NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.
SHIFT INVARIANT LINEAR MANIFOLDS

P. M. Anselone

MRC Technical Summary Report #359
December 1962

Madison, Wisconsin
ABSTRACT

This paper gives a characterization of solution manifolds of ordinary differential equations with constant coefficients as finite-dimensional shift invariant linear manifolds of continuous functions.
This paper concerns a relatively little known characterization of solution manifolds of ordinary differential equations with constant coefficients. Such a manifold consists, of course, of polynomial-exponential functions. In the main theorem, which follows, \( R \) is the real line and \( C(R) \) is the set of complex continuous functions defined on \( R \).

**Theorem 1.** Let \( M \) be a finite dimensional linear manifold in \( C(R) \) which is shift invariant, i.e.,

\[
f(t) \in M, x \in R \Rightarrow f(t + x) \in M.
\]

Then there exist non-negative integers \( m_p \) and complex numbers \( z_p \), \( p = 1, \ldots, q \), such that \( M \) is spanned by the functions

\[
t^{m} e^{z_p t}, \quad \begin{cases} m = 0, 1, \ldots, m_p, \\ p = 1, \ldots, q. \end{cases}
\]

Equivalently, \( M \) is the solution manifold of an ordinary differential equation with constant coefficients.

Sponsored by the Mathematics Research Center, United States Army, Madison, Wisconsin under the Contract No.: DA-11-022-ORD-2059.
This theorem is not really new. A generalization of it to functions defined on a topological group was given in 1948 by Laurent Schwartz \[1\]. His proof depends on the theory of group representations. The primary purpose of this paper is to present a new, more elementary, proof of Theorem 1. Both the theorem and this proof have features which ought to be of substantial mathematical and pedagogical interest. In particular, the theorem was essential in a paper \[2\] by D. Greenspan and the author on linear difference-integral equations. It was used in order to determine certain "fundamental" solutions, which are eigenfunctions of a related operator.

**Proof of Theorem 1.** Let \( \{ f_i(t) : i = 1, \ldots, m \} \) be a basis for \( M \). It follows from (1) that, for each \( i = 1, \ldots, m \) and each \( x \in \mathbb{R} \), there exist unique complex numbers \( a_{ij}(x), j = 1, \ldots, m \), such that

\[
\sum_{j=1}^{m} a_{ij}(x) f_j(t) = f_i(t+x), \quad i = 1, \ldots, m, \quad t, x \in \mathbb{R}.
\]

If we could differentiate (3) with respect to \( x \) and then let \( x = 0 \), we should obtain a system of ordinary differential equations with constant coefficients, from which the theorem would follow immediately. The rest of the proof justifies this procedure by showing that the functions \( a_{ij} \) are differentiable.

Let \( f(t) \) denote the (column) vector with components \( f_i(t), \) \( i = 1, \ldots, m \). Let \( A(x) \) denote the matrix with elements \( a_{ij}(x), i, j = 1, \ldots, m \).
Then (3) is equivalent to

\[ I(t + x) = A(x)I(t), \quad t, x \in R. \]  

(4)

Moreover, there is a unique matrix function \( A = \{a_{ij}\} \) which satisfies (4).

One implication of the uniqueness is that

\[ A(0) = I. \]  

(5)

Equation (4) yields

\[ A(y)A(x)I(t) = A(y)I(t + x) = I(t + x + y) = A(x + y)I(t) \]

for all \( x, y, t \in R \). By the uniqueness of \( A \),

\[ A(y)A(x) = A(x + y) = A(y + x) = A(x)A(y), \quad x, y \in R. \]  

(6)

Thus, \( A \) is a commutative semigroup of matrices. This fact motivates the steps which follow, but is not actually used to obtain the results.

We prove next that \( A \) is continuous, i.e., that each \( a_{ij} \), is a continuous function. For this purpose an auxiliary result is needed.

**Lemma.** Let \( g_i, i = 1, \ldots, m, \) be \( m \) linearly independent real or complex functions defined on an arbitrary (abstract) set \( P \). Then there
exists a subset of \( P \),

\[
P_m = \{ p_j : j=1, \ldots, m \},
\]

consisting of exactly \( m \) points, such that the restrictions of the functions \( g_i \) to \( P_m \) are linearly independent. Hence, the matrix

\[
\{ g_i(p_j) \}, \ i, j = 1, \ldots, m,
\]

is non-singular.

This lemma is a direct generalization of the familiar theorem in linear algebra that the row and column ranks of a matrix are equal. To see this, consider a "matrix" with rows \( i = 1, \ldots, m \) and with the columns indexed by the elements of \( P \). Let \( g_i(p) \) be the element in row \( i \) and column \( p \). Since the functions \( g_i \) are linearly independent, the row rank of this matrix is \( m \). By the same reasoning as in the classical case, the column rank of the matrix also is \( m \), so that there exists a non-singular square submatrix \( \{ g_i(p_j) \}, \ i, j = 1, \ldots, m \). Incidentally, the same lemma was used by R. C. Buck [3] in a different connection.

Since the functions \( f_i(t_k), \ i = 1, \ldots, m \), are linearly independent, the lemma implies that there exists \( t_k \in R, \ k = 1, \ldots, m \), such that the matrix \( \{ f_i(t_k) \} \) is non-singular. For each \( x \in R \), define a matrix function \( B \) by

\[
B(x) = \{ f_i(t_k + x) \}, \ i, k = 1, \ldots, m.
\] (7)
Then, by (3),

\[ B(x) = A(x)B(o), \] (8)

where \( B(o) \) is non-singular. It follows that

\[ A(x) = B(x)B^{-1}(o). \] (9)

Since \( B \) is clearly continuous, \( A \) is continuous.

We have proved that \( A \) is a continuous semigroup of matrices. One of the earlier results in semigroup theory (cf. [4], pp. 282-284, or [5]) asserts that \( A \) is necessarily differentiable. In order to keep this proof elementary and self-contained, we prove here in a few lines that \( A' \) exists.

By (6),

\[ A(x) \int_0^t A(y) \, dy = \int_0^t A(x + y) \, dy = \int_0^{x+t} A(s) \, ds. \] (10)

The continuity of \( A \) and (5) imply that

\[ \frac{1}{t} \int_0^t A(y) \, dy \to I \quad \text{as} \quad t \to 0. \] (11)

It follows that both \( \frac{1}{t} \int_0^t A(y) \, dy \) and \( \int_0^t A(y) \, dy \) are non-singular for \( t \).
sufficiently small and positive. By (10),

$$A(x) = \int_{x}^{x+t} A(s) ds \left[ \int_{0}^{t} A(y) dy \right]^{-1}$$

(12)

for some $t > 0$. Since the right member of (12) is differentiable with respect to $x$, $A'$ exists. Therefore, each $a_{ij}$ is differentiable.

Finally, differentiate (3) with respect to $x$ and then let $x = 0$ to obtain

$$f'(t) = \sum_{j=1}^{m} a_{ij}(0) f_j(t), \quad i = 1, \ldots, m, \quad t \in \mathbb{R}.$$  

(13)

This system of ordinary differential equations with constant coefficients yields the desired results.

The conditions in Theorem 1 can be relaxed. For example, it suffices to require (1) only for $x$ sufficiently small and positive. The functions in $M$ need not be defined on all of $\mathbb{R}$; they could be defined on $[0, \infty]$ or on $[0, c]$ for some $c > 0$.

A generalization of Theorem 1 to real Euclidean n-space $\mathbb{R}^n$ is given below. A typical element of $\mathbb{R}^n$ is devoted by $\mathbf{t} = (t_1, \ldots, t_n)$.

**Theorem 2.** Let $M$ be a finite dimensional linear manifold in $\mathbb{C}(\mathbb{R}^n)$ such that

$$f(\mathbf{t}) \in M, \quad \mathbf{x} \in \mathbb{R}^n \Rightarrow f(\mathbf{t} + \mathbf{x}) \in M.$$
Then there exist integers $m_{kp}$ and complex numbers $z_{kp}$, $p = 1, \ldots, q$ and $k = 1, \ldots, n$, such that $M$ is spanned by the functions

$$m_1 \leq m \leq m_n z_{lp} \leq z_{np}, \left\{ \begin{array}{l} m_k = 0, 1, \ldots, m_{kp}, \\ p = 1, \ldots, q_k. \end{array} \right\} k = 1, \ldots, n.$$ 

This theorem can be proved in much the same way as Theorem 1.

Details are omitted.
REFERENCES

[1] Schwartz, L., Théorie générale des fonctions moyenne-périodiques,


[3] Buck, R. C., Zero sets for continuous functions,

    Providence, R. I., 1957.

[5] Nagumo, M., Einige analytische Untersuchungen in linearen metrischen