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MEMORANDUM
RM-3397-PR
FEBRUARY 1963

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# A NOTE ON NONLINEAR SUMMABILITY TECHNIQUES IN INVARIANT IMBEDDING

Richard Bellman and Robert Kalaba

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PREPARED FOR:

UNITED STATES AIR FORCE PROJECT RAND

The RAND Corporation
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296 925

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**Richard Bellman and Robert Kalaba**

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## PREFACE

Part of the Project RAND research program consists of basic supporting studies. The research presented in this Memorandum concerns certain mathematical techniques as applied to problems of radiative transfer.

## SUMMARY

The use of principles of invariance, as in invariant imbedding and dynamic programming, leads characteristically to functional equations of the form

$$f_{n+1}(p) = T_n(f_n(g(p))), \quad n = 0,1,2,\ldots,$$

where $f_0(p)$ is known. The computational solution proceeds stagewise, with $f_1$ determined from a knowledge of $f_0$, $f_2$ determined by $f_1$, and so on.

In general, what is desired is the transient behavior, small $n$, and the steady-state, or asymptotic behavior as $n \to \infty$. In a number of significant processes—radiative transfer, control theory, inventory theory, and Markovian decision processes in general—only the asymptotic results are of interest. This is also the case in the application of gradient techniques.

In this paper, we shall outline the application of nonlinear summability techniques to radiative transfer.

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# A NOTE ON NONLINEAR SUMMABILITY TECHNIQUES IN INVARIANT IMBEDDING

## 1. INTRODUCTION

The use of principles of invariance, as in invariant imbedding [1] and dynamic programming [2], [3], leads characteristically to functional equations of the form

$$(1.1) \qquad f_{n+1}(p) = T_n(f_n(g(p))), \quad n = 0,1,2,\ldots,$$

where $f_0(p)$ is known. The computational solution proceeds stagewise, with $f_1$ determined from a knowledge of $f_0$, $f_2$ determined by $f_1$, and so on.

In general, what is desired is the transient behavior, small $n$, and the steady-state, or asymptotic behavior as $n \to \infty$. In a number of significant processes—radiative transfer [1], control theory [3], inventory theory [4], and Markovian decision processes in general [5], [6]—only the asymptotic results are of interest. This is also the case in the application of gradient techniques [7].

In some cases where the asymptotic results are not of primary importance, they are worth obtaining in order to test the accuracy of the numerical techniques, since the steady-state solution can often be derived by other independent means.

It is evident that direct step-by-step calculation of the asymptotic solution, using (1.1), is time consuming.

Hence, it is important to develop extrapolation techniques. A most important step in this direction is the work of Shanks [8], closely related to the QD-algorithm of Rutishauser [9]. We shall discuss below some extensions related to nonequally spaced observations or calculations. In this connection, let us mention the work of Kantorovich and Krylov [10] concerning the improvement of convergence of Fourier series and other types of sequences.

In this paper, we shall outline the application of nonlinear summability techniques to radiative transfer. In a separate communication, we discuss its use in connection with control theory [11].

## 2. MOTIVATION OF METHOD

The fundamental assumption is that we possess an asymptotic expansion of the form

$$f_n \sim f_\infty + \sum_k a_k e^{-\lambda_k n} \tag{2.1}$$

as $n \to \infty$, or that we can sensibly approximate to $f_n$ by an expression of the type appearing on the right-hand side. From theoretical considerations in invariant imbedding, dynamic programming, gradient techniques and elsewhere, we obtain rigorous demonstrations of this relation; see [12], [13]. What is remarkable, as shown by Shanks [8], is that quite accurate results are

obtained even when an asymptotic relation of this type does not hold, and even more remarkable is the fact that calculations based on small values of $n$ give excellent estimates for $f_\infty$.

If we take $f_n$ to have the simple form

$$f_n = f_\infty + ae^{bn}, \tag{2.2}$$

an immediate calculation yields

$$f_\infty = \frac{\begin{vmatrix} f_n & f_{n+1} \\ f_{n+1} & f_{n+2} \end{vmatrix}}{(f_n + f_{n+2} - 2f_n)}. \tag{2.3}$$

In the next section we shall describe some experiments with this simple nonlinear predictor. As mentioned above, many further results based upon more sophisticated approximations will be found in Shanks [8].

## 3. RADIATIVE TRANSFER

Let parallel rays of radiation be incident upon a plane-parallel slab of finite thickness which absorbs radiation and scatters it isotropically (see Fig. 1).

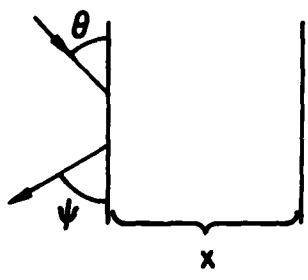


Fig. 1

Using the theory of invariant imbedding, we obtain an equation for the diffuse reflection $r(\theta,\psi,x)$ which leads to a feasible computational algorithm. Numerical and analytic results are given in [14] for a representative set of input angles $\theta$, output angles $\psi$, and thickness $x$. As pointed out above, it is of interest to obtain the results for infinite $x$ in this way in order to compare with previous results of Ambarzumian and Chandrasekhar obtained in another fashion.

If the albedo for single scattering is 0.9, a thickness of 6 mean free paths is required to saturate, i.e., to obtain a reflection coefficient equivalent to infinite thickness to about four decimal places. For the particular case of input and output angles at $60^0$ to the normal, we calculated the reflection coefficient at thicknesses of 0.00, 0.02, 0.04, ..., 1.20 mean free paths. These values are listed in the following Table 1, which is read across from left to right in each row and from the top row to the bottom row. For a thickness of 1.2 mean free paths the calculated value is 0.23295887.

Next we use (2.3) on each set of three consecutive entries in Table 1 to produce the entries in Table 2. These are predictions of the limiting value, which is 0.272389.

Using the predictions of Table 2, we can use the formula once again to produce another set of predicted values. These are shown in Table 3.

Table 1

THE SET OF REFLECTION COEFFICIENTS

| | | | | | |
|---|---|---|---|---|---|
| 0. | 0.90203642E-02 | 0.17958777E-01 | 0.26708581E-01 | 0.35219225E-01 | 0.43468159E-01 |
| 0.51446833E-01 | 0.59153822E-01 | 0.66591462E-01 | 0.73764186E-01 | 0.80677649E-01 | 0.87338272E-01 |
| 0.93752968E-01 | 0.99928968E-01 | 0.10587369E-00 | 0.11159468E-00 | 0.11709950E-00 | 0.12239571E-00 |
| 0.12749077E-00 | 0.13239207E-00 | 0.13710687E-00 | 0.14164225E-00 | 0.14600514E-00 | 0.15020226E-00 |
| 0.15424015E-00 | 0.15812514E-00 | 0.16186335E-00 | 0.16546070E-00 | 0.16892288E-00 | 0.17225537E-00 |
| 0.17546348E-00 | 0.17855225E-00 | 0.18152658E-00 | 0.18439115E-00 | 0,18715045E-00 | 0.18980878E-00 |
| 0.19237027E-00 | 0.19483887E-00 | 0.19721837E-00 | 0.19951241E-00 | 0.20172445E-00 | 0.20385781E-00 |
| 0.20591568E-00 | 0.20790109E-00 | 0.20981697E-00 | 0.21166609E-00 | 0.21345112E-00 | 0.21517462E-00 |
| 0.21683901E-00 | 0.21844663E-00 | 0.21999974E-00 | 0.22150043E-00 | 0.22295079E-00 | 0.22435274E-00 |
| 0.22570816E-00 | 0.22701886E-00 | 0.22828654E-00 | 0.22951284E-00 | 0.23069934E-00 | 0.23184753E-00 |
| 0.23295887E-00 | | | | | |

**Table 2**

THE FIRST ITERATION

| | | | | | |
|---|---|---|---|---|---|
| 0.99287306E-00 | 0.43262226E-00 | 0.33807480E-00 | 0.30347078E-00 | 0.28699413E-00 | 0.27778006E-00 |
| 0.27197126E-00 | 0.26796789E-00 | 0.26503253E-00 | 0.26280121E-00 | 0.26107164E-00 | 0.25972616E-00 |
| 0.25868013E-00 | 0.25788119E-00 | 0.25728158E-00 | 0.25684950E-00 | 0.25655429E-00 | 0.25637558E-00 |
| 0.25629469E-00 | 0.25628930E-00 | 0.25635426E-00 | 0.25647131E-00 | 0.25663887E-00 | 0.25683594E-00 |
| 0.25707100E-00 | 0.25732730E-00 | 0.25760114E-00 | 0.25789392E-00 | 0.25819670E-00 | 0.25850921E-00 |
| 0.25882599E-00 | 0.25915074E-00 | 0.25947401E-00 | 0.25979807E-00 | 0.26012429E-00 | 0.26044731E-00 |
| 0.26076716E-00 | 0.26108547E-00 | 0.26139567E-00 | 0.26170678E-00 | 0.26200978E-00 | 0.26230683E-00 |
| 0.26260211E-00 | 0.26288666E-00 | 0.26316931E-00 | 0.26344289E-00 | 0.26371384E-00 | 0.26397514E-00 |
| 0.26423369E-00 | 0.26448102E-00 | 0.26472771E-00 | 0.26496568E-00 | 0.26519617E-00 | 0.26542251E-00 |
| 0.26564503E-00 | 0.26585729E-00 | 0.26606464E-00 | 0.26626956E-00 | 0.26646777E-00 | |

Table 3

FURTHER SET OF PREDICTED VALUES

| | | | | | |
|---|---|---|---|---|---|
| 0.31887970E-00 | 0.28349463E-00 | 0.27201789E-00 | 0.26609017E-00 | 0.26206234E-00 | 0.25909086E-00 |
| 0.25696488E-00 | 0.25572946E-00 | 0.25510980E-00 | 0.25501280E-00 | 0.25502606E-00 | 0.25529800E-00 |
| 0.25547789E-00 | 0.25573519E-00 | 0.25591746E-00 | 0.25610149E-00 | 0.25622779E-00 | 0.25628892E-00 |
| 0.25629428E-00 | 0.25620828E-00 | 0.25608309E-00 | 0.25551960E-00 | 0.25561663E-00 | 0.25423604E-00 |
| 0.25332305E-00 | 0.25336742E-00 | 0.24902849E-00 | 0.24848599E-00 | 0.23524898E-00 | 0.24591999E-00 |
| 0.33054144E-00 | 0.12678152E-00 | 0.21058628E-00 | 0.29312269E-00 | 0.29283288E-00 | 0.32871063E-00 |
| 0.27323254E-00 | 0.15552984E-00 | 0.27332580E-00 | 0.27714357E-00 | 0.31166675E-00 | 0.27043446E-00 |
| 0.30527262E-00 | 0.27169371E-00 | 0.29164530E-00 | 0.27104514E-00 | 0.28857920E-00 | 0.26993456E-00 |
| 0.36083987E-00 | 0.27145021E-00 | 0.27231190E-00 | 0.27770513E-00 | 0.27866112E-00 | 0.27024904E-00 |
| 0.27479445E-00 | 0.28370491E-00 | 0.27231149E-00 | | | |

We see that we may predict a limiting value of about 0.27, which is quite accurate enough for many purposes.

Also remarkable is the fact that a double precision calculation of values at .005, .010, ..., .300, with $r(\pi/3, \pi/3, .300) = .111595$ (which is less than 50 per cent of the limiting value), predicts a limiting value of 0.27.

It is clear what a great saving in time can be obtained in this way.

Harriet Kagiwada carried out the calculations on an IBM-7090. Only a few minutes of computing time were required.

The questions of which predictor formula to use, what increment in thickness to employ, and how many increments to use in order to predict the limiting values most efficiently, are still open.

## 4. TIME-DEPENDENT PROCESSES

As indicated in a previous paper [15], time-dependent radiative-transfer problems may be resolved computationally in a multistage fashion. First we use invariant imbedding to obtain a set of nonlinear partial differential-integral equations. A Laplace transform then reduces these to equations identical in form to those encountered in the time-independent problem. These are integrated numerically for appropriately chosen values of the transform variable. Finally, we use a numerical inversion of the Laplace

transform. A disadvantage of this method is that the values of the desired function, say $u(t)$, are obtained at irregularly spaced points $t_i$, $i = 1,2,\ldots,M$, and it is not easy to make $t_M$ large. Hence, an accurate extrapolation method would be quite useful in this case.

In place of asking that we have a representation of the type appearing in (2.1), we can ask that $u(t)$ be approximated to by a function $w(t)$ satisfying the linear differential equation

$$(4.1) \qquad w^{(n)} + b_1 w^{(n-1)} + \cdots + b_n w = b_{n+1}.$$

The unknown constants $b_i$ and the initial values $w^{(i)}(0) = c_i$, $i = 0,1,\ldots,n-1$, are to be determined by the condition that

$$(4.2) \qquad \sum_{i=1}^{M} (u(t_i) - w(t_i))^2$$

is a minimum. The numerical solution of problems of this type can be carried out quite easily using the techniques of [16].

## 5. GRADIENT TECHNIQUES

The general idea of the gradient technique is to solve an equation of the form $T(u) = 0$ by imbedding it within the solutions of

$$(5.1) \qquad \frac{\partial u}{\partial t} = T(u).$$

The solutions of the original equation are taken as the steady-states of (5.1). Since in this case only the values at $t = \infty$ are desired, nonlinear summability results will save a good deal of computing time. Combined with quasilinearization [17], [18], very accurate results can be obtained. Results of this nature will be presented subsequently.

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