processes

considered in this report. In some cases our results differ somewhat from those reported by McFadden. The standard deviations resulting from the Markov assumption together with the standard deviations computed from Equations (87), (88) and (89) are compared with the experimental standard deviations in the tables accompanying Figures 18, 19, 22, 23, 26, 27, 34, 35, 50, and 51. From the comparisons of the standard deviations in the tables accompanying Figures 26, 27, 50, and 51, we conclude that successive zero-crossing intervals of some Gaussian processes do not form a Markov chain in the wide sense.

Notice that the Markov assumption implies that if $\rho_1 > 0$ then $\rho_n > 0$ for all n. Also, the Markov assumption implies the inequalities given by Equations 92 and 93. These restrictions are not clearly implied by the Gaussian character of the process and depend only on the Markov assumption. Accordingly, one might surmise that the successive zero-crossing intervals of some Gaussian processes do not form a Markov chain in the wide sense.

From Equations (90) and (91) we have that:

$$2 \sum_{n=1}^{\infty} \rho_{2n-1} = \frac{E_o(\tau)}{\sigma_o^2} \left[E_o(\tau) \left(B + \frac{1}{2}\right) - A \right] . \qquad (94)$$

As an approximation, let us assume that $\sum_{n=2}^{\infty} \rho_{2n-1} = 0$. (95)

Then,

$$\rho_1 = \frac{E_o(\tau)}{2\sigma_o^2} \left[E_o(\tau) \left(B + \frac{1}{2}\right) - A \right] . \qquad (96)$$

By applying Equation (96) to most of the Gaussian processes considered in this report and by using the experimental variances for successive intervals, the experimental correlation coefficients between two successive intervals were computed. The results are given in the tables accompanying Figures 18, 22, 26, 34, and 50.

If the variances for the sum of two successive intervals are computed from Equations 87, 88, and 89 by using the experimental variances for successive intervals and the experimental correlation coefficients between two successive intervals, the results agree with the measured variances for the sum of two successive intervals to within the experimental error. Hence the experimental values of the correlation coefficients between two successive intervals yield results consistent with experiment. This serves as partial justification for the assumption given by Equation (95).

If one assumes that $\rho_n = 0$ for $n \geq 2$, then Equations (90), (91), and (86), can be used to compute approximate theoretical values of σ_o, ρ_1 and σ_1 for Gaussian processes. For the Gaussian processes considered in this report, these approximate theoretical values compare with our experimental values slightly better than the corresponding theoretical values resulting from the Markov assumption.

For a Gaussian process, Rice (1) derived $U(\tau)d\tau$, see Appendix, the conditional probability that a zero occurs between $t + \tau$ and $t + \tau + d\tau$ given a zero at t. Hence,

$$U(\tau) = \sum_{n=0}^{\infty} P_n(\tau) \quad . \tag{97}$$

Also, for a Gaussian process, Rice (1) derived $Q(\tau)d\tau$, see Appendix, the conditional probability that a downward zero-crossing occurs between $t + \tau$ and $t + \tau + d\tau$ given an upward zero-crossing at t. Hence

$$Q(\tau) = \sum_{m=0}^{\infty} P_{2m}(\tau) \tag{98}$$

and

$$U(\tau) - Q(\tau) = \sum_{m=0}^{\infty} P_{2m+1}(\tau) \quad . \tag{99}$$

From Figure 31 and Equation (98) we see that the first portion of $Q(u_4)$ generates $P_o(u_4)$. Similarly, from Figure 32 and Equation (99) we see that the first portion of $U(u_4) - Q(u_4)$ generates $P_1(u_4)$. The first normalized portion of the $Q(u_4)$ function was used to compute approximate theoretical means and variances associated with the $P_o(u_4)$ density. Similarly, the first normalized portion of the $U(u_4) - Q(u_4)$ function was used to compute approximate theoretical means and variances associated

with the $P_1(u_4)$ density. By using these computed variances in Equation (87), the approximate theoretical correlation coefficient between two successive intervals, K_4, was computed. Because of the unfavorable propagation of errors associated with Equation (87), K_4 should be regarded as a rough estimate. These approximate theoretical results are compared with other results in the table accompanying Figures 30 and 31.

All experimental standard deviations were computed by using the theoretical mean values. In general, these standard deviations would decrease somewhat if one were to use the experimental mean values in the computations.

VI. PROBABILITIES AND PROBABILITY DENSITIES DEFINED BY THE ZERO POINTS OF GAUSSIAN PROCESSES

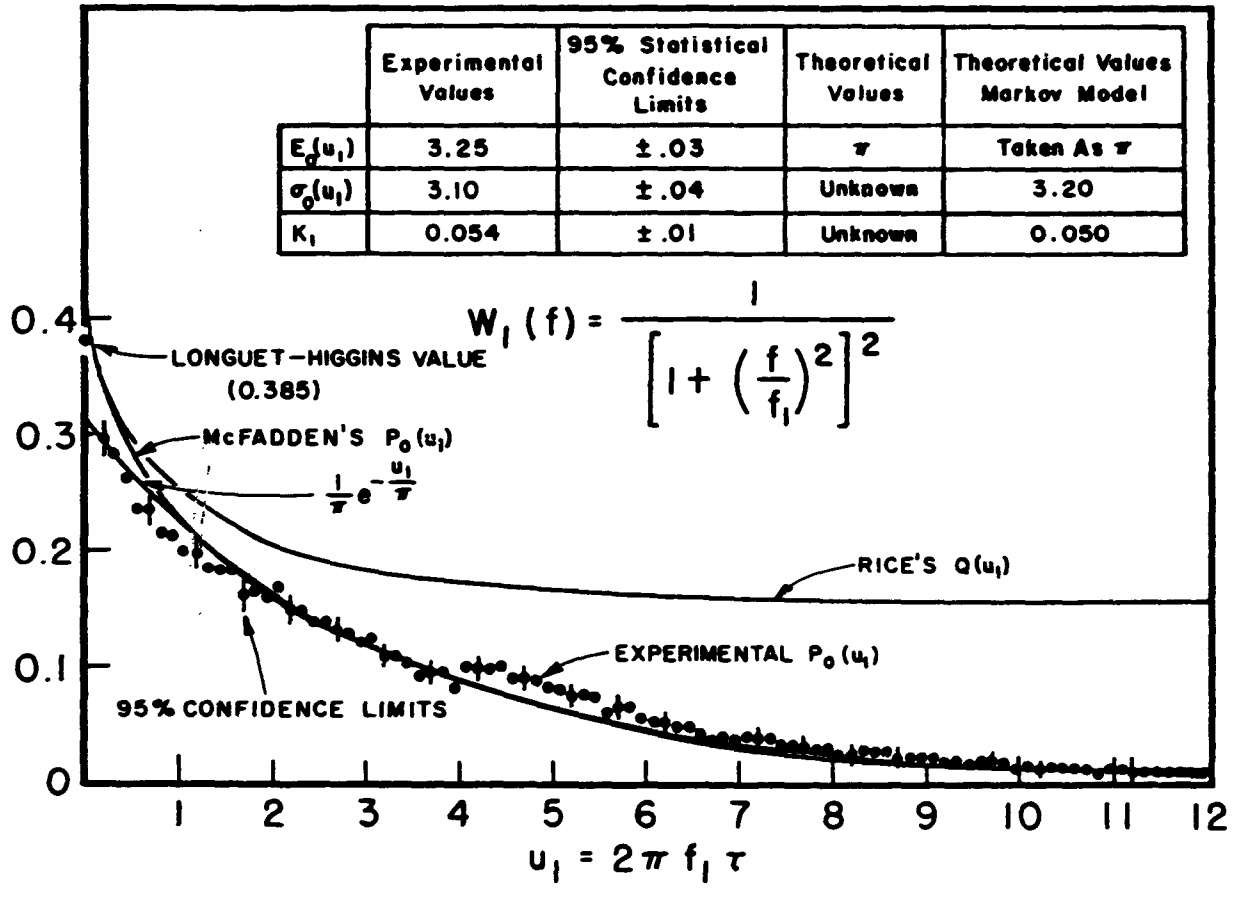
A. POWER SPECTRAL DENSITY $W_1(f)$

Probabilities and probability densities defined by the zero-crossing points of a Gaussian process having a power spectral density $W_1(f)$ are presented in Figures 18, 19, 20, and 21. Theoretical approximations derived by S. O. Rice (1) and J. A. McFadden (2) are presented in Figures 18 and 19 for comparison purposes. Figure 18 also presents for comparison an exponential density which was suggested by R. R. Favreau, H. Low, and I. Pfeffer (19) as the exact probability density. From this latter comparison we see that the experimental $P_0(u_1)$ and the exponential curve deviate significantly in the neighborhoods of $u_1 = 1$ and $u_2 = 5$. Accordingly, we conclude that the exact probability density is not the suggested exponential.

From Equation (98) we have that:

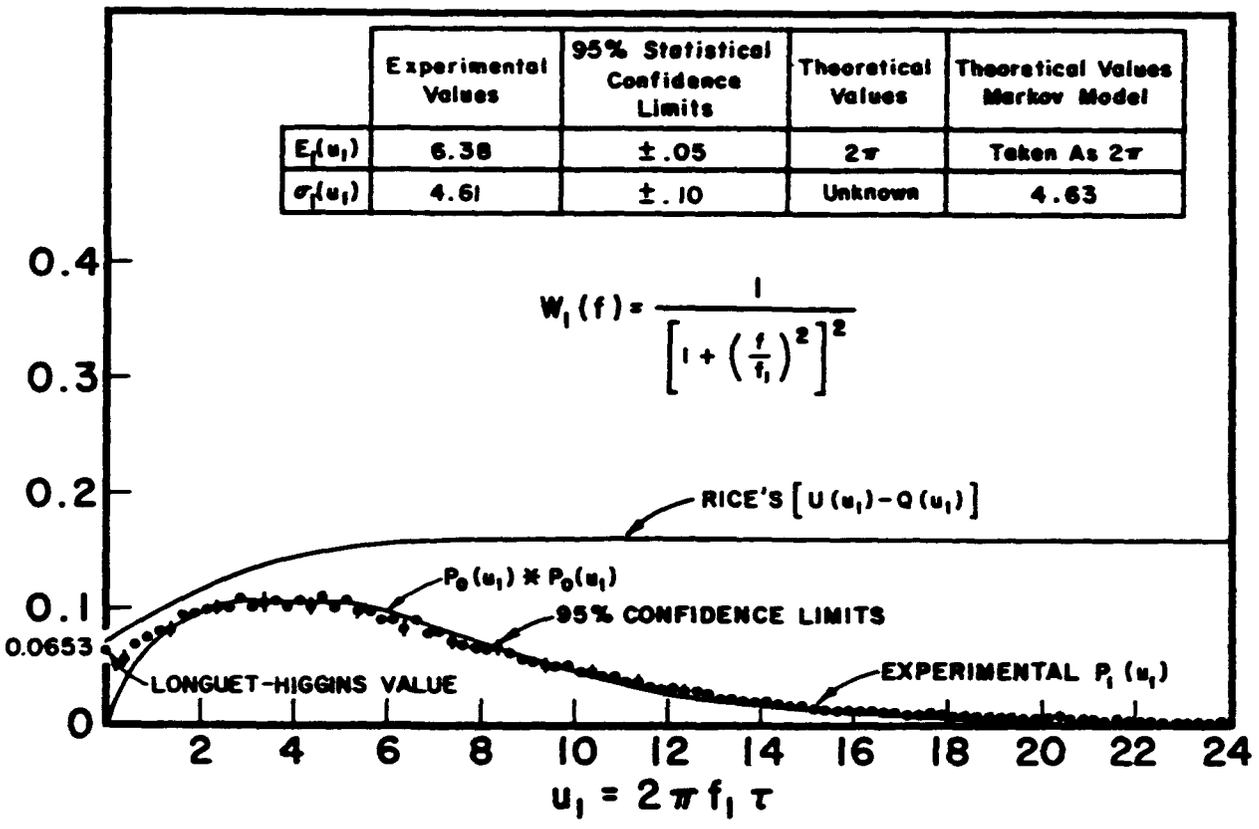
$$Q(\tau) \geq P_0(\tau) \quad \text{for all } \tau \quad . \tag{100}$$

Accordingly, Rice's $Q(u_1)$ function serves as an upper bound for $P_0(u_1)$ and is presented in Figure 18. Recently, M. S. Longuet-Higgins (8) deduced bounds for $P_0(0)$ theoretically as $0.382 < P_0(0) < Q(0) = 0.406$ and deduced the approximate value $P_0(0) \doteq 0.385$. The inequality $0.382 < P_0(0)$ was used by Longuet-Higgins (8) to disprove rigorously the exponential conjecture of Favreau et al. From Equation (99) we have that:



	Experimental Values	95% Statistical Confidence Limits	Theoretical Values	Theoretical Values Markov Model
$E_0(u_1)$	3.25	± .03	π	Taken As π
$\sigma_0(u_1)$	3.10	± .04	Unknown	3.20
K_1	0.054	± .01	Unknown	0.050

Figure 18 - $P_0(u_1)$ is the Probability Density Function for Successive Zero-Crossing Intervals of Gaussian Noise Having a Power Spectral Density $W_1(f)$. Theoretical Approximations are Compared with Experimental Results.



	Experimental Values	95% Statistical Confidence Limits	Theoretical Values	Theoretical Values Markov Model
$E_1(u_1)$	6.38	±.05	2π	Taken As 2π
$\sigma_1(u_1)$	4.61	±.10	Unknown	4.63

Figure 19 $P_1(u_1)$ is the Probability Density Function for the Sum of Two Successive Zero-Crossing Intervals of Gaussian Noise Having a Power Spectral Density $W_1(f)$. $P_0(u_1)*P_0(u_1)$ is Compared with the Experimental Result to Exhibit the Statistical Dependence of Two Successive Intervals.

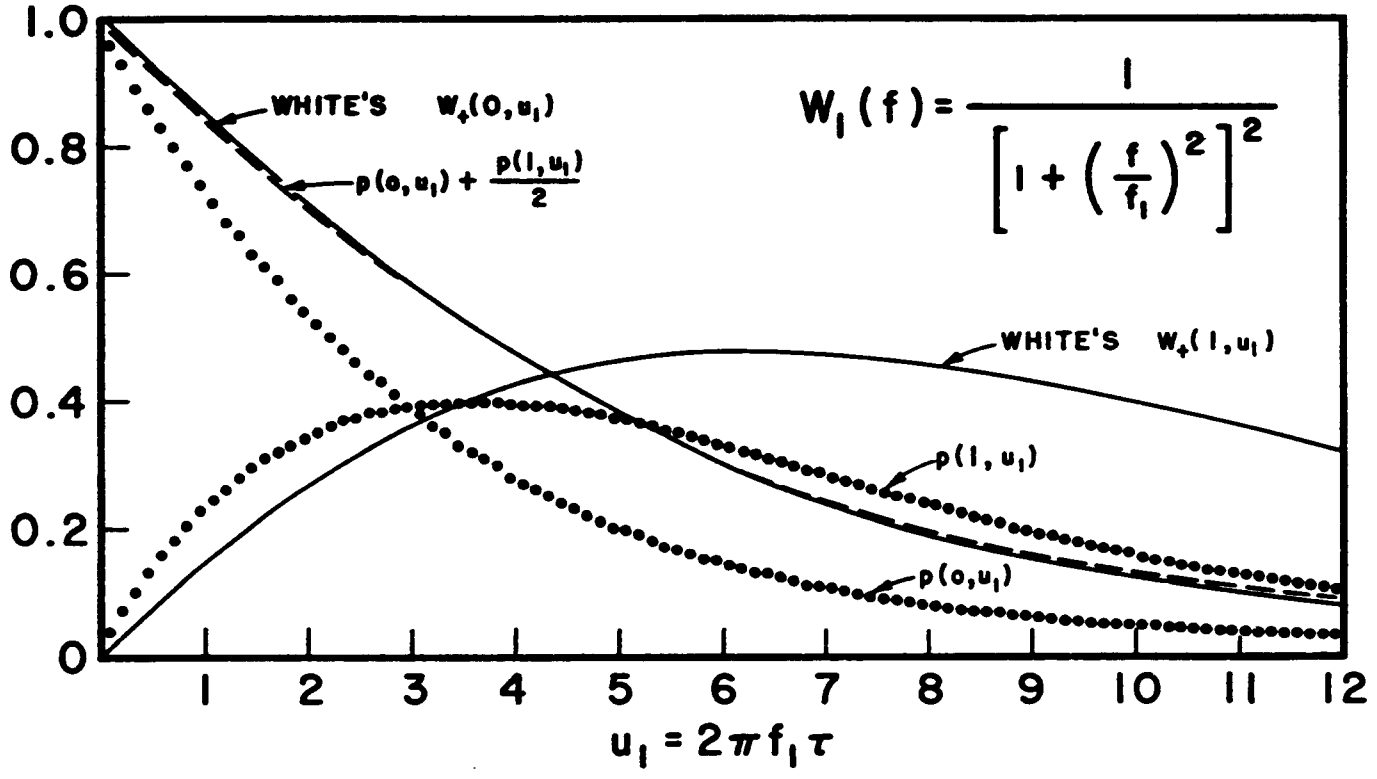


Figure 20 $p(n, u_1)$ is the Probability that a Given Interval u_1 Contains Exactly n Zero-Crossing Points. $p(n, u_1)$ and White's $W_+(n, u_1)$ Probabilities are Compared for the Case of Gaussian Noise Having a Power Spectral Density $W_1(f)$ when n = 0, 1.

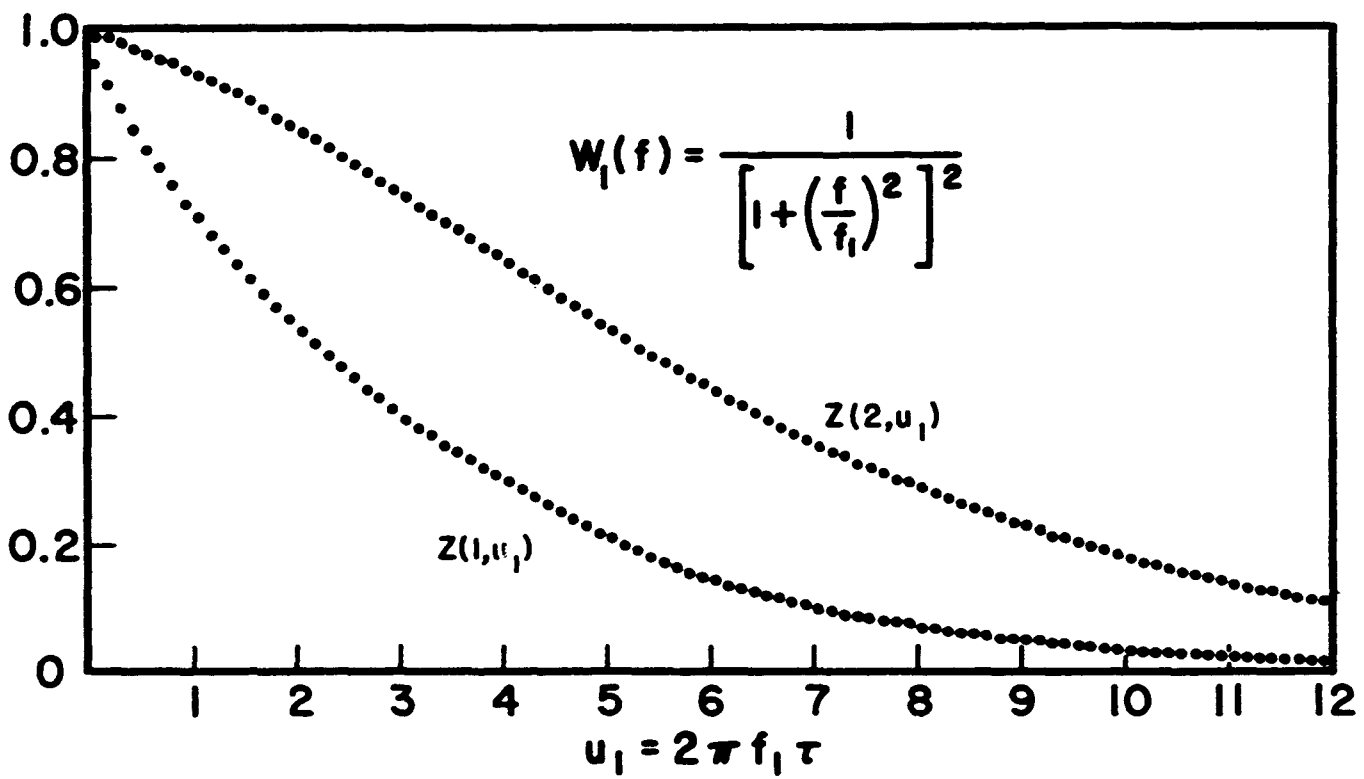


Figure 21 $Z(n, u_1)$ is the Conditional Probability that the nth Zero-Crossing Point from a Given Zero-Crossing Point in du_1 Occurs after the Time du_1+u_1. $Z(1, u_1)$ and $Z(2, u_1)$ are Compared for the Case of Gaussian Noise Having a Power Spectral Density $W_1(f)$.

$$U(\tau) - Q(\tau) \geq P_1(\tau) \text{ for all } \tau. \tag{101}$$

Accordingly, Rice's $U(u_1) - Q(u_1)$ function serves as an upper bound for $P_1(u_1)$ and is presented in Figure 19. Longuet-Higgins (8) deduced bounds for $P_1(0)$ theoretically as $0.0645 < P_1(0) < U(0) - Q(0) = 0.0726$ and deduced the approximate value $P_1(0) \doteq 0.0653$.

G. M. White (18) experimentally determined the following probabilities concerning the zero-crossings of a Gaussian process having a power spectral density $W_1(f)$:

(1) $W_+(0, u_1)$, the probability of having no zero-crossings with positive slope in a time interval u_1.

(2) $W_+(1, u_1)$, the probability of having one zero-crossing with positive slope in a time interval u_1.

In general

$$W_+(0, \tau) = p(0, \tau) + \frac{1}{2} p(1, \tau) \tag{102}$$

and

$$W_+(1, \tau) = \frac{1}{2} p(1, \tau) + p(2, \tau) + \frac{1}{2} p(3, \tau) \tag{103}$$

or

$$p(2, \tau) \leq W_+(1, \tau) - \frac{p(1, \tau)}{2} \quad . \tag{104}$$

Equation (102) states that the interval can contain no upcrossings in two ways: Either it contains no crossings at all, or it contains one. If it contains one, then with conditional probability $\frac{1}{2}$ it is either upward or downward.

Equation (103) states that the interval can contain exactly one upcrossing in three ways: Either it contains one crossing, or two, or three. If it contains one, then with conditional probability $\frac{1}{2}$ it is upward or downward. If it contains two, then one of them must be upward. If it contains three, then one or two of them will be upward, each with probability $\frac{1}{2}$.

Figure 20 compares $p(0, u_1) + \frac{1}{2} p(1, u_1)$ with $W_+(0, u_1)$. The comparison is excellent. Also, Figure 20 presents both $W_+(1, u_1)$ and $p(1, u_1)$. Accordingly, an upper bound for $p(2, u_1)$ can be readily computed from inequality (104). This upper bound agrees satisfactorily with the approximate theoretical initial behavior $p(2,u_1) \sim \frac{0.818}{6\pi} (u_1)^2$ which was deduced by Longuet-Higgins (8).

Figure 19 compares the convolution of $P_o(u_1)$ with itself, $P_o(u_1) * P_o(u_1)$, and $P_1(u_1)$. This comparison serves to demonstrate that two successive zero-crossing intervals are statistically dependent. A theorem is proved in Section IX which asserts that for Gaussian processes the sum of m + 1 successive zero-crossing intervals and the sum of the next n + 1 successive zero-crossing intervals are statistically dependent. The theorem applies for all nonnegative integral m and n.

B. POWER SPECTRAL DENSITY $W_2(f)$

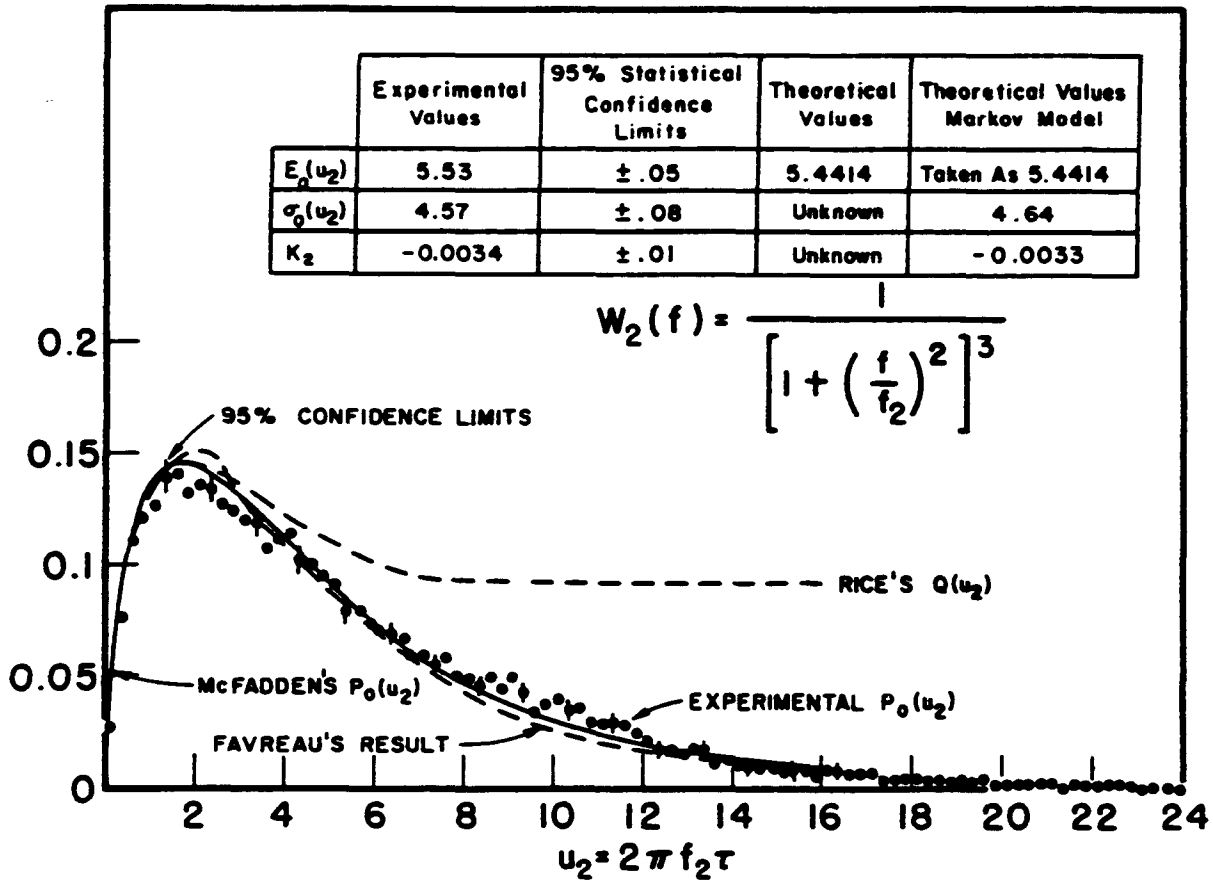
Probabilities and probability densities defined by the zero-crossing points of a Gaussian process having a power spectral

density $W_2(f)$ are presented in Figures 22, 23, 24, and 25. Theoretical approximations derived by Rice (1) and McFadden (2) are presented in Figure 22 for comparison purposes. Figure 22 also presents an experimental probability density reported by Favreau, et al (19). Rice's $U(u_2)-Q(u_2)$ function is presented in Figure 23 for comparison purposes. Figure 23 also compares the convolution of $P_0(u_2)$ with itself, $P_0(u_2) * P_0(u_2)$, and $P_1(u_2)$. This latter comparison was intended to exhibit the statistical dependence of two successive intervals. In this case the convolution function is nearly equal to $P_1(u_2)$. However, as the theorem of Section IX asserts, two successive zero-crossing intervals are still statistically dependent.

The theoretical initial values of $P_0(u_2)$ and $P_1(u_2)$ are zero since Rice's $U(u_2)$ function is initially zero.

C. POWER SPECTRAL DENSITY $W_3(f)$

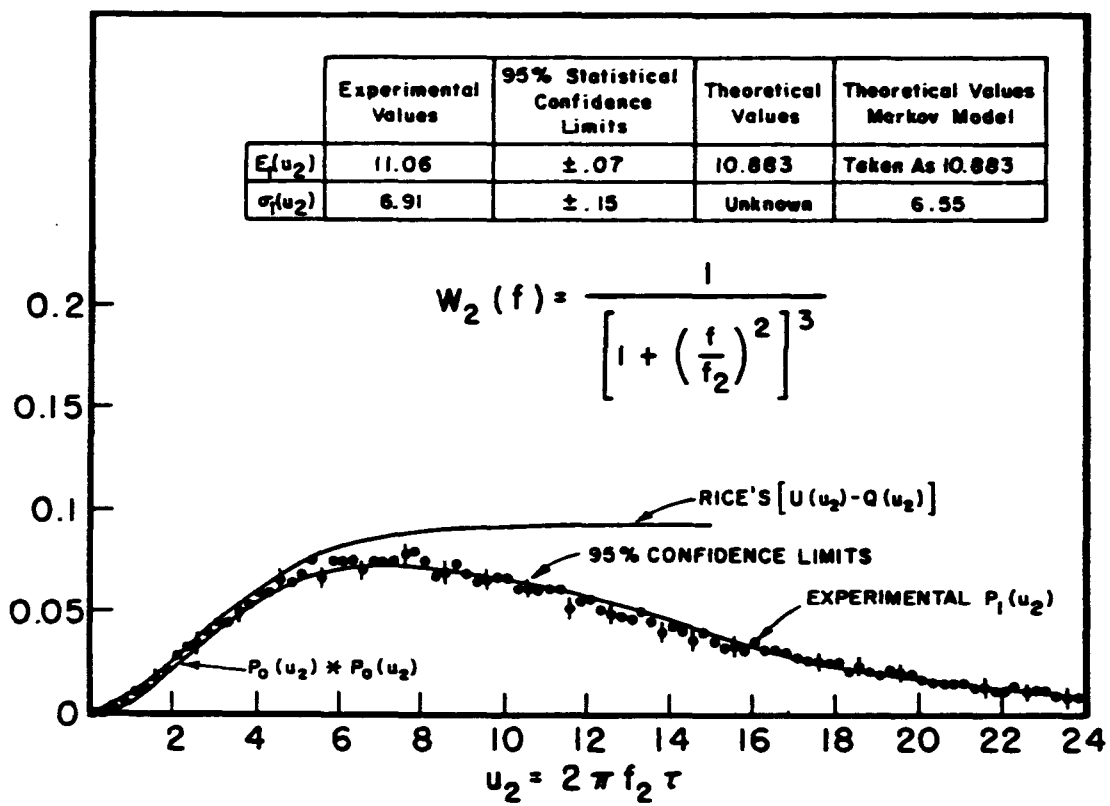
Probabilities and probability densities defined by the zero-crossing points of a Gaussian process having a power spectral density $W_3(f)$ are presented in Figures 26, 27, 28, and 29. Theoretical approximations derived by Rice (1) and McFadden (2) are presented in Figure 26 for comparison purposes. Figure 26 also presents an experimental probability density reported by Favreau, et al. (19).



	Experimental Values	95% Statistical Confidence Limits	Theoretical Values	Theoretical Values Markov Model
$E_0(u_2)$	5.53	±.05	5.4414	Taken As 5.4414
$\sigma_0(u_2)$	4.57	±.08	Unknown	4.64
K_2	-0.0034	±.01	Unknown	-0.0033

$$W_2(f) = \frac{1}{\left[1 + \left(\frac{f}{f_2}\right)^2\right]^3}$$

Figure 22 $P_0(u_2)$ is the Probability Density Function for Successive Zero-Crossing Intervals of Gaussian Noise Having a Power Spectral Density $W_2(f)$. Theoretical Approximations are Compared with Experimental Results.



	Experimental Values	95% Statistical Confidence Limits	Theoretical Values	Theoretical Values Markov Model
$E_1(u_2)$	11.06	±.07	10.883	Taken As 10.883
$\sigma_1(u_2)$	6.91	±.15	Unknown	6.55

Figure 23 $P_1(u_2)$ is the Probability Density Function for the Sum of Two Successive Zero-Crossing Intervals of Gaussian Noise Having a Power Spectral Density $W_2(f)$. $P_0(u_2)*P_0(u_2)$ is Compared with the Experimental Result to Exhibit the Practical Independence of Two Successive Intervals.

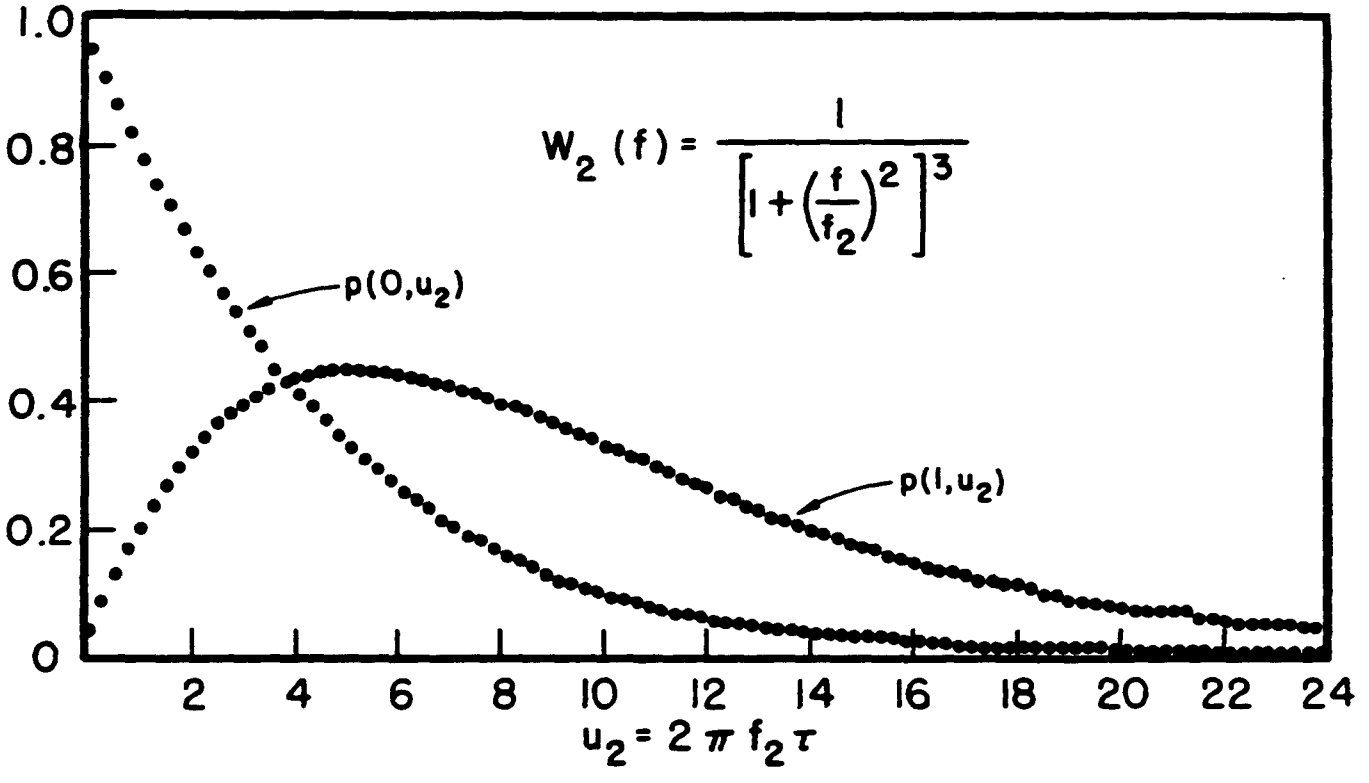


Figure 24 $p(n, u_2)$ is the Probability that a Given Interval u_2 Contains Exactly n Zero-Crossing Points. $p(0, u_2)$ and $p(1, u_2)$ are Compared for the Case of Gaussian Noise Having a Power Spectral Density $W_2(f)$.

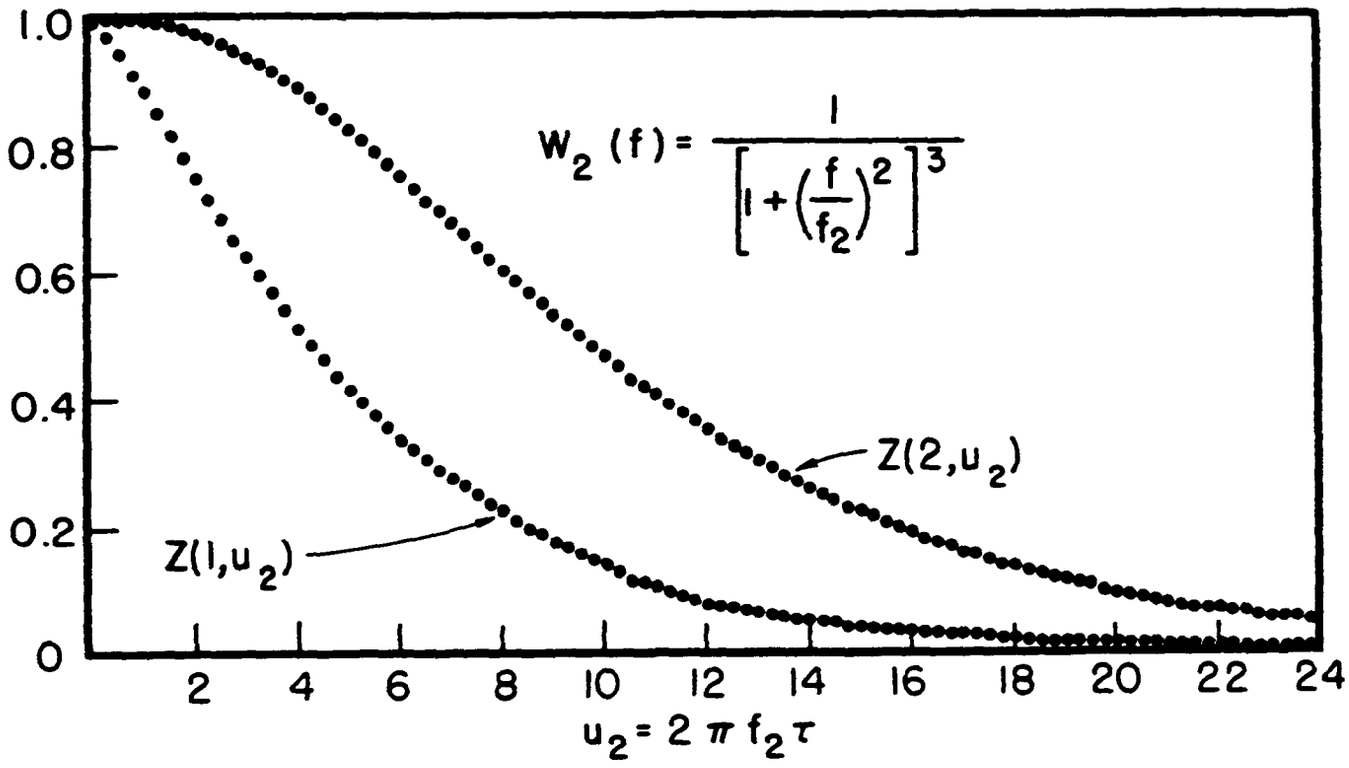
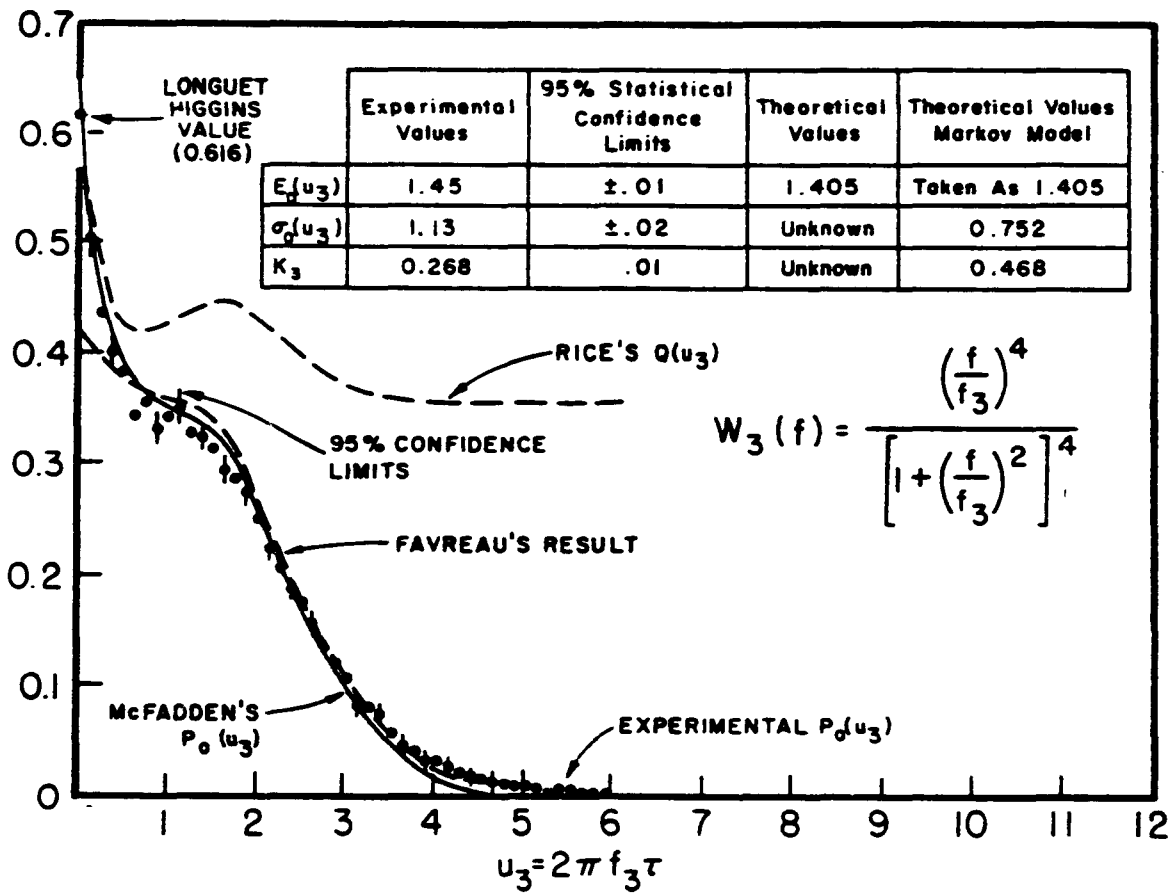


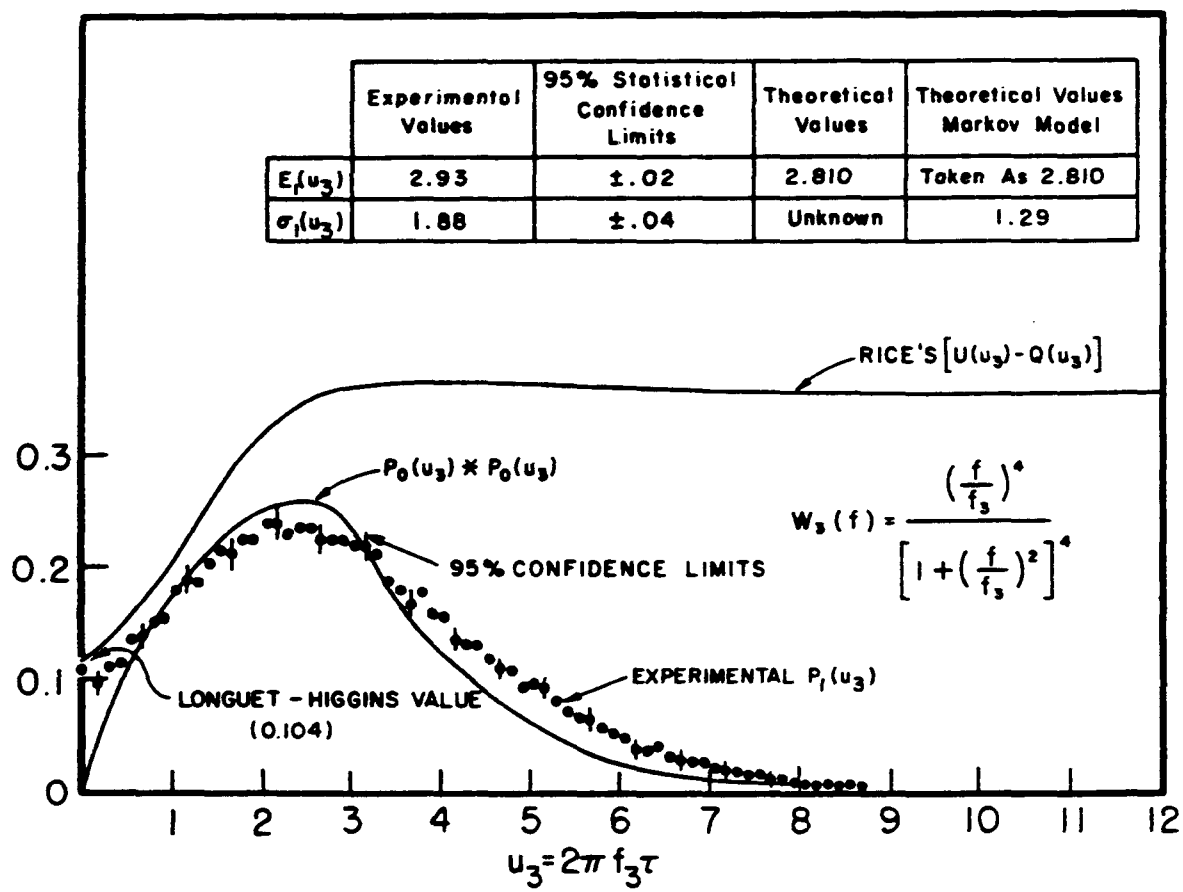
Figure 25 $Z(n, u_2)$ is the Conditional Probability that the nth Zero-Crossing Point from a Given Zero-Crossing Point in du_2 Occurs After the Time du_2+u_2. $Z(1, u_2)$ and $Z(2, u_2)$ are Compared for the Case of Gaussian Noise Having a Power Spectral Density $W_2(f)$.



	Experimental Values	95% Statistical Confidence Limits	Theoretical Values	Theoretical Values Markov Model
$E_0(u_3)$	1.45	±.01	1.405	Taken As 1.405
$\sigma_0(u_3)$	1.13	±.02	Unknown	0.752
K_3	0.268	.01	Unknown	0.468

$$W_3(f) = \frac{\left(\frac{f}{f_3}\right)^4}{\left[1+\left(\frac{f}{f_3}\right)^2\right]^4}$$

Figure 26 $P_0(u_3)$ is the Probability Density Function for Successive Zero-Crossing Intervals of Gaussian Noise Having a Power Spectral Density $W_3(f)$. Theoretical Approximations are Compared with Experimental Results.



	Experimental Values	95% Statistical Confidence Limits	Theoretical Values	Theoretical Values Markov Model
$E_1(u_3)$	2.93	±.02	2.810	Taken As 2.810
$\sigma_1(u_3)$	1.88	±.04	Unknown	1.29

Figure 27 $P_1(u_3)$ is the Probability Density Function for the Sum of Two Successive Zero-Crossing Intervals of Gaussian Noise Having a Power Spectral Density $W_3(f)$. $P_o(u_3)*P_o(u_3)$ is Compared with the Experimental Result to Exhibit the Statistical Dependence of Two Successive Intervals.

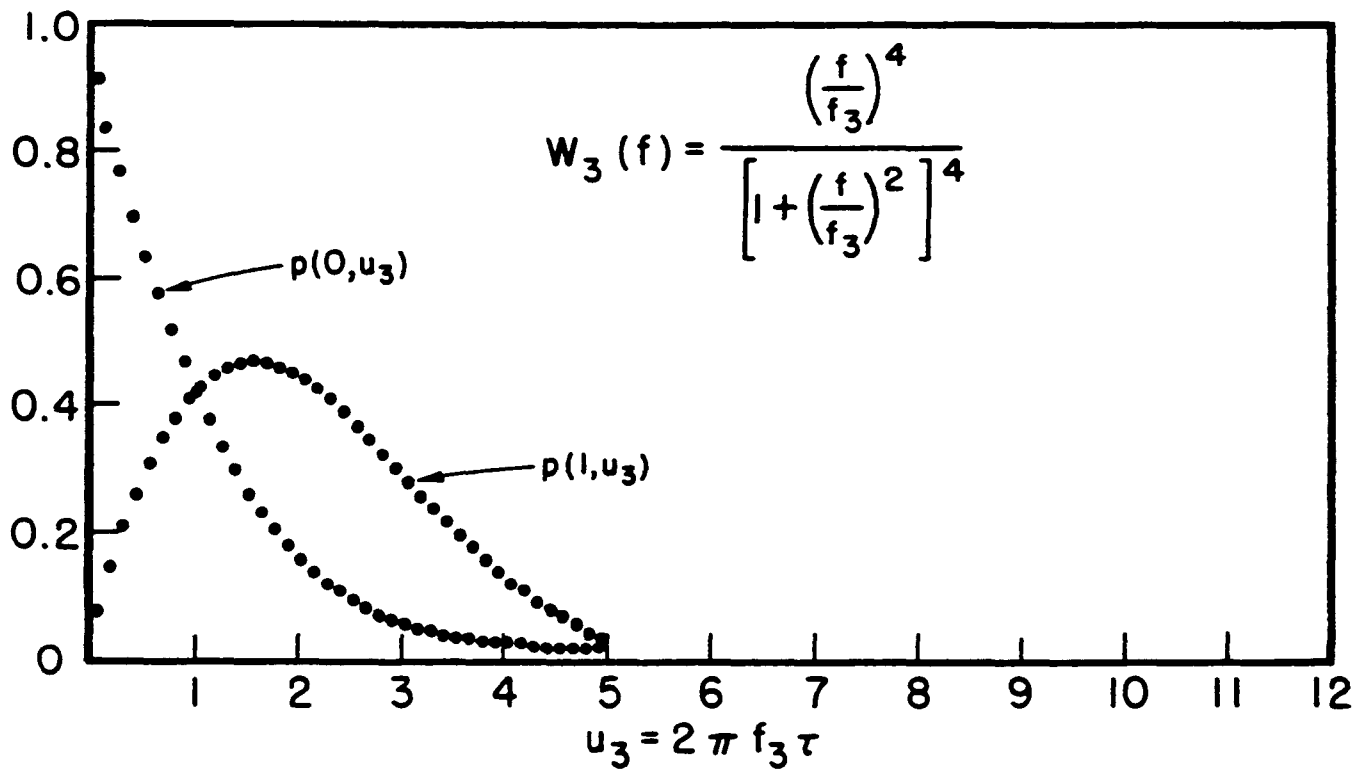


Figure 28 $p(n, u_3)$ is the Probability that a Given Interval u_3 Contains Exactly n Zero-Crossing Points. $p(0, u_3)$ and $p(1, u_3)$ are Compared for the Case of Gaussian Noise Having a Power Spectral Density $W_3(f)$.

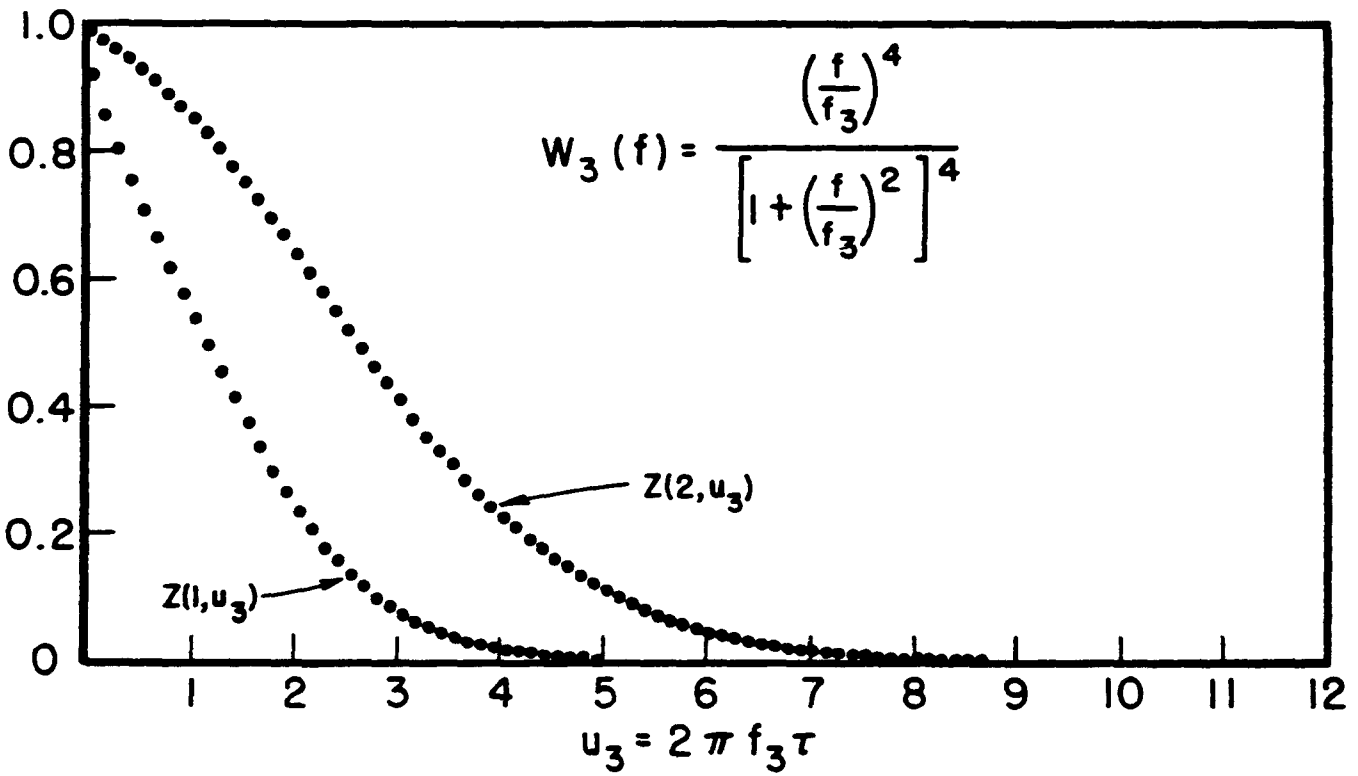


Figure 29 $Z(n, u_3)$ is the Conditional Probability that the nth Zero-Crossing Point from a Given Zero-Crossing Point in du_3 Occurs After the Time du_3+u_3. $Z(1, u_3)$ and $Z(2, u_3)$ are Compared for the Case of Gaussian Noise Having a Power Spectral Density $W_3(f)$.

Rice's $Q(u_3)$ function serves as an upper bound for $P_0(u_3)$ and is presented in Figure 26. Longuet-Higgins (8) deduced bounds for $P_0(0)$ theoretically as $0.612 < P_0(0) < Q(0) = 0.650$ and deduced the approximate value $P_0(0) \doteq 0.616$. Also, Longuet-Higgins (8) deduced bounds for $P_1(0)$ theoretically as $0.103 < P_1(0) < U(0) - Q(0) = 0.116$ and deduced the approximate value $P_1(0) \doteq 0.104$.

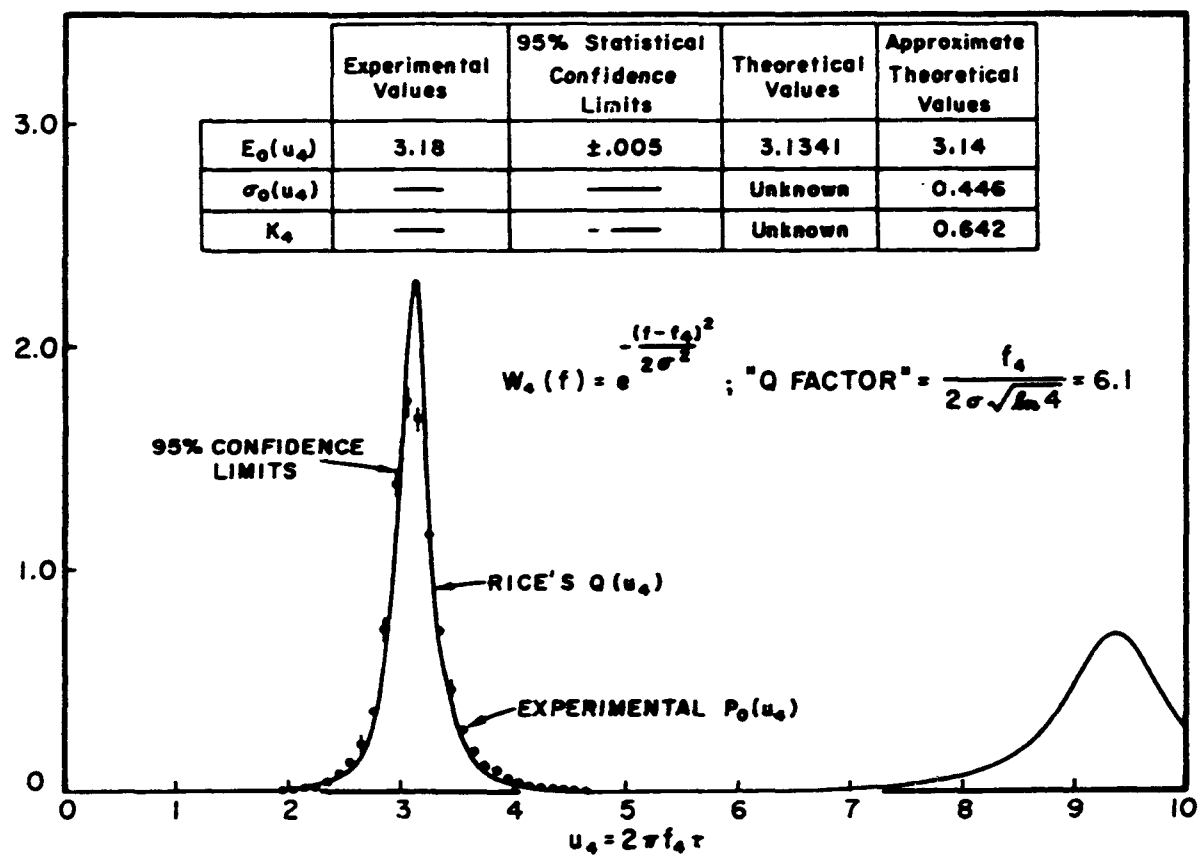
The experimental probabilities $p(0, u_n)$ for $n = 1, 2, 3$ satisfy theorems 1, 3 and Equation (22) of D. Slepian's (13) recent paper regarding bounds for these probabilities.

Rice's $\left[U(u_3)-Q(u_3)\right]$ function is presented in Figure 27 for comparison purposes. Figure 27 also compares the convolution of $P_0(u_3)$ with itself, $P_0(u_3) * P_0(u_3)$, and $P_1(u_3)$. This latter comparison serves to demonstrate that two successive zero-crossing intervals are statistically dependent.

The experimental probability densities $P_0(u_n)$ for $n = 1, 2, 3$ agree very well with theoretical approximations recently published by Longuet-Higgins (8). The approximate theoretical initial values of Longuet-Higgins were used in the normalization of the experimental densities $P_0(u_n)$ and $P_1(u_n)$ for $n = 1, 3$.

D. POWER SPECTRAL DENSITY $W_4(f)$

Probabilities and probability densities defined by the zero-crossing points of a Gaussian process having a power spectral density $W_4(f)$ are presented in Figures 30, 31, 32, and 33. Theoretical approximations derived by Rice (1) are presented in Figures 30 and 31 for



	Experimental Values	95% Statistical Confidence Limits	Theoretical Values	Approximate Theoretical Values
$E_0(u_4)$	3.18	±.005	3.1341	3.14
$\sigma_0(u_4)$	—	—	Unknown	0.446
K_4	—	—	Unknown	0.642

Figure 30 $P_0(u_4)$ is the Probability Density Function for Successive Zero-Crossing Intervals of Gaussian Noise Having a Power Spectral Density $W_4(f)$. Theoretical Approximations are Compared with Experimental Results.

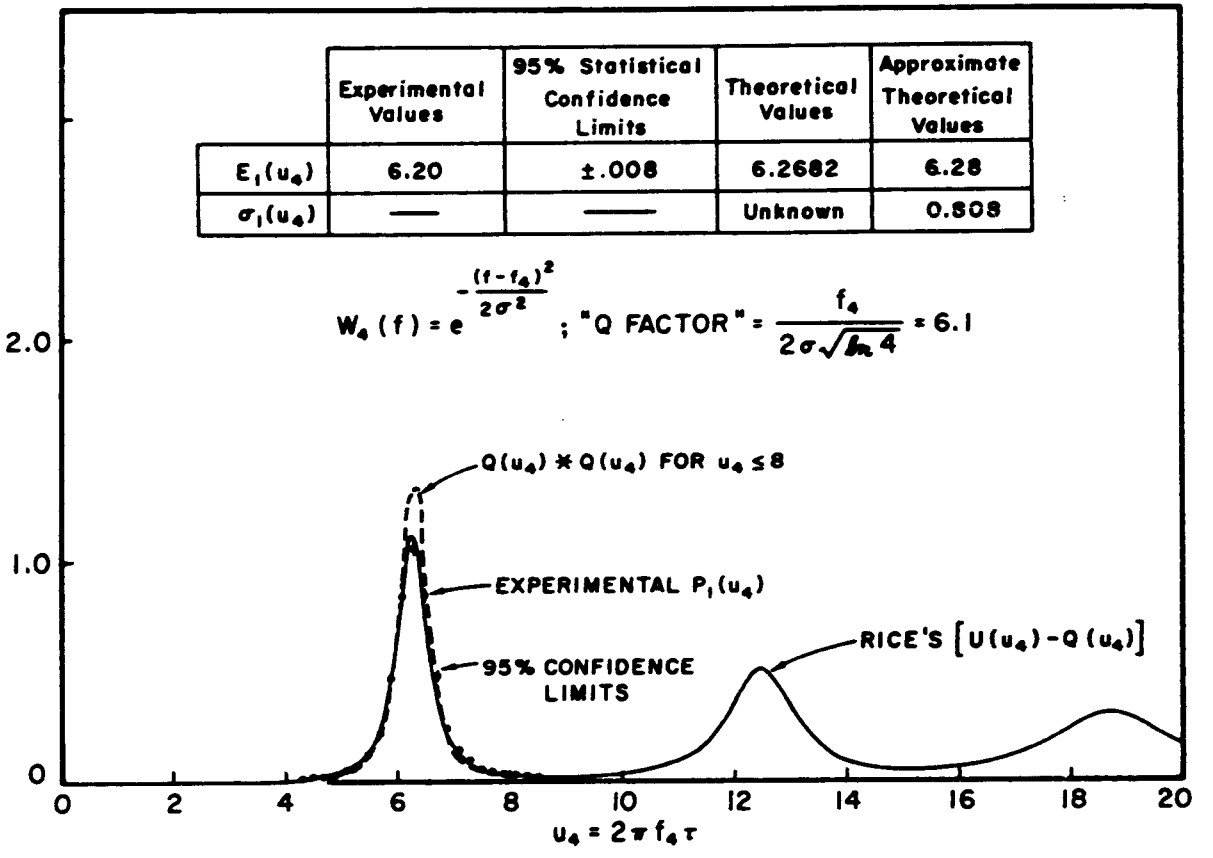


Figure 31 $P_1(u_4)$ is the Probability Density Function for the Sum of Two Successive Zero-Crossing Intervals of Gaussian Noise Having a Power Spectral Density $W_4(f)$. $Q(u_4)*Q(u_4)$ is Compared with $U(u_4)-Q(u_4)$ to Exhibit the Statistical Dependence of Two Successive Intervals.

comparison purposes. Figure 31 also compares the convolution of the first normalized portion of $Q(u_4)$ with itself, $Q(u_4)*Q(u_4)$, and the first normalized portion of $U(u_4) - Q(u_4)$. In accordance with Equations (98) and (99), this comparison serves to demonstrate that two successive zero-crossing intervals are statistically dependent. The theoretical initial values of $P_o(u_4)$ and $P_1(u_4)$ are zero since Rice's $U(u_4)$ function is initially zero.

The results in Figures 32 and 33 were computed by using Equations (39), (40), (41), and (42). The first normalized portions of $Q(u_4)$ and $U(u_4) - Q(u_4)$ served as $P_o(u_4)$ and $P_1(u_4)$ respectively.

In order to obtain good experimental estimates of $\sigma_o(u_4)$ and $\sigma_1(u_4)$, we see from Figures 30 and 31 that the experimental densities $P_o(u_4)$ and $P_1(u_4)$ must be measured with finer resolution. Accordingly, our present experimental estimates of $\sigma_o(u_4)$ and $\sigma_1(u_4)$ are not worth reporting.

Notice that a "Q factor" of 6.1 as defined in Equation (18) is approximately the minimum "Q factor" representing a "narrow-band system" as defined in communication theory. Accordingly, the problem of theoretically determining $P_o(\tau)$ and $P_1(\tau)$ for narrow-band Gaussian processes is practically solved by using Rice's $Q(\tau)$ and $\left[U(\tau) - Q(\tau)\right]$ functions as the generating functions of $P_o(\tau)$ and $P_1(\tau)$ in accordance with Equations (98) and (99).

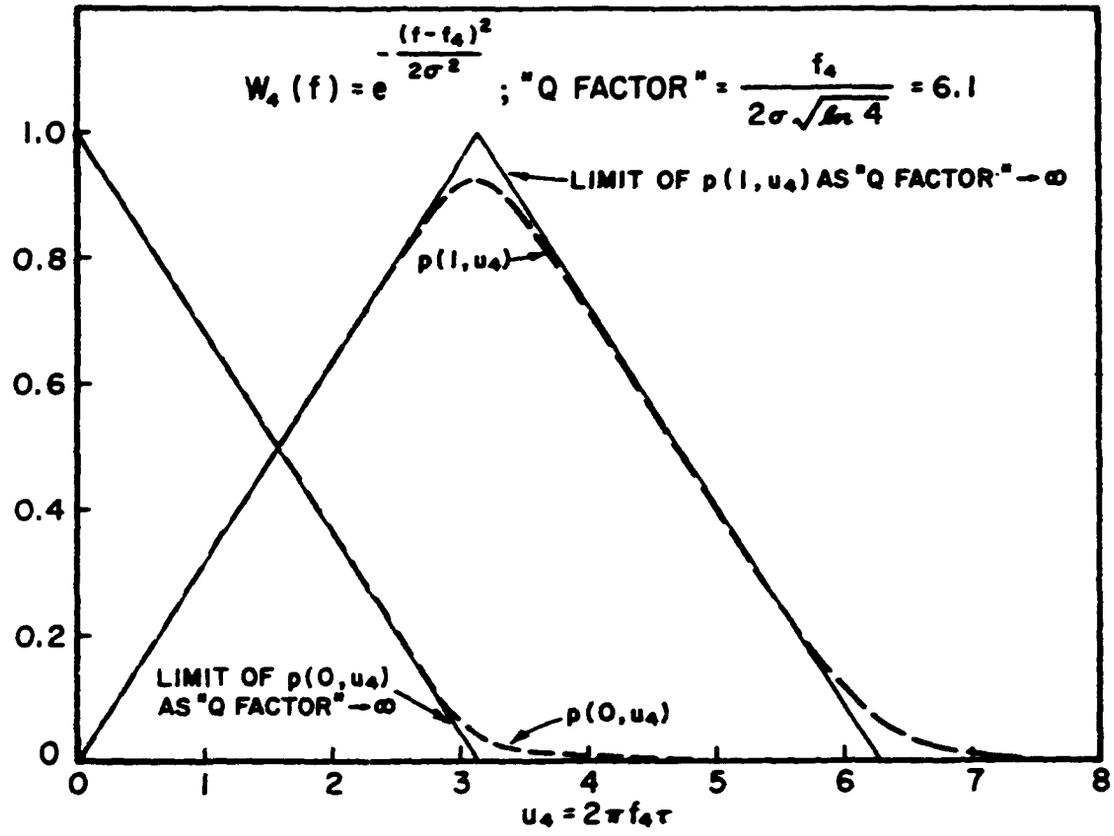


Figure 32 $p(n, u_4)$ is the Probability That a Given Interval u_4 Contains Exactly n Zero-Crossing Points. $p(0, u_4)$ and $p(1, u_4)$ are Compared for the Case of Gaussian Noise Having the Power Spectral Density $W_4(f)$.

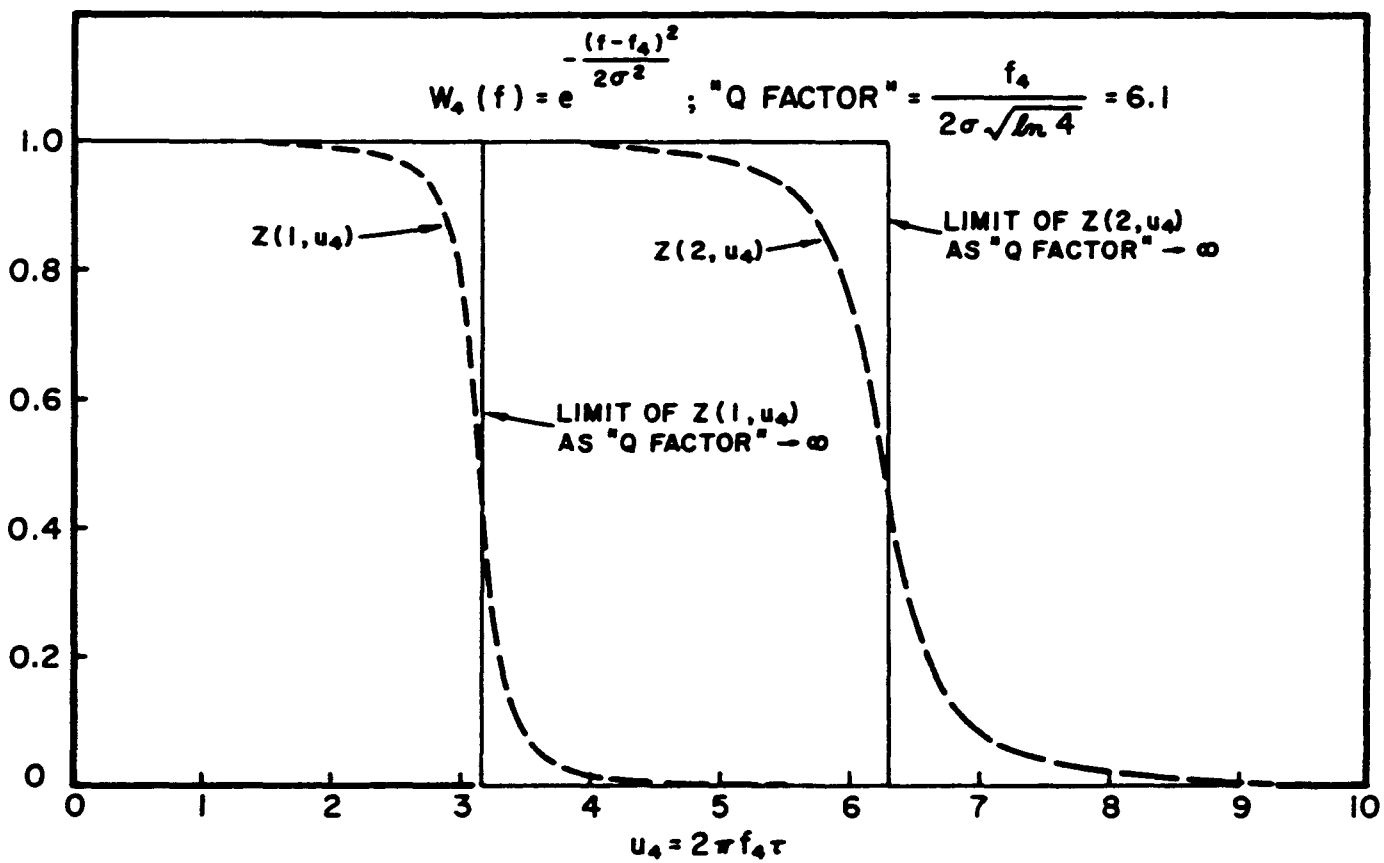
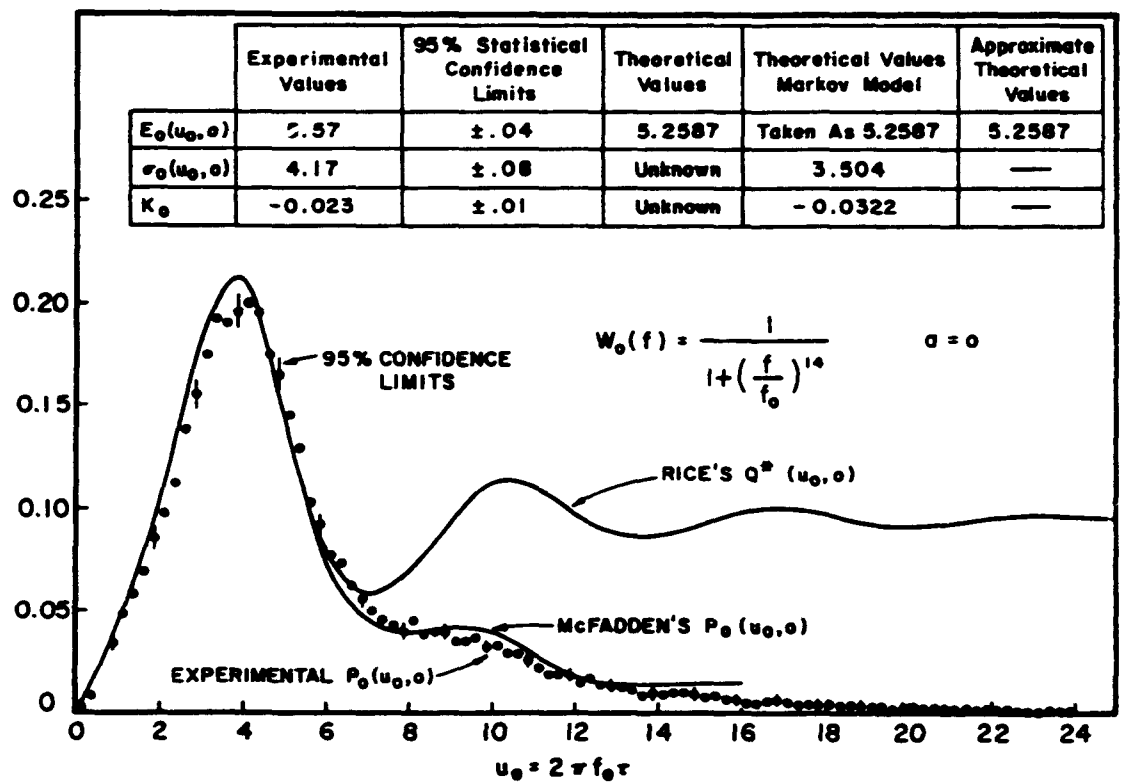


Figure 33 $Z(n, u_4)$ is the Conditional Probability that the nth Zero-Crossing Point From a Given Zero-Crossing Point in du_4 Occurs After the Time du_4+u_4. $Z(1, u_4)$ and $Z(2, u_4)$ are Compared for the Case of Gaussian Noise Having a Power Spectral Density $W_4(f)$.

A portion of the above work dealing with Gaussian processes has been published recently in the IRE Transactions on Information Theory (35).

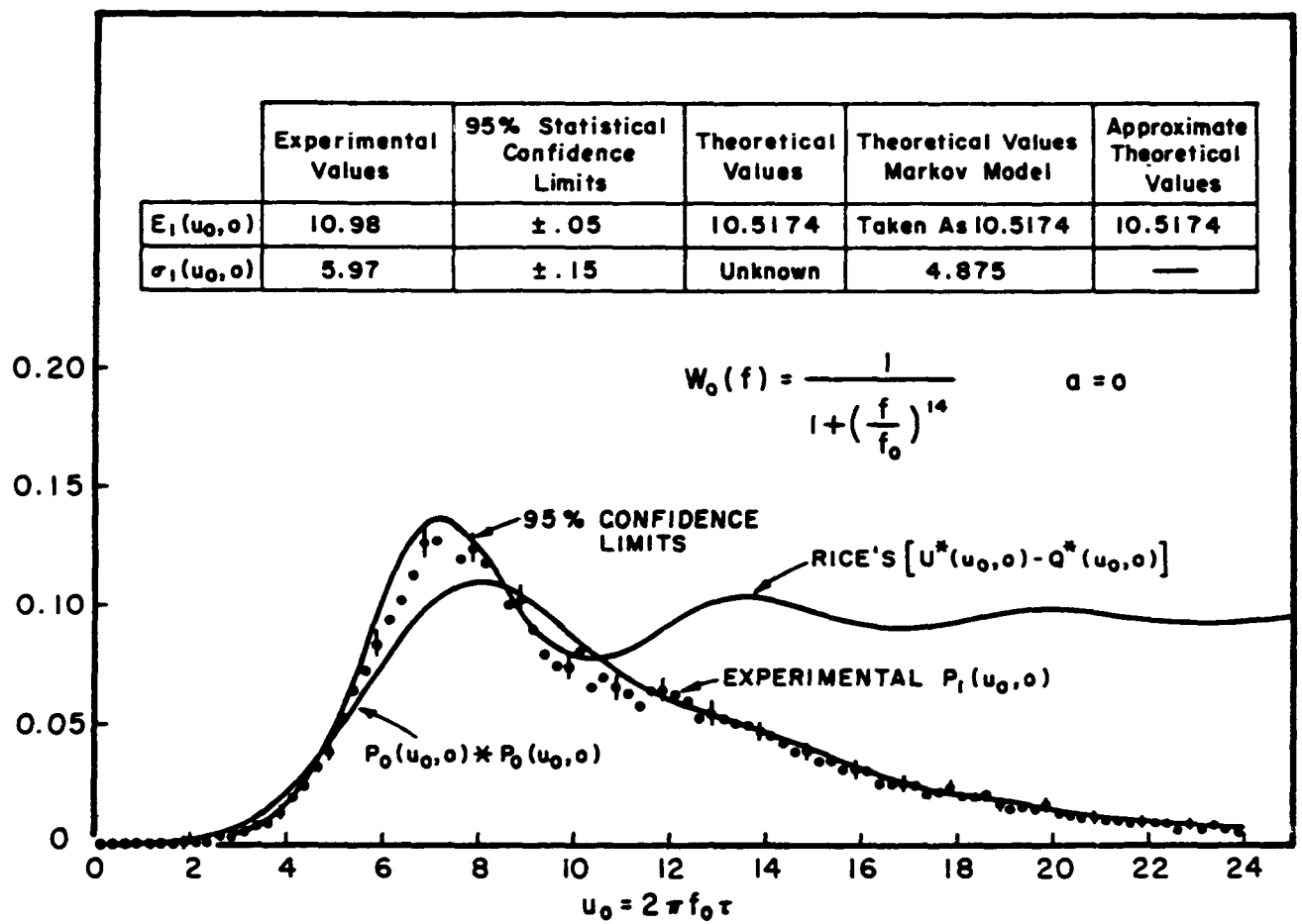
VII. PROBABILITIES AND PROBABILITY DENSITIES DEFINED BY THE ZERO POINTS OF A SINE WAVE PLUS A GAUSSIAN PROCESS

Probabilities and probability densities defined by the zero-crossing points of the random process, $\xi(u, a)$, consisting of a sine wave of frequency $\frac{f_o}{2}$ plus Gaussian noise having a power spectral density $W_o(f)$ are presented in Figures 34 thru 49. The ratio of the average sine wave power, S, to the average noise power, N, is denoted by the parameter a. For small values of $a = \frac{S}{N}$, the random process $\xi(u, a)$ is approximately Gaussian with normalized autocorrelation function, $R(u_o, a)$, given by Equation (64). Accordingly, Rice's probability function $Q^*(u_o, a)du_o$ approximates the conditional probability that a downward zero-crossing of $\xi(u, a)$ occurs between $u + u_o$ and $u + u_o + du_o$, given an upward zero-crossing of $\xi(u, a)$ at time u. In accordance with Equation (98), $Q^*(u_o, a)$ serve as approximations to the initial behavior of $P_o(u_o, a)$ for small "a" and are presented in Figures 34, 38, 42, and 46. Figure 34 also presents McFadden's theoretical approximation for $P_o(u_o, 0)$. $Q^*(u_0, a)\, du_0$ is the exact conditional probability that a downward zero-crossing of $\xi^*(u, a)$ occurs between $u + u_0$ and $u + u_0 + du_0$, given an upward zero-crossing of $\xi^*(u, a)$ at time u. The process $\xi^*(u, a)$ is defined by Equation (81) and the process $\xi(u, a)$ is defined by Equation (82). When a = 0 the processes $\xi(u, 0)$ and $\xi^*(u, 0)$ are identical.



	Experimental Values	95% Statistical Confidence Limits	Theoretical Values	Theoretical Values Markov Model	Approximate Theoretical Values
$E_o(u_o, o)$	5.57	±.04	5.2587	Taken As 5.2587	5.2587
$\sigma_o(u_o, o)$	4.17	±.08	Unknown	3.504	—
K_o	-0.023	±.01	Unknown	-0.0322	—

Figure 34 $P_o(u_o, a)$ is the Probability Density Function for Successive Zero-Crossing Intervals of a Random Process Consisting of Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a. Theoretical Approximations are Compared with Experimental Results.



	Experimental Values	95% Statistical Confidence Limits	Theoretical Values	Theoretical Values Markov Model	Approximate Theoretical Values
$E_1(u_0,0)$	10.98	±.05	10.5174	Taken As 10.5174	10.5174
$\sigma_1(u_0,0)$	5.97	±.15	Unknown	4.875	—

Figure 35 $P_1(u_0, a)$ is the Probability Density Function for the Sum of Two Successive Zero-Crossing Intervals of a Random Process Consisting of a Sine Wave of Frequency $f_0/2$ Plus Gaussian Noise Having the Power Spectral Density $W_0(f)$. The Ratio of the Average Noise Power is Denoted by the Parameter a.

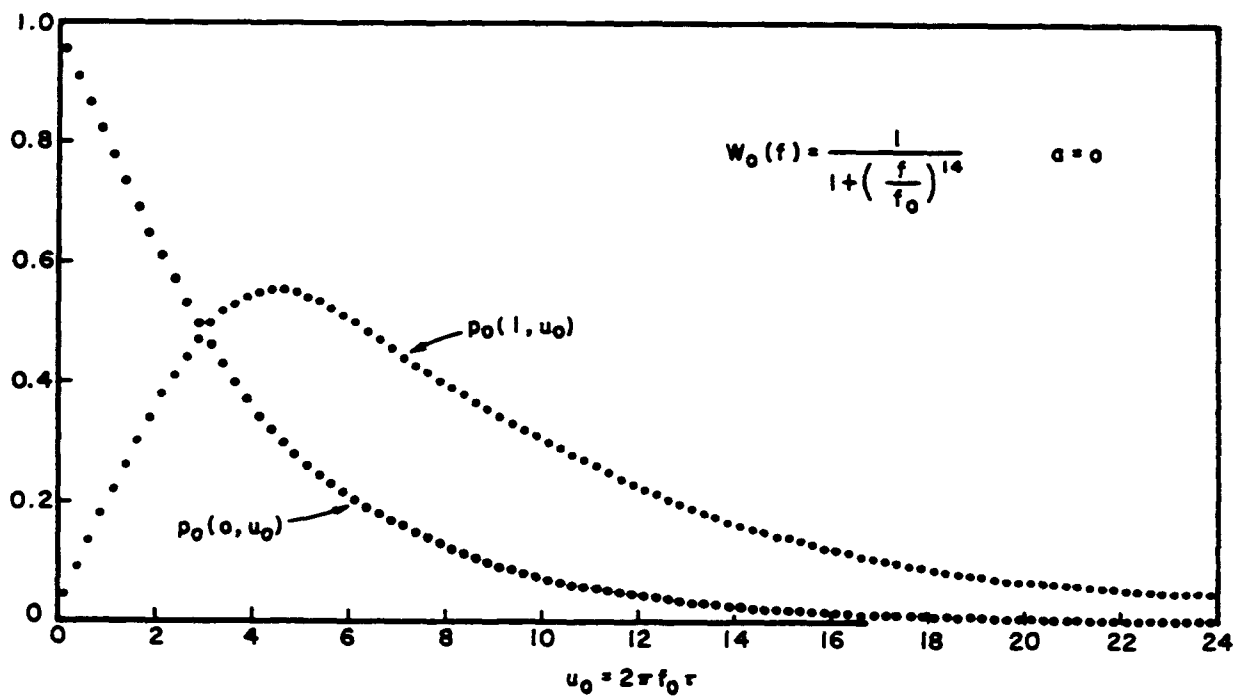


Figure 36

$p_a(n, u_o)$ is the Probability that a Given Interval u_o Contains Exactly n Zero-Crossing Points of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by The Parameter a.

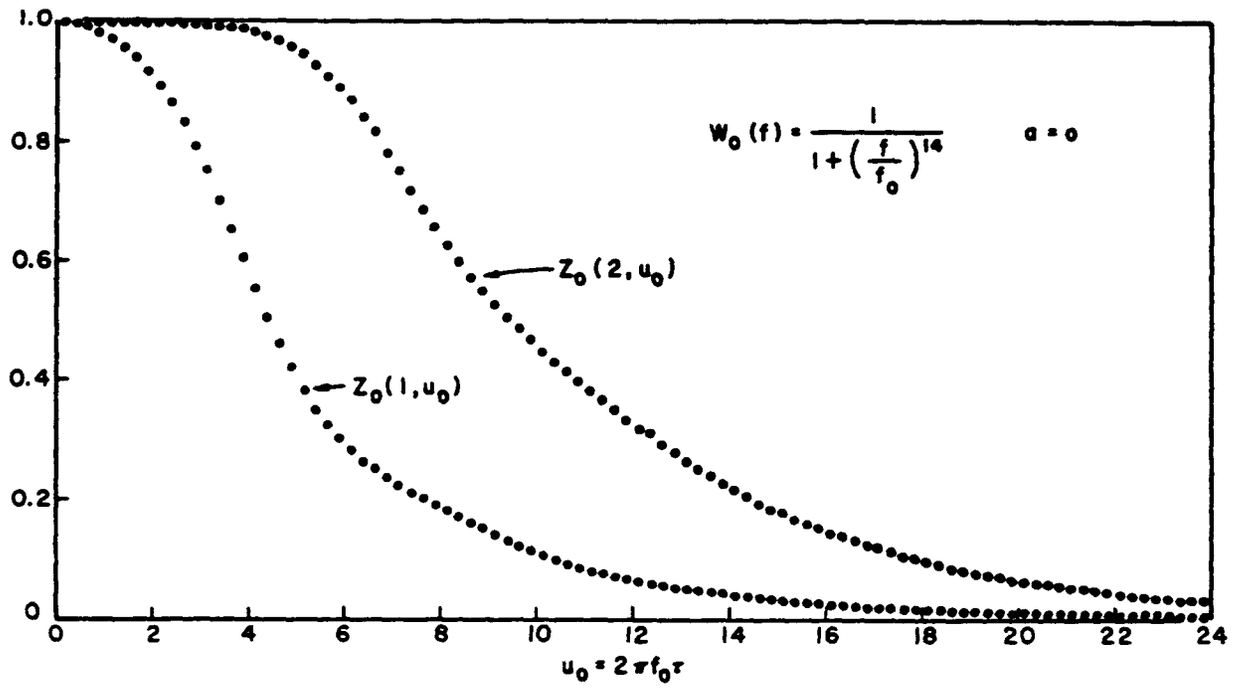
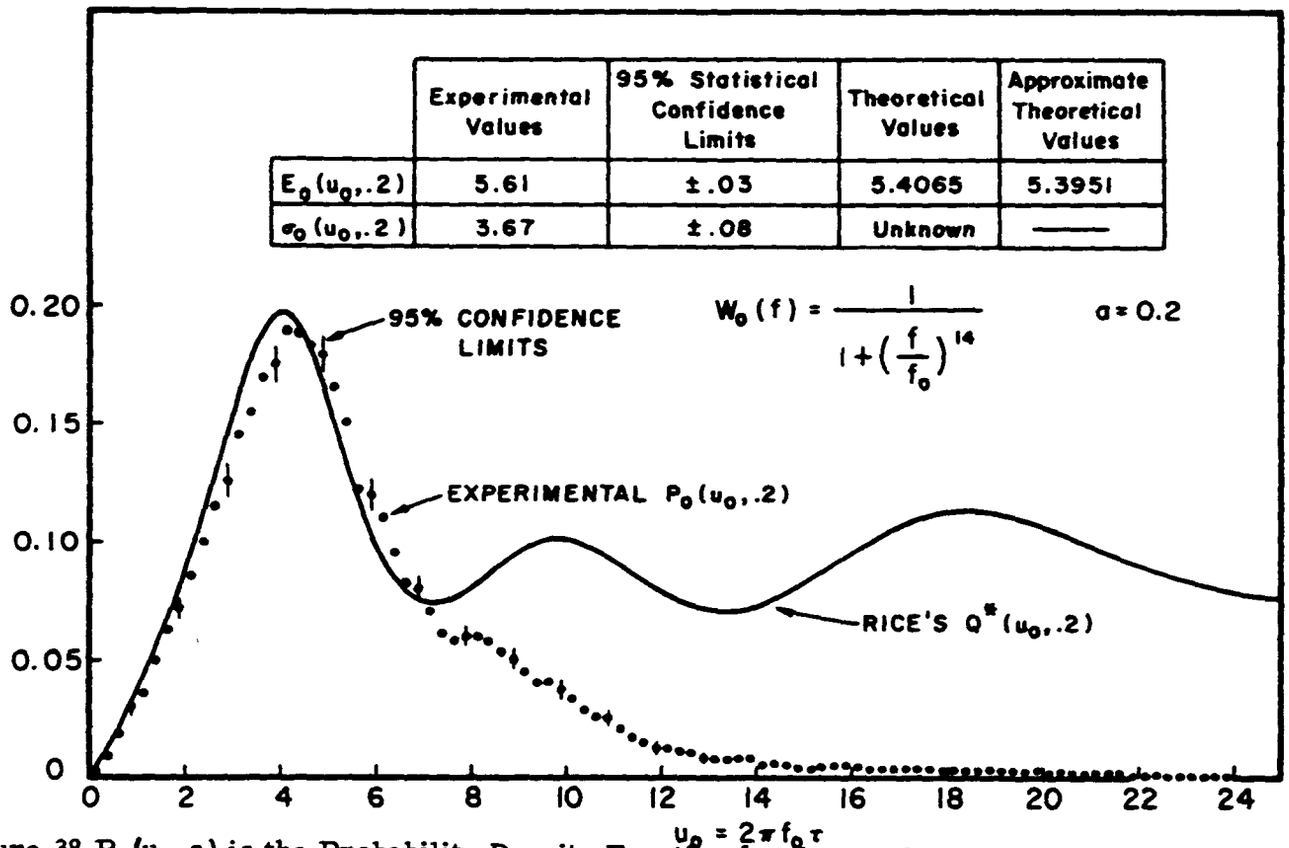
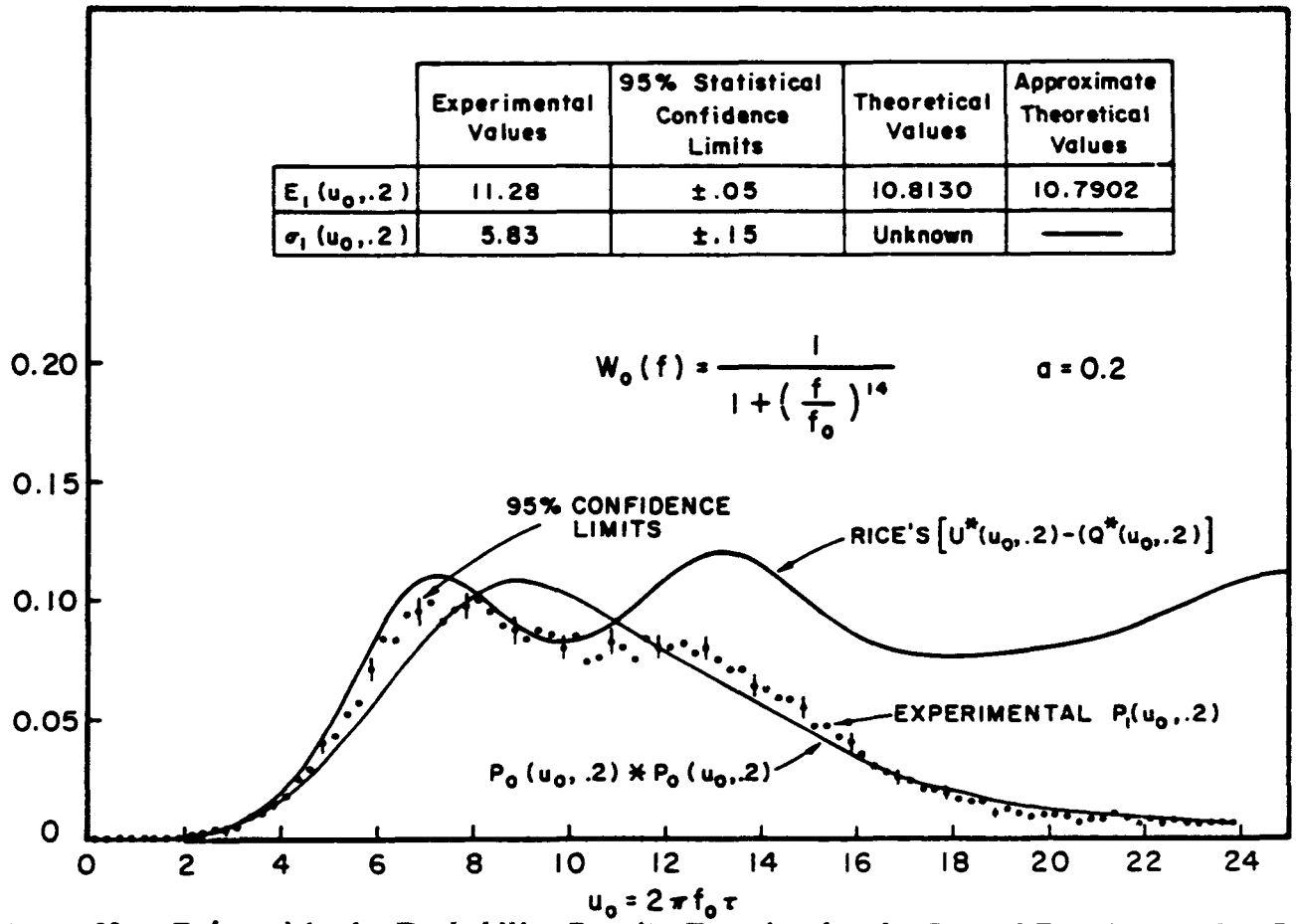


Figure 37 A Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$ Defines $Z_a(n, u_o)$, the Conditional Probability That the nth Zero-Crossing Point From a Given Zero-Crossing Point in du_o Occurs After the Time $du_o + u_o$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.



	Experimental Values	95% Statistical Confidence Limits	Theoretical Values	Approximate Theoretical Values
$E_0(u_0, .2)$	5.61	±.03	5.4065	5.3951
$\sigma_0(u_0, .2)$	3.67	±.08	Unknown	——

Figure 38 $P_0(u_0, a)$ is the Probability Density Function for Successive Zero-Crossing Intervals of a Random Process Consisting of a Sine Wave of Frequency $f_0/2$ Plus Gaussian Noise Having the Power Spectral Density $W_0(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a. Theoretical Approximations are Compared With Experimental Results.



	Experimental Values	95% Statistical Confidence Limits	Theoretical Values	Approximate Theoretical Values
$E_1(u_0,.2)$	11.28	±.05	10.8130	10.7902
$\sigma_1(u_0,.2)$	5.83	±.15	Unknown	——

Figure 39 $P_1(u_0, a)$ is the Probability Density Function for the Sum of Two Successive Zero-Crossing Intervals of a Random Process Consisting of a Sine Wave of Frequency Plus Gaussian Noise Having the Power Spectral Density $W_0(f)$. The Ratio to the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

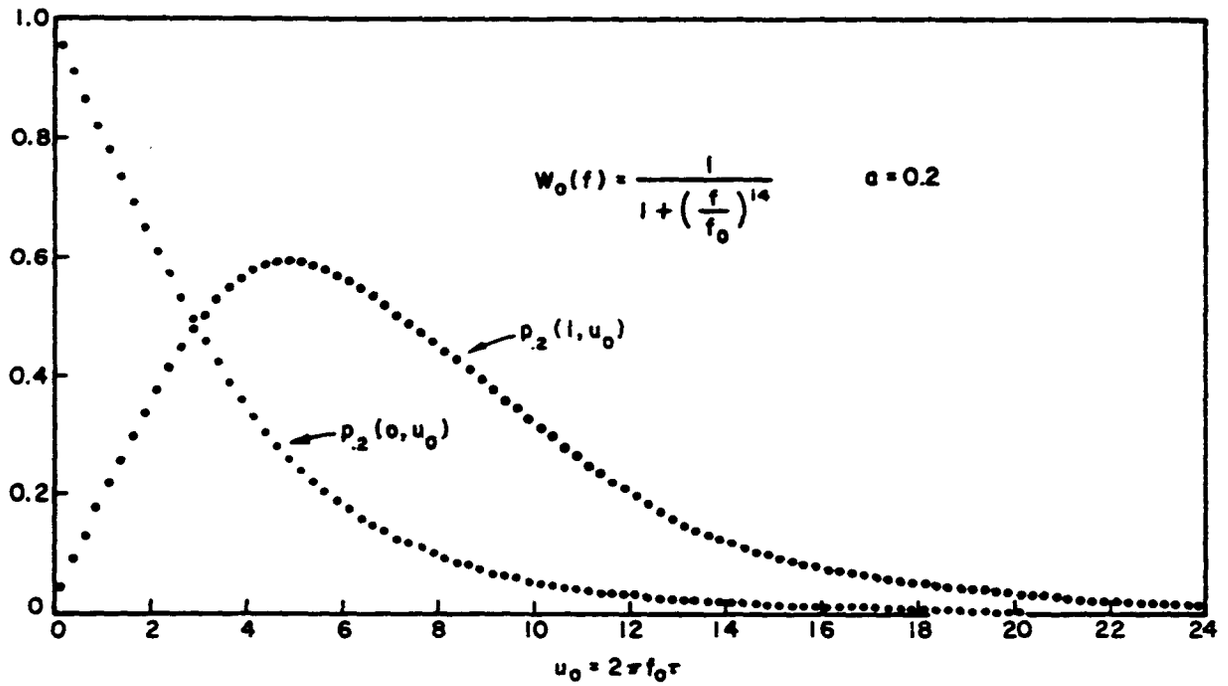


Figure 40 $p_a(n, u_o)$ is the Probability that a Given Interval u_o Contains Exactly n Zero-Crossing Points of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

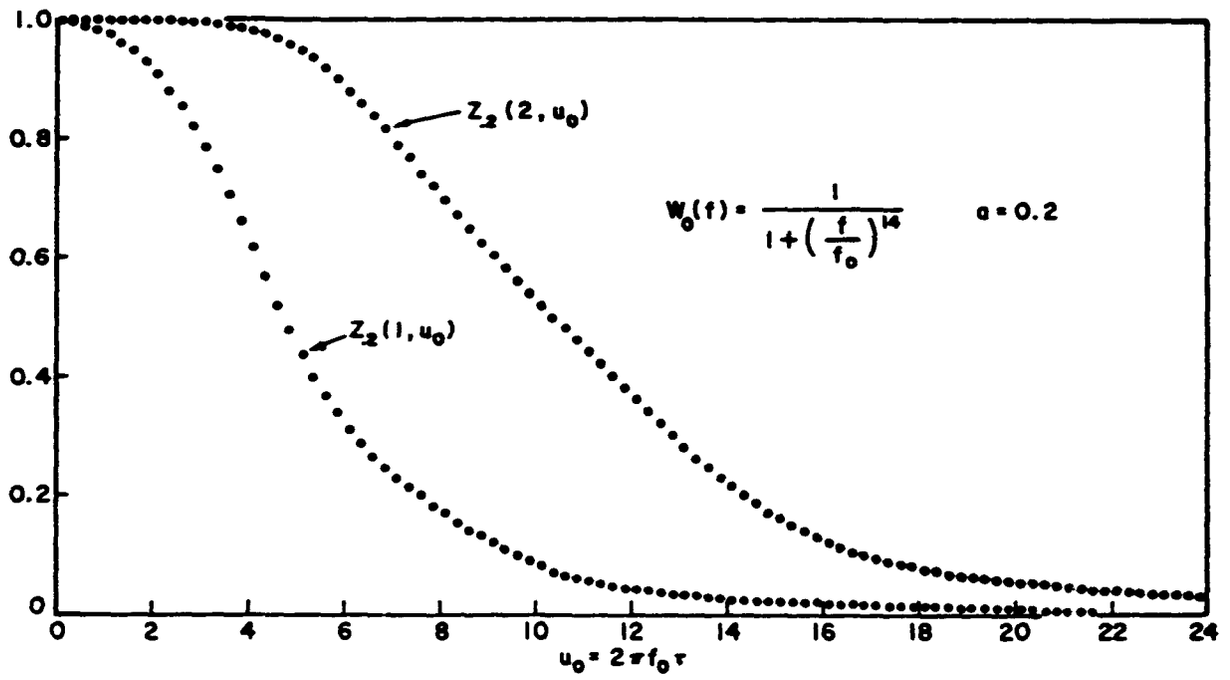
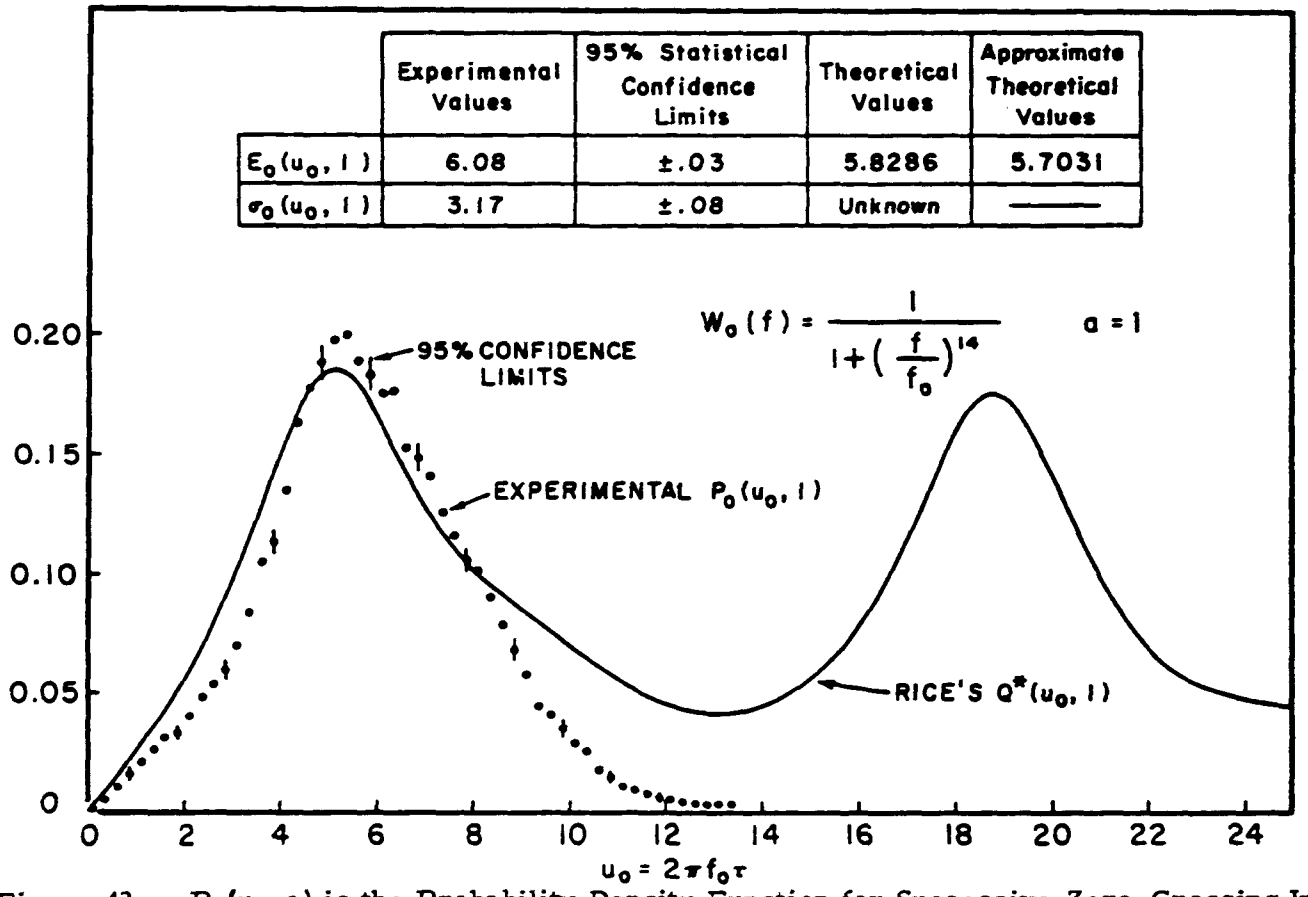
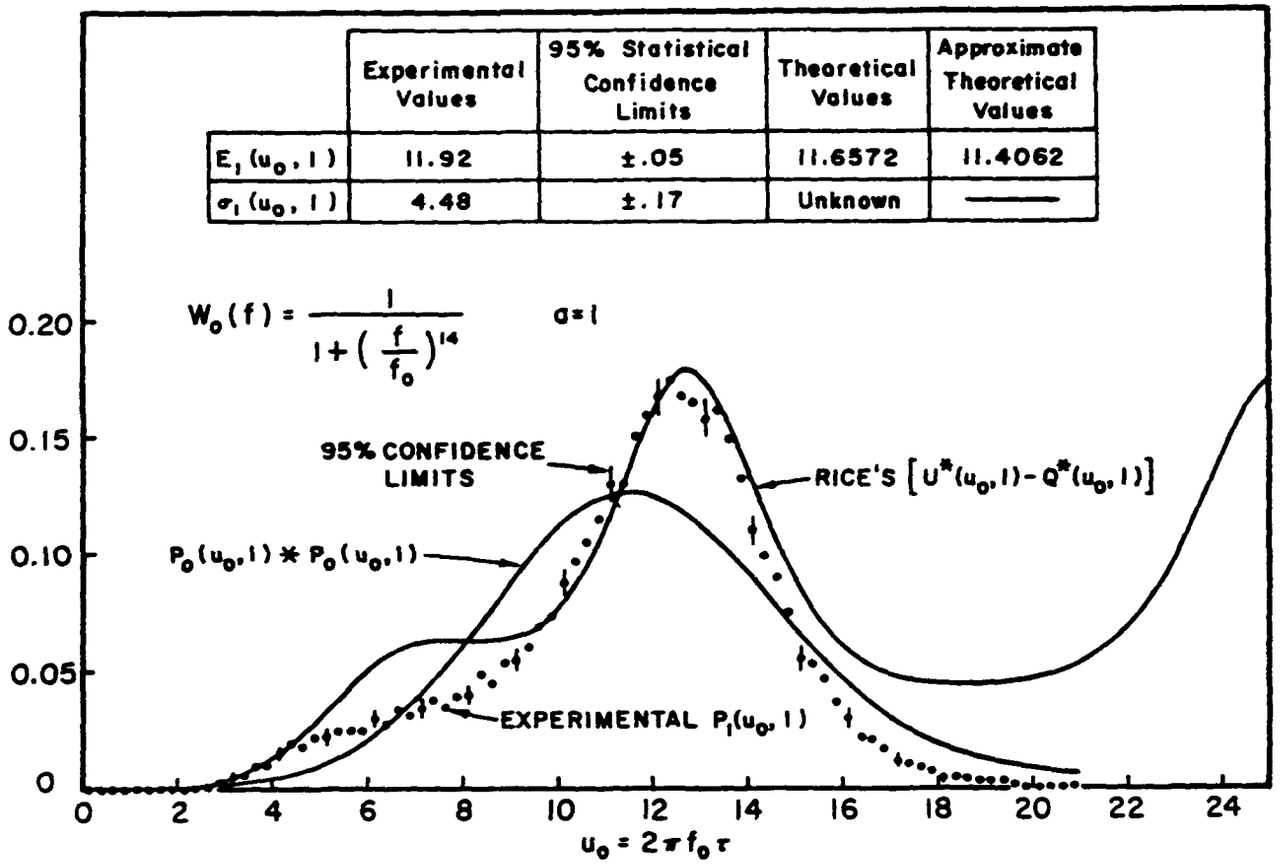


Figure 41 A Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$ Defines $Z_a(n, u_o)$, the Conditional Probability that the nth Zero-Crossing Point in du_o Occurs After the Time $du_o + u_o$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.



	Experimental Values	95% Statistical Confidence Limits	Theoretical Values	Approximate Theoretical Values
$E_0(u_0, 1)$	6.08	±.03	5.8286	5.7031
$\sigma_0(u_0, 1)$	3.17	±.08	Unknown	———

Figure 42 $P_0(u_0, a)$ is the Probability Density Function for Successive Zero-Crossing Intervals of a Random Process Consisting of a Sine Wave of Frequency $f_0/2$ Plus Gaussian Noise Having the Power Spectral Density $W_0(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a. Theoretical Approximations are Compared with Experimental Results.



	Experimental Values	95% Statistical Confidence Limits	Theoretical Values	Approximate Theoretical Values
$E_1(u_0, 1)$	11.92	±.05	11.6572	11.4062
$\sigma_1(u_0, 1)$	4.48	±.17	Unknown	———

Figure 43 $P_1(u_0, a)$ is the Probability Density Function for the Sum of Two Successive Zero-Crossing Intervals of a Random Process Consisting of a Sine Wave of Frequency $f_0/2$ Plus Gaussian Noise Having the Power Spectral Density $W_0(f)$. The Ratio of the Average Sine Wave Power to The Average Noise Power is Denoted by the Parameter a.

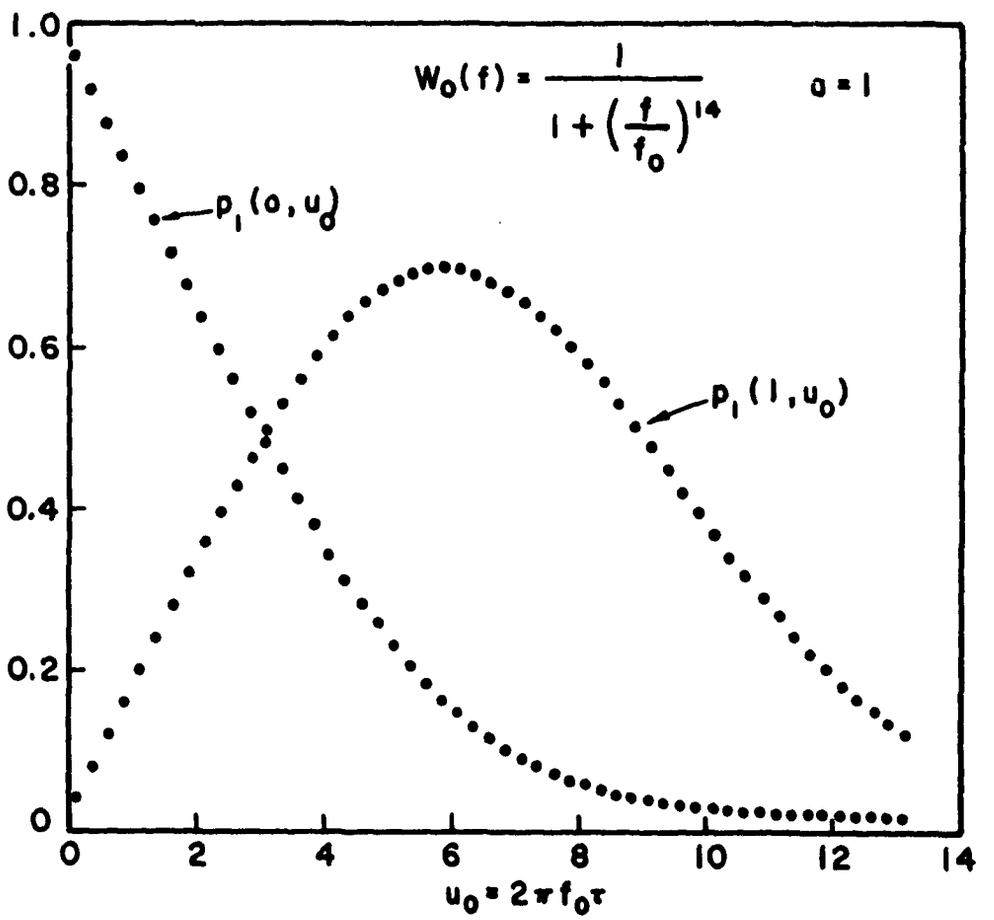


Figure 44 $p_a(n, u_o)$ is the Probability that a Given Interval u_o Contains Exactly n Zero-Crossing Points of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

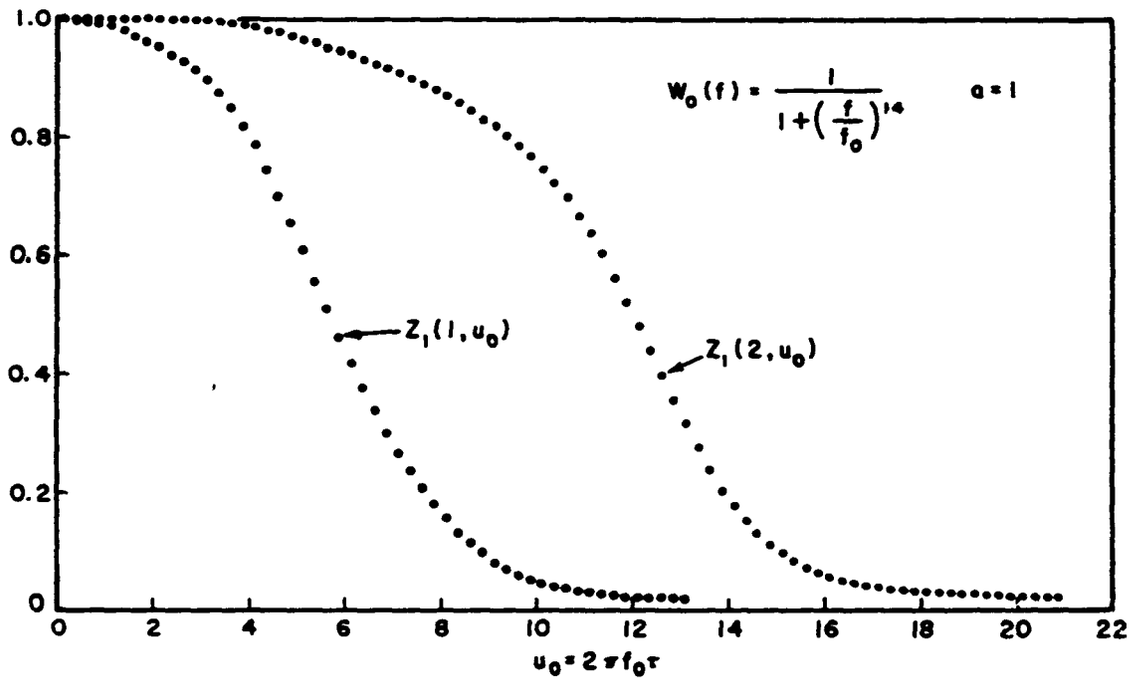
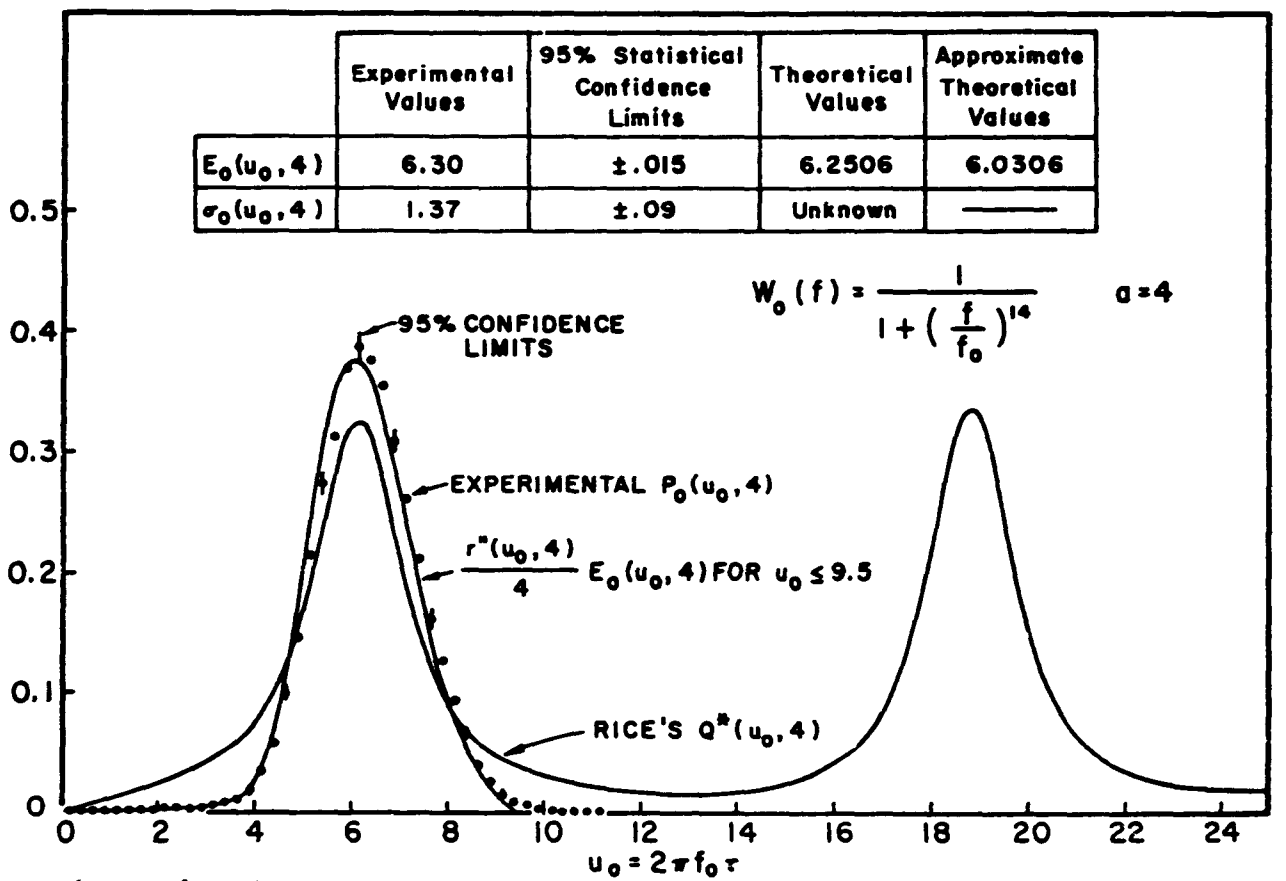
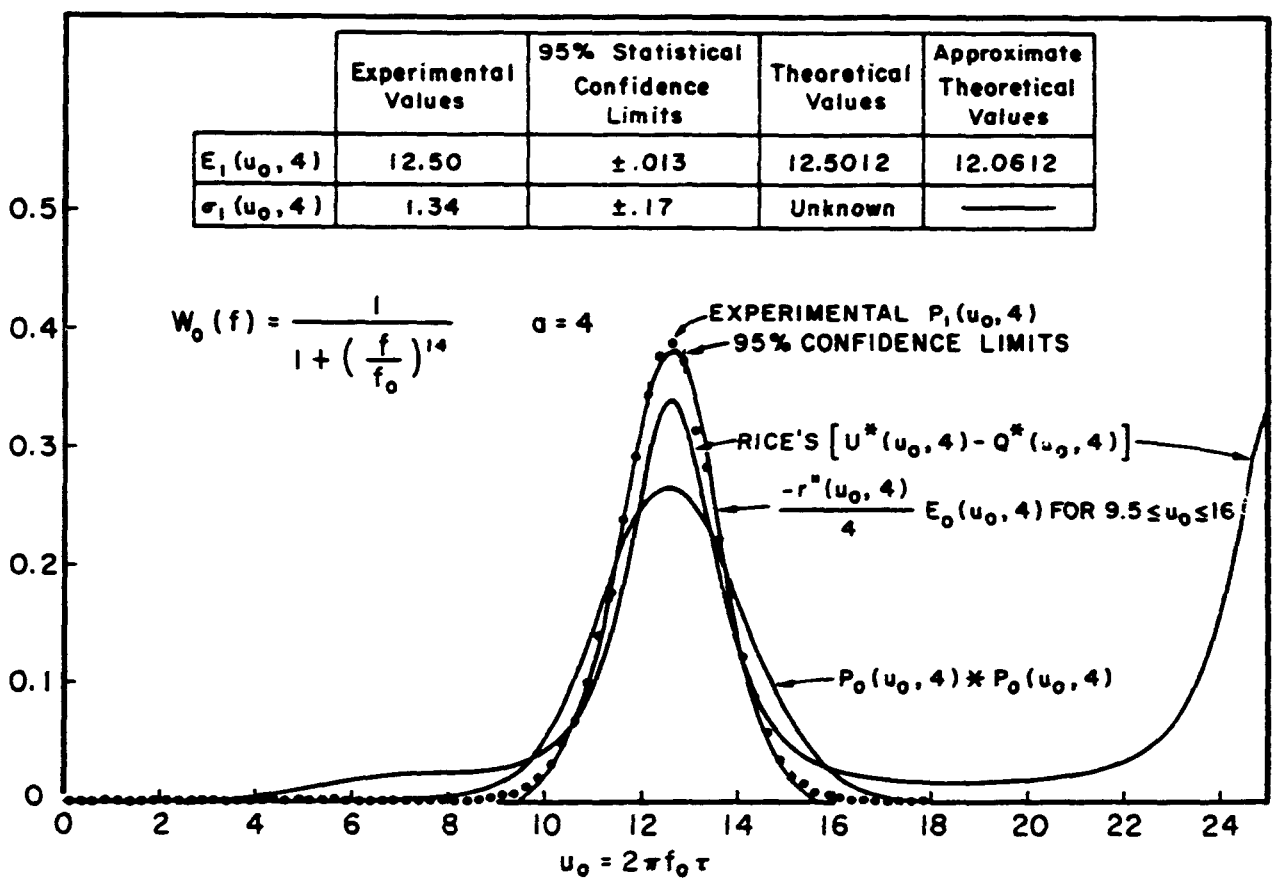


Figure 45 A Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$ Defines $Z_a(n, u_o)$, the Conditional Probability That the nth Zero-Crossing Point from a Given Zero-Crossing Point in du_o Occurs after the Time $du_o + u_o$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.



	Experimental Values	95% Statistical Confidence Limits	Theoretical Values	Approximate Theoretical Values
$E_0(u_0, 4)$	6.30	±.015	6.2506	6.0306
$\sigma_0(u_0, 4)$	1.37	±.09	Unknown	———

Figure 46 $P_0(u_0, a)$ is the Probability Density Function for Successive Zero-Crossing Intervals of a Random Process Consisting of a Sine Wave of Frequency $f_0/2$ Plus Gaussian Noise Having the Power Spectral Density $W_0(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a. Theoretical Approximations are Compared with Experimental Results.



	Experimental Values	95% Statistical Confidence Limits	Theoretical Values	Approximate Theoretical Values
$E_1(u_0, 4)$	12.50	±.013	12.5012	12.0612
$\sigma_1(u_0, 4)$	1.34	±.17	Unknown	———

Figure 47 $P_1(u_0, a)$ is the Probability Density Function for the Sum of Two Successive Zero-Crossing Intervals of a Random Process Consisting of a Sine Wave of Frequency $f_0/2$ Plus Gaussian Noise Having the Power Spectral Density $W_0(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

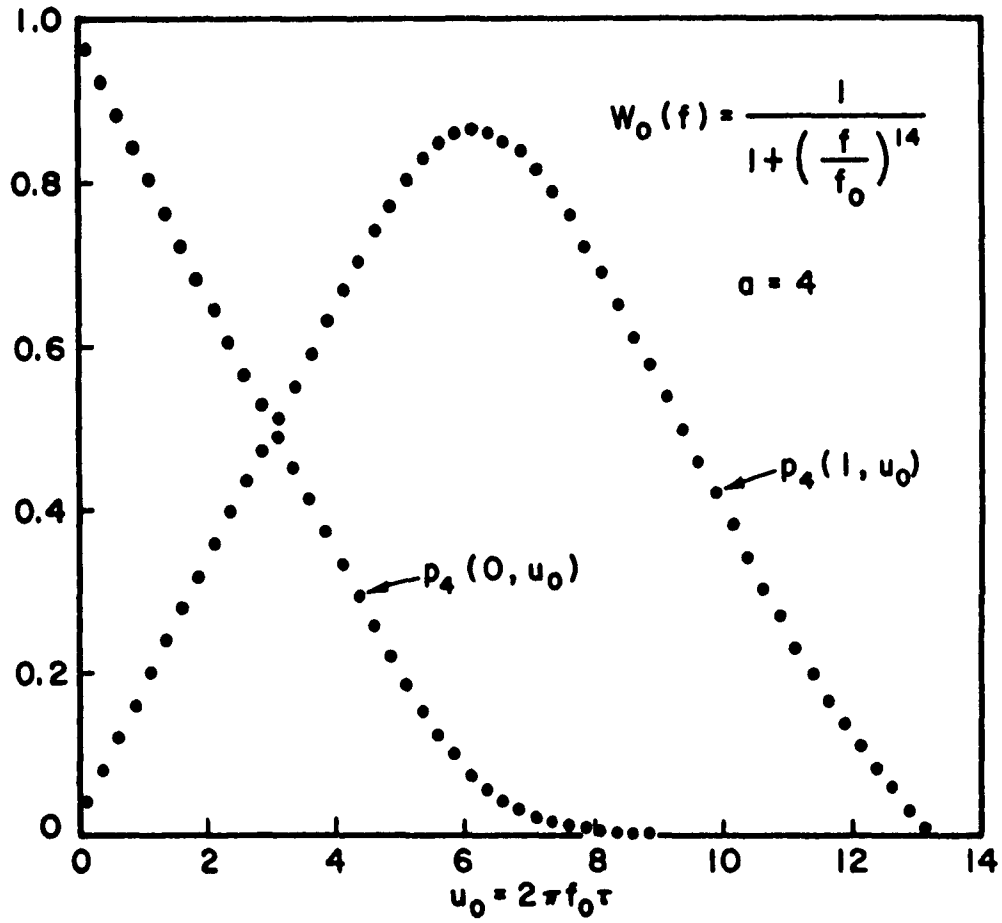


Figure 48 $p_a(n, u_o)$ is the Probability that a Given Interval u_o Contains Exactly n Zero-Crossing Points of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

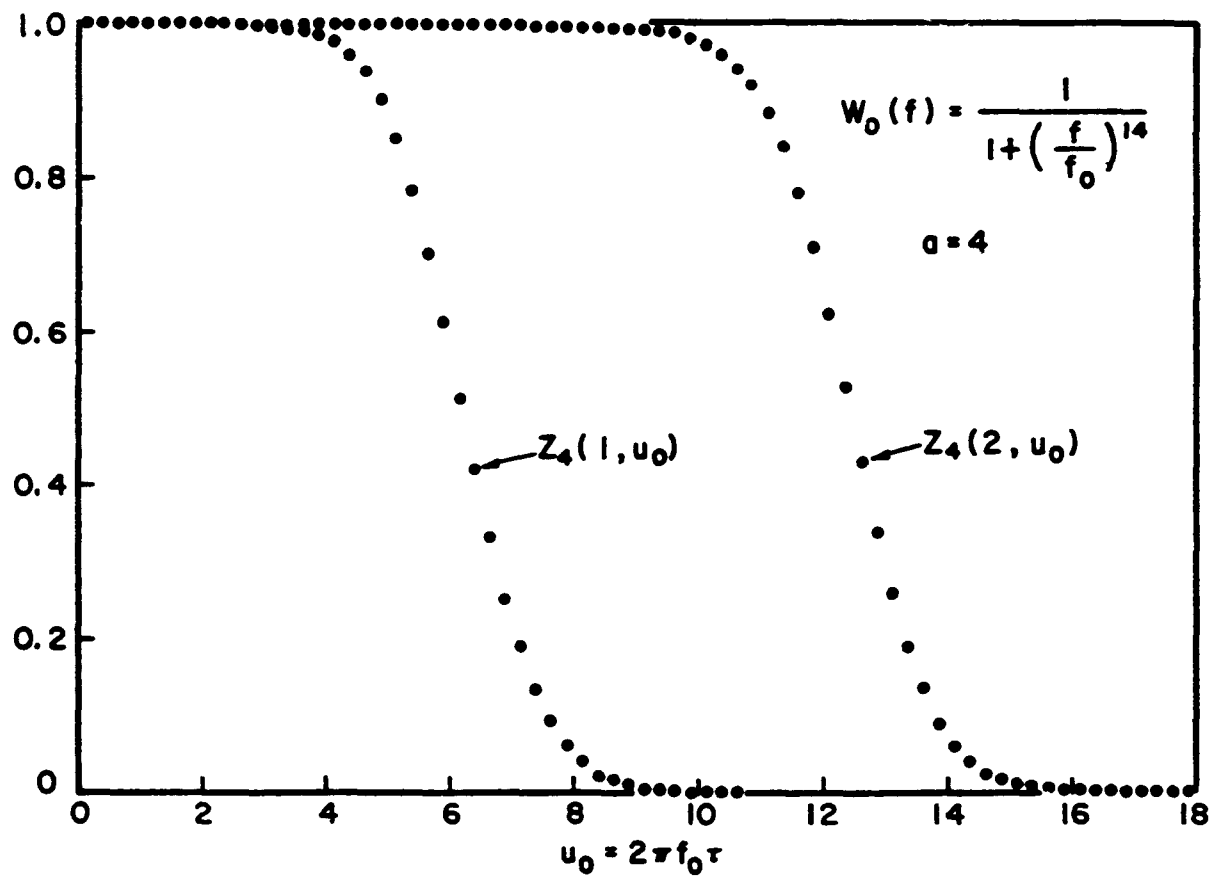


Figure 49 A Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$ Defines $Z_a(n, u_o)$, the Conditional Probability that the nth Zero-Crossing Point from a Given Zero-Crossing Point in du_o Occurs After the Time $du_o + u_o$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

Similarly, for small values of $a = \frac{S}{N}$, Rice's probability function $U^*(u_0, a)du_0$ approximates the conditional probability that a zero of $\xi(u, a)$ occurs between $u+u_0$ and $u+u_0+du_o$ given a zero of $\xi(u, a)$ at time u. In accordance with Equation (99), $U^*(u_o, a)-Q^*(u_0, a)$ serve as approximations to the initial behavior of $P_1(u_0, a)$ for small "a" and are presented in Figures 35, 39, 43, and 47. Notice that $U^*(u_0, a)du_o$ is the exact conditional probability that a zero of $\xi^*(u, a)$ occurs between $u+u_0$ and $u+u_0+du_0$ given a zero of $\xi^*(u, a)$ at time u.

Figure 35, 39, 43, and 47 also compare the convolution of $P_0(u_0, a)$ with itself, $P_0(u_0, a)*P_0(u_0, a)$, and $P_1(u_0, a)$ for a = 0, .2, 1, 4. These latter comparisons serve to demonstrate that successive zero-crossing intervals of $\xi(u, a)$ are statistically dependent.

For a general random process, McFadden (28) showed that:

$$\frac{r''(\tau)}{4\beta} = \sum_{n=0}^{\infty} (-1)^n P_n(\tau) \quad . \qquad (105)$$

$r''(\tau)$ is the second derivative of the normalized autocorrelation function of the infinitely clipped random process. The normalized autocorrelation function $r(u_0, a)$, see Appendix, for the infinitely clipped process $\tau(u, a)$ can be computed from a general result given by Davenport (40) and Middleton (3a).

Figure 46 compares the experimental $P_0(u_0, 4)$ with a first portion of $\frac{r''(u_0, 4)}{4}$ $E_0(u_0, 4)$ in accordance with Equation (105).

The comparison is excellent. Figure 47 compares the experimental $P_1(u_0, 4)$ with a second portion of $\frac{r''(u_0, 4)}{4} E_0(u_0, 4)$ in accordance with Equation (105). Again, the comparison is excellent.

VIII. PROBABILITIES AND PROBABILITY DENSITIES DEFINED BY THE STATIONARY POINTS OF A SINE WAVE PLUS A GAUSSIAN PROCESS

Probabilities and probability densities defined by the mathematical stationary points of the random process $\xi(u, a)$, defined by Equation (82), are presented in Figures 50 thru 65. For small values of $a = \frac{S}{N}$, the random process

$$\eta(u, a) = \frac{d\xi(u, a)}{du} \tag{106}$$

is approximately Gaussian with normalized autocorrelation function, $R_s(u_0, a)$, given by Equation (76). Accordingly, Rice's probability function $Q_s^*(u_0, a)\,du_o$ approximates the conditional probability that a downward zero-crossing of $\eta(u, a)$ occurs between $u + u_0$ and $u + u_0 + du_0$, given an upward zero-crossing of $\eta(u, a)$ at time u. In accordance with Equation (98), $Q_s^*(u_0, a)$ serve as approximations to the initial behavior of $M_0(u_0, a)$ for small "a" and are presented in Figures 50, 54, 58, 62. Notice that $Q_s^*(u_0, a)\,du_0$ is the exact conditional probability that a downward zero-crossing of

$$\eta^*(u, a) = \frac{d\xi^*(u, a)}{du} \tag{107}$$

occurs between $u + u_0$ and $u + u_0 + du_0$, given an upward zero-crossing of $\eta^*(u, a)$ at time u.

Similarly, for small values of $a = \frac{S}{N}$, Rice's probability function $U_s^*(u_0, a)\,du_0$ approximates the initial behavior of the

	Experimental Values	95% Statistical Confidence Limits	Theoretical Values	Theoretical Values Markov Model	Approximate Theoretical Values
$\mathcal{E}_0(u_0,0)$	3.95	±.02	3.7765	Taken As 3.7765	3.7765
$D_0(u_0,0)$	1.95	±.06	Unknown	1.199	—
κ_0	0.099	±.01	Unknown	0.245	—

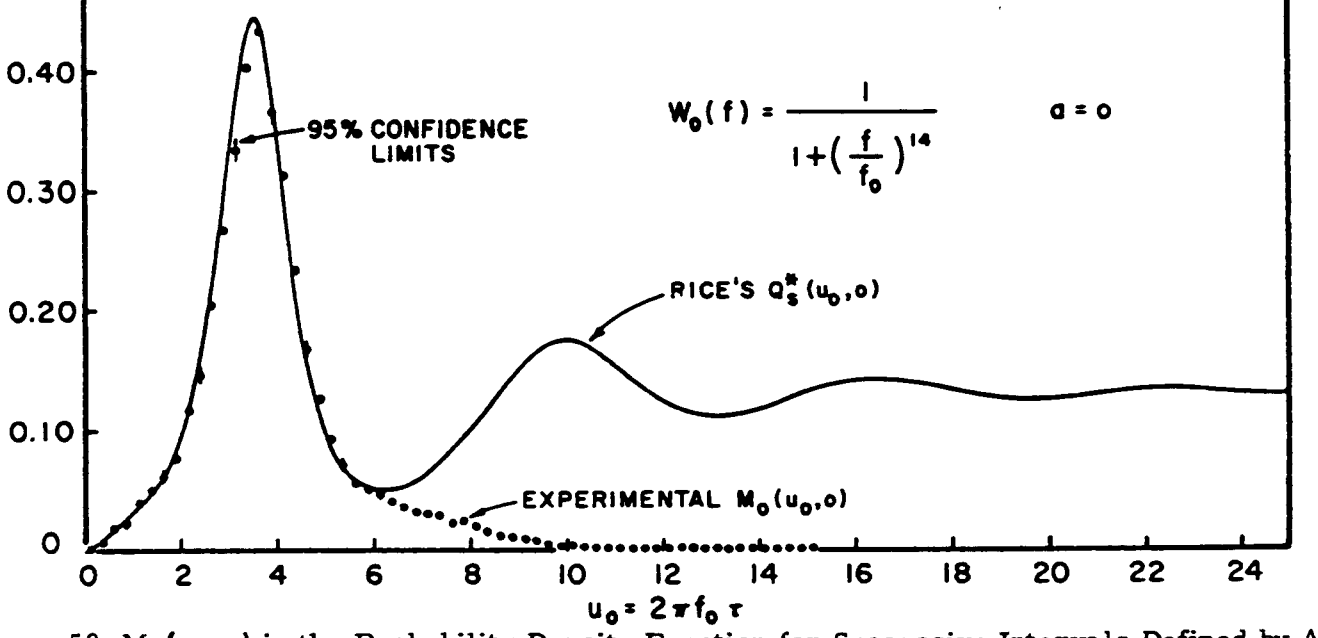


Figure 50 $M_0(u_0, a)$ is the Probability Density Function for Successive Intervals Defined by Adjacent Stationary Points of a Random Process Consisting of a Sine Wave of Frequency $f_0/2$ Plus Gaussian Noise Having the Power Spectral Density $W_0(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a. Theoretical Approximations are Compared with Experimental Results.

	Experimental Values	95% Statistical Confidence Limits	Theoretical Values	Theoretical Values Markov Model	Approximate Theoretical Values
$\mathcal{E}_1(u_0,o)$	7.71	±.03	7.5530	Taken As 7.5530	7.5530
$D_1(u_0,o)$	2.83	±.11	Unknown	1.894	—

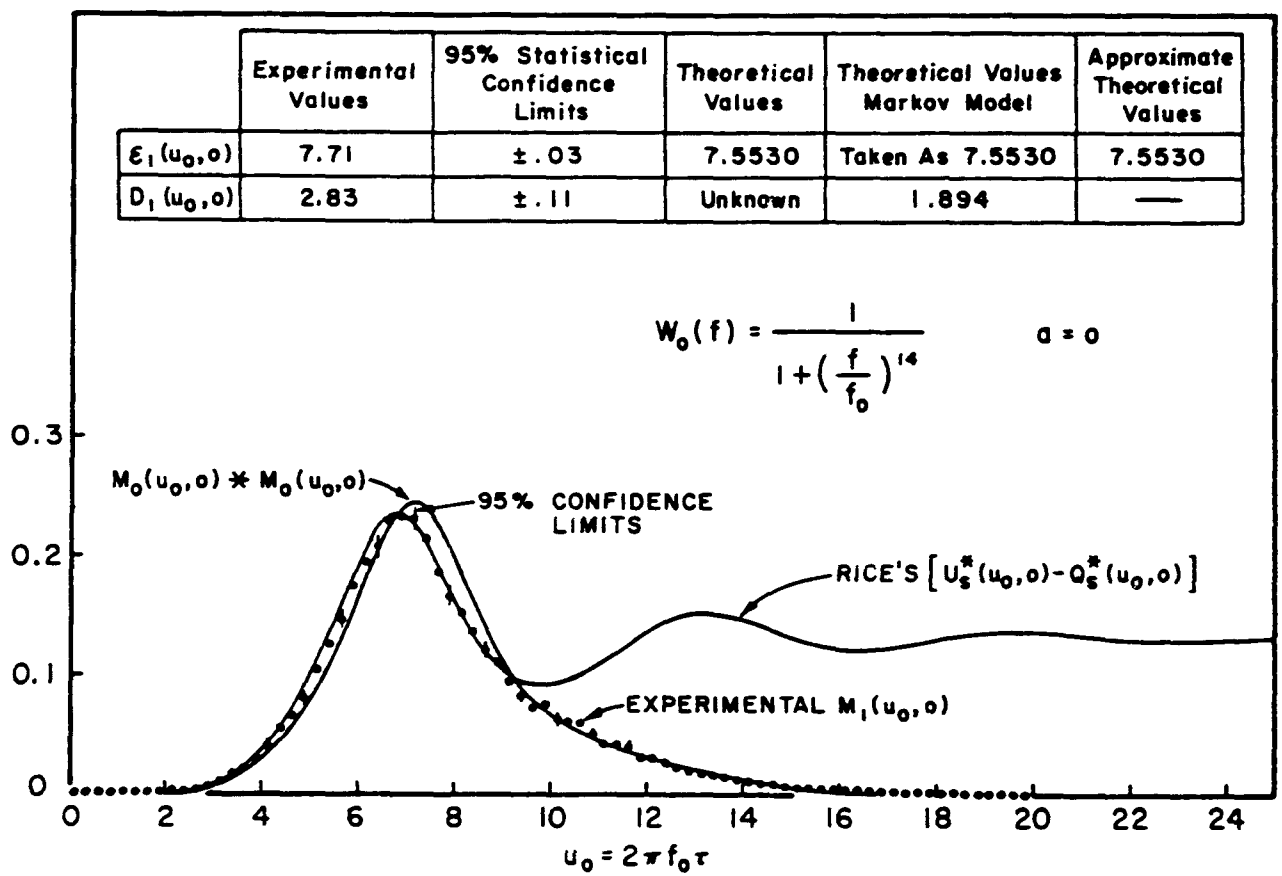


Figure 51 $M_1(u_o, a)$ is the Probability Density Function for the Sum of Two Successive Intervals Defined by Adjacent Stationary Points of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

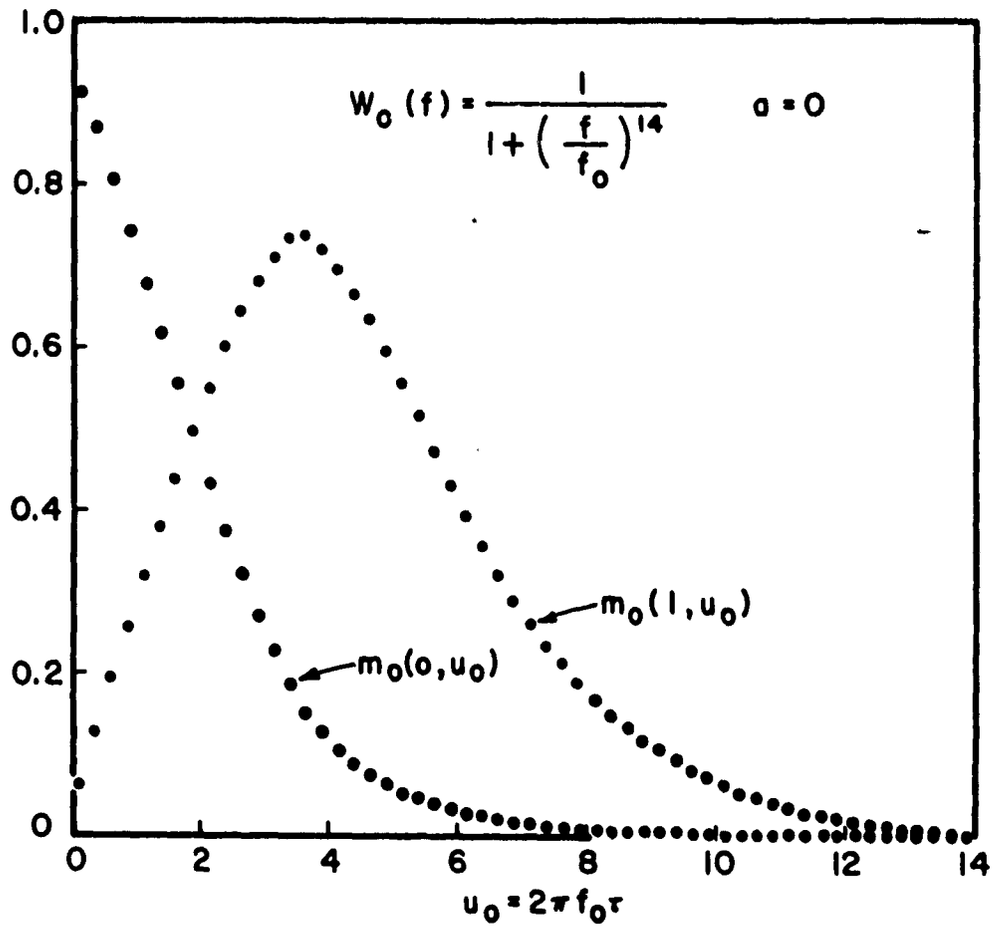


Figure 53 $m_a(n, u_o)$ is the Probability that a Given Interval u_o Contains Exactly n Stationary Points of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

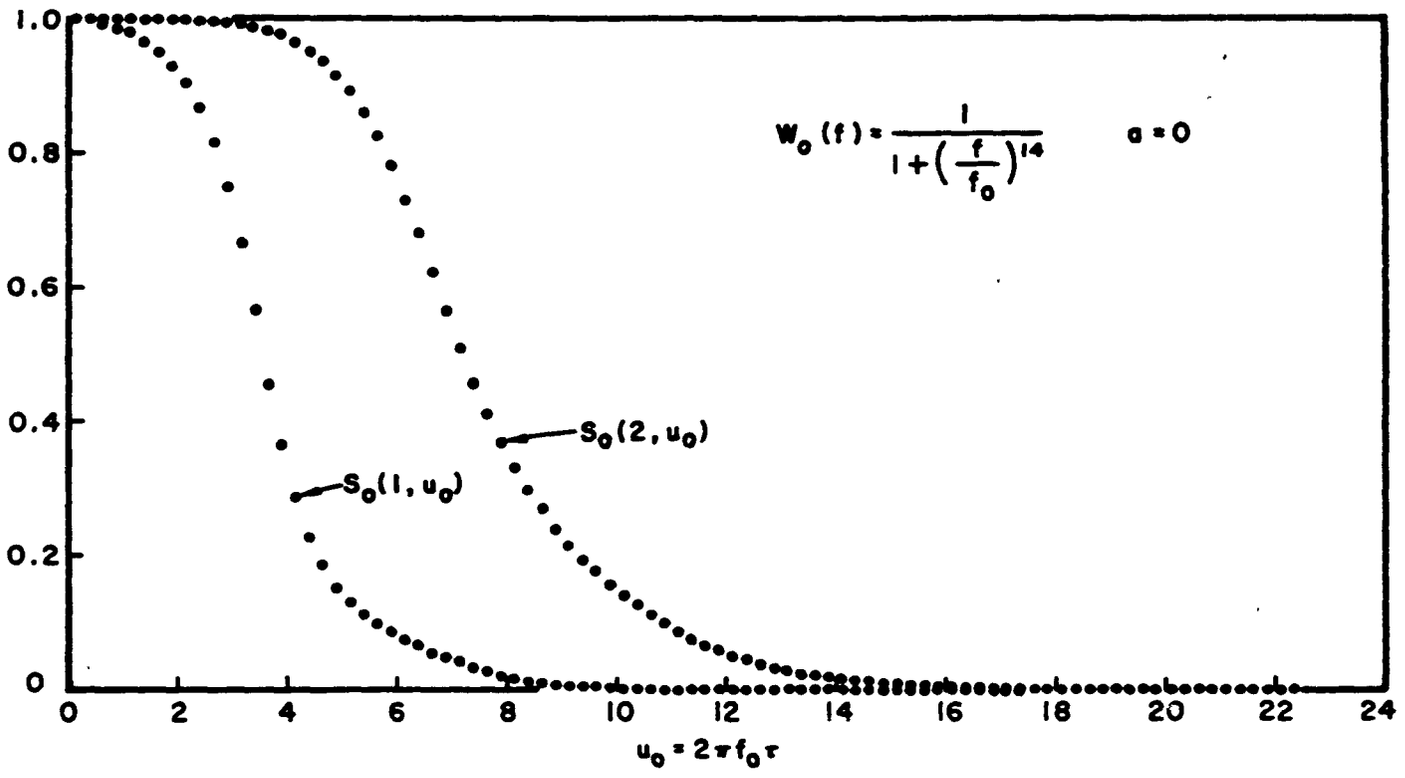
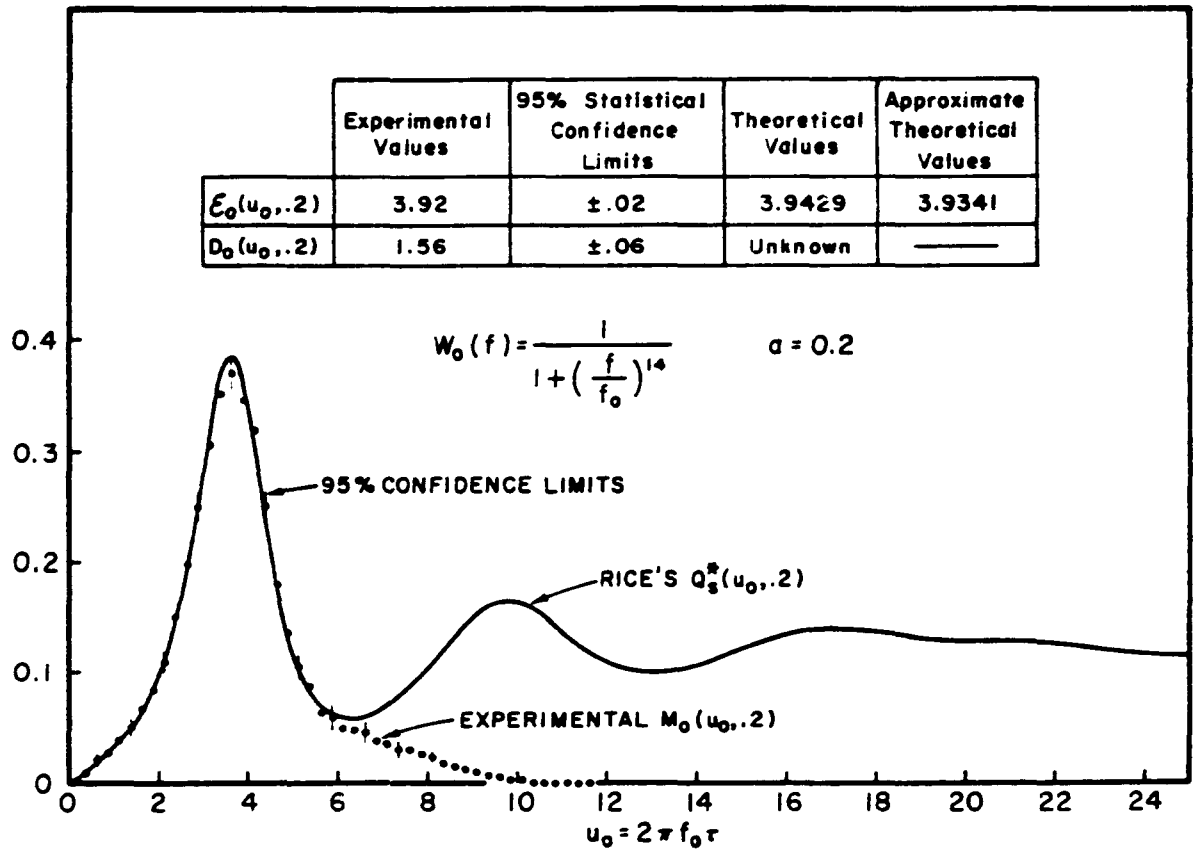
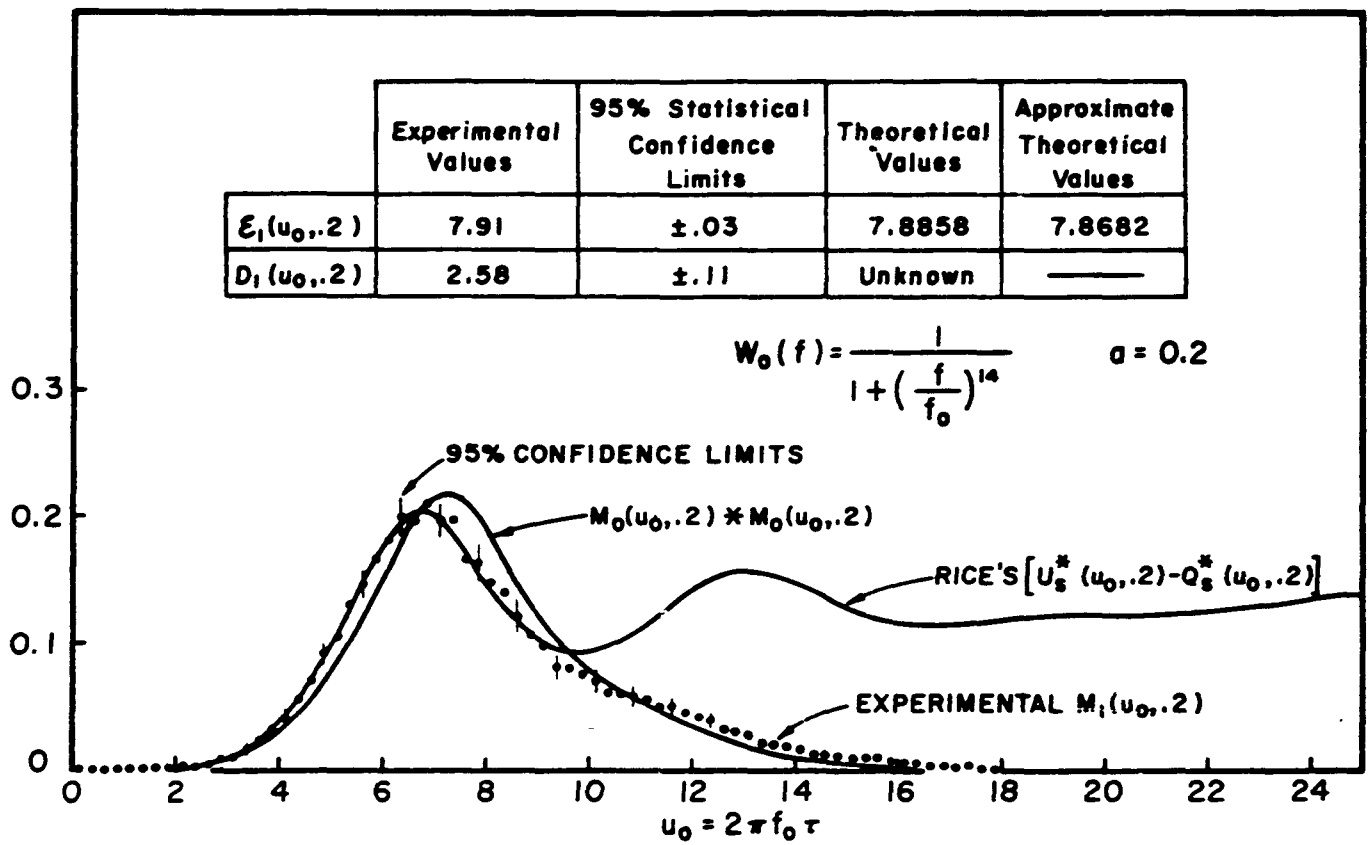


Figure 53 A Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$ Defines $S_a(n, u_o)$, the Conditional Probability that the nth Stationary Point from a Given Stationary Point in du_o Occurs After the Time $du_o + u_o$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.



	Experimental Values	95% Statistical Confidence Limits	Theoretical Values	Approximate Theoretical Values
$\mathcal{E}_0(u_0, .2)$	3.92	±.02	3.9429	3.9341
$D_0(u_0, .2)$	1.56	±.06	Unknown	———

Figure 54 $M_0(u_0, a)$ is the Probability Density Function for Successive Intervals Defined by Adjacent Stationary Points of a Random Process Consisting of a Sine Wave of Frequency $f_0/2$ Plus Gaussian Noise Having the Power Spectral Density $W_0(f)$. The Ratio of The Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a. Theoretical Approximations are Compared with Experimental Results.



	Experimental Values	95% Statistical Confidence Limits	Theoretical Values	Approximate Theoretical Values
$\mathcal{E}_1(u_0,.2)$	7.91	±.03	7.8858	7.8682
$D_1(u_0,.2)$	2.58	±.11	Unknown	———

Figure 55 $M_1(u_o, a)$ is the Probability Density Function for the Sum of Two Successive Intervals Defined by Adjacent Stationary Points of a Random Process Consisting of A Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

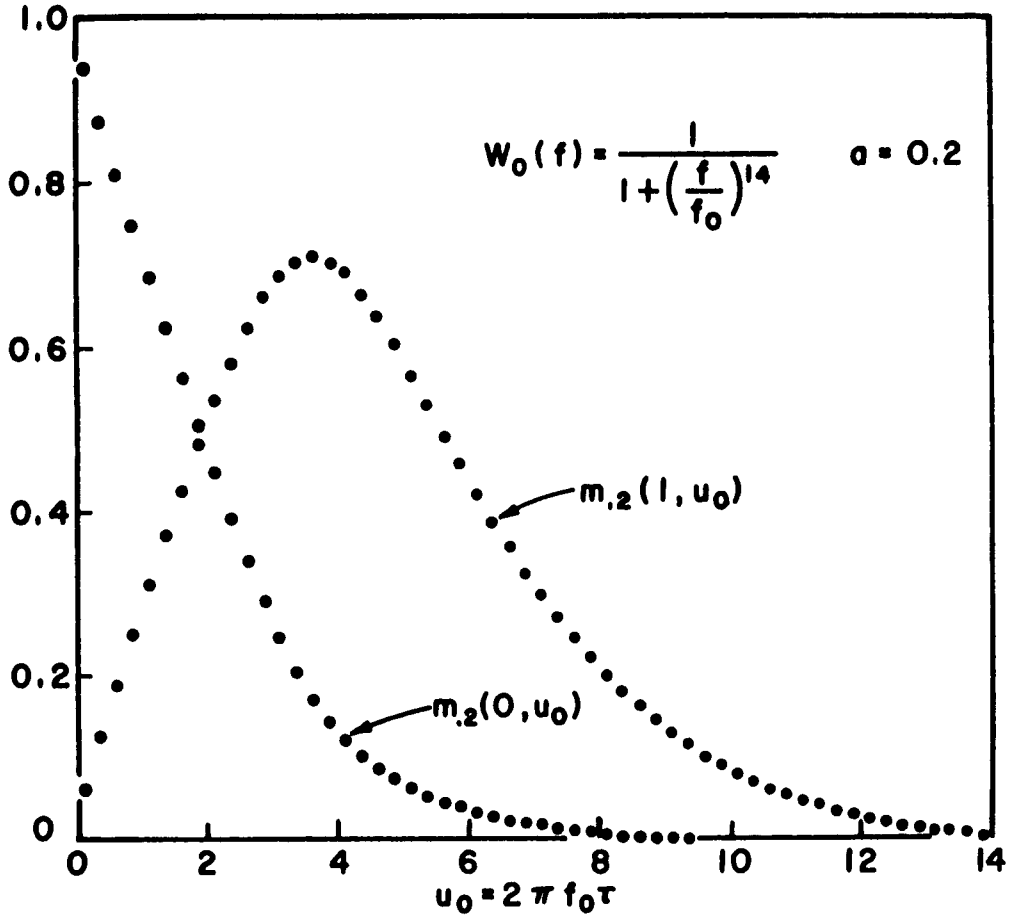


Figure 56 $m_a(n, u_o)$ is the Probability That a Given Interval u_o Contains Exactly n Stationary Points of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

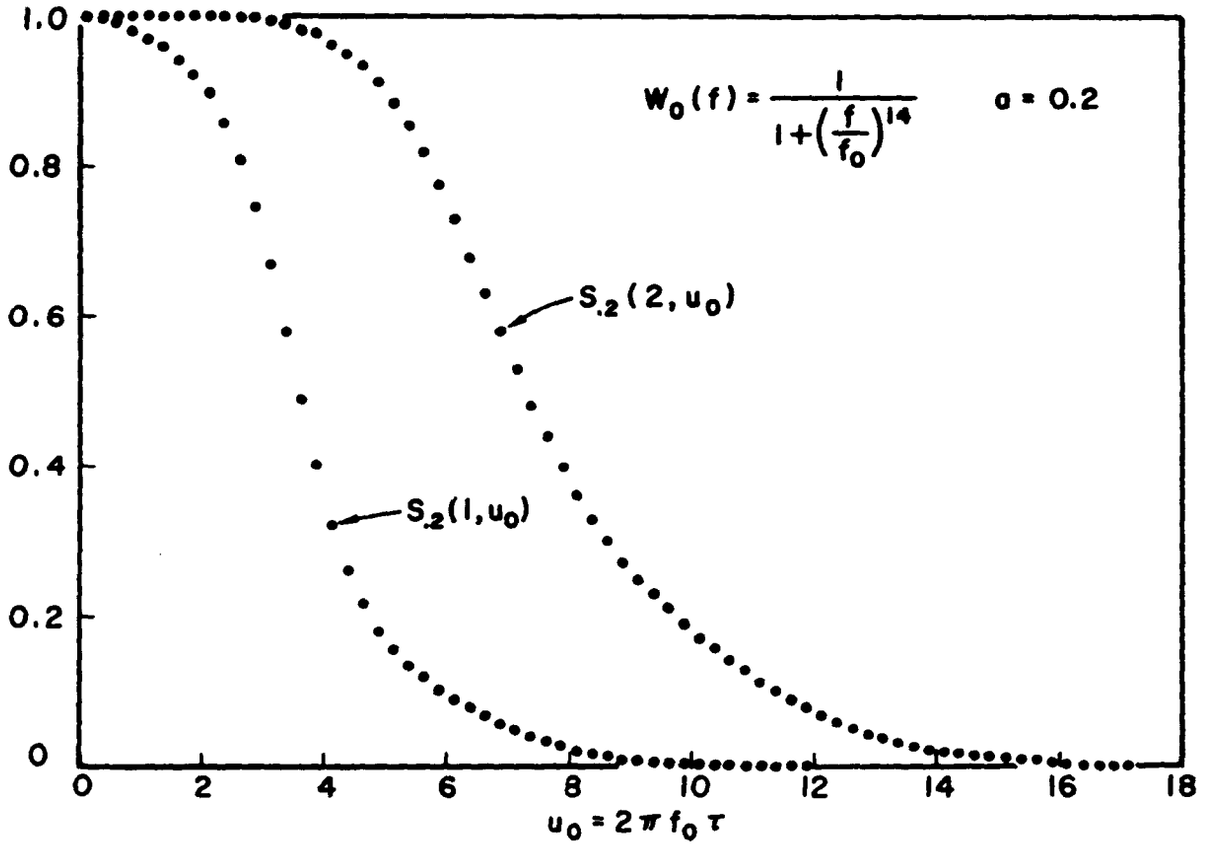
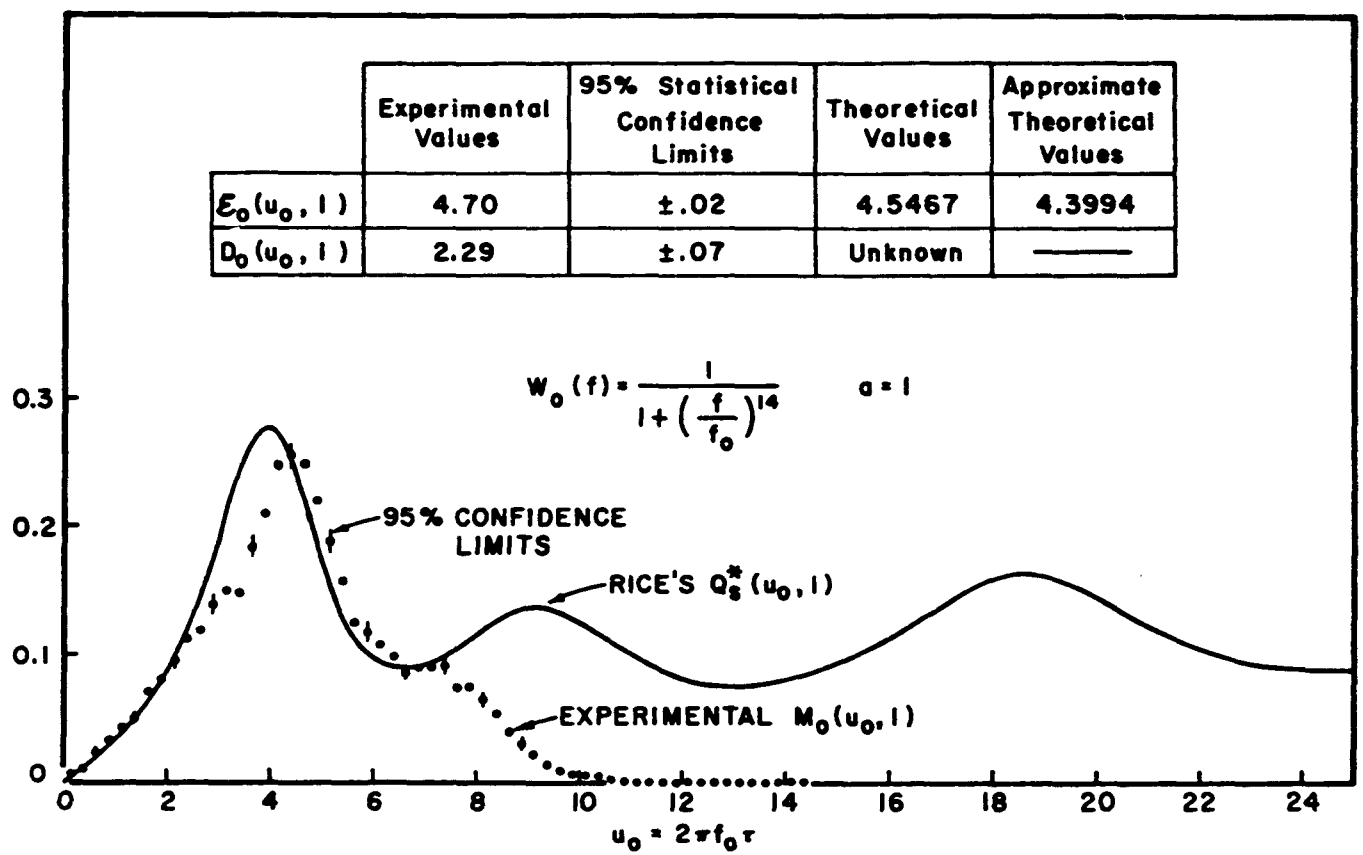
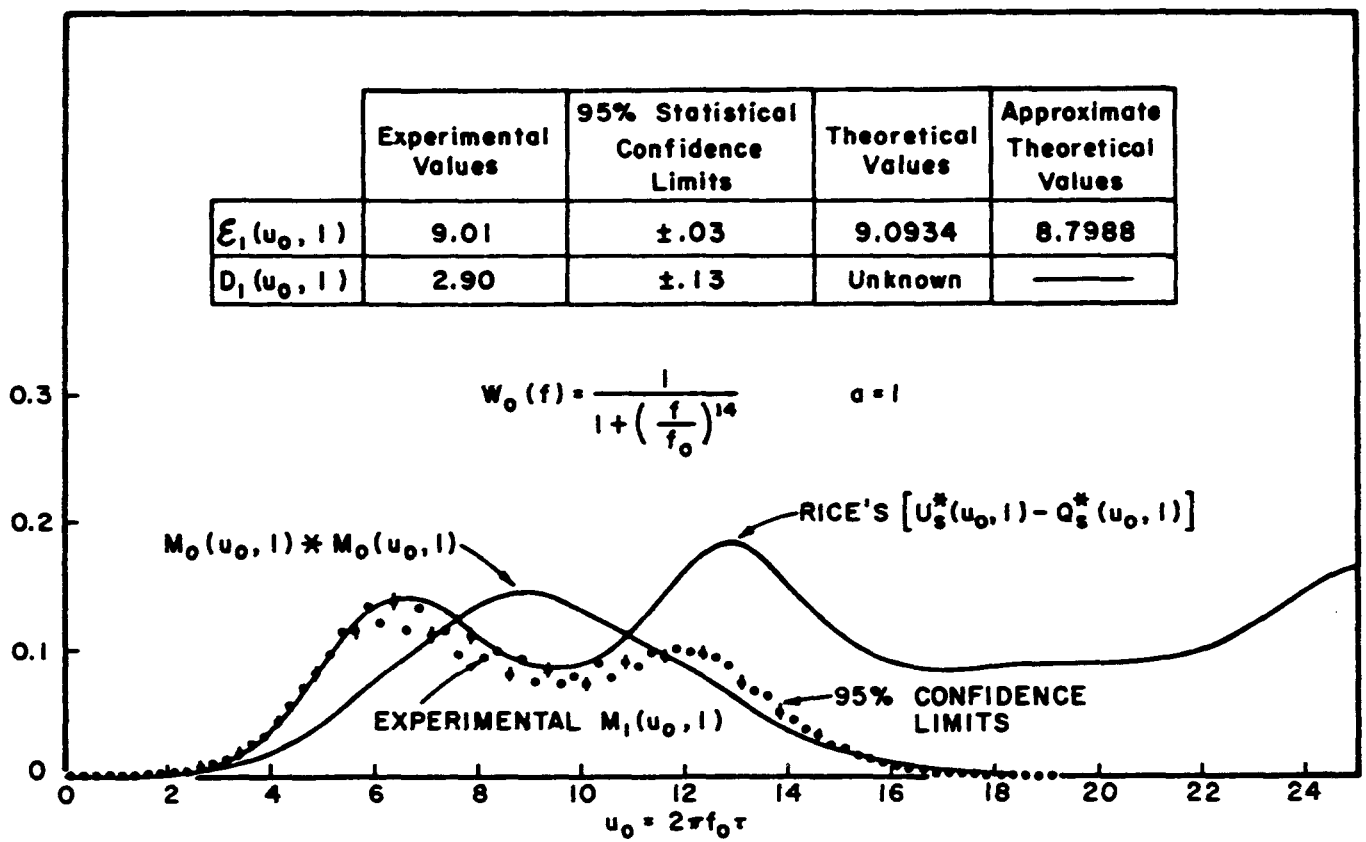


Figure 57 A Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$ Defines $S_a(n, u_o)$, the Conditional Probability that the nth Stationary Point from a Given Stationary Point in du_o Occurs after the Time $du_o + u_o$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.



	Experimental Values	95% Statistical Confidence Limits	Theoretical Values	Approximate Theoretical Values
$\mathcal{E}_0(u_0, 1)$	4.70	±.02	4.5467	4.3994
$D_0(u_0, 1)$	2.29	±.07	Unknown	———

Figure 58 $M_0(u_0, a)$ is the Probability Density Function for Successive Intervals Defined by Adjacent Stationary Points of a Random Process Consisting of a Sine Wave of Frequency $f_0/2$ Plus Gaussian Noise Having the Power Spectral Density $W_0(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a. Theoretical Approximations are Compared with Experimental Results.



	Experimental Values	95% Statistical Confidence Limits	Theoretical Values	Approximate Theoretical Values
$\mathcal{E}_1(u_0, 1)$	9.01	±.03	9.0934	8.7988
$D_1(u_0, 1)$	2.90	±.13	Unknown	———

Figure 59 $M_1(u_0, a)$ is the Probability Density Function for the Sum of Two Successive Intervals Defined by Adjacent Stationary Points of a Random Process Consisting of a Sine Wave of Frequency $f_0/2$ Plus Gaussian Noise Having the Power Spectral Density $W_0(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

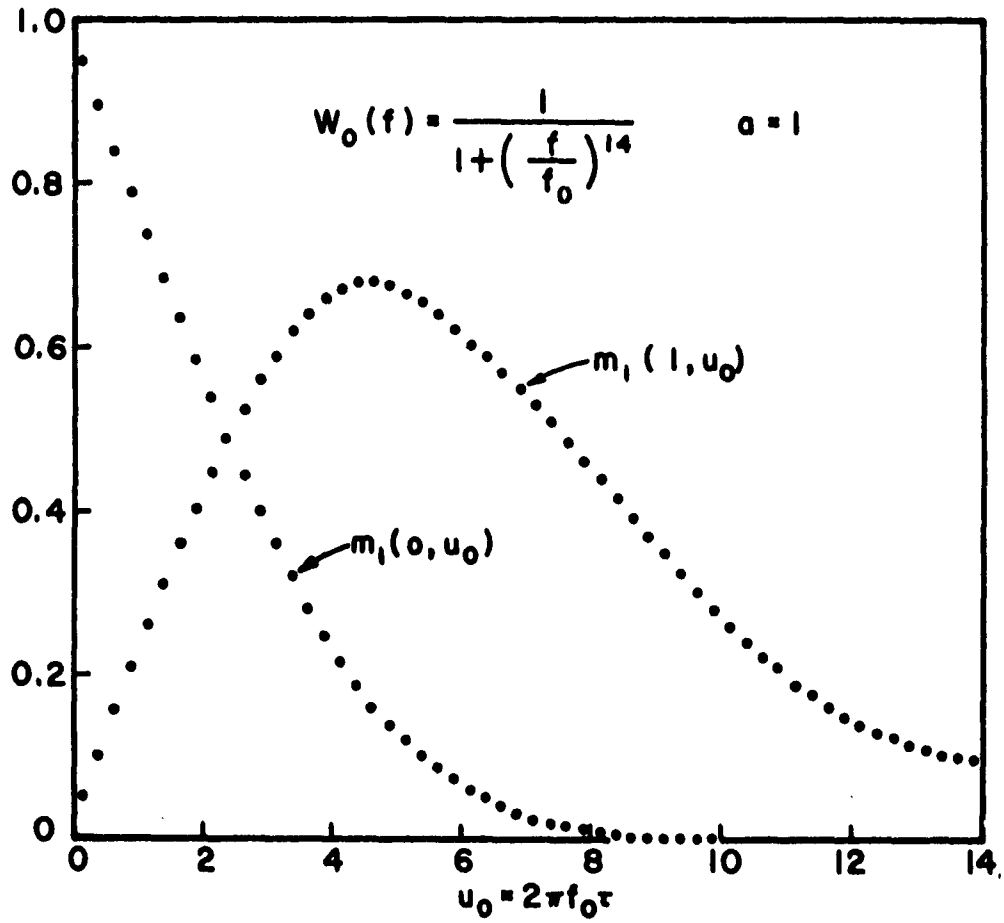


Figure 60 $m_a(n, u_o)$ is the Probability that a Given Interval u_o Contains Exactly n Stationary Points of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

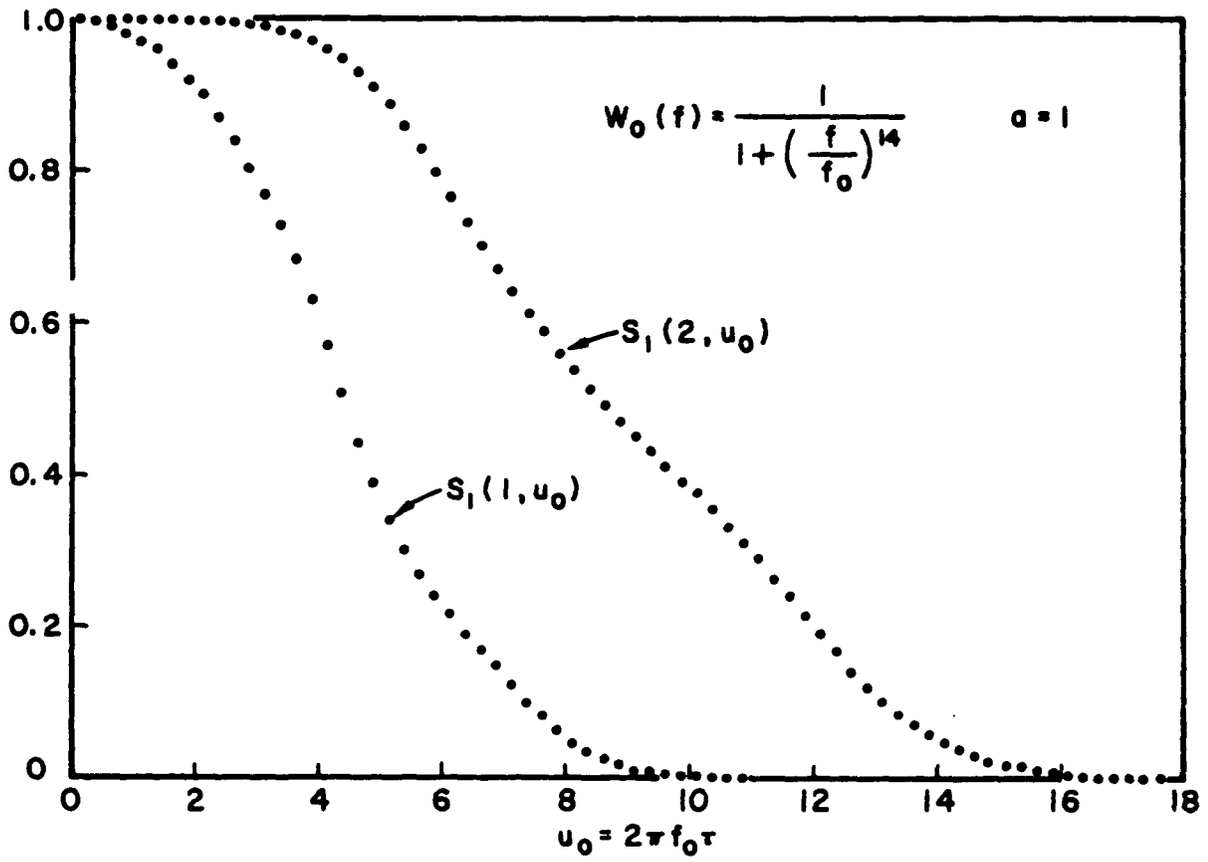
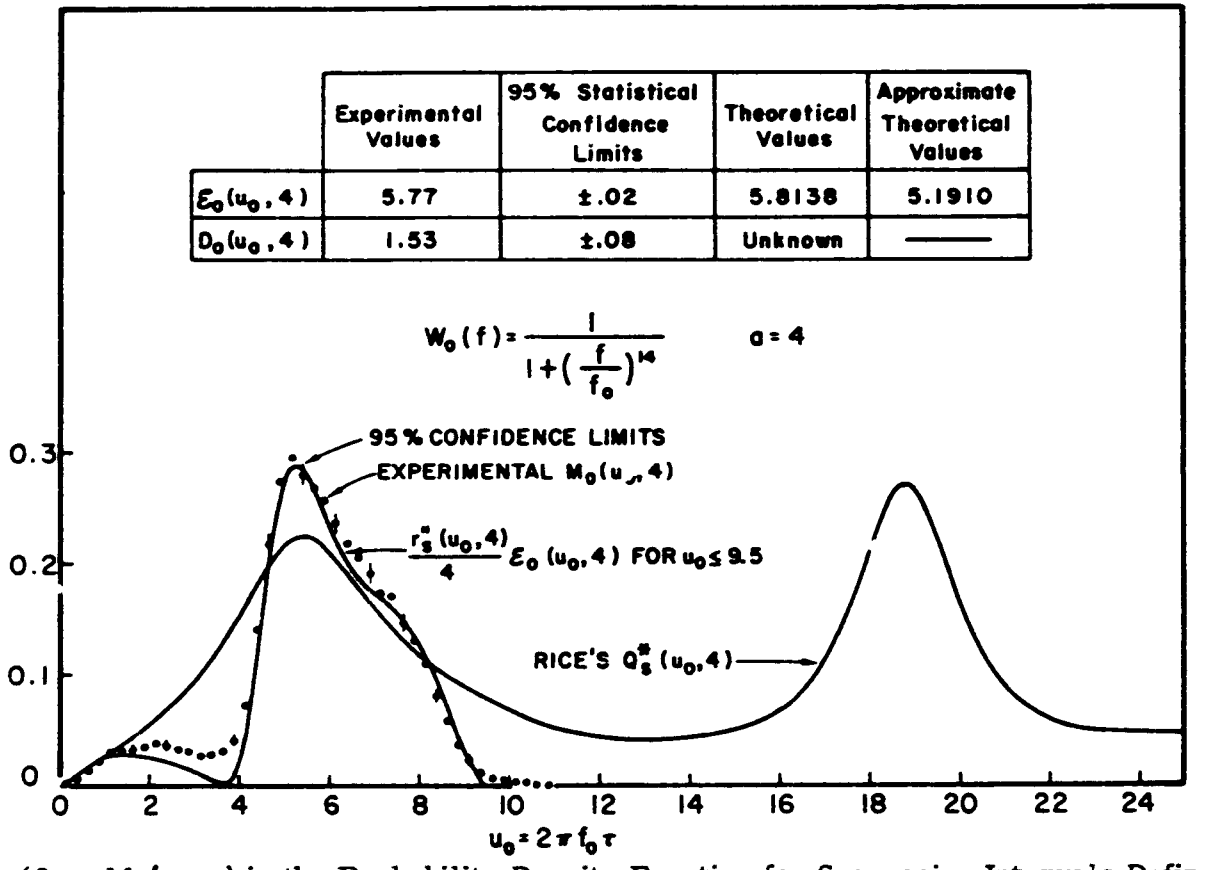


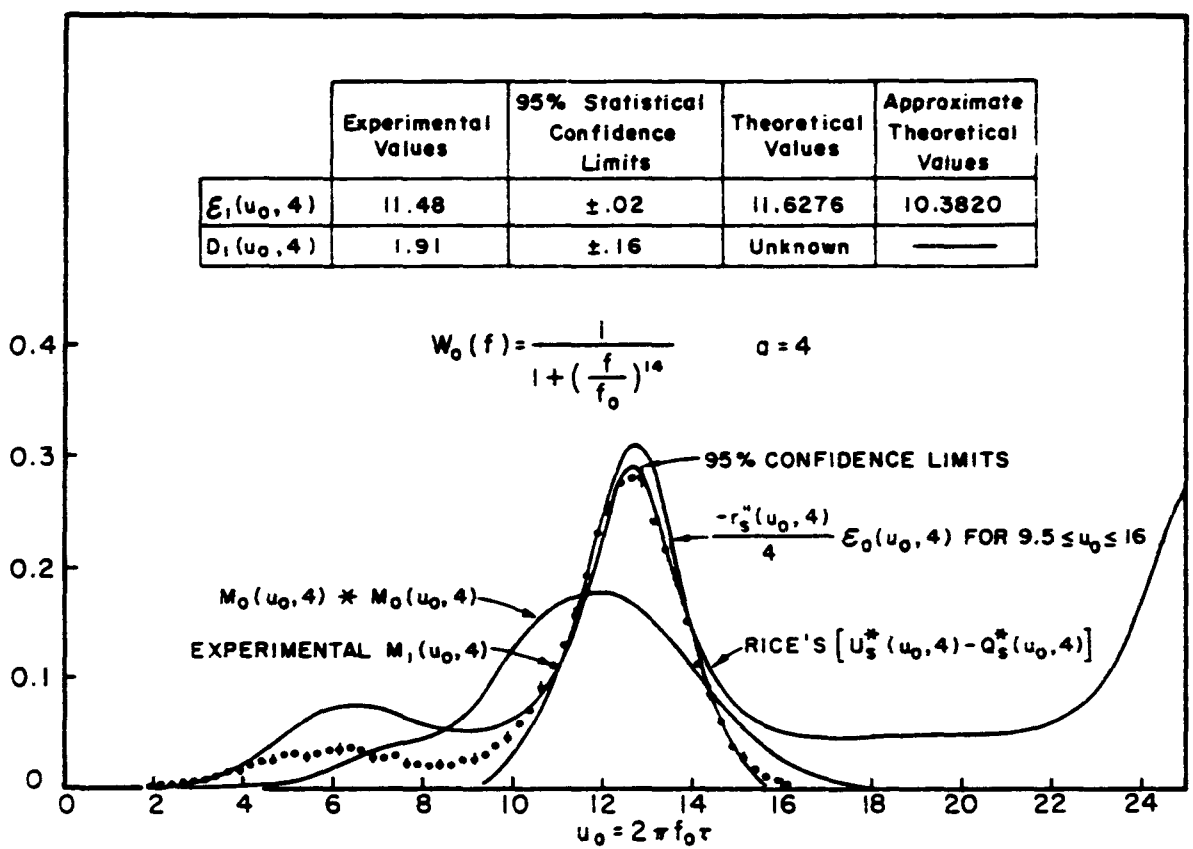
Figure 61 A Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$ Defines $S_a(n, u_o)$, the Conditional Probability that the nth Stationary Point from a Given Stationary Point in du_o Occurs After the Time $du_o + u_o$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.



	Experimental Values	95% Statistical Confidence Limits	Theoretical Values	Approximate Theoretical Values
$\mathcal{E}_0(u_0, 4)$	5.77	±.02	5.8138	5.1910
$D_0(u_0, 4)$	1.53	±.08	Unknown	———

$$W_0(f) = \frac{1}{1+\left(\frac{f}{f_0}\right)^{14}} \qquad a = 4$$

Figure 62: $M_0(u_0, a)$ is the Probability Density Function for Successive Intervals Defined by Adjacent Stationary Points of a Random Process Consisting of a Sine Wave of Frequency $f_0/2$ Plus Gaussian Noise Having the Power Spectral Density $W_0(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a. Theoretical Approximations are Compared with Experimental Results.



	Experimental Values	95% Statistical Confidence Limits	Theoretical Values	Approximate Theoretical Values
$\mathcal{E}_1(u_0, 4)$	11.48	±.02	11.6276	10.3820
$D_1(u_0, 4)$	1.91	±.16	Unknown	———

Figure 63 $M_1(u_0, a)$ is the Probability Density Function for the Sum of Two Successive Intervals Defined by Adjacent Stationary Points of a Random Process Consisting of a Sine Wave of Frequency $f_0/2$ Plus Gaussian Noise Having the Power Spectral Density $W_0(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

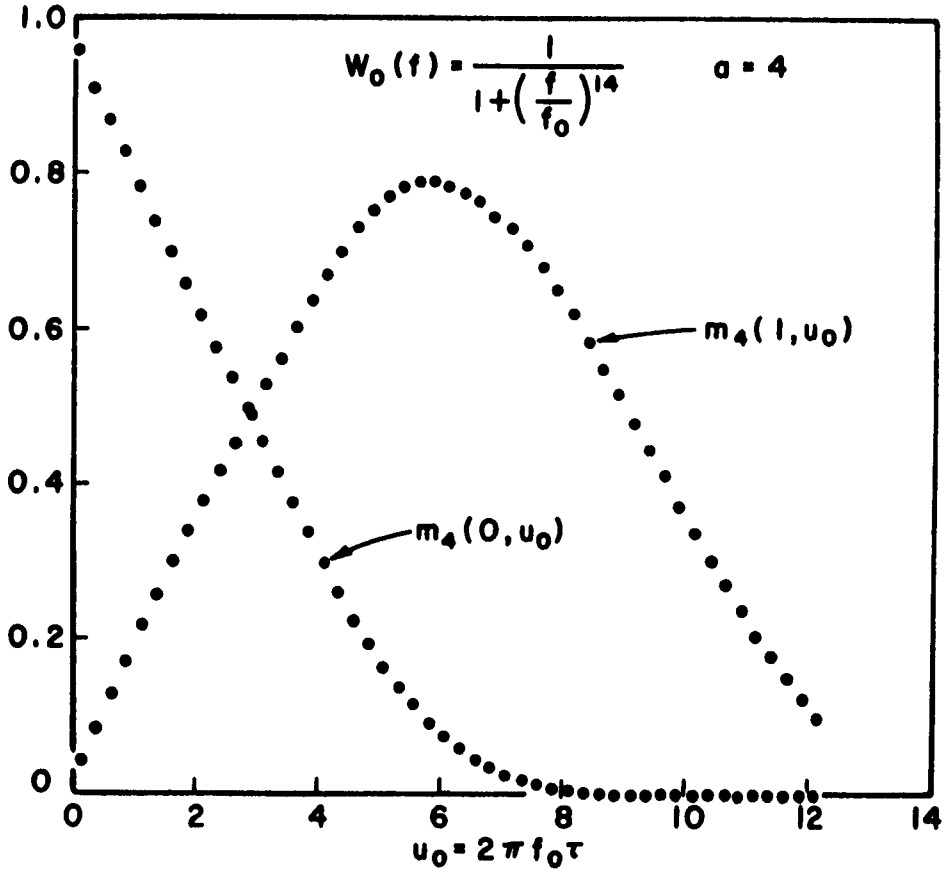


Figure 64 $m_a(n, u_o)$ is the Probability that a Given Interval u_o Contains Exactly n Stationary Points of a Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

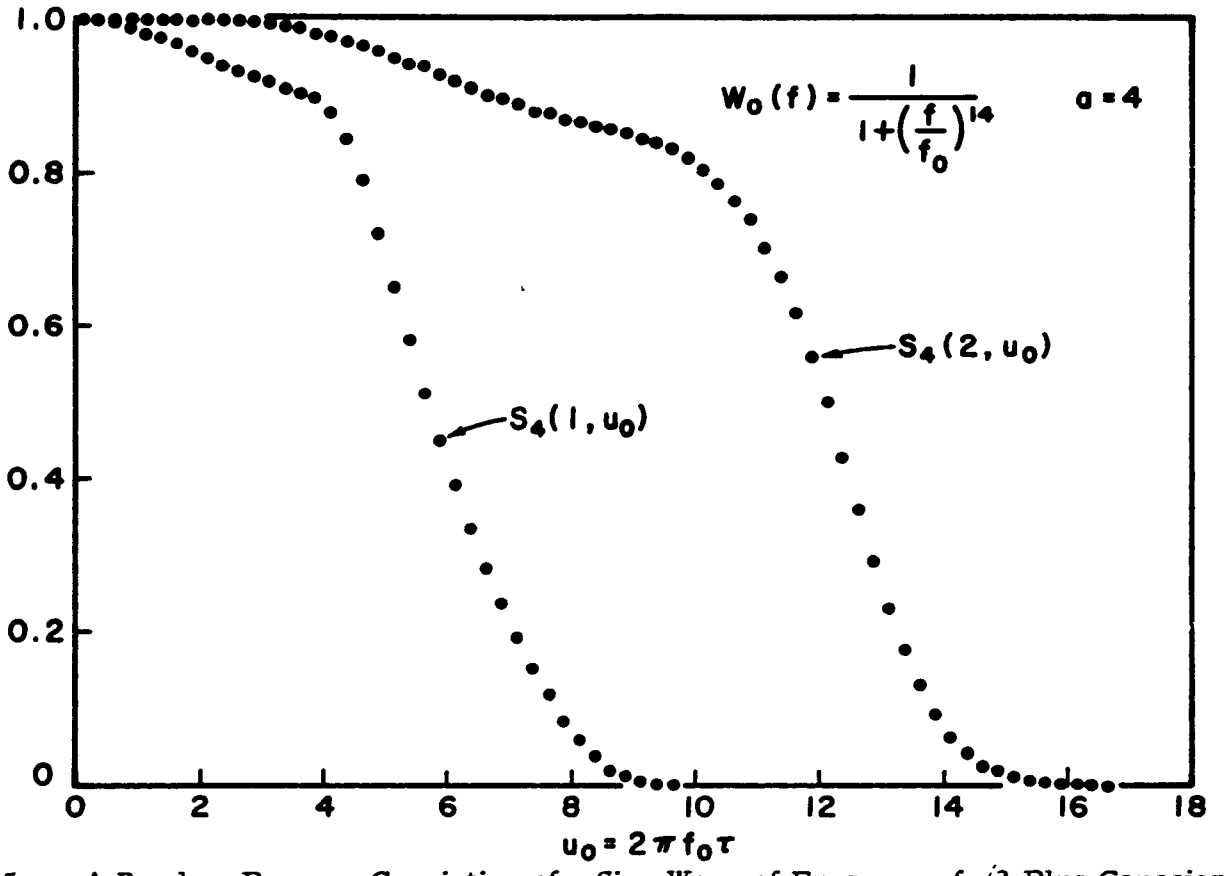


Figure 65 A Random Process Consisting of a Sine Wave of Frequency $f_o/2$ Plus Gaussian Wave of Frequency $f_o/2$ Plus Gaussian Noise Having the Power Spectral Density $W_o(f)$ Defines $S_a(n, u_o)$, The Conditional Probability that the nth Stationary Point from a Given Stationary Point in du_o Occurs After the Time $du_o + u_o$. The Ratio of the Average Sine Wave Power to the Average Noise Power is Denoted by the Parameter a.

conditional probability that a stationary point of $\xi(u, a)$ occurs between $u + u_o$ and $u + u_0 + du_0$ given a stationary point of $\xi(u, a)$ at time u. In accordance with Equation (99), $U_s^*(u_0, a)$-$Q_s^*(u_0, a)$ serve as approximations to the initial behavior of $M_1(u_0, a)$ for small "a" and are presented in Figures 51, 55, 59, and 63. Notice that $U_s^*(u_0, a)du_o$ is the exact conditional probability that a stationary point of $\xi^*(u, a)$ occurs between $u + u_0$ and $u + u_0 + du_0$ given a stationary point of $\xi^*(u, a)$ at time u.

Figures 51, 55, 59, and 63 also compare the convolution of $M_o(u_o, a)$ with itself, $M_o(u_0, a) * M_o(u_0, a)$, and $M_1(u_0, a)$ for a = 0, .2, 1, 4. These comparisons serve to demonstrate that two successive intervals defined by the adjacent stationary points of $\xi(u, a)$ are statistically dependent.

The normalized autocorrelation function $r_s(u_0, a)$, see Appendix, for the infinitely clipped process $\eta(u, a)$ can be computed by applying the transformation associated with the ideal differentiator operator to the general result given by Davenport and Middleton.

Figure 62 compares the experimental $M_o(u_0, 4)$ with a first portion of $\frac{r''_s(u_0, a)}{4} \mathcal{E}_o(u_0, 4)$ in accordance with Equation (105). The comparison is satisfactory.

Figure 63 compares the experimental $M_1(u_0, 4)$ with a second portion of $\frac{r''_s(u_0, a)}{4} \mathcal{E}_o(u_0, 4)$ in accordance with Equation (105). Again, the comparison is satisfactory.

IX. STATISTICAL DEPENDENCE OF ZERO-CROSSING INTERVALS

The recent work of Longuet-Higgins (8) leads one to the following important theorem.

Theorem: If $\xi(t)$ is a Gaussian process having a finite expected number of zeros per unit time, then the sum of (m+1) successive zero-crossing intervals of $\xi(t)$ and the sum of the next (n+1) successive zero-crossing intervals $\xi(t)$ are statistically dependent for all nonnegative integral m, n.

Proof:

For Gaussian processes having power spectral densities with asymptotic behavior f^{-4} (the singular case), Longuet-Higgins showed that $P_\ell(\tau)$, the probability density function for the sum of $(\ell + 1)$ successive zero-crossing intervals, tends to a positive value as $\tau \to 0$. Accordingly,

$$\lim_{\tau\to 0} P_{m+n+1}(\tau) \neq \lim_{\tau\to 0} P_m(\tau) * P_n(\tau) \equiv \lim_{\tau\to 0} \int_0^\tau P_m(\ell) P_n(\tau-\ell)\, d\ell = 0 \tag{108}$$

This completes the proof of the theorem for the singular case.

For Gaussian processes having power spectral densities with asymptotic behavior other than f^{-4} (the regular case), Longuet-Higgins showed that

$$P_m(\tau) = O\left(\tau^{\frac{1}{2}(m+2)(m+3)-2}\right) \text{ as } \tau \to 0 \quad . \tag{109}$$

Hence, as $\tau \to 0$

$$P_{m+n+1}(\tau) = O\left(\tau^{\frac{1}{2}(m+n+3)(m+n+4)-2}\right) \tag{110}$$

and

$$P_m(\tau) * P_n(\tau) = O\left(\tau^{\frac{1}{2}(n+2)(n+3)+\frac{1}{2}(m+2)(m+3)-3}\right) . \tag{111}$$

Suppose Equations (110) and (111) were equal for some nonnegative integral m, n. Then,

$$\frac{1}{2}(m+n+3)(m+n+4)-2 = \frac{1}{2}(n+2)(n+3)+\frac{1}{2}(m+2)(m+3)-3 \tag{112}$$

and,

$$mn + m + n + 1 = 0 \quad . \tag{113}$$

Equation (113) is clearly false for all nonnegative integral m, n. Accordingly, as $\tau \to 0$

$$P_{m+n+1}(\tau) \neq P_m(\tau) * P_n(\tau) \quad . \tag{114}$$

This completes the proof of the theorem. For m = n = 0, the content of the theorem was given by Palmer (11) and McFadden (28).

For certain Gaussian processes, the convolutions shown in Figures 19, 27, 31, 35, and 51 verify the truth of the theorem for m = n = 0.

In order to observe the statistical dependence between the ith and (i+n)th zero-crossing intervals of a random process, two of the experimental systems described in Section IIA were used to display 5000 random samples of the ith and (i+n)th zero-crossing intervals on an oscilloscope. The skipping discussed in Section

IIA was set at 11 or 12 each with probability $\frac{1}{2}$. The resulting intensity pattern on the oscilloscope represents an approximation to the joint probability density of these intervals. If the intensity pattern corresponding to $n \to \infty$ is similar to the intensity pattern corresponding to $n = n_1$, then one can conclude that the ith and $(i+n_1)$th zero-crossing intervals of a finite memory process are practically independent.

For a Gaussian process having power spectral density $W_3(f)$, Figure 66 illustrates the statistical dependence between the ith and (i+n)th zero-crossing intervals of the process for n = 1, 2, 3, 4, and ∞. $n = \infty$ represents a condition such that the two zero-crossing intervals are separated by approximately 71 average zero-crossing intervals. This condition was convenient experimentally and corresponds to a one second time separation between the zero-crossing intervals. Notice the symmetry of the intensity patterns about the straight line passing through the origin with a slope of 45°. This symmetry results from the fact that the joint probability density of the intervals is invariant under a time reversal of the random process.

The conditional mean of the (i+n)th zero-crossing interval given the ith zero-crossing interval is found from Figure 66 by considering narrow vertical strips of the intensity patterns and

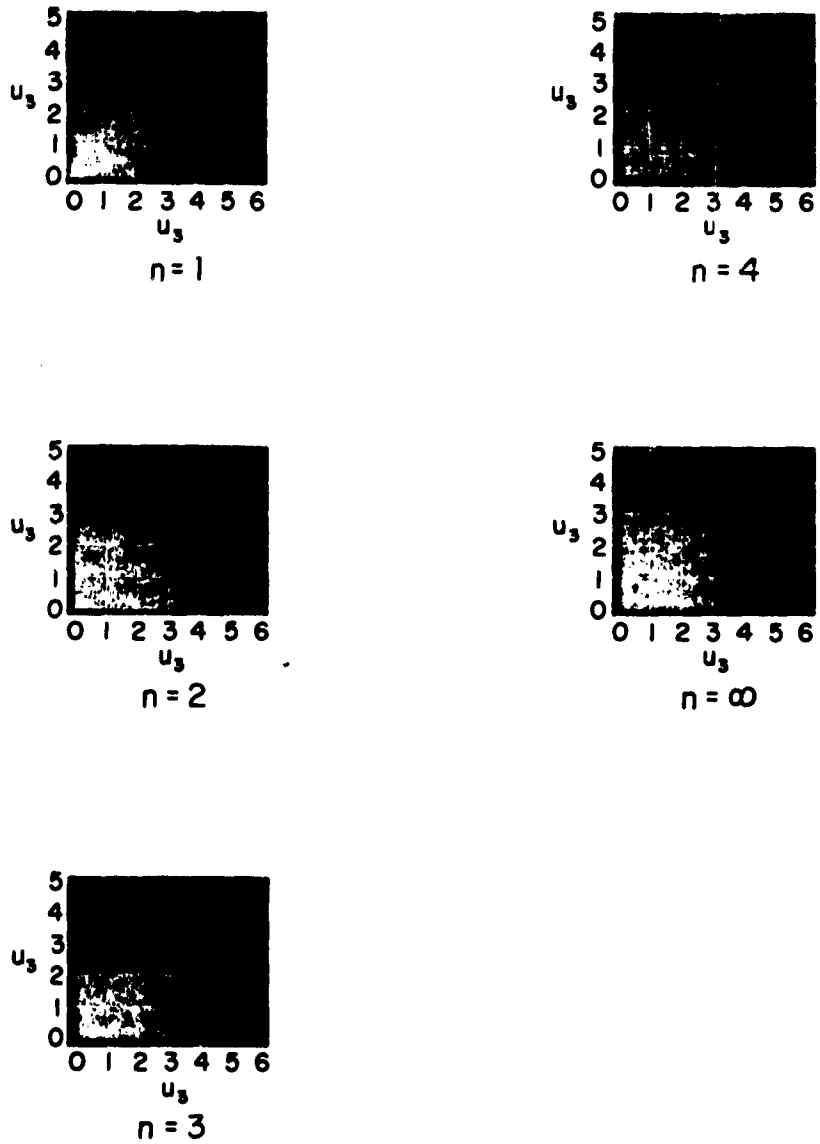


Figure 66: Each Intensity Pattern Represents an Approximation to the Joint Probability Density of the ith and (i+n)th Zero-Crossing Intervals of a Gaussian Process Having Power Spectral Density $W_3(f)$.

determining the center of gravity for each strip when unit intensity is assigned unit mass. A line joining these points is called a regression curve. By using the principle of least squares one can determine the best fitting straight line to the regression curve. The slope of this best fitting straight line represents the correlation coefficient ρ_n. From Figure 66 we see that when $n \geq 3$ the regression curve is approximately a line parallel to the axis of abscissas, and the conditional mean value is approximately equal to the mean value of the zero-crossing intervals. Thus when $n \geq 3$, $\rho_n \doteq 0$ or the zero-crossing intervals are approximately uncorrelated.

By comparing the intensity pattern for $n = \infty$ to the others we see that when $n \geq 4$ the statistical dependence practically vanishes. However, when we compare the intensity pattern for $n = \infty$ to that corresponding to $n = 1$ we see significant dependence. This dependence is in agreement with the theorem of this section. These observations may be useful to guide the development of models aimed at developing a suitable theory of the zero-crossing intervals of Gaussian processes. For example, we see immediately that a theory based on the independence of successive zero-crossing intervals is destined to be inadequate.

Similar photographs were made for all of the other random processes considered in this report. For the Gaussian process having the power spectral density $W_4(f)$, the correlation coefficient ρ_n is non-zero when $n \leq 4$, and the ith and (i+n)th zero-crossing intervals are statistically dependent when $n \leq 4$. For all of the other random processes, the correlation coefficient ρ_n practically vanishes when $n \geq 3$, and the statistical dependence between the ith and (i+n)th zero-crossing intervals practically vanishes when $n \geq 4$.

X. APPLICATIONS

The sampling technique for generating Gaussian noise which was described in Section IIIB is one application resulting from this study of the zero-crossing intervals. Another application concerns an integral technique for measuring pulse duration and has been reported in the literature (36).

In certain countermeasure applications of the zero-crossing phenomenon one requires a modulating noise waveform having an "optimum" $P_o(\tau)$, the probability density function for successive zero-crossing intervals. Usually one asks that a certain mean, $E_o(\tau)$, and a certain standard deviation, σ_o, be associated with the $P_o(\tau)$. The other required properties of $P_o(\tau)$ for "optimum" conditions are somewhat vague. Here we shall give a few results concerning the "optimum" $P_o(\tau)$.

A. THE MOST RANDOM DISTRIBUTION OF SUCCESSIVE ZERO-CROSSING INTERVALS

Consider a sequence of successive zero-crossing intervals $\{\tau_i\}$ generated by a sample function of an ergodic random process having a finite expected number of zeros per unit time. Let each of the τ_i have a common probability density $P_o(\tau)$, and let the n dimensional joint probability density for the sequence of intervals $\{\tau_i\}$ be denoted by $P(\tau_1, \tau_2, \tau, \ldots \tau_n)$. In order to determine the most random distribution of successive

zero-crossing intervals, we adopt the criterion that the relative entropy per zero-crossing interval, H', given by

$$H' = - \lim_{n \to \infty} \frac{1}{n} \int_0^\infty \cdots \int_0^\infty P(\tau_1, \tau_2, \ldots \tau_n) \ln P(\tau_1, \tau_2, \ldots \tau_n) d\tau_1 \ldots d\tau_n \tag{115}$$

is a measure of the randomness. By a well-known principle from information theory, the maximum of H' occurs when the sequence of successive zero-crossing intervals, $\{\tau_i\}$, are statistically independent. That is when

$$P(\tau_1, \tau_2, \ldots \tau_n) = P_o^n(\tau) \quad \text{for all n} \tag{116}$$

Accordingly, in order to maximize H' we need only maximize the relative entropy, H, given by

$$H = - \int_0^\infty P_o(\tau) \ln P_o(\tau) \, d\tau \quad . \tag{117}$$

Let the constraints be:

$$\int_0^\infty P_o(\tau) \, d\tau = 1 \tag{118}$$

$$\int_0^\infty \tau P_o(\tau) \, d\tau = E_o(\tau) \tag{119}$$

and

$$\int_0^\infty \tau^2 P_o(\tau) \, d\tau = E_o(\tau^2) \quad . \tag{120}$$

By applying the calculus of variations we find that the maximum relative entropy, H, results when:

$$P_o(\tau) = \frac{1}{D\sqrt{2\pi}\,\Phi(\frac{m}{D})} \; e^{-\frac{(\tau-m)^2}{2D^2}} \tag{121}$$

where

$$E_o(\tau) = m + \lambda D$$

$$E_o(\tau^2) = m^2 + \lambda Dm + D^2$$

$$\lambda = \frac{\phi(\frac{m}{D})}{\Phi(\frac{m}{D})}$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \; e^{-x^2/2}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} \, dy \quad .$$

Given $E_o(\tau)$ and $E_o(\tau^2)$ one can determine m and D and hence $P_o(\tau)$ by using the tables reported by K. Pearson (37).

Accordingly, for a given mean, $E_o(\tau)$, and a given standard deviation, $\sigma_o = \left[E_o(\tau^2) - E_o^2(\tau) \right]^{1/2}$, the most random distribution of successuve zero-crossing intervals is represented by the truncated Gaussian probability density function shown in Figure 67 and by Equation (116). Since successive zero-crossing intervals must be statistically independent in order to yield the maximum relative entropy per zero-crossing interval, the theorem of Section IX implies that a Gaussian process or an infinitely clipped Gaussian process can not generate the most random distribution of successive zero-crossing intervals. Moreover, for Gaussian

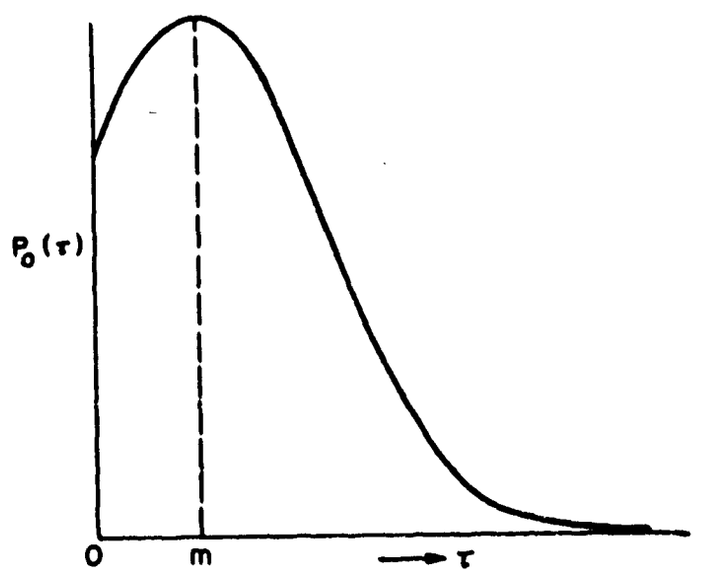


Figure 67 The Truncated Gaussian Probability Density, $P_o(\tau)$, Represents the Most Random Distribution of Successive Zero-Crossing Intervals Having a Given Mean and a Given Standard Deviation.

processes of finite memory, $P_o(\tau)$ for large τ must have an asymptotic exponential (5, 13, 38, 41) behavior such as:

$$P_o(\tau) \sim e^{-\alpha\tau} \quad . \tag{122}$$

Physically, this means that the random process having a $P_o(\tau)$ given by Equation (121) cannot be a Gaussian process or an infinitely clipped Gaussian process. That is, one cannot obtain a random process having a $P_o(\tau)$ given by Equation (121) by the familiar technique of spectrally shaping "white" Gaussian noise with a linear filter.

It is interesting to note that if the constraints are given by Equations (118) and (119) only, then the maximum entropy H occurs when

$$P_o(\tau) = \frac{1}{E_o(\tau)} \, e^{-\frac{\tau}{E_o(\tau)}} \quad . \tag{123}$$

The Gaussian process having power spectral density $W_1(f)$ generates a $P_o(\tau)$ that approximates an exponential density of the form given by Equation (123) as is seen in Figure 18. Furthermore, successive zero-crossing intervals of this Gaussian process are approximately independent as is seen in Figure 19.

B. TECHNIQUE FOR ESTIMATING THE ASYMPTOTIC EXPONENTIAL BEHAVIOR OF $P_o(\tau)$

Here we describe a technique for estimating the α in Equation (122). Lewis (39) and McFadden (32) showed that:

$$E_o(\tau^2) = 2E_o(\tau) \int_0^\infty p(0,\tau)\, d\tau \quad . \tag{124}$$

If $P_o(\tau)$ is asymptotically exponential then $p(0,\tau)$ and $Z(1,\tau)$ are also asymptotically exponential as is seen from Equations 39 and 41. Hence, in order to determine α in Equation (122) we determine a "reasonable" exponential to tie on to the measured $p(0,\tau)$ such that Equation (124) is satisfied for measured $E_o(\tau^2)$ and theoretical $E_o(\tau)$. Semi-log plots of $P_o(\tau)$, $p(0,\tau)$ and $Z(1,\tau)$ are useful for determining where to tie on the exponential. However, the point of "tie on" is still somewhat arbitrary. Accordingly, we end up with at most rough estimates of α. Using normalized time, some rough estimates of α_n associated with the asymptotic exponential behavior:

$$P_o(u_n) \sim e^{-\alpha_n u_n} \tag{125}$$

are given below.

For tie on point at $u_1 = 8$, $\alpha_1 = 0.28$

For tie on point at $u_2 = 12$, $\alpha_2 = 0.11$

For tie on point at $u_3 = 4$, $\alpha_3 = 0.47$

XI. CONCLUSIONS

The results of this study of the zero-crossing intervals of random processes should prove useful for guiding the development of some new techniques which are useful in both science and technology.

The sampling technique for generating Gaussian noise is perhaps the simplest technique available for generating such noise.

The integral technique for measuring pulse duration should be especially useful for airborne or space instrumentation applications.

At present none of the probabilities or probability densities presented in this report can be derived explicitly by analytical methods.

The probability density $P_o(\tau)$ for the case of Gaussian noise having a power spectral density $W_1(f)$ is not exactly $\frac{1}{\pi} e^{-\tau/\pi}$.

The hypothesis that successive zero-crossing intervals of Gaussian processes from a wide sense Markov chain is false for some Gaussian processes.

The experimental initial values of $P_0(u_n)$ and $P_1(u_n)$ for $n = 1, 3$ obtained by extrapolation agree satisfactorily with the approximate theoretical initial values.

The results of this study should prove useful for guiding the development of models aimed at developing a suitable theory for the zero-crossing intervals of certain random processes. In general the models must not assume that successive zero-crossing intervals of random processes are statistically independent.

For the Gaussian process having the power spectral density $W_4(f)$, the correlation coefficient ρ_n is nonzero when $n \leq 4$, and the ith and (i+n)th zero-crossing intervals are statistically dependent when $n \leq 4$. For all of the other random processes, the correlation coefficient ρ_n practically vanishes when $n \geq 3$, and the statistical dependence between the ith and (i+n)th zero-crossing intervals practically vanishes when $n \geq 4$.

XII. APPENDIX

For a Gaussian process having a normalized autocorrelation function $\rho(\tau)$, the conditional probability, $U(\tau)\,d\tau$, that a zero occurs between $t+\tau$, $t+\tau+d\tau$ given a zero at t is given by:

$$U(\tau)d\tau = \frac{d\tau}{\pi}\left[-\rho''(0)\right]^{-\frac{1}{2}}\left[\frac{M_{23}}{H}\right]\left[1-\rho^2(\tau)\right]^{-\frac{3}{2}}\left[1+H\tan^{-1}H\right] \tag{126}$$

where

$$H = M_{23}\left[M_{22}^2 - M_{23}^2\right]^{-\frac{1}{2}}$$

$$-\frac{\pi}{2} \le \tan^{-1}H \le \frac{\pi}{2}$$

$$M_{22} = -\rho''(0)\left[1-\rho^2(\tau)\right] - \left[\rho'(\tau)\right]^2$$

$$M_{23} = \rho''(\tau)\left[1-\rho^2(\tau)\right] + \rho(\tau)\left[\rho'(\tau)\right]^2 \quad .$$

For a Gaussian process having a normalized autocorrelation function $\rho(\tau)$, the conditional probability, $Q(\tau)\,d\tau$, that a downward zero-crossing occurs between $t+\tau$, $t+\tau+d\tau$ given an upward zero-crossing at time t is given by:

$$Q(\tau)\,d\tau = \frac{d\tau}{2\pi}\left[-\rho''(0)\right]^{-\frac{1}{2}}\left[\frac{M_{23}}{H}\right]\left[1-\rho^2(\tau)\right]^{-\frac{3}{2}}\left[1+H\cot^{-1}(-H)\right] \tag{127}$$

where

$$0 \le \cot^{-1}(-H) \le \pi \quad .$$

The normalized autocorrelation function, $r(u_o, a)$, for the infinitely clipped process $\xi(u, a)$, defined by Equation (82), is given by:

$$r(u_o, a) = -4 \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{h_{o,n}^2}{n!} \rho_o^n(u_o) - 8 \left[\sum_{\substack{m=1 \\ \text{odd } m}}^{\infty} h_{m,0}^2 \operatorname{Cos} \frac{m u_o}{2} + \sum_{\substack{n=1 \\ \text{odd } (m+n)}}^{\infty} \sum_{m=1}^{\infty} \frac{h_{m,n}^2}{n!} \rho_o^n(u_o) \operatorname{Cos} \frac{m u_o}{2} \right] \tag{128}$$

where

$$h_{m,n}^2 = -\frac{1}{(2m!)^2} \, 2^n a^m \, \frac{{}_1F_1{}^2\left(\frac{m+n}{2}; m+1; -a\right)}{\Gamma^2\left[\frac{2-(m+n)}{2}\right]}$$

${}_1F_1$ = confluent hypergeometric function

Γ = gamma function

a = signal-to-noise power ratio.

The normalized autocorrelation function, $r_s(u_o, a)$, for the infinitely clipped process $\eta(u, a)$, defined by Equation (106), is given by:

$$r_s(u_o, a) = -4 \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{h_{o,n}^2}{n!} \left[\frac{\rho_o''(u_o)}{\rho_o''(0)} \right]^n - 8 \left\{ \sum_{\substack{m=1 \\ \text{odd } m}}^{\infty} h_{m,o}^2 \operatorname{Cos} \frac{m u_o}{2} + \sum_{\substack{n=1 \\ \text{odd } (m+n)}}^{\infty} \sum_{m=1}^{\infty} \frac{h_{m,n}^2}{n!} \left[\frac{\rho_o''(u_o)}{\rho_o''(0)} \right]^n \operatorname{Cos} \frac{m u_o}{2} \right\} \tag{129}$$

where

$$h^2_{m,n} = -\frac{1}{(2m!)^2}\ 2^n(ba)^m\ \frac{{}_1F_1{}^2\left(\frac{m+n}{2};m+1;-ba\right)}{\Gamma^2\left[\frac{2-(m+n)}{2}\right]}$$

$$b = \frac{\operatorname{Sin}\frac{3\pi}{14}}{4\operatorname{Sin}\frac{\pi}{14}}$$

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