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GRAPH THEORY AND AUTOMATIC CONTROL

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SUMMARY

Several important classes of problems in the theory of automatic control--including time-optimal control--find a natural setting in the field of graph theory. Various aspects of formulation, analytical and numerical treatment, and implementation are sketched. The paper is intended to be quite self-contained.

1. Introduction

A problem of central importance in the theory of automatic control involves the transforming of a system from an initial state into a desired terminal state in the most efficient fashion. If, for example a gust of wind causes an aircraft to begin rolling, the automatic pilot is supposed to send control signals to the control surfaces in an effort to restore the craft, as rapidly as possible, to the horizontal position with no angular velocity about its longitudinal axis. Or consider a spacecraft which is about to re-enter the earth's atmosphere. It is desired to fly along a trajectory which will bring the craft to the surface of the earth while keeping

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the maximum temperature to which the surface of the craft is exposed as low as possible. One might also wish to select a path which will minimize the maximum deceleration during descent.

From the mathematical viewpoint it seems natural to consider such problems as belonging to the calculus of variations. Toward this end one introduces a state vector $x(t)$, with $x(0) = c$, a control vector $y(t)$, and a dynamical equation

$$\dot{x} = f(x,y). \tag{1}$$

We wish to determine the control vector $y(t)$ in such a manner that the system is transformed from the initial state c to some desired terminal state, say $x = 0$, as rapidly as possible. An extensive literature now exists concerning such problems [1,2], especially for the case in which the function f is linear in x and y and the components of the control vector y are subject to certain constraints. Alternatively, we might wish to determine the control vector $y(t)$ so that we minimize the maximum value of some function $g(x(t),y(t))$ during the course of transforming the system from the initial state to the terminal state. There is a less extensive literature associated with such problems [3,4].

If we keep in mind that the treatment of significant problems in this area will ultimately involve the use of digital computers, wherein all variables are rendered discrete, and that some control systems operate in a discrete fashion by design, it becomes of interest to consider a discrete formulation and treatment of automatic control processes, as opposed to a continuous one. The aim of this paper is to give an indication of the current state of affairs in this program.

2. Time-optimal Control and the Nature of Feedback

Let us consider a system S which may be found in any of a finite number, N, of possible states. We represent these states by the nodes of a graph [5,6] and number them $1,2,\ldots,N$. In the course of time the system changes its state; i.e., it undergoes a sequence of transformations which we call a process. When the system is in state i, we assume that a control decision can be made, the result of which is that the system is transformed into some new state. In general, when the system is in state i, only certain new states may be attained as the result of making a control decision.

In the passage from state i to state j a certain amount of a resource such as energy must be consumed during such a transformation. This assigns a number, t_{ij}, to the directed arc from i to j. Let us consider that we wish to control the system S in such a way that if the system is disturbed from its equilibrium (or most desirable) state, say the state N, then we shall return it to the equilibrium state in the least possible time.

At first glance it might seem that we wish merely to find a shortest trajectory through phase space leading from the initial state i to the desired terminal state N. Actually this is not so. Frequently we shall have no prior knowledge of what the particular initial state i will be, so that we must be prepared to transform the system from any initial state i into the desired terminal state, and do so in minimum time. Now we may go one step further and note that if the system is in state i, it is not necessary to specify the entire trajectory from i to N. Indeed, all we need do is specify that if the system is in state i, then the next state into which to transform the system is a particular state j.

These considerations enable us to distinguish between what is called open-loop and feedback control. In an open-loop process the entire sequence of transformations is specified ahead of time. This type of control is useful when one has great confidence that the system will perform as specified and that random external influences are negligible. In practice this means that all control influences will be exerted in a prescribed fashion as a function of time, i.e., according to the reading of a clock, no notice being taken of the actual state of the system. A simple example of this is the control of the temperature of a room by turning a furnace on and off at certain fixed times. Open-loop control is usually economical to provide, but in turn, may not provide satisfactory control due to the neglect of external factors.

A more sophisticated type of control is achieved by carrying out the following cycle of operation: (1) determine the state of the system (measure the temperature of the room); (2) decide on a control action (turn the furnace on or off); (3) return to (1). A control system which operates on this principle is termed a feedback control system. There is an extensive theory of such processes going back to Maxwell [7,8,9]. For the most part this is a theory describing how a system would act if controlled in a certain way, the notion of stability playing a major role [10]. It is clear, though, that we wish to control in optimal fashion, so that emphasis has turned from stability to optimality considerations, though, of course, there is an intimate relation between the two.

3. Formulation [11]

We introduce the following nomenclature:

(1) t_{ij} = the time required to transform the system from state i to state j over a direct link.

(2) u_i = the time to transform the system from state i to state N (the desired state) in optimal fashion, $i = 1,2,\ldots,N$.

To derive relations among these quantities we note that if the system is in state i and the decision to transform it into state j is made, $j \neq i$, then the process will have to continue optimally from state j to state N if the entire process is to be optimal. Furthermore, the choice of the next state, state j, is made on the basis of minimizing the sum

(3) $$t_{ij} + u_j,$$

for t_{ij} is the time to pass directly from state i to state j, and u_j is the minimal time required to pass from state j to the terminal state N. These observations, manifestations of Bellman's principle of optimality [1], lead to the nonlinear system of equations

(4) $$u_i = \min_{j \neq i} (t_{ij} + u_j), \quad i = 1,2,\ldots,N - 1,$$
$$u_N = 0.$$

4. Uniqueness [11]

Since we contemplate solving the equations (3.4) via various successive approximation techniques, it is important to establish the uniqueness of the solution. Suppose that $\{u_i\}$ and $\{U_i\}$ represent two solutions of the system of equations (3.4), and that k is an index where the difference

(1) $$u_i - U_i$$

achieves its maximum. Let

(2) $$u_k = \min_{j \neq k} \{t_{kj} + u_j\} = t_{kr} + u_r$$

and

(3) $$U_k = \min_{j \neq k} \{t_{kj} + U_j\} = t_{ks} + U_s.$$

Since we are assuming that

(4) $$t_{ij} > 0,$$

it is clear that

(5) $$r \neq k, \quad s \neq k.$$

We have the inequalities

(6) $$u_k = t_{kr} + u_r \leq t_{ks} + u_s,$$

(7) $$U_k = t_{ks} + U_s \leq t_{kr} + U_r,$$

which lead to the relation

(8) $$u_k - U_k \leq u_s - U_s.$$

Since k is an index for which the difference $u_j - U_j$ is maximized, equality must hold in relation (8),

(9) $$u_k - U_k = u_s - U_s,$$

which implies that equality also holds in relation (6)

(10) $$u_k = t_{ks} + u_s.$$

But now we may repeat the same reasoning for the state s and establish that there is another state, state m, for which

(11) $$u_m - U_m = u_s - U_s = u_k - U_k.$$

Furthermore $m \neq s$ and $m \neq k$ since

(12) $$u_k = t_{ks} + t_{sm} + u_m.$$

Continuing in this way we must eventually come upon the state N, for which

(13) $$u_N - U_N = 0,$$

which completes the proof.

5. Successive Approximations

We may use Picard's method of successive approximations to establish the existence of a solution of the system of equations (3.4) and to provide a practical computational scheme. As our initial approximation, $u_i^{(0)}$, we make the decision to transform the system directly from state i to state N. Of course, if no such direct link exists we assume that the time that elapses is M, a suitably large number,

$$(1) \qquad u_i^{(0)} = t_{iN}, \quad i = 1,2,\ldots,N.$$

The higher order approximations are obtained in the usual way,

$$(2) \qquad \begin{cases} u_i^{(k+1)} = \min\limits_{j \neq i} \{t_{ij} + u_j^{(k)}\}, & i = 1,2,\ldots,N-1, \\ u_N^{(k+1)} = 0, & \end{cases}$$

for $k = 0,1,2,\ldots$. The physics of the problem enables us to see some of the properties of the successive approximations. For example, since

$$(3) \qquad u_i^{(1)} = \min_{j \neq i} \{t_{ij} + t_{jN}\}, \quad i = 1,2,\ldots,N-1,$$

$$u_i^{(1)} = 0,$$

we see that

(4) $u_i^{(1)}$ = the minimal time to transform the system from state i to state N via at most one intermediate state, $i = 1,2,\ldots,N$.

And, in general,

(5) $u_i^{(k)}$ = the minimal time to transform the system from state i to state N via at most k intermediate states.

From this it follows that the approximations are monotone decreasing,

(6) $$u_i^{(k+1)} \le u_i^{(k)}, \quad i = 1,2,\ldots,N,$$

which is easy to establish via induction. Since $u_i^{(k)}$ is bounded from below,

(7) $$u_i^{(k)} \ge 0, \quad i = 1,2,\ldots,N,$$

the convergence of the approximating sequence is established. As a matter of fact, though, an optimal trajectory from any state i to state N has at most $N - 2$ intermediate states, since an optimal trajectory does not cross itself to form a loop. Thus the convergence of the process is assumed after at most $N - 2$ stages. That the limiting values satisfy equations (3.4) is clear.

6. Observations on the Approximation Scheme

Use of the equations (5.2) involves only addition and comparison of numbers, two operations for which a digital computer is well-suited. Furthermore, in calculating the value of $u_i^{(k+1)}$ only the i-th row of the matrix (t_{ij}) and the vector $(u_1^{(k)}, u_2^{(k)}, \ldots, u_N^{(k)})$ need be in high-speed storage. In this way problems for which N is of the order of several thousand can be solved by an IBM-7090 in several minutes. Efficient programs will exploit special features of a given problem. This is certainly true if each state is directly connected to only several of its neighbors. In some instances the values of the successive approximations are more important than the solution of the original problem. If the optimal trajectory from state i to state N has many intermediate states, then it might require a complex instrumentation to achieve, so that a knowledge of both the successive approximations and the limiting values may be of importance in designing a control system.

Finally, let us note that our solution consists not so much in the production of the values $u_1, u_2, \ldots, u_N$, as in the knowledge of the value of j which minimizes the expression $t_{ij} + u_j$, for each value of i. This is precisely the knowledge that is required for determining optimal feedback control.

7. Other Approaches

Many other approaches to this problem have been devised, in particular by Dantzig [12] and Ford and Fulkerson [13]. In particular it is possible to fan out from the destination and determine, one after the other, the nearest, second nearest, and so on, states to the terminal state.

A variety of analogue devices can be used. Some of these are described in the paper [14], where many references are provided.

8. Arbitrary Terminal States

In the event that we wish to determine optimal trajectories from any initial state to any terminal state, we might apply the procedures referred to above N times, in each case letting a different state be the desired terminal state. Alternatively, we may let

(1) $u_{ij}^{(k)}$ = the time to transform a system in state i into state j using a trajectory with at most k intermediate states.

Using the principle of optimality we see that

$$(2) \qquad u_{ij}^{(2k+1)} = \min_{m \neq i} \{u_{im}^{(k)} + u_{mj}^{(k)}\}, \quad i \neq j.$$

Since k will be at most $N - 1$, and since we can easily determine the matrices $(u_{ij}^{(0)})$, $(u_{ij}^{(1)})$, $(u_{ij}^{(3)})$, $(u_{ij}^{(7)})$, $(u_{ij}^{(15)}),\ldots,$ the problem is readily handled, at least computationally.

9. Preferred Suboptimal Trajectories [15]

One of the key difficulties in applying these ideas to a concrete physical situation lies in deciding on the number and nature of the physical states of the system to be considered. In particular if the number of states chosen is too small, i.e., the grid in phase space is too coarse, the time to traverse a second shortest path may be considerably larger than the time to traverse an optimal path. On the other hand, if these times are not too different, one's confidence in the reasonableness of the mathematical model of the physical process may be increased.

In addition, we must recognize that even if we determine an exact solution to the mathematical solution, due to the neglect of a variety of physical factors we have only an approximate solution of the physical solution. If a mathematically optimal trajectory which we have found is unsatisfactory from the physical viewpoint, we may either find a near optimal trajectory for the mathematical problem, in hopes that it will be better from the physical viewpoint, or we may reformulate the mathematical problem. Let us show how we may determine second best trajectories, once having determined optimal trajectories to the terminal state N.

Under the assumption that there is at least one trajectory from state i to state N that is not optimal, $i = 1,2,\ldots,N-1$, let us define the variables v_i, $i = 1,2,\ldots,N-1$,

(1) v_i = the time that it takes to transform a system from state i to state N using a second best trajectory, $i = 1,2,\ldots,N-1$,

$$v_N = 0.$$

Of course, we are assuming that there are no loops in the trajectory. Next notice that if we make the decision to transform the system from state i directly to state j, then the continuation must be along a trajectory from state j to state N which is either optimal or second best. This leads to the relations

(2)
$$v_i = \min_{2\, j \neq i} [t_{ij} + u_j, t_{ij} + v_j], \quad i = 1,2,\ldots,N-1,$$

$$v_N = 0,$$

where we have used the notation

(3) $\min_2(a_1,a_2,\ldots,a_R)$ = the second smallest of $a_1,a_2,\ldots,a_R$ (under the assumption that they are not all equal).

Generalizations are given in the paper [15], and are discussed at length in [16].

Pollack [17] has also observed that we may find second best paths by first finding the optimal trajectory (assumed unique) from state i to state N. Then we eliminate the first link in the optimal path from the network and determine an optimal trajectory from state i to state N using only the remaining links in the network. Then this is done for each of the remaining links in the optimal trajectory being considered. Since there are at most $N - 1$ such links in the optimal trajectory, at most $N - 1$ such problems need be solved. A trajectory which yields the smallest of the numbers so found is a second shortest trajectory. The proof is given by noting that a second shortest trajectory must differ from an optimal trajectory in at least one link. With this method there is no difficulty concerning the possible formation of trajectories with loops.

10. A Stochastic Time-optimal Control Process [14]

Next let us assume that the physical situation is such that the time involved in transforming the system from state i to state j, t_{ij}, is not known precisely. Suppose, though, that we may consider it to be a random variable with a known probability density function $p_{ij}(t)$. This is a great assumption which may or may not be justified in a given situation. Furthermore, we shall assume that the time to traverse any link in a trajectory is independent of the time to traverse any other link in the trajectory, another severe restriction.

Under the assumptions just stated, our aim is to find the optimal feedback control decision to make when the system is in state i. Let us now explain carefully what we mean by this. We shall assume that our objective is to maximize the probability of transforming the system into the desired state N in a time t or less, where

(1) $\quad t \geq 0.$

The sequence of operations to be carried out is this: First the current state of the system and the time are measured (by sensing equipment). Secondly, a decision is made as to the next state that the system is to occupy. Then a random time elapses until the system reaches this state. Next a measurement is made of the new time and state. On the basis of this knowledge of the new state and the new time, a decision is made as to what the next state is to be, and so on. Notice particularly that we do not attempt to lay out the entire sequence of decisions leading from state i to state N initially (as would be the case for open loop control); rather we observe, decide, and act over and over again. We aim to determine the optimal decision to make under given circumstances, i.e., for a given state and given time remaining in the process. While for deterministic processes open loop and closed loop (feedback) control lead to identical results, for stochastic decision processes they are conceptually quite different.

Let us now define the functions $u_i(t)$, $i = 1,2,\ldots,N$, by the equations

(2) $\quad u_i(t)$ = the probability of transforming a system from the initial state i to the desired terminal state N in time t or less using an optimal feedback control policy.

If we use the principle of optimality once again, we can write the equations

$$(3)\qquad \begin{cases} u_i(t) = \max\limits_{j \neq i} \int_0^t p_{ij}(t-s)u_j(s)ds, & i = 1,2,\ldots,N-1, \\ u_N(t) = 1. \end{cases}$$

Equations (3) appear to be quite difficult to handle both analytically and computationally. The appearance of the convolution integrals suggests use of Laplace transforms, but the occurrence of the maximum operator militates against this. The maximum transform discussed by Bellman and Karush [18] and others might be useful.

Some other closely related stochastic control processes are discussed in references [14] and [19].

11. Minimax Control Processes

In some circumstances, such as we have discussed in Sec. 1, it is desirable to transform a system from an initial state to a terminal state in such a way as to minimize the maximum stress to which the system is exposed during the course of the process. We shall refer to such control processes as "minimax control processes." Once again we shall consider a system which may be in any of a finite number of states, N, the states being numbered from 1 to N. We consider state N to be the desired terminal state. When the system is transformed directly from state i to state j, a maximum stress, s_{ij}, is encountered. Thus the stress s_{ij} is associated with the link (i,j). Our basic problem consists in finding a trajectory from state i to state N which is such that the maximum stress along this trajectory is as small as possible.

12. Use of Functional Equations

Let us introduce the variables u_i, $i = 1,2,\ldots,N-1$, by the relations

(1) u_i = the maximum stress along an optimal trajectory from state i to the terminal state N, $i = 1,2,\ldots,N-1$,

$$u_N = 0.$$

Then use of the principle of optimality immediately leads to the relations

(2) $$u_i = \min_{j \neq i} \{\max(s_{ij}, u_j)\}, \quad i = 1,2,\ldots,N-1,$$

$$u_N = 0.$$

These are, of course, the analogues of equations (3.4). The results of the following sections could be obtained directly from these equations. Rather than pursue this path, we shall keep the network itself in the foreground.

13. A Special Case

A very important and interesting special case arises when we stipulate the reversibility equality

(1) $$s_{ij} = s_{ji}.$$

Thus one stress, s_{ij}, is connected with the arcs (i,j) and (j,i). We can carry through the analysis in some detail and establish a relationship with a seemingly unrelated problem in graph theory.

We first observe that the maximum stress encountered along any trajectory is one of the numbers $\{s_{ij}\}$. Let us then arrange the (positive) numbers $\{s_{ij}\}$ in ascending order of magnitude and denote this sequence by $s_1, s_2, \ldots, s_R$ $(R \leq (N/2)(N + 1))$. Call the corresponding arcs $S_1, S_2, \ldots, S_R$. For convenience, we assume that the stresses associated with the arcs are all different from one another, a condition which can be attained by adding suitably small quantities to the stresses given, if necessary. Then we observe that the states joined by the arc having the stress s_1 cannot be joined by any trajectory with a smaller maximum stress. This arc, S_1, constitutes an optimal trajectory for those states. Next we observe that the states joined by the arc of stress s_2 are also joined optimally by this arc, S_2.

The situation becomes a little more complicated insofar as the states joined by arc S_3 are concerned. If the arcs S_1, S_2 and S_3 do not form a loop, then the states joined by arc S_3 are joined optimally by it. However, if arcs S_1, S_2, and S_3 do form a loop, then the arcs S_1 and S_2, and not S_3, connect these states optimally.

Furthermore, we see that if we continue this process of selecting arcs from the sequence of arcs $S_1, S_2, \ldots, S_R$, making sure that no arc selected forms a loop with any of the arcs already selected, we shall eventually select $N - 1$ arcs containing no loops.

We shall thus have formed a particular spanning tree in the network, and the unique trajectory in this tree which connects any two states is the minimax trajectory between those states.

14. Minimal Spanning Trees

But the construction which we have just indicated is well known to provide the solution to another seemingly unrelated problem: Under the conditions stated in Sec. 13, find the tree for which the sum of the stresses in its branches (arcs) is as small as possible. This tree is called the minimal spanning tree. In 1956 Kruskal [20] showed that the construction given solves this problem. Various other algorithms are given in references [20,14].

The algorithm described is not satisfactory from the computational viewpoint, for testing to see whether or not an arc completes a loop when added to another set of arcs can be quite time-consuming. Prim [21] has given some very effective computational procedures.

15. Comments and Interconnections

Our result on minimax control processes, subject to the restrictions in Sec. 13, may be formulated thusly: optimal minimax trajectories lie in the minimal spanning tree. Let us now illuminate this result in several other ways.

First we give another proof. Consider an optimal minimax trajectory from state i to state N. Suppose that is is not the trajectory from state i to state N which lies in the minimal spanning tree. Then there is at least one arc in this trajectory which does not lie in the minimal spanning tree. Denote this as arc A. If we add this arc to the set of arcs in the minimal spanning tree, then exactly one loop is formed, say the loop with arcs $(A, A_1, \ldots, A_J)$. But the following inequality must hold

(1) $$\text{stress}(A) \geq \max(\text{stress}(A_1), \text{stress}(A_2), \ldots, \text{stress}(A_J)).$$

If it did not, then it would be possible to lower the sum of the stresses in the branches of the minimal spanning tree by adding arc A to the minimal spanning tree and deleting an arc A_2 for which

(2) $$\text{stress}(A) < \text{stress}(A_2).$$

Inequality (1) shows that the arc (A) in the supposed minimax trajectory may be replaced by arcs in the minimal spanning tree without increasing the maximal stress encountered. Since arc (A) could be any arc in the minimax trajectory, the proof is complete.

In his paper [21], Prim observed that the minimal spanning tree not only minimizes the sum of the stresses in its branches but also minimizes, among all trees, any monotone increasing and symmetric function of the stresses in the branches. In particular, we note, it minimizes the function

(3) $$S_p = [s_1{}^p + s_2{}^p + \ldots + s_{N-1}{}^p]^{1/p},$$

for $p = 1, 2, \ldots,$ and the limit function

(4) $$\max(s_1, s_2, \ldots, s_{N-1}) = \lim_{p \to \infty} S_p,$$

which is easily proved.

This suggests that we consider the problem of determining a trajectory from state i to state N which minimizes the sum of the p-th powers of the stresses in its arcs. Clearly for $p = 1$ this is equivalent to the problem of determining a time-optimal trajectory, which was considered earlier. On the other hand, in view of equation (4), for p sufficiently large this is the problem of determining a minimax trajectory. More precisely, we can show that for p sufficiently large a trajectory which minimizes the sum of the p-th powers of the stresses in its arcs lies in the minimal spanning tree. For consider an arc S which is in the optimal trajectory but which does not lie in the minimal spanning tree. Let its stress be s, and let the stresses of the arcs in the minimal spanning tree with which it forms a loop be $s_1, s_2, \ldots, s_r$. Then, as we observed earlier,

$$s > \max(s_1, s_2, \ldots, s_r), \tag{5}$$

and for p sufficiently large,

$$1 > (s_1/s)^p + (s_2/s)^p + \ldots + (s_r/s)^p, \tag{6}$$

or

$$s^p > s_1^p + s_2^p + \ldots + s_r^p. \tag{7}$$

This inequality shows that the arc S may be replaced by arcs in the minimal spanning tree and establishes the result.

In reference [22] M. Pollack posed the problem of determining a maximum capacity path between two stations in a communications network and provided several solutions. This problem is equivalent to ours. Furthermore, D. R. Fulkerson has pointed out to the author that T. C. Hu [23], commenting on Pollack's paper, has obtained our result on minimax trajectories.

16. Multiple Stresses

It frequently happens that stresses arise during a process from several different causes, i.e,, mechanical and thermal. While, in general, a minimax trajectory for one is not a minimax trajectory for the other, we can attempt to determine trajectories which are optimal in the sense that no change in the trajectory can lower both maximal stresses.

On link (i,j) let the maximal thermal stress be t_{ij} and the maximal mechanical stress be m_{ij}. Then, if we introduce the Lagrange multipliers n_1 and n_2, we can associate the generalized stress s_{ij},

$$(1) \qquad s_{ij} = n_1 t_{ij} + n_2 m_{ij}$$

with each arc. By letting, e.g.,

$$(2) \qquad \begin{aligned} n_1 &= 1, \\ n_2 &= a, \end{aligned}$$

and determining a minimax trajectory between two particular states, for which the maximum mechanical stress is, for example, s, we can guarantee, that among all trajectories for which the maximal mechanical stress between the states is s, we will have found a trajectory for which the thermal stress is minimal. Then a parameter study, involving the determination of many minimal spanning trees, might yield useful design information concerning the trade-offs that are possible.

17. Discussion

Our primary aim has been to show the close connection between several important classes of automatic control problems and graph theory. These considerations have raised many additional questions. Let us conclude by stating some of these.

Formulation. How are we to decide how many states of a system need be considered, what the stresses or times are, and what the criterion is?

Analytical and Computational Treatment. What are the connections between the solution to the discrete and continuous problems? Myriad other problems, too varied to catalogue, arise.

Implementation. After the optimal feedback control decisions have been determined, how can controllers be realized to carry out the programs?

It is felt that the answers to these questions would provide both automatic control and mathematics with additional interesting chapters.

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