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CANONICAL FORM OF TWO TANDEM-CONNECTED FOUR-PORTS

by

Allan L. Reynolds

Research Report No. PIBMRI-1092-62

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MRI

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44 Pages of Text

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ABSTRACT

Under the rather general conditions it is possible to represent a 4-port by means of an ideal directional-coupler together with certain 2-ports in each of its lines. Such a representation is called the " canonical-form" of the given 4-port.

The canonical-form of two tandem-connected 4-ports and the coupling coefficient of the associated ideal directional coupler are determined.

The presentation is of a theoretical nature.

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Introduction and Summary

An early result by Kyhl [Ref. 2] demonstrated that any " non-degenerate" lossless, reciprocal 4-port may be represented as an ideal directional coupler with certain lossless, reciprocal 2-ports in each of its lines. Such a form will be referred to as the " canonical form" of the given 4-port. Further work by Kahn and Kyhl [Ref. 1] yielded formulae which expressed the parameters of the associated ideal directional coupler and the appropriate 2-ports in terms of the characteristics of the given 4-port.

The transfer-scattering formalism was adopted because the cascading of 2N-ports is equivalent to the multiplication of the corresponding transfer matrices.

It will develop that the question of whether or not the determinants of certain submatrices of the transfer-scattering matrix are real numbers is of fundamental importance. Kahn and Kyhl imply their reality by assuming the existance of the canonical form. Herein a proof of their reality is given based solely on the restrictions of losslessness and reciprocity.

The transfer matrix of an ideal directional coupler can be of only three possible forms corresponding to any permutation of port designations. This report centers on the problem of determining the canonical form of two tandem connected 4-ports in terms of the parameters of the canonical forms of each of them. Nine cases were considered which exhaust all possibilities.

The results of this endeavor are presented in table III. It is noteworthy that the form of ideal directional coupler associated with two type three couplers is always a type three.

(i) Some Definitions and Notations

Throughout this work certain matrices and products of these matrices recur with sufficient frequency to justify the construction of a table of matrix products and other pertinent information.

There are essentially three such matrices:

$$j \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \equiv \rho$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \equiv \sigma$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \equiv \hat{\rho}$$

The table of interest appears below.

K	K^{-1}	K^2	K^+	det K
σ	σ	I	σ	-1
ρ	ρ	I	ρ	-1
$\hat{\rho}$	$-\hat{\rho}$	$-I$	$-\hat{\rho}$	1

(ii) Some Matrix Conventions

a) The inverse of a matrix A will be denoted by A^{-1}

b) If A is the N x N matrix

$$\begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{N1} & \cdots & A_{NN} \end{bmatrix}$$

the trace of A is $\mathrm{Tr}\, A = \sum_{i=1}^{N} Aii$

c) If A is an N x N matrix, the transpose of A is the N x N matrix A^t formed by letting the ith column of A be the ith row of A^t.

d) If $Z = r\epsilon^{j\theta}$ is a complex scaler, the complex conjugate of Z is denoted by Z^* and $Z^* = r\epsilon^{-j\theta}$.

e) If A is an N x N matrix the matrix

$$\begin{bmatrix} A_{11}^{*} & \cdots & A_{1N}^{*} \\ \vdots & & \vdots \\ A_{N1}^{*} & & A_{NN}^{*} \end{bmatrix}$$

will be denoted by A^{*}.

f) The matrix $(A^{t})^{*}$ will be denoted by A^{+}.

g) The N x N matrix with ones along its principal diagonal and zeros everywhere else will be denoted by I.

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

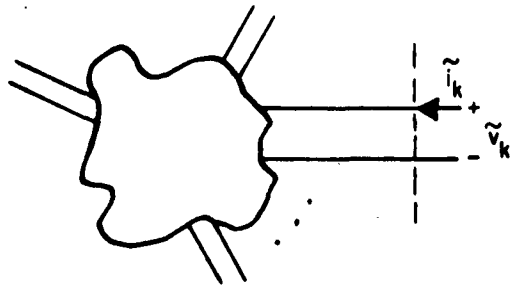
(i) Definition of an N-port

In this study we shall assume that all voltages and currents are of constant frequency and that the frequency is "high enough" so that a "transmission line approach" becomes convenient.

An N-port is a structure, access to which is gained by means of N transmission lines. On each line we shall assume a position is available at which it is possible to measure a current and a voltage. There are N such places available in an N-port and each such position will be called a port.

(ii) Impedance Representation of an N-port

On the $k^{\underline{th}}$ line the voltage and current, whose reference directions are as shown, are related by the following pair of first order equations:



$$-\frac{d}{dx}\tilde{v}_k(x) = j\,\gamma_k\,\tilde{z}_k\,\tilde{i}_k$$

$$-\frac{d}{dx}\tilde{i}_k(x) = j\,\gamma_k\,\tilde{y}_k\,\tilde{v}_k \qquad (1)$$

$$k = 1, \ldots, N$$

where $\tilde{v}_k(x)$ and $\tilde{i}_k(x)$ are the voltage and current measured at the point x on the $k^{\underline{th}}$ line; γ_k is the propagation constant of the $k^{\underline{th}}$ line and $\tilde{z}_k = \tilde{y}_k^{-1}$ is the (real positive) characteristic impedance of the $k^{\underline{th}}$ line.

It will be useful to define a new set of so-called "normalized" voltages and currents v_k and i_k, respectively by

$$\begin{bmatrix} \sqrt{\tilde{y}_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\tilde{y}_2} & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & \sqrt{\tilde{y}_N} \end{bmatrix} \begin{bmatrix} \tilde{v}_1 \\ \cdot \\ \cdot \\ \cdot \\ \tilde{v}_N \end{bmatrix} = \begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_N \end{bmatrix} \tag{2}$$

and

$$\begin{bmatrix} \sqrt{\tilde{z}_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\tilde{z}_2} & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 0 & 0 & \cdots & \sqrt{\tilde{z}_N} \end{bmatrix} \begin{bmatrix} \tilde{i}_1 \\ \cdot \\ \cdot \\ \cdot \\ \tilde{i}_N \end{bmatrix} = \begin{bmatrix} i_1 \\ \cdot \\ \cdot \\ \cdot \\ i_N \end{bmatrix} \tag{3}$$

In particular the set of normalized voltages and currents are related to the old set by

$$\begin{bmatrix} \sqrt{\tilde{y}_k} & 0 \\ 0 & \sqrt{\tilde{z}_k} \end{bmatrix} \begin{bmatrix} \tilde{v}_k \\ \tilde{i}_k \end{bmatrix} = \begin{bmatrix} v_k \\ i_k \end{bmatrix} \tag{4}$$

If the new set is substituted into (1) there results

$$
\begin{aligned}
-\frac{d}{dx} v_k(x) &= j\,\gamma_k\, i_k(x) \\
-\frac{d}{dx} i_k(x) &= j\,\gamma_k\, v_k(x)
\end{aligned}
\tag{5}
$$

which is equivalent to saying that the $k^{\underline{th}}$ line has a characteristic impedance of unity corresponding to v_k and i_k. It is convenient to deal exclusively with the normalized quantities v_k and i_k because every N-port having lines of characteristic impedance $\tilde{z}_k$ corresponding to $\tilde{v}_k$ and $\tilde{i}_k$ can be transformed into one having characteristic impedance of unity and terminal quantities v_k and i_k. The converse is also true.

The normalized impedance representation is

$$
\begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_N \end{bmatrix}
=
\begin{bmatrix} z_{11} & \cdots & z_{1N} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ z_{N1} & & z_{NN} \end{bmatrix}
\begin{bmatrix} i_1 \\ \cdot \\ \cdot \\ \cdot \\ i_N \end{bmatrix}
\tag{6}
$$

where

$$
z_{k\ell} = \frac{v_k}{i_\ell} \qquad \begin{matrix} k = 1, \ldots, N \\ \ell = 1, \ldots, N \end{matrix}
\tag{7}
$$

with $i_m = 0$, $m = 1, \ldots, N$, $m \neq \ell$

Although we shall not be concerned with the impedance representation as such, its presentation complements the "scattering" representation next to be discussed.

(iii) Scattering Representation

The normalized scattering parameters a_k and b_k associated with the kth line are defined by

$$\frac{1}{Z}\left(v_k(x) + i_k(x) \right) = a_k(x)$$
$$\frac{1}{Z}\left(v_k(x) - i_k(x) \right) = b_k(x) \qquad (8)$$
$$k = 1, \ldots, N .$$

Using these relations, (5) is transformed into

$$\frac{d}{dx} a_k(x) = -j\, \gamma_k\, a_k(x)$$
$$\frac{d}{dx} b_k(x) = j\, \gamma_k\, b_k(x) \qquad (9)$$
$$k = 1, \ldots, N ,$$

whose solutions are

$$a_k(x) = a_k(0)\, \varepsilon^{-j \gamma_k x}$$
$$b_k(x) = b_k(0)\, \varepsilon^{j \gamma_k x} \qquad (10)$$
$$k = 1, \ldots, N .$$

The quantities $a_k(x)$ and $b_k(x)$ represent amplitudes associated with wave motion in the x and -x directions respectively, at the point whose coordinate is x on the kth line. Treating the quantities a_k as independent parameters and the quantities b_k as dependent we can write

$$\begin{bmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_N \end{bmatrix} = \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ A_{N1} & \cdots & A_{NN} \end{bmatrix} \begin{bmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_N \end{bmatrix} \qquad (11)$$

where

$$s_{ij} = \frac{b_i}{a_j} \qquad \begin{array}{l} i = 1, \ldots, N \\ j = 1, \ldots, N \end{array}$$

$$a_k = 0, \quad k = 1, \ldots, N, \quad k \neq j .$$

The utility of the scattering matrix representation resides in the concise expressions which result in scattering terms when constraints imposed by losslessness and lorentz reciprocity, among others, are imposed.

We now state without proof two fundamental results:

" Conservation of energy and Lorentz Reciprocity respectively imply

$$SS^{+} = I \qquad \text{and} \qquad S^{t} = S . \tag{12}$$

where S is the N x N matrix

$$\begin{bmatrix} s_{ij} & \cdots & A_{1N} \\ \vdots & & \vdots \\ A_{N1} & & A_{NN} \end{bmatrix} .$$

(iv) Partitioned Matrices

It will be found convenient to partition a matrix into submatrices and to consider it as a matrix whose elements themselves are submatrices.

For example the 4 x 4 matrix T

$$T = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{bmatrix} \tag{13}$$

may be partitioned into

$$T = \left[\begin{array}{c|c} T_{11} & T_{12} \\ \hline T_{21} & T_{22} \end{array}\right] \tag{14}$$

where

$$T_{11} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}, \ldots$$

We shall denote a partitioned matrix by the horizontal and verticle lines as shown by (14). Let U be the matrix

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \\ u_{31} & u_{32} & u_{33} & u_{34} \\ u_{41} & u_{42} & u_{43} & u_{44} \end{bmatrix} \tag{15}$$

and let it be partitioned

$$U = \left[\begin{array}{c|c} U_{11} & U_{12} \\ \hline U_{21} & U_{22} \end{array}\right] \tag{16}$$

where

$$U_{11} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, \ldots \tag{17}$$

Then the product of partitioned matrices

$$TU = \left[\begin{array}{c|c} T_{11} & T_{12} \\ \hline T_{21} & T_{22} \end{array}\right] \left[\begin{array}{c|c} U_{11} & U_{12} \\ \hline U_{21} & U_{22} \end{array}\right] \left[\begin{array}{c|c} T_{11} U_{11} + T_{12} U_{21} & T_{11} U_{12} + T_{12} U_{22} \\ \hline T_{21} U_{11} + T_{22} U_{21} & T_{21} U_{12} + T_{22} U_{22} \end{array}\right]$$

when expanded, is equivalent to the product TU when T is of form (13) and U of form (15).

(V) Transfer - Scattering Representation

The (normalized) transfer-scattering matrix or simply the transfer matrix of an N-port arises as an outgrowth of the basic scattering concept. It affords a natural way of providing a scattering description of n tandem connected 2 N-ports because the process of connecting 2 N-ports in tandem is equivalent to multiplication of the representative transfer matrices of each 2 N-port.

Let Λ be a 2 N-port and suppose each port is assigned a number from 1 to 2N. Further suppose that the 2N-ports are grouped into two sets of N ports such that ports 1 to N will be called inputs; ports N + 1 to 2N will be called outputs. Then a transfer matrix of the 2N-port is the 2N x 2N matrix (t_{ij}) defined by

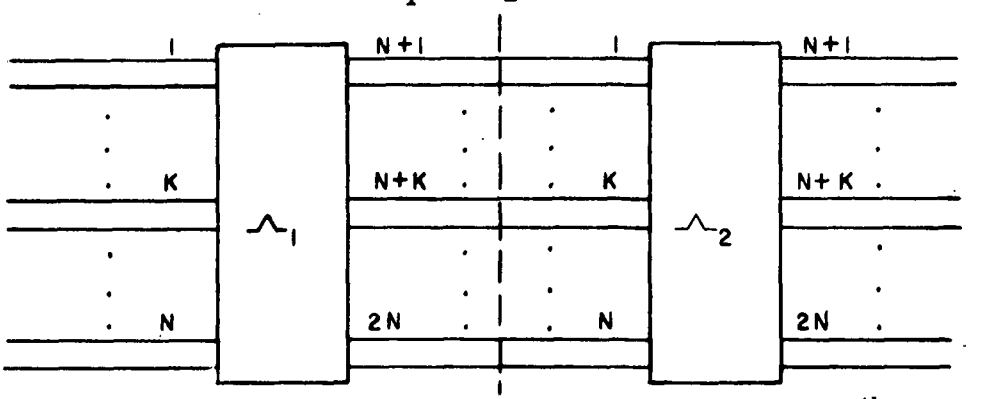
$$\begin{bmatrix} b_{N+1} \\ a_{N+1} \\ b_{N+2} \\ a_{N+2} \\ \vdots \\ b_{2N} \\ a_{2N} \end{bmatrix} = \begin{bmatrix} t_{11} & \cdots & t_{1,2N} \\ \vdots & & \vdots \\ & & \\ \vdots & & \vdots \\ t_{2N,1} & \cdots & t_{2N,2N} \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \\ \vdots \\ a_N \\ b_N \end{bmatrix} \tag{18}$$

Clearly the essence of the transfer matrix point of view is that it treats the 2N terminal quantities a_k, b_k of N selected ports called the input as independent variables while the remaining 2N terminal quantities associated with the N output ports are dependent variables. It follows that another possible transfer matrix of Λ is

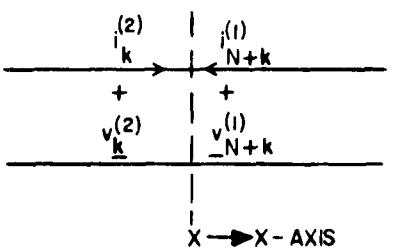
$$\begin{bmatrix} b_{N+1} \\ \vdots \\ b_{2N} \\ a_{N+1} \\ \\ \\ a_{2N} \end{bmatrix} = \begin{bmatrix} u_{11} & \cdots & u_{1,2N} \\ \vdots & & \vdots \\ & & \\ & & \\ \vdots & & \vdots \\ u_{2N,1} & \cdots & u_{2N,2N} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_N \\ b_1 \\ \vdots \\ b_N \end{bmatrix} \tag{19}$$

However we shall deal exclusively with the transfer matrix defined by (18).

Consider two 2N-ports Λ_1 and Λ_2 connected in tandem as shown



and in particular let us study in detail the connection of the $(N + k)^{\underline{th}}$ port of Λ_1 to the $k^{\underline{th}}$ port of Λ_2. As was defined earlier the reference directions at the $(N + k)^{\underline{th}}$ port of Λ_1 and the $k^{\underline{th}}$ port of Λ_2 are schematically



where the superscripts 1 and 2 refer respectively to Λ_1 and Λ_2.

This implies

$$i_{N+k}(x) = -i_k(x)$$
$$v_{N+k}(x) = v_k(x) \tag{20}$$

Now from (8)

$$a^{(1)}_{N+k}(x) = \tfrac{1}{2}\left(v_{N+k}(x) + i_{N+k}(x)\right)$$
$$b^{(1)}_{N+k}(x) = \tfrac{1}{2}\left(v_{N+k}(x) - i_{N+k}(x)\right) \tag{21}$$

and

$$a^{(2)}_k(x) = \tfrac{1}{2}\left(v_k(x) + i_k(x)\right)$$
$$b^{(2)}_k(x) = \tfrac{1}{2}\left(v_k(x) - i_k(x)\right) \tag{22}$$

When (20) is substituted into (21) and (22) we find

$$a^{(1)}_{N+k}(x) = b^{(2)}_k(x)$$
$$b^{(1)}_{N+k}(x) = a^{(2)}_k(x) \tag{23}$$

Suppose T_1 and T_2 are the transfer matrices of Λ_1 and Λ_2 respectively. Then

$$\begin{bmatrix} b^1_{N+1} \\ a^1_{N+1} \\ \vdots \\ b^1_{2N} \\ a^1_{2N} \end{bmatrix} = T_1 \begin{bmatrix} a^1_1 \\ b^1_1 \\ \vdots \\ a^1_N \\ b^1_N \end{bmatrix} \tag{24}$$

and

$$\begin{bmatrix} b^2_{N+1} \\ a^2_{N+1} \\ \vdots \\ b^2_{2N} \\ a^2_{2N} \end{bmatrix} = T_2 \begin{bmatrix} a^2_1 \\ b^2_1 \\ \vdots \\ a^2_N \\ b^2_N \end{bmatrix} \tag{25}$$

But (23) implies

$$\begin{bmatrix} b^1_{N+1} \\ a^1_{N+1} \\ \vdots \\ b^1_{2N} \\ a^1_{2N} \end{bmatrix} = \begin{bmatrix} a^2_1 \\ b^2_1 \\ \vdots \\ a^2_N \\ b^2_N \end{bmatrix} \tag{26}$$

or

$$\begin{bmatrix} b^2_{N+1} \\ a^2_{N+1} \\ \vdots \\ b^2_{2N} \\ a^2_{2N} \end{bmatrix} = T_2\, T_1 \begin{bmatrix} a^1_1 \\ b^1_1 \\ \vdots \\ a^1_N \\ b^1_N \end{bmatrix} \tag{27}$$

It follows that if n 2N-ports are connected in tandem then

$$\begin{bmatrix} b_{N+1}^{n} \\ a_{N+1}^{n} \\ \vdots \\ b_{2N}^{n} \\ a_{2N}^{n} \end{bmatrix} = T_n \cdots T_1 \begin{bmatrix} a_1^{1} \\ b_1^{1} \\ \vdots \\ a_N^{1} \\ b_N^{1} \end{bmatrix} \tag{28}$$

We shall now consider a special class of 2N-ports namely the set of all four - ports.

$$\begin{bmatrix} b_3 \\ a_3 \\ b_4 \\ a_4 \end{bmatrix} = \begin{bmatrix} t_{11} & \cdots & t_{14} \\ \vdots & & \vdots \\ t_{41} & \cdots & t_{44} \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{bmatrix} \tag{29}$$

Suppose the scattering representation

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} s_{11} & \cdots & s_{14} \\ \vdots & & \vdots \\ s_{41} & \cdots & s_{44} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \tag{30}$$

is given. It will be useful to determine the parameters t_{ij} in terms of the parameters s_{ij}. From (30) it follows that

$$\begin{bmatrix} b_3 \\ a_3 \\ b_4 \\ a_4 \end{bmatrix} = \begin{bmatrix} s_{31} & s_{32} & s_{33} & s_{34} \\ 0 & 0 & 1 & 0 \\ s_{41} & s_{42} & s_{43} & s_{44} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}, \quad \begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ s_{11} & s_{12} & s_{13} & s_{14} \\ 0 & 1 & 0 & 0 \\ s_{21} & s_{22} & s_{23} & s_{24} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \tag{31}$$

and

$$\begin{bmatrix} b_3 \\ a_3 \\ b_4 \\ a_4 \end{bmatrix} = \begin{bmatrix} s_{31} & s_{32} & s_{33} & s_{34} \\ 0 & 0 & 1 & 0 \\ s_{41} & s_{42} & s_{43} & s_{44} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ s_{11} & s_{12} & s_{13} & s_{14} \\ 0 & 1 & 0 & 0 \\ s_{21} & s_{22} & s_{23} & s_{24} \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{bmatrix} \tag{32}$$

(Vi) <u>Directional Couplers</u>

An "ideal directional coupler" henceforth abbreviated by I. D. C. is a lossless, reciprocal four - port, the scattering matrix of which has zeros along the main diagonal when each port is terminated in the characteristic impedance of the corresponding transmission line.

From (18) the transfer representation of a four - port is

$$T = \frac{1}{S_{13}S_{24}-S_{14}S_{23}} \left[\begin{array}{cc|cc} +S_{31}\,S_{13}\,S_{24} & +S_{33}\,S_{24} & +S_{32}\,S_{13}\,S_{24} & -S_{33}\,S_{14} \\ -S_{31}\,S_{14}\,S_{23} & -S_{34}\,S_{23} & -S_{32}\,S_{14}\,S_{23} & +S_{34}\,S_{13} \\ -S_{11}\,S_{33}\,S_{24} & & -S_{12}\,S_{33}\,S_{24} & \\ +S_{11}\,S_{34}\,S_{23} & & +S_{12}\,S_{34}\,S_{23} & \\ +S_{21}\,S_{33}\,S_{14} & & +S_{22}\,S_{33}\,S_{14} & \\ -S_{21}\,S_{34}\,S_{13} & & -S_{22}\,S_{34}\,S_{13} & \\ -S_{24}\,S_{11} & +S_{24} & -S_{24}\,S_{12} & -S_{14} \\ +S_{14}\,S_{21} & & +S_{14}\,S_{22} & \\ \hline +S_{41}\,S_{13}\,S_{24} & +S_{43}\,S_{24} & +S_{42}\,S_{13}\,S_{24} & -S_{43}\,S_{14} \\ -S_{41}\,S_{14}\,S_{23} & -S_{44}\,S_{23} & -S_{42}\,S_{14}\,S_{23} & +S_{44}\,S_{13} \\ -S_{11}\,S_{43}\,S_{24} & & -S_{12}\,S_{43}\,S_{24} & \\ +S_{11}\,S_{44}\,S_{23} & & +S_{12}\,S_{44}\,S_{23} & \\ +S_{21}\,S_{43}\,S_{14} & & +S_{22}\,S_{43}\,S_{14} & \\ -S_{21}\,S_{44}\,S_{13} & & -S_{22}\,S_{44}\,S_{13} & \\ +S_{23}\,S_{11} & -S_{23} & +S_{23}\,S_{12} & +S_{13} \\ -S_{13}\,S_{21} & & -S_{13}\,S_{22} & \end{array}\right] \tag{33}$$

To each numbering of the ports of a device which is an I. D. C. there corresponds a form of scattering matrix. Since there are twenty-four ways of numbering the four ports there are twenty-four possible "forms" of scattering matrix associated with a given I. D. C. We shall say a "form" of scattering matrix is defined by the distribution of zeros in the matrix. If the twenty-four possible forms of scattering matrix associated with an I. D. C. are written and compared there appear to be only three different forms. The scattering matrix of an I. D. C. must assume one of the three following forms:

$$S_1 = \begin{bmatrix} 0 & 0 & \alpha & j\beta \\ 0 & 0 & j\beta & \alpha \\ \alpha & j\beta & 0 & 0 \\ j\beta & \alpha & 0 & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & \alpha & 0 & j\beta \\ \alpha & 0 & j\beta & 0 \\ 0 & j\beta & 0 & \alpha \\ j\beta & 0 & \alpha & 0 \end{bmatrix}, \quad S_3 = \begin{bmatrix} 0 & j\beta & \alpha & 0 \\ j\beta & 0 & 0 & \alpha \\ \alpha & 0 & 0 & j\beta \\ 0 & \alpha & j\beta & 0 \end{bmatrix} \tag{34}$$

where α and β are positive real numbers and

$$\alpha^2 + \beta^2 = 1 \tag{35}$$

Note that forms 1 and 2 collapse into

$$\begin{bmatrix} 0 & 0 & 0 & j \\ 0 & 0 & j & 0 \\ 0 & j & 0 & 0 \\ j & 0 & 0 & 0 \end{bmatrix} \tag{36}$$

for $\alpha = 0$ and forms 1 and 3 collapse into

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{37}$$

when $\alpha = 1$.

Corresponding to each form of scattering matrix there is a transfer matrix which similarly assumes one of the three forms:

$$T_1 = \left[\begin{array}{cc|cc} \alpha & 0 & j\beta & 0 \\ 0 & \alpha & 0 & -j\beta \\ \hline j\beta & 0 & \alpha & 0 \\ 0 & -j\beta & 0 & \alpha \end{array}\right], \tag{38}$$

$$T_2 = \left[\begin{array}{cc|cc} 0 & -j\frac{\alpha}{\beta} & j\left(\beta+\frac{\alpha^2}{\beta}\right) & 0 \\ j\frac{\alpha}{\beta} & 0 & 0 & -j\frac{1}{\beta} \\ \hline j\left(\beta+\frac{\alpha^2}{\beta}\right) & 0 & 0 & -j\frac{\alpha}{\beta} \\ 0 & -j\frac{1}{\beta} & j\frac{\alpha}{\beta} & 0 \end{array}\right],$$

$$T_3 = \left[\begin{array}{cc|cc} \frac{\alpha^2+\beta^2}{\alpha} & 0 & 0 & j\frac{\beta}{\alpha} \\ 0 & \frac{1}{\alpha} & -j\frac{\beta}{\alpha} & 0 \\ \hline 0 & j\frac{\beta}{\alpha} & \frac{\alpha^2+\beta^2}{\alpha} & 0 \\ -j\frac{\beta}{\alpha} & 0 & 0 & \frac{1}{\alpha} \end{array}\right]$$

When partitioned appropriately these forms become

$$T_1 = \left[\begin{array}{c|c} \alpha I & j\beta\sigma \\ \hline j\beta\sigma & \alpha I \end{array}\right], \quad T_2 = \left[\begin{array}{c|c} \frac{\alpha}{\beta}\rho & j\beta\sigma \\ \hline j\beta\sigma & \frac{\alpha}{\beta}\rho \end{array}\right], \quad T_3 = \frac{1}{\alpha}\left[\begin{array}{c|c} I & -\beta\rho \\ \hline -\beta\rho & I \end{array}\right] \tag{39}$$

(i) A Proof That the Determinant of a Partitioned Submatrix of T is Real.

We shall now derive a result which will be of fundamental importance in our later work. If the matrix T in (29) is partitioned naturally into four 2 x 2 submatrices as shown, then the determinants of these submatrices are real. This we now prove.

A direct computation shows that

$$\det T_{11} = \frac{S_{24}\,S_{31} - S_{21}\,S_{34}}{S_{13}\,S_{24} - S_{14}\,S_{23}}$$

$$\det T_{22} = \frac{S_{13}\,S_{42} - S_{12}\,S_{43}}{S_{13}\,S_{24} - S_{14}\,S_{23}}$$

$$\det T_{12} = \frac{S_{14}\,S_{32} - S_{12}\,S_{34}}{S_{14}\,S_{23} - S_{13}\,S_{24}} \tag{40}$$

$$\det T_{21} = \frac{S_{23}\,S_{41} - S_{21}\,S_{43}}{S_{23}\,S_{14} - S_{24}\,S_{13}}$$

Invoking the principle of reciprocity, it can be seen at once that

$$\text{(a)} \quad \det T_{11} = \det T_{22}$$

$$\text{(b)} \quad \det T_{12} = \det T_{21} \tag{41}$$

and

$$\text{(c)} \quad 1 - \det T_{11} = \det T_{12}$$

From (12) we find

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{43} \end{bmatrix} \begin{bmatrix} S_{11}^* & S_{21}^* & S_{31}^* & S_{41}^* \\ S_{12}^* & S_{22}^* & S_{32}^* & S_{42}^* \\ S_{13}^* & S_{23}^* & S_{33}^* & S_{43}^* \\ S_{14}^* & S_{24}^* & S_{34}^* & S_{44}^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{42}$$

By multiplying appropriately, the following expressions can be obtained:

$$S_{31}\, S_{23}^{*}\, (S_{21}\, S_{11}^{*} + S_{22}\, S_{12}^{*} + S_{23}\, S_{13}^{*} + S_{24}\, S_{14}^{*}) = 0 \tag{43}$$

$$S_{21}\, S_{23}^{*}\, (S_{31}\, S_{11}^{*} + S_{32}\, S_{12}^{*} + S_{33}\, S_{13}^{*} + S_{34}\, S_{14}^{*}) = 0 \tag{44}$$

$$S_{21}\, S_{13}^{*}\, (S_{31}\, S_{21}^{*} + S_{32}\, S_{22}^{*} + S_{33}\, S_{23}^{*} + S_{34}\, S_{24}^{*}) = 0 \tag{45}$$

where the factor outside the bracket has been added as a guide to subsequent manipulation, and also

$$S_{23}\, S_{21}^{*}\, (S_{11}\, S_{31}^{*} + S_{12}\, S_{32}^{*} + S_{13}\, S_{33}^{*} + S_{14}\, S_{34}^{*}) = 0 \tag{46}$$

$$S_{13}\, S_{21}^{*}\, (S_{21}\, S_{31}^{*} + S_{22}\, S_{32}^{*} + S_{23}\, S_{33}^{*} + S_{24}\, S_{34}^{*}) = 0 \tag{47}$$

$$S_{23}\, S_{31}^{*}\, (S_{11}\, S_{21}^{*} + S_{12}\, S_{22}^{*} + S_{13}\, S_{23}^{*} + S_{14}\, S_{24}^{*}) = 0 \tag{48}$$

The set of equations (43) to (45) can be put in the following form:

$$\begin{aligned} & S_{21}\, S_{23}^{*}\, S_{34}\, S_{14}^{*} - S_{24}\, S_{14}^{*}\, S_{31}\, S_{23}^{*} - S_{21}\, S_{13}^{*}\, S_{34}\, S_{24}^{*} \\ = \; & S_{22}\, S_{12}^{*}\, S_{31}\, S_{23}^{*} + S_{31}\, S_{23}^{*}\, S_{23}\, S_{13}^{*} - S_{21}\, S_{23}^{*}\, S_{32}\, S_{12}^{*} \\ + \; & S_{21}\, S_{13}^{*}\, S_{31}\, S_{21}^{*} + S_{21}\, S_{13}^{*}\, S_{32}\, S_{22}^{*} \end{aligned} \tag{49}$$

and similarly the set (46) to (48) yield

$$\begin{aligned} & S_{23}\, S_{14}\, S_{21}^{*}\, S_{34}^{*} - S_{24}\, S_{13}\, S_{21}^{*}\, S_{34}^{*} - S_{23}\, S_{14}\, S_{24}^{*}\, S_{31}^{*} \\ = \; & - S_{23}\, S_{21}^{*}\, S_{12}\, S_{22}^{*} + S_{12}\, S_{21}^{*}\, S_{21}\, S_{31}^{*} + S_{13}\, S_{21}^{*}\, S_{22}\, S_{32}^{*} \\ + \; & S_{23}\, S_{31}^{*}\, S_{12}\, S_{22}^{*} + S_{23}\, S_{31}^{*}\, S_{13}\, S_{23} \end{aligned} \tag{50}$$

Due to the reciprocity condition it is apparent that

$$\begin{aligned} & S_{21}\, S_{23}^{*}\, S_{34}\, S_{14}^{*} - S_{24}\, S_{14}^{*}\, S_{31}\, S_{23}^{*} - S_{21}\, S_{13}^{*}\, S_{34}\, S_{24}^{*} \\ = \; & S_{23}\, S_{14}\, S_{21}^{*}\, S_{34}^{*} - S_{24}\, S_{13}\, S_{21}^{*}\, S_{34}^{*} - S_{23}\, S_{14}\, S_{24}^{*}\, S_{31}^{*} \end{aligned} \tag{51}$$

Now

$$S_{21}\, S_{23}^{*}\, S_{34}\, S_{14}^{*} - S_{24}\, S_{14}^{*}\, S_{31}\, S_{23}^{*} - S_{21}\, S_{13}^{*}\, S_{24}\, S_{24}^{*}$$
$$= (S_{24}\, S_{31} - S_{21}\, S_{34})\ (S_{24}^{*}\, S_{13}^{*} - S_{23}^{*}\, S_{14}^{*}) - |\, S_{24}\, S_{31}\,|^{2} \tag{52}$$

and

$$S_{23}\, S_{14}\, S_{21}^{*}\, S_{34}^{*} - S_{24}\, S_{13}\, S_{21}^{*}\, S_{34}^{*} - S_{23}\, S_{14}^{*}\, S_{24}\, S_{31}^{*}$$
$$= (S_{24}\, S_{31} - S_{23}\, S_{14})\ (S_{24}^{*}\, S_{31}^{*} - S_{21}^{*}\, S_{34}^{*}) - |\, S_{24}\, S_{31}\,|^{2} \tag{53}$$

Therefore from (52) and (53)

$$(S_{24}\, S_{31} - S_{21}\, S_{34})\ (S_{24}^{*}\, S_{13}^{*} - S_{23}^{*}\, S_{14}^{*})$$
$$= (S_{24}\, S_{13} - S_{23}\, S_{14})\ (S_{24}^{*}\, S_{31}^{*} - S_{21}^{*}\, S_{34}^{*}) \tag{54}$$

and finally

$$\frac{S_{24}\, S_{31} - S_{21}\, S_{34}}{S_{24}\, S_{13} - S_{22}\, S_{14}} = \frac{S_{24}^{*}\, S_{31}^{*} - S_{21}^{*}\, S_{34}^{*}}{S_{24}^{*}\, S_{13}^{*} - S_{23}^{*}\, S_{14}^{*}} = \left(\frac{S_{24}\, S_{31} - S_{21}\, S_{34}}{S_{24}\, S_{13} - S_{22}\, S_{14}}\right)^{*} \tag{55}$$

or

$$\det\ T_{11} \text{ is real.}$$

From this and equation (4b), it follows immediately that det T_{22} is real.

As a consequence of (41)

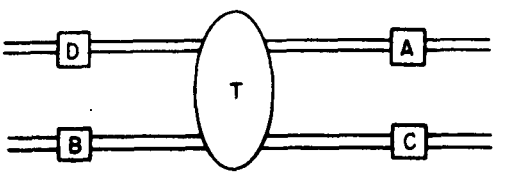
$$(\det T_{12})^{*} = (1 - \det T_{11})^{*} = 1 - (\det T_{11})^{*}$$
$$= 1 - \det T_{11} \tag{56}$$
$$= \det T_{12}$$

and det T_{12} and det T_{21} are also real.

(ii) "Canonical Form" of a Four-Port

Under rather general conditions it is possible to represent a four-port by means of an I. D. C. with a certain two-port in each of its lines: it is necessary that the given four-port be linear, lossless and reciprocal, and not belong to a set of degenerate structures which will be defined later. In what follows we shall assume, unless otherwise stated, that a "given arbitrary four-port" will satisfy the conditions stated above, and hence can be represented as described.

Let us first consider the transfer matrix of an I. D. C. with a lossless, reciprocal two-port in each line. Assume that the transfer matrix of each two-port is non-singular.



A, B, C, D, denote the transfer matrices of each of the two-ports and t denotes the partitioned transfer matrix of an I. D.C.

The transfer matrix we are seeking is given by

$$T = T_2 \; t \; T_1 \tag{57}$$

where

$$T_1 = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & B \end{array}\right], \quad t = \left[\begin{array}{c|c} t_{11} & t_{12} \\ \hline t_{21} & t_{22} \end{array}\right], \quad T_2 = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & C \end{array}\right] \tag{58}$$

or

$$T = \left[\begin{array}{c|c} A\, t_{11}\, D & A\, t_{12}\, B \\ \hline C\, t_{21}\, D & C\, t_{22}\, B \end{array}\right]$$

We will now try to define the matrices A, B, C, D, t_{11}, t_{12}, t_{21}, so that T will be equal to the transfer matrix of a given arbitrary four-port.

$$\left[\begin{array}{c|c} T_{11} & T_{12} \\ \hline T_{21} & T_{22} \end{array}\right] = \left[\begin{array}{c|c} A\, t_{11}\, D & A\, t_{12}\, B \\ \hline C\, t_{21}\, D & C\, t_{22}\, B \end{array}\right] \tag{59}$$

For our purposes one two-port, say A, can be chosen initially in an arbitrary manner. We find

$$
\begin{aligned}
A &= X \\
B &= (t_{12})^{-1} X^{-1} T_{12} \\
C &= T_{22} (T_{12})^{-1} X t_{12} (t_{22})^{-1} \\
D &= (t_{11})^{-1} X^{-1} T_{11}
\end{aligned}
\tag{60}
$$

where X is arbitrary.

The submatrices t_{ij}, $i = 1, 2$, $j = 1, 2$, are as yet unknown. From a consideration of equations (39) it is apparent that for

$$
\begin{aligned}
0 \leqq \det t_{11} \leqq 1 \\
0 \leqq \det t_{12} \leqq 1
\end{aligned}
\qquad \text{form 1}
\tag{61}
$$

$$
\begin{aligned}
\infty < \det t_{11} \leqq 0 \\
1 \leqq \det t_{12} < \infty
\end{aligned}
\qquad \text{form 2}
$$

$$
\begin{aligned}
1 \leqq \det t_{11} < \infty \\
-\infty < \det t_{12} \leqq 0
\end{aligned}
\qquad \text{form 3}
$$

and

$$
\begin{aligned}
\det t_{11} = \det t_{22} &= 1 - \det t_{12} \\
\det t_{12} &= \det t_{21}
\end{aligned}
\tag{62}
$$

The values that $\det t_{ij}$ can assume for a fixed i and j is the real line and furthermore the real line may be partitioned into three intervals each interval corresponding to a distinct form of I. D. C. The intervals are disjoint except for the values 0 and 1. This seems to indicate that for these values of the determinant there may correspond two forms of I. D. C. However it will be shown later that for the value of $\det t_{ij}$ equal to 0 or 1 the transfer matrices

corresponding to two different forms collapse into the same form.

Suppose we are given the transfer matrix of a four-port. The determinant of some submatrix, say det T_{11}, must lie in one of the three disjoint intervals or else assume the values 0 or 1 because each sub-determinant is real. See above, equations (40) to (56). Furthermore for a linear, lossless reciprocal four-port it was shown earlier that

$$\begin{aligned} \det T_{11} &= \det T_{22} = 1 - \det T_{12} \\ \det T_{12} &= \det T_{21} \end{aligned} \tag{63}$$

Therefore a subdeterminant of an arbitrary four-port must satisfy the inequalities associated with one and only one form of I. D. C. A consequence of this is that to every four-port there corresponds a unique form of I. D. C.

Once the form of coupler corresponding to a given four-port has been determined, its coupling coefficient α^2 can be found from one of the following formulae:

$$\begin{aligned} 0 \leqq \det T_{11} \leqq 1 \qquad & \text{form 1 and } \alpha^2 = \det T_{11} \\ -\infty \leqq \det T_{11} \leqq 0 \qquad & \text{form 2 and } \alpha^2 = \frac{\det T_{11}}{\det T_{11} - 1} \\ 1 \leqq \det T_{11} < \infty \qquad & \text{form 3 and } \alpha^2 = \frac{1}{\det T_{11}} \end{aligned} \tag{64}$$

It should be observed that although the form of the I. D. C. associated with arbitrary four-port is unique, the form of the representing structure is not unique due to the arbitrary choice of a two-port.

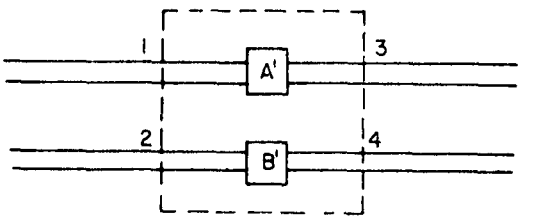
Once α^2 has been determined the matrix t is known.

Henceforth a structure of the form of fig. 4 which represents a given arbitrary four-port will be called <u>the canonical form</u> of the four-port .

(iii) Illustrations

As an illustration of the technique we shall try to find the canonical forms of some rather special four-ports.

Consider the device shown below.



where A^1 and B^1 are the transfer matrices of the indicated two-ports. The transfer matrix of this device is easily seen to be of the form

$$\left[\begin{array}{c|c} A^1 & 0 \\ \hline 0 & B^1 \end{array}\right]$$

From equation (59)

$$A^1 = A\, t_{11}\, D \tag{68}$$

$$0 = A\, t_{12}\, B \tag{69}$$

$$0 = C\, t_{21}\, D \tag{70}$$

$$B^1 = C\, t_{22}\, B \tag{71}$$

$$\det A^1 = \det t_{11} = 1$$

Now from equation (61) the I. D. C. associated with the given device appears to be of either form 1 or form 3. But we know from (64) that if it is of form 1 then

$$\alpha^2 = \det A^1 = 1$$

and if of form 3

$$\alpha^2 = \frac{1}{\det A^1} = 1$$

Also it is obvious that equations (63) are satisfied.

Hence we find that the scattering transfer matrix of the I. D. C. associated with the given device assumes the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

for both type 1 and type 3

For the determination of the two-ports let $A = X$, where X is an arbitrary two-port. Since $t_{12} = 0$, B is also arbitrary Let $B = Y$, where Y is some arbitrary two-port. Then the two-ports are

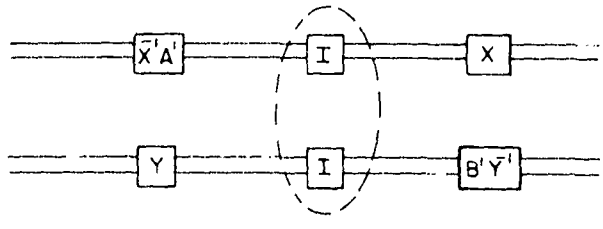
$$A = X$$

$$B = Y$$

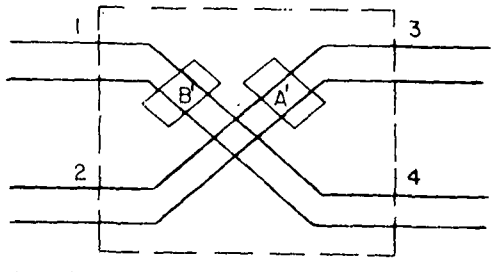
$$C = B^{1} Y^{-1}$$

$$D = X^{-1} A^{1}$$

The canonical form of fig. 5 is



As our second illustration consider the following device.



The transfer matrix of this device is

$$\left[\begin{array}{c|c} 0 & A^{1} \\ \hline B^{1} & 0 \end{array}\right]$$

We require

$$0 = A\, t_{11}\, D$$

$$A^1 = A\, t_{12}\, B$$

$$B^1 = C\, t_{21}\, D$$

$$0 = C\, t_{22}\, B$$

$$\det 0 = \det t_{11} = 0$$

Hence the corresponding I. D. C. will be either of form 1 or 2 This implies $a^2 = 0$ and we see that both form 1 and form 2 collapse into

$$\begin{bmatrix} 0 & 0 & j & 0 \\ 0 & 0 & 0 & -j \\ j & 0 & 0 & 0 \\ 0 & -j & 0 & 0 \end{bmatrix}$$

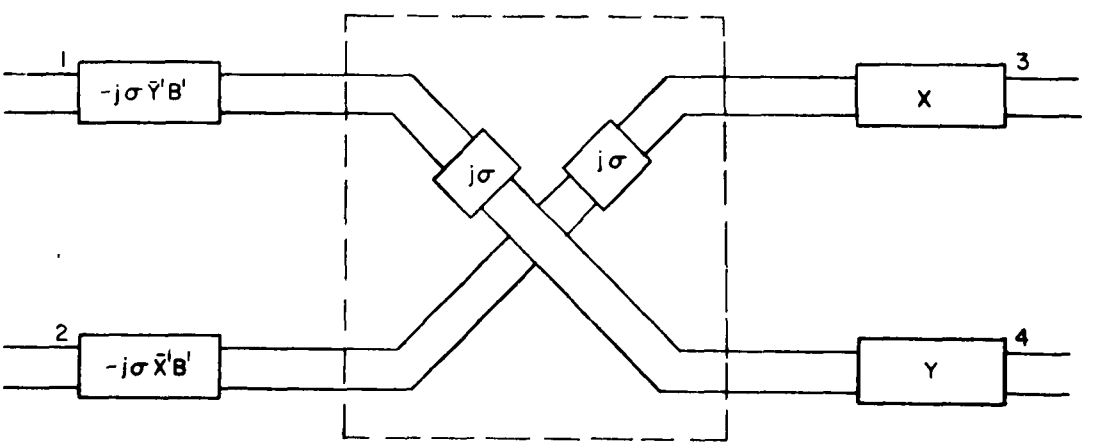
To determine the appropriate two ports let $A = X$

then $B = -j\,\sigma\, X^{-1} A^1$

$C = Y$, Y arbitrary

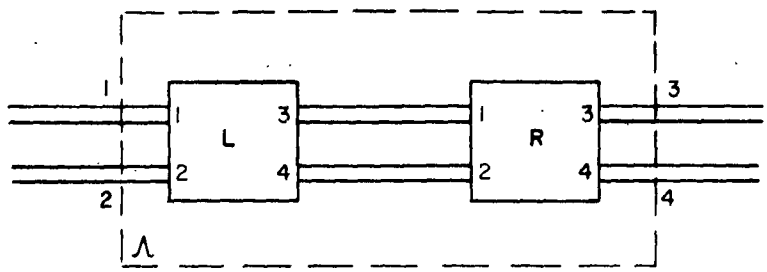
$D = -j\,\sigma\, Y^{-1} B^1$

The canonical form is



(iv) Canonical Form of Two Four-Ports.

Suppose two arbitrary four-ports are connected as shown below.

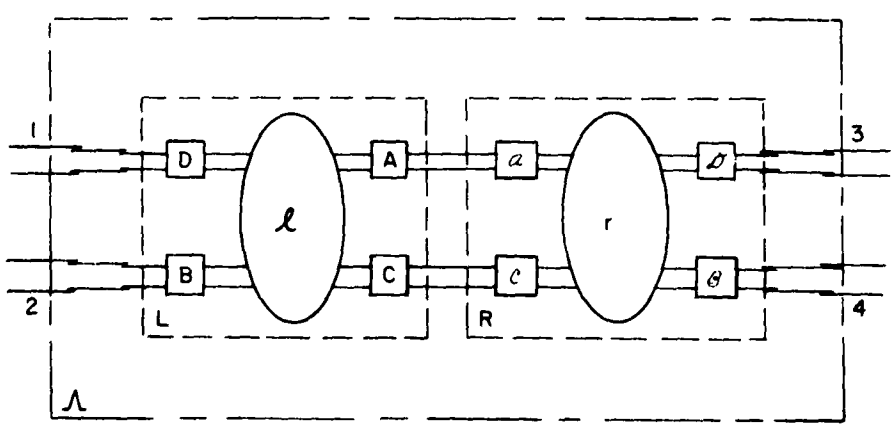


The composite four-port defined by the dashed lines will be denoted by Λ We now consider the problem of relating the coupling coefficient α^2 of the I. D. C associated with Λ to the coupling coefficients α_1^2 and α_2^2 of the I. D. C.'s associated with L and R respectively.

We know that any four-port can be associated with just one of three possible forms of I. D. C Consequently there are nine different combinations of forms that can be associated with L and R For each of the nine possible cases we shall evaluate α^2 in the following four steps:

a) compute the transfer matrix T of Λ
b) evaluate det T_{11}
c) determine upper and lower limits for det T_{11}
d) α^2 will assume one of the values of either equation (64) or (65) or (66)

If L and R are put in their respective canonical forms Λ is transformed into



where ℓ and r are the I. D. C. 's associated with L and R respectively. The transfer matrix T of Λ can be given as

$$T = T_6 T_5 T_4 T_3 T_2 T_1 \tag{72}$$

where

$$T_1 = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & B \end{array}\right] \qquad T_2 = \left[\begin{array}{c|c} \ell_{11} & \ell_{12} \\ \hline \ell_{21} & \ell_{22} \end{array}\right]$$

$$T_3 = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & C \end{array}\right] \qquad T_4 = \left[\begin{array}{c|c} a & 0 \\ \hline 0 & C \end{array}\right]$$

$$T_5 = \left[\begin{array}{c|c} r_{11} & r_{12} \\ \hline r_{21} & r_{22} \end{array}\right] \qquad T_6 = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & B \end{array}\right]$$

or

$$T = \frac{Dr_{11}aA\ell_{11}D + Dr_{12}eC\ell_{21}D \quad Dr_{11}aA\ell_{12}B + Dr_{12}eC\ell_{22}B}{Br_{21}aA\ell_{11}D + Br_{22}eC\ell_{21}D \quad Br_{21}aA\ell_{12}B + Br_{22}eC\ell_{22}B}$$

$$\begin{aligned} \det T_{11} &= \det (Dr_{11}\, aA\ell_{11}D + Dr_{12}\, eC\ell_{21}D) \\ &= \det (r_{11}\, aA\ell_{11} + r_{12}\, eC\ell_{21}) \\ &= \det (r_{11}\, aA\ell_{11})\ (I + \ell_{11}^{-1}A^{-1}\, a^{-1}\, r_{11}^{-1}\, r_{12}\, eC\ell_{21}) \end{aligned} \tag{73}$$

Now

$$\det (I + K) \cong \det K + \mathrm{Tr}\, K + 1 \tag{74}$$

where K is a 2 x 2 matrix.

Define

$$\ell_{11}^{-1}\, A^{-1}\, a^{-1}\, r_{11}^{-1}\, r_{12}\, eC\ell_{21} = K .$$

Then

$$\det T_{11} = \det (r_{11}\, \ell_{11})\ \det (I + K) \tag{75}$$

or

(76)

$$\det T_{11} = \det (r_{11}\, \ell_{11}) \left[\det (\ell_{11}^{-1}\, r_{11}^{-1}\, r_{12}\, \ell_{21}) + \mathrm{Tr}\, \ell_{11}^{-1} (aA)^{-1}\, r_{11}^{-1} r_{12} e C \ell_{21}) + 1 \right]$$

For each of the nine possible combinations of forms of ℓ and r, equation (76) has been evaluated. The results of these computations appear in table I.

The next step in our determination of a^2 is to determine the range of values that det T_{11} can assume for each of the nine cases.

Observe that in each entry of det T_{11} in table I the trace of a complicated matrix product is required. We shall now discuss an abstract matrix M that is assigned certain properties. This consideration will facilitate the evaluation of the traces in question.

(v) Range of Tr M

Let M be a 2 x 2 matrix. Suppose there exists a scaler λ for which

$$MX = \lambda X \tag{77}$$

where

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

The values of λ which satisfy this equation are called the characteristic values of M. If λ is a characteristic value of M, a non-zero vector X which satisfies (20) is called a characteristic vector of M corresponding to the characteristic value λ For brevity we shall denote "characteristic value" by c. v. For a 2 x 2 matrix M there are two c. v.'s.

Two cases arise:

case I . the c. v are distinct

case II the c v. are equal

Case I When the c. v of a 2 x 2 matrix M are distinct then it can be demonstrated that there exists a non-singular 2 x 2 matrix P such that

$$M = P^{-1} \mathcal{M} P \tag{78}$$

where

$$\mathcal{M} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{79}$$

and λ_1 and λ_2 are the c v. of M.

Case II . When the c. v. of M are not distinct, there exists a non-singular matrix Q such that

$$M = Q^{-1} \mathcal{M} Q \tag{80}$$

where

$$\mathcal{M} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \tag{81}$$

with λ being a c. v. of M.

Consider a matrix M_1 for which

$$\det M_1 = 1 \tag{82}$$

$$M_1^+ + \sigma M_1 = \sigma \tag{83}$$

and a matrix M_2 for which

$$\det M_2 = -1 \tag{84}$$

$$M_2^+ + \sigma M_2 = -\sigma \tag{85}$$

We will now try to obtain bounds for the range of values that the traces of M_1 and M_2 can assume

For M_1 there are two cases to consider.

Case I The c. v. of M_1 are distinct.

$$M_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{86}$$

where $\lambda_1 \neq \lambda_2$.

Two matrices A and B are said to be similar if there exists a non-singular matrix P such that

$$A = P^{-1} BP \tag{87}$$

It is apparent that

$$\det A = \det B \tag{88}$$

Also it can be shown that

$$\mathrm{Tr}\, A = \mathrm{Tr}\, B \tag{89}$$

Using this information it follows from (29) that

$$\lambda_2 = \frac{1}{\lambda_1} \tag{90}$$

and

$$\mathrm{Tr}\, M_1 = \lambda_1 + \lambda_2 = \lambda_1 + \frac{1}{\lambda_1} \tag{91}$$

Since λ_1 is a c.v. of M_1

$$M_1 X = \lambda_1 X \tag{92}$$

Now

$$(M_1 X)^+ \sigma (M_1 X) = X^+ (M_1{}^+ \sigma M_1) X = X^+ \sigma X \tag{93}$$

$$(\lambda_1 X)^+ \sigma (\lambda_1 X) = X^+ \sigma X$$

$$\lambda_1^* \lambda_1 X^+ \sigma X = X^+ \sigma X$$

$$|\lambda_1|^2 X^+ \sigma X = X^+ \sigma X$$

If $X^+ \sigma X \neq 0$, then

$$|\lambda_1| = 1 \tag{94}$$

For convenience denote λ_1 by $\varepsilon^{j\theta}$ where θ is some real number .

$$\mathrm{Tr}\, M_1 = \lambda_1 + \frac{1}{\lambda_1} = 2 \cos \theta$$

$$\mathrm{Tr}\, M_1 = 2 \cos \theta \qquad -\infty < \theta < \infty \tag{95}$$

θ real

and

$$-2 \leqq \mathrm{Tr}\, M_1 \leqq 2 \tag{96}$$

If $X^+ \sigma X = 0$.

From (83) write

$$M_1^+ = \sigma M_1^{-1} \sigma^{-1} \tag{97}$$

Since M_1^+ is similar to M_1^{-1}

$$\mathrm{Tr}\, M_1^+ = \mathrm{Tr}\, M_1^{-1} \tag{98}$$

Also since det $M_1 = 1$

$$\mathrm{Tr}\, M_1^{-1} = \mathrm{Tr}\, M_1 \tag{99}$$

$$\mathrm{Tr}\, M_1^+ \equiv (\mathrm{Tr}\, M_1)^* \tag{100}$$

We conclude

$$(\mathrm{Tr}\, M_1)^* = \mathrm{Tr}\, M_1 \tag{101}$$

or $\mathrm{Tr}\, M_1$ is real.

From (91)

$$\mathrm{Tr}\, M_1 = \lambda_1 + \frac{1}{\lambda_1} \tag{102}$$

Since $\mathrm{Tr}\, M_1$ is real

$$(\lambda_1 + \frac{1}{\lambda_1}) = (\lambda_1 + \frac{1}{\lambda_1})^* \qquad \lambda_1 \neq 0 \tag{103}$$

This can be written as

$$(\lambda_1 - \lambda_1^*) \left(1 - \frac{1}{|\lambda_1|^2}\right) = 0 \tag{104}$$

and for $\lambda_1 \neq 1$ it follows that

$$\lambda_1 = \lambda_1^* \tag{105}$$

or that λ_1 is real

Therefore

$$\mathrm{Tr}\, M_1 = \lambda_1 + \frac{1}{\lambda_1} \qquad \lambda_1 \text{ real} \qquad |\lambda_1| \neq 1, \ \lambda_1 \neq 0$$

For convenience denote λ_1 by ε^{ϕ} where ϕ is real and $\phi \neq 0$.

$$\mathrm{Tr}\, M_1 = \varepsilon^{\phi} + \varepsilon^{-\phi} = 2 \cosh \phi \tag{106}$$

$$2 < \mathrm{Tr}\, M_1 < \infty \tag{107}$$

We now summarize our results:

Case I The c. v. of M_1 are distinct

From (96) and (107)

$$-2 \leqq \mathrm{Tr}\, M_1 < \infty$$

Now we shall consider

Case II The c. v. of M_1 are equal.

From (81)

$$\mathrm{Tr}\, M_1 = 2\lambda \tag{109}$$

where λ is a c. v. of M_1. Since det $M_1 = 1$, it follows that

$$\lambda = \pm 1 \tag{110}$$

Therefore M_1 can be of the forms

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \tag{111}$$

and

$$\mathrm{Tr}\, M_1 = \pm 2 \tag{112}$$

However since we are interested primarily in the range of values that Tr M_1 can assume we see that the information contained in (112) is already contained in (108).

We conclude that for a matrix M_1 if

$$\det M_1 = 1 \tag{113}$$

$$M_1^{+} \sigma M_1 = \sigma \tag{114}$$

then

$$-2 \leq \mathrm{Tr}\, M_1 < \infty \tag{115}$$

We now consider the matrix M_2 which has the properties that

$$\det M_2 = -1 \tag{116}$$

$$M_2^{+}\, \sigma M_2 = -\sigma \tag{117}$$

Case I. the c. v. of M_2 are distinct

$$\mathcal{M}_2 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

From (88)

$$\det M_2 = \lambda_1 \lambda_2 = -1 \tag{118}$$

$$\lambda_2 = -\frac{1}{\lambda_1} \qquad (\lambda_1 \neq 0)$$

$$\mathrm{Tr}\, M_2 = \lambda_1 - \frac{1}{\lambda_1} \tag{119}$$

Since λ_1 is a c. v of M_2

$$M_2 X = \lambda_1 X \tag{120}$$

In a manner similar to (93 a) we find

$$|\lambda_1|^2 X + \sigma X = -X^{+} \sigma X \tag{121}$$

which implies $X^{+} \sigma X = 0$ for otherewise there would be a contradiction.

Since

$$M_2^{+} = -\sigma M_2^{-1} \sigma^{-1} \tag{122}$$

it can be shown in a manner analogous to that used in deriving (101) that

$$\mathrm{Tr}\, M_2 = (\mathrm{Tr}\, M_2)^{*} \tag{123}$$

or $\mathrm{Tr}\, M_2$ is real.

Then

$$\left(\lambda_1 - \frac{1}{\lambda_1}\right) = \left(\lambda_1 - \frac{1}{\lambda_1}\right)^* \tag{124}$$

or

$$(\lambda_1 - \lambda_1^*)\left(1 + \frac{1}{|\lambda_1|^2}\right) = 0 \qquad \lambda_1 \neq 0 \tag{125}$$

from which we conclude

$$\lambda_1 = \lambda_1^* \quad \text{for all } \lambda_1 ,\ \lambda_1 \neq 0 \tag{126}$$

or that λ_1 is real.

For convenience denote λ_1 by ε^θ where θ is real, $-\infty < \theta < \infty$.

Then

$$\mathrm{Tr}\, M_2 = \lambda_1 - \frac{1}{\lambda_1} = 2 \sinh \theta \qquad -\infty < \theta < \infty \tag{127}$$

and conclude that

$$-\infty < \mathrm{Tr}\, M_2 < \infty \tag{128}$$

(vi) Determination of the Form of I. D. C. Associated With the Composite Structure.

Using the results of the previous development it will be an easy matter to determine the traces of each of the nine matrix products appearing in table I.

Let us denote the matrix product corresponding to a type r and type ℓ device appearing in table I by

$$M_{r\ell} \qquad r = 1, 2, 3 \qquad \ell = 1, 2, 3$$

For example $M_{21} = (\mathcal{A}A)^{-1} \rho \sigma \mathcal{C} c \sigma$.

The product

$$(M_{r\ell})^{+} \sigma (M_{r\ell}) \tag{129}$$

and

$$\det M_{r\ell} \tag{130}$$

for each of the nine cases has been evaluated and these results appear in table II.

From (115) and (118) we find

$$-2 \leqq \operatorname{Tr} M_{r\ell} < \infty \tag{131}$$

when $M_{r\ell}$ is any one of the matrices M_{11}, M_{22}, M_{33}, M_{32}, M_{23} .

When $M_{r\ell}$ is any one of M_{21}, M_{31}, M_{12}, M_{13} we find

$$-\infty < \operatorname{Tr} M_{r\ell} < \infty \ . \tag{132}$$

Using these two inequalities and the expressions for det T_{11} in table I , the range of det T_{11} may be determined. The results of this effort are shown in table III . Also listed in table III is the possible form of the I. D. C. associated with Λ which has been determined through the use of equations (64) , (65) and (66) .

Observe that if r and ℓ are each of type - three form then the form of the coupler associated with Λ is certain to be of type - three.

The coupling coefficient α^2 can then be written down at once ,

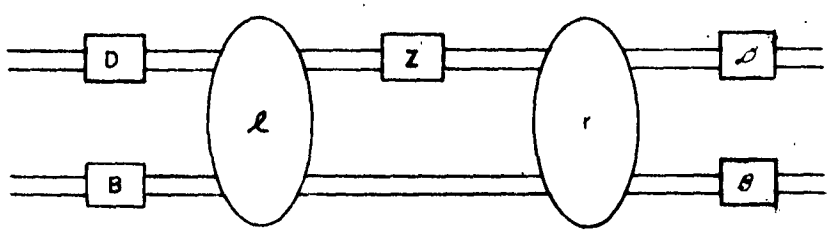
$$\alpha^2 = \frac{(\alpha_1 \alpha_2)^2}{(\beta_1 \beta_2)^2 + 1 - \beta_1 \beta_2 \operatorname{Tr} M_{33}} \tag{133}$$

where

$$M_{33} \equiv (\mathcal{A}A)^{-1} \rho \mathcal{C} c \rho \ . \tag{134}$$

It will be useful at this time to consider a transformation of the structure comprising fig. 10, which is interesting in its own right but which also has special significance for our present discussion.

We shall now show that every device possessing the structure indicated in fig. 4 can be transformed into the following system .



It will be recalled that in our derivation of the canonical form of a given four-port, one of the two-ports could be chosen arbitrarily.

Let L be a given four-port. In constructing the canonical form of L suppose we choose A arbitrarily and let A = X . Assume that the I. D. C. , ℓ , associated with L is known. Then the other three two-ports namely B, C, D, are uniquely determined in terms of L and X or equivalently in terms of ℓ, L and X . From equation (60) and corresponding to our new notation

$$C = L_{22} (L_{12})^{-1} X \ell_{12} (\ell_{22})^{-1} \tag{135}$$

Similarly let $\mathcal{C}$ be the arbitrary two-port in the canonical form of the four-port R. This time we shall choose

$$\mathcal{C} = C^{-1} = \ell_{22} (\ell_{12})^{-1} X^{-1} L_{12} (L_{22})^{-1} \tag{136}$$

With this choice $\mathcal{C}$ becomes

$$\mathcal{C} = (r_{11})^{-1} r_{12} \ell_{22} (\ell_{12})^{-1} X L_{12} (L_{22})^{-1} (R_{12})^{-1} R_{11} \tag{137}$$

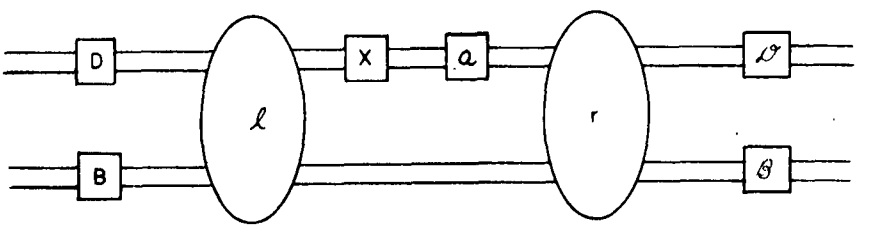
From equation (72) it is seen that

$$T_4 T_3 = \left[\begin{array}{c|c} \mathcal{A}X & 0 \\ \hline 0 & \mathcal{C}C \end{array}\right] \tag{138}$$

becomes

$$T_4 T_3 = \left[\begin{array}{c|c} \mathcal{a} X & 0 \\ \hline 0 & I \end{array}\right] \tag{139}$$

Therefore fig. 10 can be expressed as



and if we define

$$\mathcal{a} X = Z \tag{140}$$

then fig. 11 obtains.

For certain forms of r and ℓ the matrix product

$$(r_{11})^{-1} (r_{12}) \ell_{22} (\ell_{12})^{-1} \tag{141}$$

can be expressed in the form

$$k I \tag{142}$$

where k is a scaler. One such case is when ℓ and r are both of type-three. For this case

$$k = \frac{\beta_2}{\beta_1} \tag{143}$$

where β_2 refers to "r" and β_1 refers to "ℓ".

Under these conditions Z can be expressed as

$$Z = X^{-1} k K X \tag{144}$$

where

$$K \equiv L_{12} (L_{22})^{-1} (R_{12})^{-1} R_{11} \tag{145}$$

and k is some known constant. Thus we observe that Z is similar to the matrix k K (see fig. 13) .

This leads to an interesting result namely that if (141) can be expressed as (142) then it is not possible to choose X such that Z will be of any desired form.

Proof: Suppose it were possible to choose X so that Z could have any desired form. In particular require that

$$\mathrm{Tr}\, Z \neq \mathrm{Tr}\, k K \tag{146}$$

But from (144)

$$\mathrm{Tr}\, Z = \mathrm{Tr}\, k K \tag{147}$$

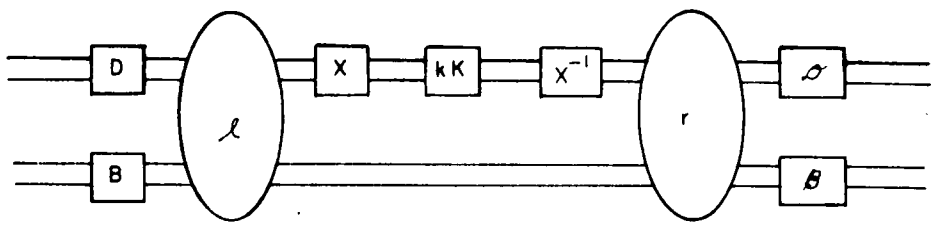
which contradicts our assertion. The result follows immediately.

Observe that a simple choice for X is

$$X = I \,. \tag{148}$$

With this choice

$$Z = \mathcal{a} \,. \tag{149}$$



If M_{33} of equation (134) is computed under these conditions we find

$$M_{33} = \mathcal{a}^{-1} \tag{150}$$

Since $\mathrm{Tr}\,\mathcal{a}^{-1} = \mathrm{Tr}\,\mathcal{a}$, (because det $\mathcal{a} = 1$)

$$\mathrm{Tr}\, M_{33} = \mathrm{Tr}\, \mathcal{a} \,. \tag{151}$$

Equation (133) can then be written as

$$a^2 = \frac{(a_1 a_2)^2}{(\beta_1\beta_2)^2 + 1 - \beta_1\beta_2 \operatorname{Tr} \mathcal{a}} \tag{152}$$

We shall now consider some special values of a^2 and the corresponding condition that is imposed on $\operatorname{Tr} \mathcal{a}$.

A useful relation is

$$a^2 = (a_1 a_2)^2 \tag{153}$$

which requires

$$\operatorname{Tr} \mathcal{a} = \beta_1 \beta_2 \tag{154}$$

Another special case is

$$a^2 = 1 \tag{155}$$

which requires

$$\operatorname{Tr} \mathcal{a} = \frac{\beta_1}{\beta_2} + \frac{\beta_2}{\beta_1} \tag{156}$$

This result has an interesting interpretation. When the c. v. of $\mathcal{a}$ are distinct then a is similar to a matrix of the form

$$\begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix}$$

where λ and $\frac{1}{\lambda}$ are the two c. v. of $\mathcal{a}$. If we define

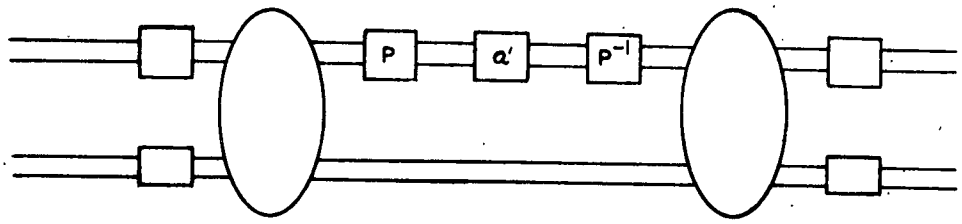
$$\frac{\beta_1}{\beta_2} = \gamma \tag{157}$$

Then $\lambda = \gamma$ or $\lambda = \frac{1}{\gamma}$ with $\beta_1 \neq \beta_2$.

When $\beta_1 = \beta_2$ then $\mathcal{a}$ is similar to a matrix of the form

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then fig. 13 assumes the special form



where

$$\bar{\mathcal{a}}^1 = \begin{bmatrix} \dfrac{\beta_1}{\beta_2} & 0 \\ 0 & \dfrac{\beta_2}{\beta_1} \end{bmatrix}, \qquad \beta_1 \neq \beta_2 \tag{158}$$

and

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \beta_1 = \beta_2 \tag{159}$$

for some matrix P .

TABLE II

"1" refers to ℓ
"2" " " r

Form of r	Form of ℓ	det T_{11}
1	1	$= (\alpha_1 \alpha_2)^2 + (\beta_1 \beta_2)^2 - \alpha_1 \alpha_2 \beta_1 \beta_2 \, \mathrm{Tr} \left[(\mathcal{a} A)^{-1} \sigma \mathcal{C} C \sigma \right]$
	2	$= \frac{1}{\beta_2} \left[(\alpha_1 \alpha_2)^2 - \beta_1^2 - \alpha_1 \alpha_2 \beta_1 \, \mathrm{Tr} \left[\rho (\mathcal{a} A)^{-1} \sigma \mathcal{C} C \sigma \right] \right.$
	3	$= \frac{1}{\alpha_2^2} \left[\alpha_1^2 - (\beta_1 \beta_2)^2 - \alpha_1 \beta_1 \beta_2 \, \mathrm{Tr} \left[(\mathcal{a} A)^{-1} \sigma \mathcal{C} C \hat{\rho} \right] \right]$
2	1	$= \frac{1}{\beta_2^2} \left[\beta_1^2 - (\alpha_1 \alpha_2)^2 + \alpha_1 \alpha_2 \beta_1 \, \mathrm{Tr} \left[(\mathcal{a} A)^{-1} \rho \sigma \mathcal{C} C \sigma \right] \right]$
	2	$= \frac{1}{(\beta_1 \beta_2)^2} \left[(\alpha_1 \alpha_2)^2 + 1 - (\alpha_1 \alpha_2) \, \mathrm{Tr} \left(\rho (\mathcal{a} A)^{-1} \rho \sigma \mathcal{C} C \sigma \right) \right]$
	3	$= -\left(\frac{\alpha_2}{\beta_2 \alpha_1} \right)^2 \left[1 + \left(\frac{\beta_1}{\alpha_2} \right)^2 - \frac{\beta_1}{\alpha_2} \, \mathrm{Tr} \left((\mathcal{a} A)^{-1} \rho \sigma \mathcal{C} C \hat{\rho} \right) \right]$
3	1	$= \frac{1}{\beta_2^2} \left[1 - (\beta_1 \beta_2)^2 - \alpha_1 \beta_1 \beta_2 \, \mathrm{Tr} \left((\mathcal{a} A)^{-1} \hat{\rho} \mathcal{C} C \sigma \right) \right]$
	2	$= -\left(\frac{\alpha_1}{\alpha_2 \beta_1} \right)^2 \left[1 + \left(\frac{\beta_2}{\alpha_1} \right)^2 - \frac{\beta_2}{\alpha_1} \, \mathrm{Tr} \left(\rho (\mathcal{a} A)^{-1} \hat{\rho} \mathcal{C} C \sigma \right) \right]$
	3	$= \frac{1}{(\alpha_1 \alpha_2)^2} \left[1 + (\beta_1 \beta_2)^2 + \beta_1 \beta_2 \, \mathrm{Tr} \left((\mathcal{a} A)^{-1} \rho \mathcal{C} C \rho \right) \right]$

TABLE III

C_{11} $\det C_{11} = 1$ $C_{11}^{+}\sigma C_{11} = \sigma$	C_{12} $\det C_{12} = -1$ $C_{12}^{+}\sigma C_{12} = -\sigma$	C_{13} $\det C_{13} = -1$ $C_{13}^{+}\sigma C_{13} = -\sigma$
C_{21} $\det C_{21} = -1$ $C_{21}^{+}\sigma C_{21} = -\sigma$	C_{22} $\det C_{22} = 1$ $C_{22}^{+}\sigma C_{22} = \sigma$	C_{23} $\det C_{23} = 1$ $C_{23}^{+}\sigma C_{23} = \sigma$
C_{31} $\det C_{31} = -1$ $C_{31}^{+}\sigma C_{31} = -\sigma$	C_{32} $\det C_{32} = 1$ $C_{32}^{+}\sigma C_{32} = \sigma$	C_{33} $\det C_{33} = 1$ $C_{33}^{+}\sigma C_{33} = \sigma$

TABLE IV

r / ℓ	1	2	3
1	$-\infty < \det T_{11} \leqq 1$ 1, 2,	$-\infty < \det T_{11} < \infty$ 1, 2, 3,	$-\infty < \det T_{11} < \infty$ 1, 2, 3,
2	$-\infty < \det T_{11} < \infty$ 1, 2, 3,	$0 \leqq \det T_{11} < \infty$ 1, 3,	$-\infty < \det T_{11} < \infty$ 1, 2, 3,
3	$-\infty < \det T_{11} < \infty$ 1, 2, 3,	$-\infty < \det T_{11} < \infty$ 1, 2, 3,	$1 \leqq \det T_{11} < \infty$ 3

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