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LINEARIZED THEORY OF THE UNSTEADY MOTION OF A PARTIALLY CAVITATED HYDROFOIL

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ABSTRACT

A linearized theory for the unsteady motion of a partially cavitated flat hydrofoil in two dimensions is carried out. A second linearization procedure is used, based on ideas of Timman and Guerst, to obtain the unsteady pressure distribution around the hydrofoil and the resulting force and moment, as functions of the cavitation number and Strouhal number for given pitch and/or heave.

NOMENCLATURE

ϕ	normalized pressure (or acceleration potential) = $\frac{p_\infty - p}{d}$, where p is the actual pressure and d is the density
σ	normalized cavity pressure (cavitation number $= \sigma/\frac{1}{2}U^2$)
G	Green's function for regular term for pressure
ℓ	one-half the length of the hydrofoil
c	position of the rear end of the cavity (assuming hydrofoil between $-\ell$ and ℓ)
α	$\sqrt{\frac{\ell - c}{\ell + c}}$
Φ	velocity potential
U	free stream velocity
ν	oscillation frequency
Ω	$\frac{\nu \ell}{U}$ (reduced frequency)
M(t)	slope of hydrofoil
B(t)	y intercept of hydrofoil
f	force perpendicular to hydrofoil
m	moment on hydrofoil

HYDROELASTIC EFFECTS OF UNSTEADY MOTION OF A HYDROFOIL

1. INTRODUCTION

In order to analyze the hydroelastic effects of unsteady motion of a hydrofoil it is necessary to determine the pressure distribution on the hydrofoil. The problem of particular concern here is that of a partially cavitated hydrofoil, experiencing simple harmonic oscillations in vertical position (heaving) or angle of attack (pitching).

In order to solve this problem, it is first necessary to determine the flow pattern around the oscillating hydrofoil. Furthermore, in order to produce a solution relatively quickly, it is desirable to make certain simplifying assumptions. First a two-dimensional flow pattern will be assumed, i.e., the hydrofoil will be assumed to have infinite span. Secondly, the fluid will be assumed infinite in all directions. Third, thickness effects and initially, camber effects will be neglected. Moreover, the angle of attack will be assumed sufficiently small that linearized theory may be used. Finally, the unsteady motion will be assumed to consist of small oscillations around a steady motion so that a "second linearization" may be made. Therefore,

the flow pattern is assumed to fluctuate at the same frequency as the hydrofoil oscillation, i.e., higher harmonics are neglected.

Timman [3] has made an attempt at solving this problem. In his work the assumptions made to complete the definition of the problem have consequences which are physically unacceptable. Guerst [1] has formulated the problem in such a way that this contradiction is resolved. However, he has not carried through the problem to any extent. In our treatment, Timman's analytical machinery is used throughout, but the physical assumptions used to determine the unique solution are those of Guerst. Although Guerst's set of conditions is not beyond question we felt it is the best possible in the light of present knowledge.

To solve this problem, it was first necessary to obtain a steady state solution by Timman's method. This was done, and the result compared to that of Guerst [2]. Preliminary calculations indicate complete agreement, except for discrepancies which can be attributed to numerical inaccuracies (of less than 1%) in Guerst's calculations. Timman's approach, as developed here, gives much simpler closed-form expressions for the pressure and velocity potentials than Guerst was able to obtain.

The next step in solving this problem was to develop an explicit mathematical representation for the physical assumptions

made. This was done, and resulted in six simultaneous linear equations, involving six unknowns. The coefficients are functions of reduced frequency and cavity length. Some of them are defined by integrals which must be evaluated numerically. To do this a program is being written for the IBM 7090, which will compute these integrals as functions of the two parameters. The expressions for pressure and velocity are simple combinations of these coefficients and elementary functions.(see Section 4)

Finally the lift and moment of the hydrofoil were calculated as linear functions of the above mentioned coefficients.

2. GENERAL STATEMENT OF PROBLEM

Timman has derived the general expression for the acceleration potential Φ to be:

$$\Phi = \sigma - \frac{1}{2\pi}\int_{\substack{\text{wetted}\\ \text{surface}}} f(x',t)G(x',o,x,y)dx' + c_1\Phi_1 + c_2\Phi_2$$

(see Figure 1).

In this expression c_1 and c_2 are functions of time which are determined, together with the position of the rear end of the cavity, by additional conditions given by the physical model used.

If $h(x,t)$ = profile of hydrofoil

then $w(x,t) = h_t + Uh_x$

and $f(x,t) = w_t + Uw_x$

For a flat plate, the above integration may be done explicitly. (See Appendix I).

Let $h = M(t)x + B(t)$.

Then $w = M'x + B' + MU$

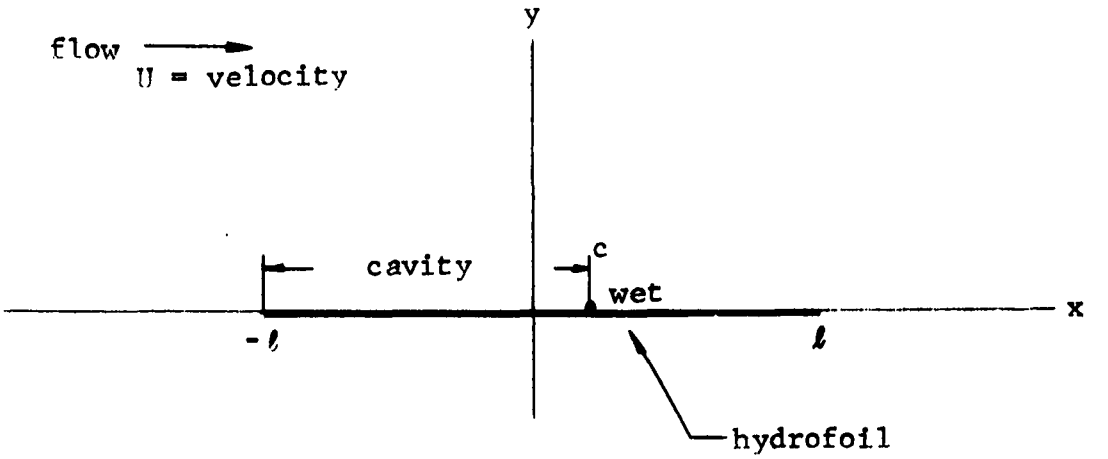
and $f = M''x + B'' + 2M'U$

Therefore

$$\Phi = \sigma + \ell^2 M'' F_2 + \ell(B'' + 2M'U)F_1 + c_1\Phi_1 + c_2\Phi_2$$

M'', $(B'' + 2M'U)$, c_1, c_2 are functions of t only. F_k, Φ_k are harmonic

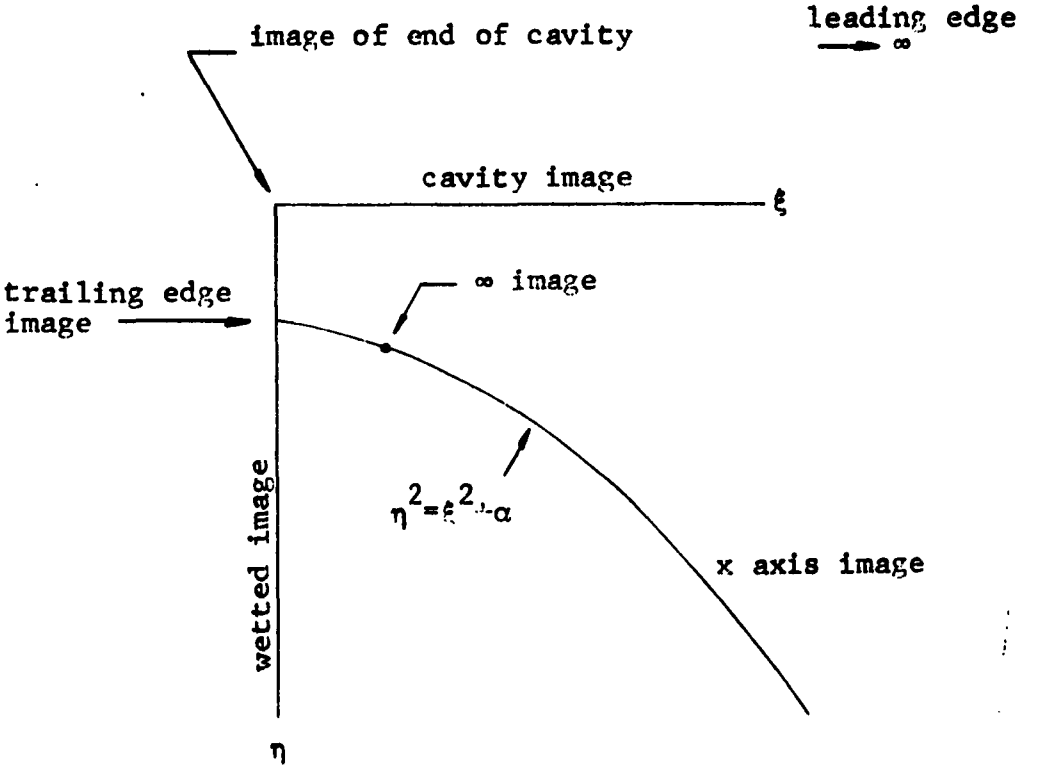
PHYSICAL PLANE



$z = x + iy$

$$\zeta = \sqrt{\sqrt{\frac{\ell - z}{\ell + z}} - \sqrt{\frac{\ell - c}{\ell + c}}}$$

IMAGE PLANE



$\zeta = \xi + i\eta$

$$\alpha = \sqrt{\frac{\ell - c}{\ell + c}}$$

Figure 1

functions of x and y in the plane slit along the hydrofoil.

$$\text{Let } \zeta = \sqrt{\sqrt{\frac{\ell - z}{\ell + z}} - \alpha}$$

where $z = x + iy$

and $\zeta = \xi + i\eta$

2ℓ = length of hydrofoil

c = x-coordinate of rear end of cavity

$$\alpha = \sqrt{\frac{\ell - c}{\ell + c}}$$

F_k, ϕ_k (see Figure 1) are harmonic and rational functions of ξ and η only, and are time dependent only through α.

The solution of the general problem requires specifying three conditions to determine the three parameters c_1, c_2, and α. There are four conditions which may be used:

Boundary conditions (general)

(1) $\phi(\infty) = 0$

(2) $\Phi_y = w(x,t)$ on wetted surface, where Φ = the velocity potential.

(3) $h(c,t) - h(-\ell,t) = \frac{1}{U}\int_{-\ell}^{c} \Phi_y(x',0^+,t - \frac{c-x'}{U})dx'$ (closure)

(4) $\Phi_y(x,0^+,t) = \Phi_y(x,0^-,t)$, $x > \ell$ (continuity of vertical component of velocity)

Conditions (1) and (2) seem quite satisfactory on physical and mathematical grounds, so that the choice lies between (3) and (4). For the steady state problem (4) is automatically satisfied so that (3) is the only condition left. However in the unsteady problem, the physics of the problem seem to us more reasonable if (4) is used rather than (3). This is the view of Guerst, Wu and others, and this approach is used here.

Using condition (1), and following Timman, we write

$$\Xi = \frac{1}{U} \int_{-\infty}^{x} \sigma + \sum_{k=1}^{4} c_k\left(t - \frac{x-x'}{U}\right) \phi_k(\xi,\eta)\,dx'$$

where

$$c_4 = \ell^2 M'', \quad c_3 = \ell(B'' + 2M'U), \quad \phi_3 = F_1, \quad \phi_4 = F_2$$

$$\Xi_y = \frac{1}{U} \int_{-\infty}^{x} \sum_{k=1}^{4} c_k\, \phi_{ky}\, dx' = \frac{1}{U} \int_{-\infty}^{x} \sum_{k=1}^{4} c_k\, \psi_{kx'}\, dx'$$

But

$$\psi_{kx'} = \frac{\partial \psi_k}{\partial x'} = \frac{d\psi_k}{dx'} - \frac{1}{U}\frac{\partial \psi_k}{\partial \alpha}\frac{d\alpha}{d\tau}, \quad \left(\tau = t - \frac{x-x'}{U}\right)$$

$$\Xi_y = \frac{1}{U}\left\{\sum_{k=1}^{4} \left([c_k \psi_k]_{-\infty}^{x} - \frac{1}{U}\int_{-\infty}^{x} \left[c_k' \psi_k + c_k \frac{\partial \psi_k}{\partial \alpha}\frac{d\alpha}{d\tau}\right] dx'\right)\right\}$$

Therefore

$$U\Phi_y = \left\{ \sum_{k=1}^{4} c_k(t)\psi_k(x,y,\alpha(t)) - \frac{1}{U}\frac{d}{dt}\int_{-\infty}^{x} c_k(\tau)\psi_k(x',y,\alpha(\tau))dx' \right\}$$

The harmonic functions are listed in Appendix VI, Table I where those with subscripts 1 and 2 were derived by Timman and those with subscripts 3 and 4 are calculated in Appendix I.

3. STEADY STATE SOLUTION

Since $c_3 = c_4 = 0$, we have

$$\phi = \sigma + c_1\phi_1 + c_2\phi_2$$

$$\xi_y = \frac{1}{U}\,(c_1\psi_1 + c_2\psi_2)$$

The boundary conditions become (1, 2, 3)

$$(1)\quad \frac{c_1 A}{\sqrt{\alpha^2+1}} - c_2 A + \sigma = 0$$

$$(2)\quad MU^2 + \frac{c_1 B}{\sqrt{\alpha^2+1}} + c_2 B = 0 \quad (\xi = 0 \text{ on wetted surface})$$

$$(3)\quad U^2 M(c+\ell) = c_1 \int_{-\ell}^{c} \left(\frac{\xi}{\eta^2+\xi^2} - \frac{B}{\sqrt{\alpha^2+1}}\right)dx + c_2 \int_{-\ell}^{c} (\xi - B)\,dx\ ,$$

where A and B are defined in Appendix VI.

If we substitute (2) in (3) we find

$$(3')\qquad c_1 \int_{-\ell}^{c} \frac{\xi}{\eta^2+\xi^2}\,dx + c_2 \int_{-\ell}^{c} \xi\,dx = 0$$

Since $x = \ell\,\dfrac{1-(\xi^2+\alpha)^2}{1+(\xi^2+\alpha)^2}$

and $\eta = 0$ on the cavity,

(3') is replaced by

$$c_1 \int_0^\infty \frac{1}{\xi} \frac{dx}{d\xi} \, d\xi + c_2 \int_0^\infty \xi \frac{dx}{d\xi} \, d\xi = 0$$

or

$$c_1 \int_0^\infty \frac{\xi^2 + \alpha}{[1+(\xi^2+\alpha)^2]^2} \, d\xi + c_2 \int_0^\infty \frac{\xi^2(\xi^2+\alpha)}{[1+(\xi^2+\alpha)^2]^2} \, d\xi = 0$$

The integrals are evaluated in Appendix III and we obtain:

$$(3^*) \qquad (\sqrt{\alpha^2+1} + 2\alpha) \frac{c_1}{\sqrt{\alpha^2+1}} + (\sqrt{\alpha^2+1}) \, c_2 = 0$$

The solution of (1), (2), and (3*) is as follows:

Let $\rho = -\frac{2U^2M}{\sigma}$

Consider the equation $y^3 - y + \rho = 0$

Then $\alpha = \frac{1}{2}(\frac{1}{y} - y)$

$$c_1 = \frac{\sigma}{4}(y + \frac{1}{y})(\frac{\rho}{\sqrt{2y}} - \sqrt{2y})$$

$$c_2 = \frac{\sigma}{2}(\frac{\rho}{\sqrt{2y}} + \sqrt{2y}$$

Let $\lambda = \frac{-\rho + i\sqrt{\frac{4}{27} - \rho^2}}{2}$, $0 \leq \rho^2 \leq \frac{4}{27}$

Then $$y = \lambda^{1/3} + \frac{1}{3\lambda^{1/3}}$$

The only allowed roots are those which give positive y. In general there are two solutions y_1, y_2 where $\frac{1}{\sqrt{3}} \geq y_1 \geq 0$ and $1 \geq y_2 \geq \frac{1}{\sqrt{3}}$. The only solution which is physically meaningful (see [2]) is given by y_1.

Special Cases:

1) $\rho^2 = \frac{4}{27}$ (apparently the maximum angle of attack for which linearized theory will hold.)

$y = \frac{1}{\sqrt{3}}$ (2 positive roots identical)

$\alpha = \frac{1}{\sqrt{3}}$

$c = \frac{\ell}{2}$

$\frac{\ell+c}{2\ell} = .75$ (cf. Geurst value of .7485)

2) $\rho = 0$ (no angle of attack)

y	α	c	
0	∞	$-\ell$	(no cavity, physically real solution)
1	0	ℓ	(cavity full length; anomalous)

Since y is only an intermediate variable, it is of interest to express everything in terms of α.

By solving $\alpha = \frac{1}{2}\left(\frac{1}{y} - y\right)$, we obtain

$$y = \sqrt{\alpha^2+1} - \alpha$$

Since $\rho = y - y^3$, we have

$$c_1 = -\sigma B(1+\alpha^2)(\sqrt{\alpha^2+1} - \alpha)$$

$$c_2 = \sigma B(\alpha\sqrt{\alpha^2+1} + 1 - \alpha^2)$$

$$\rho = 2\alpha(\sqrt{\alpha^2+1} - \alpha)^2$$

$$\sigma = \frac{-U^2M}{\alpha}(\sqrt{\alpha^2+1} + \alpha)^2$$

Therefore

$$c_1 = \frac{U^2MB}{\alpha}(\alpha^2+1)(\alpha + \sqrt{\alpha^2+1})$$

$$c_2 = \frac{-U^2MB}{\alpha}(3\alpha^2 + 1 + 3\,\alpha\sqrt{\alpha^2+1})$$

4. UNSTEADY CASE

The general expression for the velocity Φ_y is given by

$$U\Phi_y = \sum_k \left\{ c_k(t)\psi_k(x,y,\alpha(t)) - \frac{1}{U}\frac{d}{dt}\int_{-\infty}^{x} c_k\left(t-\frac{x-x'}{U}\right)\psi_k\left(x',y,\alpha\left(t-\frac{x-x'}{U}\right)\right)dx' \right\}$$

Assume the time dependence is given by a small simple harmonic oscillation around the steady state

$$\alpha = \alpha_o + \alpha_1 \cos(\nu t-\theta_\alpha)$$

$$c_k = c_{ko} + c_{k1} \cos(\nu t-\theta_k)$$

where α_o, c_{10}, c_{20} are the steady state values.

Let $M(t) = M_o + M_1 \cos \nu t$,

$$B(t) = B_1 \cos(\nu t-\theta_B)$$

where M_o, M_1, B_1, θ_B are all given quantities.

Then $M'(t) = -\nu M_1 \sin \nu t$

$$M''(t) = -\nu^2 M_1 \cos \nu t$$

$$B''(t) = -\nu^2 B_1 \cos(\nu t-\theta_B)$$

$$c_3 = -\nu\ell(\nu B_1 \cos(\nu t-\theta_B) + 2M_1U \sin \nu t)$$

$$= -\nu\ell(\nu B_1 \cos\theta_B \cos\nu t + (2M_1U + \nu B_1 \sin\theta_B) \sin \nu t)$$

$$= -\nu\ell \sqrt{(\nu B_1)^2 + 4(M_1U)^2 + 4\nu B_1 M_1 U \sin\theta_B}\ \cos\left(\nu t-\tan^{-1}\left(\tan\theta_B + \frac{2M_1U}{\nu B_1\cos\theta_B}\right)\right)$$

$$c_{30} = 0$$

$$c_{31} = -\nu\ell\sqrt{(\nu B_1)^2 + 4(M_1U)^2 + 4\nu B_1 M_1 U \sin\theta_B}$$

$$\theta_3 = \tan^{-1}\left(\tan\theta_B + \frac{2M_1U}{\nu B_1\cos\theta_B}\right)$$

$$c_4 = -\nu^2\ell^2 M_1 \cos\nu t$$

$$c_{40} = 0$$

$$c_{41} = -\nu^2\ell^2 M_1$$

$$\theta_4 = 0$$

The problem is to find α_1, c_{11}, c_{21}, θ_1, θ_2, θ_α.

We now carry out the second linearization:

($\wedge$ over a symbol denotes the perturbation in that quantity given by the symbol without the $\wedge$)

$$c_k(t)\psi_k(x,y,\alpha(t)) = c_{k0}\psi_k(x,y,\alpha_o) + c_{k1}\cos(\nu t-\theta_k)\psi_k(x,y,\alpha_o)$$

$$+ c_{k0}\frac{\partial\psi_k}{\partial\alpha_o}(x,y,\alpha_o)\alpha_1\cos(\nu t-\theta_\alpha)$$

$$U\,\hat{\Phi}_y = \sum_{k=1}^{4} \Big[c_{k1}\cos(\nu t-\theta_k)\psi_k(x,y,\alpha_o) + c_{k0}\frac{\partial\psi_k}{\partial\alpha_o}(x,y,\alpha_o)\alpha_1\cos(\nu t-\theta_\alpha)$$

$$+\frac{\nu}{U}\int_{-\infty}^{x}\Big\{c_{k1}\sin\big(\nu(t-\tfrac{x-x'}{U})-\theta_k\big)\psi_k(x',y,\alpha_o)$$

$$+c_{k0}\frac{\partial\psi_k}{\partial\alpha_o}(x',y,\alpha_o)\alpha_1\sin\big(\nu(t-\tfrac{x-x'}{U})-\theta_\alpha\big)\Big\}\,dx'\Big]$$

The above integrations are carried out explicitly for the cases where $y = 0^+$ or $y = 0^-$, thus giving the vertical component of the perturbation velocity along the hydrofoil. The steady state term has been derived previously. Similarly the following equation gives the perturbation term for the pressure distribution along the hydrofoil when $y = 0^+$ or $y = 0^-$:

$$\hat{\Phi} = \sum_{k=1}^{4}\Big(c_{k1}\cos(\nu t-\theta_k)\phi_k(x,y,\alpha_o) + c_{k0}\frac{\partial\phi_k}{\partial\alpha_o}\alpha_1\cos(\nu t-\theta_\alpha)\Big)$$

$\frac{\partial}{\partial\alpha_o}$ of ϕ_1, ψ_1, ϕ_2, and ψ_2 are given in Appendix VI, Table I.

$\frac{\partial}{\partial\alpha_o}$ of ϕ_3, ψ_3, ϕ_4, and ψ_4 are not needed since $c_{30}=c_{40}=0$.

The boundary conditions (1), (2), (4) of Section 2 are explicitly given as follows:

Condition (1)

$$\sum_{k=1}^{4} \left(c_{k1}\cos(\nu t-\theta_k)\phi_k(\infty) + c_{k0}\frac{\partial\phi_k(\infty)}{\partial\alpha_o}\alpha_1\cos(\nu t-\theta_\alpha)\right)=0$$

where the values at ∞ of ϕ_k and $\frac{\partial\phi_k}{\partial\alpha}$ are given in Appendix VI, Table II.

Condition (2)

$$\hat{w} = M'x + B' + U(M-M_o)$$

$$= -x\nu M_1 \sin\nu t - B\sin(\nu t-\theta_B) + UM_1\cos\nu t$$

$$= -x\nu M_1 \sin\nu t + (UM_1 + \nu B_1 \sin\theta_B)\cos\nu t - \nu B_1\cos\theta_B \sin\nu t$$

$$= \frac{c_{41}}{\nu\ell^2}\left(x\sin\nu t + \frac{U}{\nu}\cos\nu t\right) + \frac{c_{31}}{\nu\ell}\sin(\nu t-\theta_3)$$

Therefore Condition (2) becomes

$$\frac{Uc_{41}}{\nu\ell}\left(\frac{x}{\ell}\sin\nu t + \frac{U}{\nu\ell}\cos\nu t\right) + \frac{Uc_{31}}{\nu\ell}\sin(\nu t-\theta_3) =$$

$$\sum_{k=3}^{4} c_{k1}\left\{\psi_k(x,0^-,\alpha_o)\cos(\nu t-\theta_k) + \frac{\nu}{U}\int_{-\infty}^{x}\psi_k(x',0^-,\alpha_o)\sin\left(\nu\left(t-\frac{x-x'}{U}\right)-\theta_k\right)dx'\right\}$$

$$+ \sum_{k=1}^{2}\left[c_{k1}\cos(\nu t-\theta_k)\psi_k(0,0^-,\alpha_o) + c_{k0}\frac{\partial\psi_k}{\partial\alpha_o}(0,0^-,\alpha_o)\alpha_1\cos(\nu t-\theta_\alpha)\right.$$

$$+ \frac{\nu}{U}\int_{-\infty}^{x}\left[c_{k1}\sin\left(\nu\left(t-\frac{x-x'}{U}\right)-\theta_k\right)\psi_k(x',0^-,\alpha_o)\right.$$

$$\left.\left. + c_{k0}\frac{\partial\psi_k}{\partial\alpha_o}(x',0^-,\alpha_o)\alpha_1\sin\left(\nu\left(t-\frac{x-x'}{U}\right)-\theta_\alpha\right)\,dx'\right]\right.$$

$$- \sum_{k=1}^{2} c_{k1}\psi_k(0,0^-,\alpha_o)\left[\cos(\nu t-\theta_k)-\cos(\nu(t-\tfrac{x+\ell}{U})-\theta_k)\right]$$

$$+ c_{k0}\frac{\partial\psi_k}{\partial\alpha_o}(0,0^-,\alpha_o)\alpha_1\left[\cos(\nu t-\theta_\alpha)-\cos\nu(t-\tfrac{x+\ell}{U})-\theta_\alpha)\right]$$

$\left[\psi_k, \frac{\partial\psi_k}{\partial\alpha}\right.$ are constants along the wetted surface for $k=1,2\left.\right]$.

This may be simplified to

$$\sum_{k=3}^{4} c_{k1}(\psi_k(x,0^-,\alpha_o)\cos(\nu t-\theta_k) + \frac{\nu}{U}\int_{-\infty}^{x}\sin(\nu(t-\tfrac{x-x'}{U})-\theta_k)\psi_k(x',0^-,\alpha_o)dx')$$

$$+ \sum_{1}^{2}\left[\frac{\nu}{U}\int_{-\infty}^{-\ell}\left\{c_{k1}\psi_k(x',0,\alpha_o)\sin(\nu(t-\tfrac{x-x'}{U})-\theta_k)\right.\right.$$

$$\left. + \alpha_1 c_{k0}\frac{\partial\psi_k}{\partial\alpha_o}(x',0,\alpha_o)\sin(\nu(t-\tfrac{x-x'}{U})-\theta_\alpha)\right\}dx'$$

$$\left. + c_{k1}\psi_k(0,0^-,\alpha_o)\cos(\nu(t-\tfrac{x+\ell}{U})-\theta_k)+\alpha_1 c_{k0}\frac{\partial\psi_k}{\partial\alpha_o}(0,0^-,\alpha_o)\cos(\nu(t-\tfrac{x+\ell}{U})-\theta_\alpha)\right]$$

$$= \frac{Uc_{41}}{\nu\ell}\left(\frac{x}{\ell}\sin\nu t + \frac{U}{\nu\ell}\cos\nu t\right) + \frac{Uc_{31}}{\nu\ell}\sin(\nu t-\theta_3)$$

Let $J_{k,s} = \frac{\nu}{U}\int_{-\infty}^{-\ell}\psi_k(x',o,\alpha_o)\sin(\frac{\nu}{U}(x'+\ell))dx'$

$$J_{k,c} = \frac{\nu}{U}\int_{-\infty}^{-\ell}\psi_k(x',o,\alpha_o)\cos(\frac{\nu}{U}(x'+\ell))dx'$$

$J^*_{k,c}$, $J^*_{k,s}$ defined using $\frac{\partial\psi_k}{\partial\alpha_o}$

These integrals must be evaluated numerically.

The ψ_k and $\frac{\partial\psi_k}{\partial\alpha_o}$ needed are listed in Appendix VI, Table V (in the form $\tilde{\psi}_k$ and $\frac{\partial\tilde{\psi}_k}{\partial\alpha_o}$, the non-constant parts of ψ_k and $\frac{\partial\psi_k}{\partial\alpha_o}$, defined in Appendix I).

Let

$$\rho = \nu(t - \frac{x}{U}),$$

$$W = \int_{-\ell}^{x} \sin(\rho + \frac{\nu x'}{U} - \theta_3)\tilde{\psi}_3(x', o^-, \alpha_o)dx'$$

$$= \frac{1}{\ell}\int_{-\ell}^{x} \sin(\rho + \frac{\nu x'}{U} - \theta_3)(x'+\ell)dx'$$

$$= -\frac{U}{\nu}\cos(\rho + \frac{\nu x'}{U} - \theta_3)(\frac{x'+\ell}{\ell})\Big]_{-\ell}^{x} + \frac{U}{\nu\ell}\int_{-\ell}^{x}\cos(\rho + \frac{\nu x'}{U} - \theta_3)dx'$$

$$= -\frac{U}{\nu}(\frac{x+\ell}{\ell})\cos(\nu t - \theta_3) + \frac{U^2}{\nu^2\ell}(\sin(\nu t - \theta_3) - \sin(\nu(t - \frac{x+\ell}{U}) - \theta_3))$$

Then

$$\int_{-\ell}^{x} \sin(\rho + \frac{x'}{U} - \theta_3)\psi_3(x', o, \alpha_o)dx'$$

$$= W - \frac{U}{\nu}\psi_3(-\ell, o^-, \alpha_o)(\cos(\rho + \frac{\nu x}{U} - \theta_3) - \cos(\rho - \frac{\nu\ell}{U} - \theta_3))$$

$$= -\frac{U}{\nu}\psi_3(x, o^-, \alpha_o)\cos(\nu t - \theta_3) + \frac{U^2}{\nu^2\ell}(\sin(\nu t - \theta_3) - \sin(\nu(t - \frac{x+\ell}{U}) - \theta_3))$$

$$+ \frac{U}{\nu}\psi_3(-\ell, o^-, \alpha_o)\cos(\nu(t - \frac{x+\ell}{U}) - \theta_3)$$

Let $T = \int_{-\ell}^{x} \sin(\rho + \frac{\nu x'}{U})\, \tilde{\psi}_4(x', 0^-, \alpha_o)\, dx'$

$$= 2 \int_{-\ell}^{x} \sin(\rho + \frac{\nu x'}{U}) \left[(\frac{x'+\ell}{2\ell})^2 - (\frac{x'+\ell}{2\ell}) \right] dx'$$

$$= -2 \frac{U}{\nu} \cos(\rho + \frac{\nu x'}{U}) \left[(\frac{x'+\ell}{2\ell})^2 - (\frac{x'+\ell}{2\ell}) \right]_{-\ell}^{x}$$

$$+ 2 \frac{U}{\nu} \int_{-\ell}^{x} \cos(\rho + \frac{\nu x'}{U}) \left[\frac{x+\ell}{2\ell^2} - \frac{1}{2\ell} \right] dx'$$

$$= -2 \frac{U}{\nu} \cos(\rho + \frac{\nu x}{U}) \left[(\frac{x+\ell}{2\ell})^2 - (\frac{x+\ell}{2\ell}) \right]$$

$$+ 2 \frac{U}{\nu} \int_{-\ell}^{x} \cos(\rho + \frac{\nu x'}{U}) \frac{x'}{2\ell^2} dx' = T_1 + T_2$$

$$T_2 = \frac{U^2}{\ell^2 \nu^2} \left\{ x' \sin(\rho + \frac{\nu x'}{U}) \Big]_{-\ell}^{x} - \int_{-\ell}^{x} \sin(\rho + \frac{\nu x'}{U})\, dx' \right\}$$

$$= \frac{U^2}{\ell^2 \nu^2} \left\{ x \sin(\rho + \frac{\nu x}{U}) + \ell \sin(\rho - \frac{\nu \ell}{U}) + \frac{U}{\nu} (\cos(\rho + \frac{\nu x}{U}) - \cos(\rho - \frac{\nu \ell}{U})) \right\}$$

Note that $T_1 = -\frac{U}{\nu} \cos(\rho + \frac{\nu x}{U})\, \tilde{\psi}_4$

Therefore

$$\int_{-\ell}^{x} \sin(\rho + \frac{\nu x'}{U})\, \psi_4(x', o^-, \alpha_o)\, dx'$$

$$= T_1 + T_2 - \frac{U}{\nu} \psi_4(-\ell, 0^-, \alpha_o)(\cos(\rho + \frac{\nu x}{U}) - \cos(\rho - \frac{\nu \ell}{U}))$$

$$= -\frac{U}{\nu}\psi_4(x,o^-,\alpha_o)\cos\nu t + \frac{U^2}{\ell\nu^2}\left(\frac{x}{\ell}\sin\nu t + \frac{U}{\nu\ell}\cos\nu t\right)$$

$$+ \frac{U^2}{\nu^2\ell}\left(\sin(\rho - \frac{\nu\ell}{U}) - \frac{U}{\nu\ell}\cos(\rho - \frac{\nu\ell}{U})\right) + \frac{U}{\nu}\psi_4(-\ell,o^-,\alpha_o)\cos(\rho - \frac{\nu\ell}{U}))$$

Condition (2) then becomes

where $s = t - \frac{x+\ell}{U}$

$$0 = \sum_1^4 c_{k1}\left[(J_{k,s} + \psi_k(-\ell,o^-,\alpha_o))\cos(\nu s-\theta_k) + J_{k,c}\sin(\nu s-\theta_k)\right]$$

$$+ \alpha_1 c_{k0}\left[(J^*_{k,s} + \frac{\partial\psi_k}{\partial\alpha_o}(-\ell,o^-,\alpha_o))\cos(\nu s-\theta_\alpha) + J^*_{k,c}\sin(\nu s-\theta_\alpha)\right]$$

$$- c_{31}\frac{U}{\nu\ell}\sin(\nu s-\theta_3) + c_{41}\left(\frac{U}{\nu\ell}\sin(\nu s) - (\frac{U}{\nu\ell})^2\cos(\nu s)\right)$$

Condition (4)

$$0 = \sum_{k=1}^4 \int_{-\ell}^{\ell} c_{k1}\sin(\nu(s + \frac{x'+\ell}{U}) - \theta_k)\left[\psi_k(x',o^+,\alpha_o) - \psi_k(x',o^-,\alpha_o)\right]$$

$$+ c_{k0}\alpha_1\sin(\nu(s + \frac{x'+\ell}{U}) - \theta_\alpha)\left[\frac{\partial\psi_k}{\partial\alpha_o}(x',o^+,\alpha_o) - \frac{\partial\psi_k}{\partial\alpha_o}(x',o^-,\alpha_o)\right]dx'$$

Integration along the wetted surface in the z plane is equivalent to integrating along the η axis in the ζ plane. Similarly integrating along the cavity surface in the z plane is equivalent to integrating along the ξ axis in the ζ plane. Therefore (4) can be stated as follows:

$$\sum_{1}^{4} \int_{-\infty}^{o} c_{k1}\sin(\nu(s + \frac{x'+\ell}{U})-\theta_k)\psi_k(o,\eta)+c_{k0}\alpha_1\sin(\nu(s + \frac{x'+\ell}{U})-\theta_\alpha)\frac{\partial\psi_k}{\partial\alpha_o}(o,\eta)\frac{dx'}{d\eta}\, d\eta$$

$$+ \int_{o}^{\infty} c_{k1}\sin(\nu(s+\frac{x'+\ell}{U})-\theta_k)\psi_k(\xi,o)+c_{k0}\alpha_1\sin(\nu(s+\frac{x'+\ell}{U})-\theta_\alpha)\frac{\partial\psi_k}{\partial\alpha_o}(\xi,o)\frac{dx'}{d\xi}d\xi=0$$

Let $\tilde{\psi}_k$ = non-constant part of ψ_k. Since the above equation is satisfied by the constant part, it is sufficient to consider only the non-constant parts, which are given in Appendix VI, Tables III and IV, for the wetted cavity surfaces respectively.

On inspection of these tables we see that along the wetted surface ψ_k and $\frac{\partial\psi_k}{\partial\alpha}$ all equal 0 for k=1 and 2. Furthermore for k=3 and 4, $\tilde{\psi}_k$ on the wetted surface are both polynomials in x, while on the cavity surface $\tilde{\psi}_k$ are equal to ψ_k^*, a quantity defined in Appendix I, plus the same polynomials.

Therefore condition (4) can be simplified to the following:

$$o = \sum_{1}^{4} \int_{-\infty}^{\infty} [c_{k1}\tilde{\psi}_k\sin(\nu s-\theta_k + \frac{2\Omega}{P}) + \frac{\partial\tilde{\psi}_k}{\partial\alpha} c_{k0}\alpha_1\sin(\nu s-\theta_\alpha + \frac{2\Omega}{P})]\frac{\xi(\xi^2+\alpha)}{P^2}\, d\xi$$

where $\Omega = \frac{\nu\ell}{U}$ (reduced frequency)

$$P = 1 + (\xi^2 + \alpha)^2 ,$$

and $\tilde{\psi}_k$ is replaced by ψ_k^* for k=3 and 4.

Let $I_{k,c} = \int_{-\infty}^{\infty} \tilde{\psi}_k \cos\left(\frac{2\Omega}{P}\right) \frac{\xi(\xi^2+\alpha)}{P^2}\, d\xi$

$$I_{k,s} = \int_{-\infty}^{\infty} \tilde{\psi}_k \sin\left(\frac{2\Omega}{P}\right) \frac{\xi(\xi^2+\alpha)}{P^2}\, d\xi$$

where ψ_k^* is used instead of $\tilde{\psi}_k$ for k=3 and 4.

Let $I_{k,s}^*$ be the corresponding integrals involving $\frac{\partial\psi_k}{\partial\alpha_o}$. These integrals are reduced to infinite series in Appendix IV.

Then condition (4) can be stated

$$\sum_1^4 \Big\{c_{k1}(I_{k,c}\sin(\nu s-\theta_k) + I_{k,s}\cos(\nu s-\theta_k))$$

$$+ \alpha_1 c_{k0}(I_{k,c}^*\sin(\nu s-\theta_\alpha) + I_{k,s}^*\cos(\nu s-\theta_\alpha))\Big\} = 0$$

Since the conditions must hold for all time, each condition is expressed as two linear equations, given by the coefficients of the cosine and sine of νt or νs, respectively.

Let $d_{k,c} = c_{k1}\cos\theta_k \qquad k = 1,4$

$d_{k,s} = c_{k1}\sin\theta_k$

$\alpha_c = \alpha_1\cos\theta_\alpha$

$\alpha_s = \alpha_1\sin\theta_\alpha$

Condition (1) becomes

$$\sum_{1}^{4} \left\{ d_{k,c}\ \phi_k(\infty) + c_{k0} \frac{\partial \phi_k}{\partial \alpha_o} (\infty)\ \alpha_c \right\} = 0$$

$$\sum_{1}^{4} \left\{ d_{k,s}\ \phi_k(\infty) + c_{k0} \frac{\partial \phi_k}{\partial \alpha_o} (\infty)\ \alpha_s \right\} = 0$$

Condition (2) becomes

$$\sum_{1}^{4} \left\{ d_{k,c}\ J_{k,c} + d_{k,s}(J_{k,s} + \psi_k(-\ell, o^-, \alpha_o)) + c_{k0}\ J^*_{k,c}\ \alpha_c \right.$$

$$\left. + c_{k0}\ (J^*_{k,s} + \frac{\partial \psi_k}{\partial \alpha_o} (-\ell, o^-, \alpha_o))\alpha_s \right\}$$

$$- \frac{U}{\nu \ell}\ d_{3,c} + \frac{U}{\nu \ell}\ d_{4,c} + (\frac{U}{\nu \ell})^2\ d_{4,s} = 0$$

$$\sum_{1}^{4} \left\{ -d_{k,s}\ J_{k,c} + d_{k,c}(J_{k,s} + \psi_k(-\ell, o^-, \alpha_o)) - c_{k0}\ J^*_{k,c}\ \alpha_s \right.$$

$$\left. + c_{k0}(J^*_{k,s} + \frac{\partial \psi_k}{\partial \alpha_o} (-\ell, o^-, \alpha_o))\alpha_c \right\}$$

$$+ \frac{U}{\nu \ell}\ d_{3,s} - \frac{U}{\nu \ell}\ d_{4,s} + (\frac{U}{\nu \ell})^2\ d_{4,c} = 0$$

Condition (4) becomes

$$\sum_{1}^{4} \left\{ d_{k,c} I_{k,c} + d_{k,s} I_{k,s} + c_{k0} I^*_{k,c} \alpha_c + c_{k0} I^*_{k,s} \alpha_s \right\} = 0$$

$$\sum_{1}^{4} \left\{ -d_{k,s} I_{k,c} + d_{k,c} I_{k,s} - c_{k0} I^*_{k,c} \alpha_s + c_{k0} I^*_{k,s} \alpha_c \right\} = 0$$

After solving the above set of equations for $d_{1,c}$, $d_{1,s}$, $d_{2,c}$, $d_{2,s}$, α_c, and α_s, we then can explicitly write the unsteady pressure term as:

$$\hat{\phi} = \left[\sum_{k=1}^{4} d_{k,c}\phi_k + \alpha_c\left(c_{k0} \frac{\partial\phi_k}{\partial\alpha_o}\right)\right]\cos\nu t$$

$$+\left[\sum_{k=1}^{4} d_{k,s}\phi_k + \alpha_s\left(c_{k0} \frac{\partial\phi_k}{\partial\alpha_o}\right)\right]\sin\nu t$$

From this the unsteady lift and moment can be obtained. These are explicitly developed in Section 5.

Since the closure condition was dropped from the calculation it is of interest to obtain the cavity shape. To obtain the horizontal component of velocity along the hydrofoil, needed for such a calculation, we would have to evaluate certain integrals which are not otherwise needed. Therefore the solution of this problem is postponed.

5. THE FORCE AND MOMENT ON THE HYDROFOIL (UNSTEADY CASE)

Let $g(x,t) = \phi(x,0^-,t) - \phi(x,0^+,t)$. Since $\phi = (p_\infty - p)/d$, where d is the density, the pressure difference at each point x of the hydrofoil at time t is given by $-dg(x,t)$. Thus the force f normal to the foil and the moment m (around the leading edge) are given by (see Figure 2):

$$f = -d \int_{-\ell}^{\ell} g(x,t)dx$$

$$m = -d \int_{-\ell}^{\ell} (x+\ell)g(x,t)dx.$$

The lift $= f\cos\beta$, where β is the angle of attack (given by $\cos\beta = 1/\sqrt{1+M^2}$). The moment m' around any other point x_0 is given by:

$$m'(x_0) = m - (x_0 + \ell)f.$$

Since $\phi = \phi_0 + \hat{\phi}$,

and since $\hat{\phi}$ can be expressed by

$$\hat{\phi} = \phi_c \cos\nu t + \phi_s \sin\nu t$$

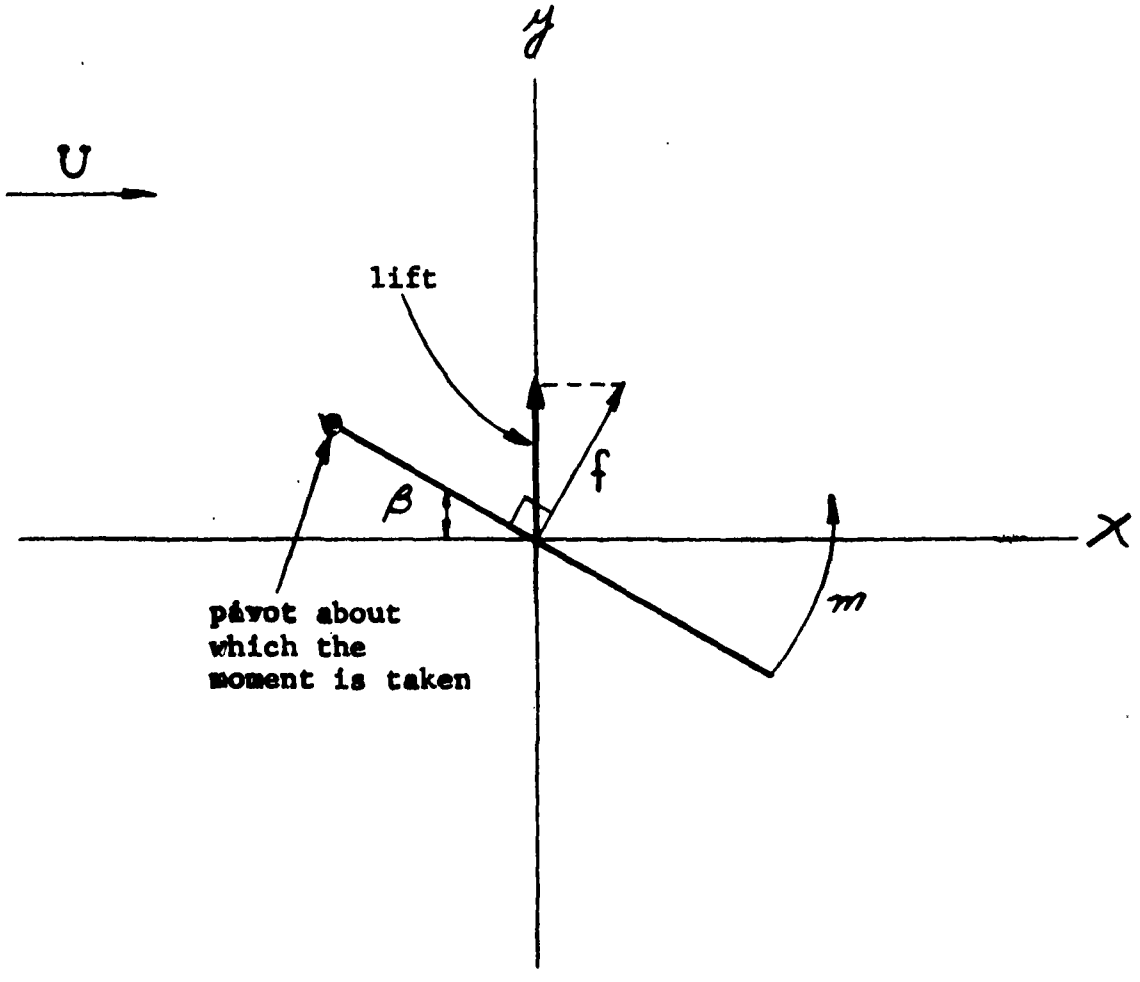
where

$$\phi_c = \sum_{k=1}^{4} d_{k,c}\phi_k + \alpha_c\left(c_{k0} \frac{\partial\phi_k}{\partial\alpha_o}\right)$$

$$\phi_s = \sum_{k=1}^{4} d_{k,s}\phi_k + \alpha_s\left(c_{k0} \frac{\partial\phi_k}{\partial\alpha_o}\right)$$

it follows that we can similarly resolve $g(x,t)$ and the resultant f and m:

FORCE AND MOMENT



Force (f) and moment (m) are positive in direction of arrow.

Figure 2

$$g(x,t) = g_o(x) + g_c(x)\cos\nu t + g_s(x)\sin\nu t$$

$$f = f_o + f_c\cos\nu t + f_s\sin\nu t$$

$$m = m_o + m_c\cos\nu t + m_s\sin\nu t$$

For problems of heave alone or pitch alone, f_c is the magnitude of the unsteady force in phase with the displacement (assume for heave alone, $\theta_B = 0$), while $-f_s$ is the magnitude of the component of force in phase with the velocity. Similarly m_c and $-m_s$ are the components of the moment in phase with the displacement and velocity. We have

$$g_o = \sum_{k=1}^{2} c_{k0}(\phi_k^- - \phi_k^+)$$

$$g_c = \sum_{k=1}^{4} d_{k,c}(\phi_k^- - \phi_k^+) + \alpha_c c_{k0}\left(\frac{\partial\phi_k^-}{\partial\alpha_o} - \frac{\partial\phi_k^+}{\partial\alpha_o}\right)$$

$$g_s = \sum_{k=1}^{4} d_{k,s}(\phi_k^- - \phi_k^+) + \alpha_s c_{k0}\left(\frac{\partial\phi_k^-}{\partial\alpha_o} - \frac{\partial\phi_k^+}{\partial\alpha_o}\right)$$

Let $R_k = 2\int_{-\ell}^{\ell}(\phi_k^- - \phi_k^+)dx$

$$S_k = 2\int_{-\ell}^{\ell}(x+\ell)(\phi_k^- - \phi_k^+)dx$$

and let R_k^*, S_k^* be defined similarly using $\frac{\partial\phi_k}{\partial\alpha_o}$ instead of ϕ_k. The integrals above can be considered as line integrals (in the (x,y) plane) over the surface of the hydrofoil in a counter clockwise direction. This is equivalent to integrating (in the (ξ,η) plane - see Figure 1) from $-\infty$ to 0

along the η axis and then integrating from 0 to ∞ along the ξ axis. Since all ϕ_k and $\frac{\partial\phi_k}{\partial\alpha_o}$ are zero along the ξ axis, R_k, S_k, R_k^* and S_k^* can be expressed by

$$R_k = 2\int_{-\infty}^{0} \phi_k(0,\eta)\,\frac{dx}{d\eta}\,d\eta$$

$$= \int_{-\infty}^{\infty} \phi_k(0,\eta)\,\frac{dx}{d\eta}\,d\eta$$

$$S_k = \int_{-\infty}^{\infty} (x+\ell)\phi_k(0,\eta)\,\frac{dx}{d\eta}\,d\eta$$

R_k^* and S_k^* are found by replacing ϕ_k by $\frac{\partial\phi_k}{\partial\alpha_o}$ in the above. Note that all ϕ_k, $\frac{\partial\phi_k}{\partial\alpha_o}$, and $\frac{dx}{d\eta}$ are odd, while $x+\ell$ is even as functions of η. For purposes of integration:

$$x + \ell = \frac{2\ell}{(\eta^2-\alpha_o)^2+1}$$

$$\frac{dx}{d\eta} = \frac{-8\ell\eta(\eta^2-\alpha_o)}{((\eta^2-\alpha_o)^2+1)^2}$$

The integrations are carried out in Appendix V.

Let $R_\alpha = \sum_{k=1}^{2} c_{k0}R_k^*$

$$S_\alpha = \sum_{k=1}^{2} c_{k0}S_k^*$$

Then

$$f_o = \frac{-d}{2} \sum_{k=1}^{2} c_{k0} R_k$$

$$f_c = \frac{-d}{2} (\alpha_c R_\alpha + \sum_{k=1}^{4} d_{k,c} R_k)$$

$$f_s = \frac{-d}{2} (\alpha_s R_\alpha + \sum_{k=1}^{4} d_{k,s} R_k)$$

and m_o, m_c, and m_s defined similarly using S_k and S_α instead of R_k and R_α. f_o, m_o, R_α, and S_α are calculated in Appendix V. The steady state terms are:

$$f_o = -\pi d \ell U^2 M \left(1 + \sqrt{1 + \frac{1}{\alpha^2}}\right)$$

$$m_o = -\frac{\pi d \ell^2 U^2 M}{4\alpha(\alpha^2+1)^{3/2}} \left(4 + \alpha\sqrt{\alpha^2+1}\,(\sqrt{\alpha^2+1} + \alpha)^2\right)$$

6. NUMERICAL ANALYSIS FOR UNSTEADY SOLUTION

The "I" integrals may be evaluated using the results of Appendix IV, while the "J" integrals require direct numerical integration, using Euler's method for increasing the rate of convergence of alternating series. Given these integrals in terms of α_o and Ω, the problem is reduced to solving the system:

$$Qv = q$$

where

$$Q = \begin{pmatrix} Q_1 & Q_2 \\ -Q_2 & Q_1 \end{pmatrix}$$

and Q_1 and Q_2 are each 3 x 3 matrices:

$$Q_1 = \begin{pmatrix} \phi_1(\infty) & \phi_2(\infty) & \phi_\alpha(\infty) \\ J_{1,s}-\tilde{\psi}_1(\infty) & J_{2,s}-\tilde{\psi}_2(\infty) & J_{\alpha,s}-\tilde{\psi}_\alpha(\infty) \\ I_{1,s} & I_{2,s} & I_{\alpha,s} \end{pmatrix}$$

$$Q_2 = \begin{pmatrix} 0 & 0 & 0 \\ J_{1,c} & J_{2,c} & J_{\alpha,c} \\ I_{1,c} & I_{2,c} & I_{\alpha,c} \end{pmatrix}$$

where

$$\phi_\alpha(\infty) = \frac{1}{U^2 M_o} \sum_1^2 c_{k0} \frac{\partial \phi_k(\infty)}{\partial \alpha}$$

$$\tilde{\psi}_\alpha(\infty) = \frac{1}{U^2 M_o} \sum_1^2 c_{k0} \frac{\partial \tilde{\psi}_k(\infty)}{\partial \alpha}$$

$$J_{\alpha,c} = \frac{1}{U^2 M_o} \sum_1^2 c_{k0} J^*_{k,c}$$

Similarly $J_{\alpha,s}$, $I_{\alpha,c}$, and $I_{\alpha,s}$ are defined in terms of $J^*_{k,s}$, $I^*_{k,c}$, and $I^*_{k,s}$. The vector v is $(d^*_{1,s}, d^*_{2,s}, \alpha_s, d^*_{1,c}, d^*_{2,c}, \alpha_c)$ and the vector q is given by:

$$q = - \sum_{k=3}^{4} \begin{bmatrix} d^*_{k,s}\phi_k(\infty) \\ d^*_{k,c}(J_{k,c} + \frac{(-1)^k}{\Omega}) + d^*_{k,s}(J_{k,s} + \frac{3-k}{\Omega^2} - \tilde{\psi}_k(\infty)) \\ d^*_{k,c}I_{k,c} + d^*_{k,s}I_{k,s} \\ d^*_{k,c}\phi_k(\infty) \\ d^*_{k,c}(J_{k,s} + \frac{3-k}{\Omega^2} - \tilde{\psi}_k(\infty)) - d^*_{k,s}(J_{k,c} + \frac{(-1)^k}{\Omega}) \\ d^*_{k,c}I_{k,s} - d^*_{k,s}I_{k,c} \end{bmatrix}$$

where

$$d^*_{k,c} = \frac{d_{k,c}}{U^2 M_o}$$

$$d^*_{k,s} = \frac{d_{k,s}}{U^2 M_o}$$

Since any solution to a problem of pitching and heaving can be expressed as a combination (taking into account phase) of pitch alone and heave alone, it is necessary to find only the solutions for unit amplitude pitch alone and unit amplitude heave alone. Let q_p and q_h be the unit pitch and heave vectors respectively, given by $M_1 = M_o$ for pitch alone and $B_1 = \ell M_o$ for heave alone.

For pitch alone:

$$c_{31} = -2\nu\ell M_1 U = -2\Omega(M_1 U^2)$$

$$\theta_3 = \pi/2$$

$$c_{41} = -\nu^2\ell^2 M_1 = -\Omega^2(M_1 U^2)$$

Therefore (for $M_1 = M_o$),

$$d^*_{3,c} = 0$$

$$d^*_{3,s} = -2\Omega$$

$$d^*_{4,c} = -\Omega^2$$

$$d^*_{4,s} = 0$$

$$q_p = \begin{bmatrix} 2\Omega\phi_3(\infty) \\ \Omega^2(J_{4,c} + \frac{1}{\Omega}) + 2\Omega(J_{3,s} - \tilde{\psi}_3(\infty)) \\ \Omega^2 I_{4,c} + 2\Omega I_{3,s} \\ \Omega^2\phi_4(\infty) \\ \Omega^2(J_{4,s} - \frac{1}{\Omega^2} - \tilde{\psi}_4(\infty)) - 2\Omega(J_{3,c} - \frac{1}{\Omega}) \\ \Omega^2 I_{4,s} - 2\Omega I_{3,c} \end{bmatrix}$$

For heave alone:

$$c_{31} = -\nu^2 \ell B_1 = -\Omega^2(\frac{B_1}{\ell} U^2)$$

$$\theta_3 = 0$$

$$c_{41} = 0$$

Therefore (for $B_1 = \ell M_o$)

$$d^*_{3,c} = -\Omega^2$$

$$d^*_{3,s} = d^*_{4,c} = d^*_{4,s} = 0$$

$$q_h = \begin{bmatrix} 0 \\ \Omega^2(J_{3,c} - \frac{1}{\Omega}) \\ \Omega^2 I_{3,c} \\ \Omega^2 \phi_3(\infty) \\ \Omega^2(J_{3,s} - \tilde{\psi}_3(\infty)) \\ \Omega^2 I_{3,s} \end{bmatrix}$$

Let

$$v_s = (d^*_{1,s}, d^*_{2,s}, \alpha_s)$$

$$v_c = (d^*_{1,c}, d^*_{2,c}, \alpha_c)$$

and let $v_{s,p}$, $v_{c,p}$ be the unit solutions for pitch alone and $v_{s,h}$, $v_{c,h}$ the unit solutions for heave alone. Let $\tilde{v}_{s,h}$, $\tilde{v}_{c,h}$ be the unit solutions for heave alone and $\theta_3 = \pi/2$. Then $\tilde{v}_{c,h} = -v_{s,h}$ and $\tilde{v}_{s,h} = v_{c,h}$. The unit solution for heave alone at an arbitrary phase θ_B is given by

$$v_s = v_{s,h}\cos\theta_B + v_{c,h}\sin\theta_B$$

$$v_c = v_{c,h}\cos\theta_B - v_{s,h}\sin\theta_B$$

Therefore for unsteady pitch $= M_1\cos\nu t$ and unsteady heave $= B_1\cos(\nu t-\theta_B)$

$$v_s = \frac{M_1}{M_o} v_{s,p} + \frac{B_1}{M_o \ell} (v_{s,h}\cos\theta_B + v_{c,h}\sin\theta_B)$$

$$v_c = \frac{M_1}{M_o} v_{c,p} + \frac{B_1}{M_o \ell} (v_{c,h}\cos\theta_B - v_{s,h}\sin\theta_B)$$

APPENDIX I - ψ_3, ϕ_3, ψ_4, ϕ_4

Calculation of ψ_3, ϕ_3, ψ_4, ϕ_4

A. General

$$\phi_k = \frac{-1}{2\pi\ell^{k-2}} \int_{\substack{\text{wetted}\\ \text{surface}}} x^{k-3} G(\zeta,\zeta')\,dx$$

$$\text{Where } G = \mathrm{Re}\left(\ell n\left(\frac{\zeta'-\zeta}{\zeta'+\zeta} \cdot \frac{\zeta'+\bar{\zeta}}{\zeta'-\bar{\zeta}}\right)\right)$$

and $$\zeta' = \sqrt{\sqrt{\frac{\ell-z}{\ell+z}} - \alpha}$$

Along wetted surface ζ' is imaginary

$$\zeta' = i\sqrt{\alpha - \sqrt{\frac{\ell-\eta}{\ell+\eta}}} = iu$$

and $$x = \frac{2\ell}{1+(u^2-\alpha)^2} - \ell$$

Therefore $G = \ell n\left(\frac{u+i\zeta}{u-i\zeta} \cdot \frac{u-i\bar{\zeta}}{u+i\bar{\zeta}}\right)$, since argument of ℓn is a positive real number.

As a result, $G = F(\zeta) + \bar{F}(\zeta)$

where $F(\zeta) = \ell n\left(\frac{u+i\zeta}{u-i\zeta}\right)$.

Since as a function of ζ, the ϕ_k are the imaginary parts of an analytic function of ζ, where ψ_k are the real parts, each ϕ_k can be expressed as:

$$\phi_k = \frac{1}{2i} \quad (f_k - \bar{f}_k)$$

Let $H(\zeta) = 2iF(\zeta)$.

Then $G(\zeta) = \frac{1}{2i} (H(\zeta) - \bar{H}(\zeta))$.

Therefore $f_k' = \frac{-1}{2\pi\ell^{k-2}} \int_{\substack{\text{wetted}\\ \text{surface}}} x^{k-3} H(\zeta, iu)\, dx$

$$= \frac{-i}{\pi\ell^{k-2}} \int x^{k-3} \ln\left(\frac{u+i\zeta}{u-i\zeta}\right) dx$$

where $f_k' = \psi_k + i\phi_k$, being determined only up to an additive constant, so that $\psi_k(x = -\infty) = 0$.

The wetted surface of the hydrofoil corresponds to the interval $(0,\infty)$ of u.

Therefore $f_k' = \frac{-i}{\pi\ell^{k-2}} \int_0^\infty x^{k-3} \frac{dx}{du} \ln\left(\frac{u+i\zeta}{u-i\zeta}\right) du$

$$= -i \left. \frac{(x^{k-2} - (-\ell)^{k-2})}{(k-2)\pi\ell^{k-2}} \ln\left(\frac{u+i\zeta}{u-i\zeta}\right) \right]_0^\infty$$

$$+ \frac{i}{(k-2)\pi\ell^{k-2}} \int_0^\infty x^{k-2} - (-\ell)^{k-2} \left(\frac{1}{u+i\zeta} - \frac{1}{u-i\zeta}\right) du$$

At $u = \infty$, $x = -\ell$, $\ln\left(\frac{u+i\zeta}{u-i\zeta}\right) = 0$

at $u = 0$, $x = \ell \frac{1-\alpha^2}{1+\alpha^2}$, $\quad \ln\left(\frac{u+i\zeta}{u-i\zeta}\right) = \pi i$

Since ψ_k is only known up to an additive constant, the first term for f_k' may be dropped.

Let $\tilde{\psi}_k$ be the non-constant part of ψ_k.

Since $\frac{1}{u+i\zeta} - \frac{1}{u-i\zeta} = \frac{2i\zeta}{u^2+\zeta^2}$, the integrand is even and f_k (= f'_k - above constant) can be represented by:

$$f_k = \frac{i}{2(k-2)\pi} \int_{-\infty}^{\infty} \left\{ \left(\frac{2}{1+(u^2-\alpha)^2} - 1\right)^{k-2} - (-1)^{k-2} \right\} \left(\frac{1}{u+i\zeta} - \frac{1}{u-i\zeta}\right) du$$

In particular:

$$f_3 = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+(u^2-\alpha)^2} \left(\frac{1}{u+i\zeta} - \frac{1}{u-i\zeta}\right) du$$

$$f_4 = \frac{i}{\pi} \int_{-\infty}^{\infty} \left[\left(\frac{1}{1+(u^2-\alpha)^2}\right)^2 - \frac{1}{1+(u^2-\alpha)^2} \right] \left(\frac{1}{u-i\zeta} - \frac{1}{u-i\zeta}\right) du$$

Let $t = A+Bi$, A and B positive.

where $t^2 = \alpha + i$

Then $1 + (u^2-\alpha)^2 = (u^2-t^2)(u^2-\bar{t}^2)$

Since $\zeta = \xi+i\eta$, $i\zeta = -\eta+i\xi$, where $\xi > 0$.

Therefore for purposes of determining residues to evaluate f_3 and f_4 by contour integration, the upper half plane poles are t, $-\bar{t}$, and $i\zeta$.

Since $$\frac{1}{(u^2-t^2)(u^2-\bar{t}^2)} = \frac{1}{t^2-\bar{t}^2} \left(\frac{1}{u^2-t^2} - \frac{1}{u^2-\bar{t}^2}\right)$$

$$= \frac{1}{2i} \left(\frac{1}{u^2-t^2} - \frac{1}{u^2-\bar{t}^2}\right)$$

$$f_3 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1}{u^2-t^2} - \frac{1}{u^2-\bar{t}^2}\right) \left(\frac{1}{u+i\zeta} - \frac{1}{u-i\zeta}\right) du$$

$$f_4 = -\frac{i}{\pi} \int_{-\infty}^{\infty} \left[\frac{1}{4} \left(\frac{1}{u^2-t^2} - \frac{1}{u^2-\bar{t}^2}\right)^2 + \frac{1}{(u^2-t^2)(u^2-\bar{t}^2)}\right] \left(\frac{1}{u+i\zeta} - \frac{1}{u-i\zeta}\right) du$$

$$= \frac{-i}{4\pi} \int_{-\infty}^{\infty} \left(\frac{1}{u^2-t^2} + \frac{1}{u^2-\bar{t}^2}\right)^2 \left(\frac{1}{u+i\zeta} - \frac{1}{u-i\zeta}\right) du$$

In addition to the general expressions, the following special cases are desired

1) at $x = \infty$ (or ξ=B and η=-A), Φ and $\tilde{\psi}$,

since the desired ψ is 0 at this point and the function must be adjusted accordingly.

2) at $\xi = 0$, (the wetted surface) Φ and $\tilde{\psi}$

3) at $\eta = 0$, (the cavity surface), $\tilde{\psi}$, and separated into odd and even functions of ξ.

4) $\eta^2 = \xi^2 + \alpha$, (for $J_{k,c}$ and $J_{k,s}$ integration) $\tilde{\psi}$

B f_3 Calculation

$$f_3 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1}{2t}\left(\frac{1}{u-t} - \frac{1}{u+t}\right) - \frac{1}{2\bar{t}}\left(\frac{1}{u-\bar{t}} - \frac{1}{u+\bar{t}}\right)\right)\left(\frac{1}{u+i\zeta} - \frac{1}{u-i\zeta}\right) du$$

$$= i \left[\frac{1}{2t} \left\{\frac{2}{t+i\zeta}\right\} - \frac{1}{2\bar{t}} \left\{\frac{-2}{i\zeta-\bar{t}}\right\}\right]$$

$$= \frac{i}{t\bar{t}} \left[\frac{\bar{t}(\bar{t}-i\bar{\zeta})}{(t+i\zeta)(\bar{t}-i\bar{\zeta})} - \frac{t(t+i\bar{\zeta})}{(\bar{t}-i\zeta)(t+i\bar{\zeta})}\right]$$

$$|t+i\zeta|^2 = |A+Bi-\eta+i\xi|^2 = (A-\eta)^2 + (B+\xi)^2$$

$$|t+i\bar{\zeta}|^2 = |A+Bi+\eta+i\xi|^2 = (A+\eta)^2 + (B+\xi)^2$$

$$t(t+i\bar{\zeta}) = (A+Bi)(A+\eta+i(B+\xi)) = A(A+\eta)-B(B+\xi)+i(2AB+A\xi+B\eta)$$

$$= \alpha + A\eta-B\xi+i(1+A\xi+B\eta)$$

$$\bar{t}(\bar{t}-i\bar{\zeta}) = (A-Bi)(A-\eta-i(B+\xi)) = \alpha - A\eta-B\xi-i(1+A\xi-B\eta)$$

$$t\bar{t} = \sqrt{\alpha^2+1}$$

Therefore $$f_3 = \frac{1}{\sqrt{\alpha^2+1}}\left(\frac{1 + A\xi - B\eta}{(A-\eta)^2 + (B+\xi)^2} + \frac{1 + A\xi + B\eta}{(A+\eta)^2 + (B+\xi)^2}\right)$$

$$+ \frac{i}{\sqrt{\alpha^2+1}}\left[\frac{\alpha-A\eta-B\xi}{(A-\eta)^2 + (B+\xi)^2} - \frac{\alpha + A\eta - B\xi}{(A+\eta)^2 + (B+\xi)^2}\right]$$

Therefore $$\tilde{\psi}_3 = \frac{1 + A\xi}{\sqrt{\alpha^2+1}}\left(\frac{1}{(A-\eta)^2 + (B+\xi)^2} + \frac{1}{(A+\eta)^2 + (B+\xi)^2}\right)$$

$$- \frac{B\eta}{\sqrt{\alpha^2+1}}\left(\frac{1}{(A-\eta)^2 + (B+\xi)^2} - \frac{1}{(A+\eta)^2 + (B+\xi)^2}\right)$$

$$\phi_3 = \frac{\alpha-B\xi}{\sqrt{\alpha^2+1}}\left(\frac{1}{(A-\eta)^2 + (B+\xi)^2} - \frac{1}{(A+\eta)^2 + (B+\xi)^2}\right)$$

$$- \frac{A\eta}{\sqrt{\alpha^2+1}}\left(\frac{1}{(A-\eta)^2 + (B+\xi)^2} + \frac{1}{(A+\eta)^2 + (B+\xi)^2}\right)$$

Special cases

1) $$\tilde{\psi}_3 = \frac{3}{2\sqrt{\alpha^2+1}}\left(\frac{1}{4(A^2+B^2)} + \frac{1}{4B^2}\right) + \frac{1}{2\sqrt{\alpha^2+1}}\left(\frac{1}{4(A^2+B^2)} - \frac{1}{4B^2}\right)$$

$$= \frac{1}{2(\alpha^2+1)} + \frac{1}{2\sqrt{\alpha^2+1}(\sqrt{\alpha^2+1} - \alpha)}$$

$$= \frac{2\sqrt{\alpha^2+1} - \alpha}{2(\alpha^2+1)(\sqrt{\alpha^2+1} - \alpha)}$$

$$= \frac{\alpha^2+2 + \alpha\sqrt{\alpha^2+1}}{2(\alpha^2+1)}$$

$$\phi_3 = \frac{3\alpha - \sqrt{\alpha^2+1}}{2\sqrt{\alpha^2+1}}\left(\frac{1}{4\sqrt{\alpha^2+1}} - \frac{1}{4B^2}\right) + \frac{\sqrt{\alpha^2+1} + \alpha}{2\sqrt{\alpha^2+1}}\left(\frac{1}{4\sqrt{\alpha^2+1}} + \frac{1}{4B^2}\right)$$

$$= \frac{\alpha}{2\sqrt{\alpha^2+1}} + \frac{2\sqrt{\alpha^2+1} - 2\alpha}{4\sqrt{\alpha^2+1}\,(\sqrt{\alpha^2+1} - \alpha)}$$

$$= \frac{\alpha}{2(\alpha^2+1)} + \frac{1}{2\sqrt{\alpha^2+1}}$$

$$= \frac{\alpha + \sqrt{\alpha^2+1}}{2(\alpha^2+1)}$$

2) $$\tilde{\psi}_3 = \frac{1}{\sqrt{\alpha^2+1}}\left(\frac{1}{B^2 + (A-\eta)^2} + \frac{1}{B^2 + (A+\eta)^2}\right) - \frac{B\eta}{\alpha^2+1}\left(\frac{1}{B^2 + (A-\eta)^2} - \frac{1}{B^2 + (A+\eta)^2}\right)$$

$$= \frac{2(B^2+A^2+\eta^2) - 4AB\eta^2}{\sqrt{\alpha^2+1}((A^2+B^2+\eta^2)^2 - 4A^2\eta^2)}$$

$$\tilde{\psi}_3 = \frac{2}{\alpha^2+1+\eta^4-2\eta^2\alpha}$$

$$= \frac{2}{(\eta^2-\alpha)^2+1} = \frac{x+\ell}{\ell}$$

$$\Phi_3 = \frac{\alpha}{\sqrt{\alpha^2+1}}\left(\frac{1}{B^2+(A-\eta)^2} - \frac{1}{B^2+(A+\eta)^2}\right) - \frac{A\eta}{\sqrt{\alpha^2+1}}\left(\frac{1}{B^2+(A-\eta)^2} + \frac{1}{B^2+(A+\eta)^2}\right)$$

$$= \frac{1}{\sqrt{\alpha^{12}+1}\ ((\eta^2-\alpha)^2+1)}\left(4A\eta\alpha + 2A\eta\ (\sqrt{\alpha^2+1} + \eta^2)\right)$$

$$= \frac{2A\eta(2\alpha - \sqrt{\alpha^2+1} - \eta^2)}{\sqrt{\alpha^2+1}\ ((\eta^2+\alpha)^2+1)}$$

3) $$\tilde{\psi}_3 = \frac{2(1+A\xi)}{\sqrt{\alpha^2+1}\ ((B+\xi)^2+A^2)}$$

$$= \frac{2(1+A\xi)}{\sqrt{\alpha^2+1}\ (\sqrt{\alpha^2+1} + \xi^2+2B\xi)} \cdot \frac{(\sqrt{\alpha^2+1} + \xi^2-2B\xi)}{(\sqrt{\alpha^2+1} + \xi^2-2B\xi)}$$

$$= \frac{2\left[\sqrt{\alpha^2+1} + \xi^2 - 2AB\xi^2 + A\xi\ (\sqrt{\alpha^2+1} + \xi^2 - 4B^2)\right]}{\sqrt{\alpha^2+1}\ ((\xi^2 + \sqrt{\alpha^2+1})^2 - 2(\sqrt{\alpha^2+1} - \alpha)\xi^2)}$$

$$= \frac{2}{(\xi^4+\alpha^2+1+2\alpha\xi^2)} + \frac{2A\xi(\xi^2+2\alpha - \sqrt{\alpha^2+1})}{\sqrt{\alpha^2+1}((\xi^2+\alpha)^2+1)}$$

$$= \frac{x+\ell}{\ell} - \psi_3^*$$

where $\psi_3^* = \dfrac{2A\xi(\xi^2 + 2\alpha - \sqrt{\alpha^2+1})}{\sqrt{\alpha^2+1}(1+(\xi^2+\alpha)^2)}$

4) Let $D = ((A-\eta)^2+(B+\xi)^2)((A+\eta)^2+(B+\xi)^2) = (A^2-\eta^2)^2+(B+\xi)^4+2(B+\xi)^2(A^2+\eta^2)$

Substitute $\xi^2 + \alpha$ for η^2

$$D = (B+\xi)^4 + 2(B+\xi)^2(A^2+\xi^2+\alpha) + (A^2-\alpha-\xi^2)^2$$

$$= B^4 + 4B^3\xi + 6B^2\xi^2 + 4B\xi^3 + \xi^4 + 2(B^2+2B\xi+\xi^2)(\xi^2+\alpha+A^2)$$

$$+ A^4 + \alpha^2 + \xi^4 + 2\alpha\xi^2 - 2A^2\alpha - 2A^2\xi^2$$

$$= \alpha^2 + 1 + \alpha^2 - 2\alpha^2 + 4B\xi(\alpha+\sqrt{\alpha^2+1}) + \xi^2(6B^2+2\alpha+2A^2+2B^2+2\alpha-2A^2)$$

$$+ 8B\xi^2 + 4\xi^4$$

$$= 1 + 4B\xi(\alpha+\sqrt{\alpha^2+1}) + \xi^2(4\sqrt{\alpha^2+1}) + 8B\xi^3 + 4\xi^4$$

$$\tilde{\psi}_3 = \frac{2\,((1+A\xi)(\sqrt{\alpha^2+1} + 2\xi^2+\alpha+2B\xi) - (\xi^2+\alpha))}{\sqrt{1+\alpha^2}(1+4B\xi(\alpha+\sqrt{\alpha^2+1}) + 4\xi^2(\sqrt{\alpha^2+1}) + 8B\xi^3 + 4\xi^4)}$$

$$= \frac{2(2\xi^2+\sqrt{\alpha^2+1} + 2A(\sqrt{\alpha^2+1}-\alpha)\xi + A\xi(\sqrt{\alpha^2+1}+\alpha+2\xi^2))}{\sqrt{1+\alpha^2}\,(1+4B\xi(\alpha+\sqrt{\alpha^2+1}) + 4\xi^2\sqrt{\alpha^2+1} + 8B\xi^3 + 4\xi^4}$$

$$= \frac{2(2\xi^2+\sqrt{\alpha^2+1} + A\xi(3\sqrt{\alpha^2+1}-\alpha) + 2A\xi^3)}{\sqrt{1+\alpha^2}(1+4B\xi(\alpha+\sqrt{\alpha^2+1}) + 4\xi^2\sqrt{\alpha^2+1} + 8B\xi^3 + 4\xi^4)}$$

$$\tilde{\psi}_3 = \frac{2}{\sqrt{1+\alpha^2}}\left[\frac{(\sqrt{\alpha^2+1}+2\xi^2+A\xi(3\sqrt{\alpha^2+1}-\alpha+2\xi^2))(1+4\xi^2\sqrt{\alpha^2+1}+4\xi^4-4B\xi(\alpha+\sqrt{\alpha^2+1}+2\xi^2))}{(1+4\xi^2\sqrt{\alpha^2+1}+4\xi^4)^2-16B^2\xi^2(\alpha+\sqrt{\alpha^2+1}+2\xi^2)^2}\right]$$

$$\frac{2\ell}{x+\ell} = 1 - 4\alpha\xi^2 - 4\xi^4$$

$$(1+4\xi^2\sqrt{\alpha^2+1}+4\xi^4)^2 = (4\xi^4+4\alpha\xi^2-1+4(\sqrt{\alpha^2+1}-\alpha)\xi^2+2)^2$$

$$= (4\xi^4+4\alpha\xi^2-1)^2 + 4(4\xi^4+4\alpha\xi^2-1)(2(\sqrt{\alpha^2+1}-\alpha)\xi^2+1)$$

$$+ 4(2(\sqrt{\alpha^2+1}-\alpha)\xi^2+1)^2$$

$$= (4\xi^4+4\alpha\xi^2-1)^2 + 4(2(\sqrt{\alpha^2+1}-\alpha)\xi^2+1)(4\xi^4+2(\sqrt{\alpha^2+1}+\alpha)\xi^2)$$

$$= (4\xi^4+4\alpha\xi^2-1)^2 + 8\,\xi^2(\sqrt{\alpha^2+1}-\alpha)(2\xi^2+\sqrt{\alpha^2+1}+\alpha)^2$$

$$\tilde{\psi}_3 = \frac{2}{\sqrt{1+\alpha^2}}\left(\frac{x+\ell}{2\ell}\right)^2\Big[(\sqrt{\alpha^2+1}+2\xi^2)(1+4\xi^2\sqrt{\alpha^2+1}+4\xi^4) - 2\xi^2(3\sqrt{\alpha^2+1}-\alpha+2\xi^2)(\alpha+\sqrt{\alpha^2+1}+2\xi^2)$$

$$+ A\xi((3\sqrt{\alpha^2+1}-\alpha+2\xi^2)(1+4\sqrt{\alpha^2+1}\,\xi^2+4\xi^4) - 4(\sqrt{\alpha^2+1}-\alpha)(\alpha+\sqrt{\alpha^2+1}+2\xi^2)(\sqrt{\alpha^2+1}+2\xi^2)$$

$$= \frac{2}{\sqrt{1+\alpha^2}}\left(\frac{x+\ell}{2\ell}\right)^2\left(\sqrt{\alpha^2+1}+\xi^2(2+4(\alpha^2+1)-2(3(\alpha^2+1)-\alpha^2+2\alpha\sqrt{\alpha^2+1})\right)$$

$$+ \xi^4(4\sqrt{\alpha^2+1}+8\sqrt{\alpha^2+1}-4(4\sqrt{\alpha^2+1}))$$

$$+ A\xi(3\sqrt{\alpha^2+1} - \alpha - 4\sqrt{\alpha^2+1} + \xi^2(12(\alpha^2+1) - 4\alpha\sqrt{\alpha^2+1} + 2 - 8(1+\alpha^2+1-\alpha\sqrt{\alpha^2+1}))$$

$$+ \xi^4(12\sqrt{\alpha^2+1} - 4\alpha + 8\sqrt{\alpha^2+1} - 16(\sqrt{\alpha^2+1} - \alpha)) + 8\xi^6)$$

$$\tilde{\psi}_3 = \frac{2}{\sqrt{1+\alpha^2}} \left(\frac{x+\ell}{2\ell}\right)^2 (\sqrt{\alpha^2+1}(1 - 4\alpha\xi^2 - 4\xi^4)$$

$$+ A\xi(-\sqrt{\alpha^2+1} - \alpha + \xi^2(4\alpha^2 - 2 + 4\alpha\sqrt{\alpha^2+1}) + \xi^4(4\sqrt{\alpha^2+1} + 12\alpha) + 8\xi^6)$$

$$= \frac{1}{\sqrt{1+\alpha^2}} \left(\frac{x+\ell}{\ell}\right) (\sqrt{\alpha^2+1} - A\xi(2\xi^2 + (\sqrt{\alpha^2+1} + \alpha))$$

C. f_4 Calculation

$$\left(\frac{1}{u^2-t^2} + \frac{1}{u^2-\bar{t}^2}\right)^2 = \left(\frac{1}{2t}\left(\frac{1}{u-t} - \frac{1}{u+t}\right) + \frac{1}{2\bar{t}}\left(\frac{1}{u-\bar{t}} - \frac{1}{u+\bar{t}}\right)\right)^2$$

$$= \frac{1}{4t^2}\left(\frac{1}{(u-t)^2} + \frac{1}{(u+t)^2} - \frac{1}{t}\left(\frac{1}{u-t} - \frac{1}{u+t}\right)\right)$$

$$+ \frac{1}{4\bar{t}^2}\left(\frac{1}{(u-\bar{t})^2} + \frac{1}{(u+\bar{t})^2} - \frac{1}{\bar{t}}\left(\frac{1}{u-\bar{t}} - \frac{1}{u-\bar{t}}\right)\right)$$

$$+ \frac{1}{2t\bar{t}}\left(\frac{1}{t-\bar{t}}\left(\frac{1}{u-t} - \frac{1}{u-\bar{t}} + \frac{1}{u+\bar{t}} - \frac{1}{u+t}\right)\right)$$

$$+ \frac{1}{t+\bar{t}}\left(\frac{1}{u+\bar{t}} - \frac{1}{u-t} + \frac{1}{u+t} - \frac{1}{u-\bar{t}}\right)$$

Let $f^* = \frac{-i}{16\pi} \int_{-\infty}^{\infty} \left\{\frac{1}{t^2}\left(\frac{1}{(u-t)^2} + \frac{1}{(u+t)^2}\right) + \frac{1}{\bar{t}^2}\left(\frac{1}{(u-\bar{t})^2} + \frac{1}{(u+\bar{t})^2}\right)\right\}\left(\frac{1}{u+i\zeta} - \frac{1}{u-i\zeta}\right) du$

Let $h^* = f_4 - f^*$

$$h^* = \frac{-i}{4\pi} \int_{-\infty}^{\infty} \left\{ \frac{1}{(u+i\zeta)(u-t)} \left(\frac{-1}{4t^3} + \frac{1}{2t\bar{t}} \left(\frac{1}{t-\bar{t}} - \frac{1}{t+\bar{t}} \right) \right) \right.$$

$$+ \frac{1}{(u+i\zeta)(u+\bar{t})} \left(\frac{1}{4\bar{t}^3} + \frac{1}{2t\bar{t}} \left(\frac{1}{t-\bar{t}} + \frac{1}{t+\bar{t}} \right) \right)$$

$$+ \frac{1}{(u-i\zeta)(u+t)} \left(\frac{-1}{4t^3} + \frac{1}{2t\bar{t}} \left(\frac{1}{t-\bar{t}} - \frac{1}{t+\bar{t}} \right) \right)$$

$$\left. + \frac{1}{(u-i\zeta)(u-\bar{t})} \left(\frac{1}{4\bar{t}^3} + \frac{1}{2t\bar{t}} \left(\frac{1}{t-\bar{t}} + \frac{1}{t+\bar{t}} \right) \right) \right\} du$$

$$= \frac{1}{t+i\zeta} \left(\frac{-1}{4t^3} + \frac{1}{t(t^2-\bar{t}^2)} \right) + \frac{1}{i\zeta-\bar{t}} \left(\frac{1}{4\bar{t}^3} + \frac{1}{\bar{t}(t^2-\bar{t}^2)} \right)$$

$$f^* = \frac{-1}{8} \left(\frac{1}{t^2(i\zeta+t)^2} + \frac{1}{\bar{t}^2(i\zeta-\bar{t})^2} \right) + \frac{-i}{16\pi} \int_{-\infty}^{\infty} \left[\frac{1}{t^2(u-t)^2} + \frac{1}{\bar{t}^2(u+\bar{t})^2} \right] \left(\frac{1}{u+i\zeta} - \frac{1}{u+i\zeta} \right) du$$

$$= m^* + n^*$$

$$n^* = \frac{-i}{16\pi} \int_{-\infty}^{\infty} \left\{ \frac{1}{t^2(u-t)} \left(\frac{1}{t+i\zeta} \left(\frac{1}{u-t} - \frac{1}{u+i\zeta} \right) \right) - \frac{1}{t-i\zeta} \left(\frac{1}{u-t} - \frac{1}{u-i\zeta} \right) \right)$$

$$\left. + \frac{1}{\bar{t}^2(u+\bar{t})} \left(\frac{1}{\bar{t}-i\zeta} \left(\frac{-1}{u+\bar{t}} + \frac{1}{u+i\zeta} \right) - \frac{1}{\bar{t}+i\zeta} \left(\frac{-1}{u+\bar{t}} + \frac{1}{u-i\zeta} \right) \right) \right\} du$$

$$n^* = \frac{1}{8} \left(\frac{-1}{t^2(t+i\zeta)^2} - \frac{1}{\bar{t}^2(i\zeta-\bar{t})^2} \right)$$

Therefore $f^* = -\frac{1}{4} \left(\frac{1}{t^2(t+i\zeta)^2} + \frac{1}{\bar{t}^2(\bar{t}-i\zeta)^2} \right)$

$$h^* = -\frac{1}{4}\left(\frac{1}{t^3(t+i\zeta)} + \frac{1}{\bar{t}^3(\bar{t}-i\zeta)}\right) - \frac{i}{2}\left(\frac{1}{t(t+i\zeta)} - \frac{1}{\bar{t}(\bar{t}-i\zeta)}\right)$$

$$= p^* + q^*$$

$$f^* = \frac{-1}{4(t\bar{t})^2}\left[\frac{\bar{t}^2(t-i\bar{\zeta})^2}{((t+i\zeta)(\bar{t}+i\bar{\zeta}))^2} + \frac{t^2(t+i\bar{\zeta})^2}{((\bar{t}-i\zeta)(t+i\bar{\zeta}))^2}\right]$$

$$= \frac{1}{4(\alpha^2+1)}\left[\frac{[\alpha-A\eta-B\xi-i(1+A\xi-B\eta)]^2}{[(A-\eta)^2+(B+\xi)^2]^2} + \frac{[\alpha+A\eta-B\xi+i(1+A\xi+B\eta)]^2}{[(A+\eta)^2+(B+\xi)^2]^2}\right.$$

$$= \frac{1}{4(\alpha^2+1)}\left(\frac{(\alpha-A\eta-B\xi)^2-(1+A\xi-B\eta)^2}{((A-\eta)^2+(B+\xi)^2)^2} + \frac{(\alpha+A\eta-B\xi)^2-(1+A\xi+B\eta)^2}{((A+\eta)^2+(B+\xi)^2)^2}\right)$$

$$+ \frac{i}{2(\alpha^2+1)}\left(\frac{(\alpha-A\eta-B\xi)(1+A\xi-B\eta)}{((A-\eta)^2+(B+\xi)^2)^2} + \frac{(\alpha+A\eta-B\xi)(1+A\xi+B\eta)}{((A+\eta)^2+(B+\xi)^2)^2}\right)$$

$$f^* = -\frac{1}{4(\alpha^2+1)}\left(\frac{\alpha^2-1+\alpha(\eta^2-\xi^2)+2\eta\xi-2\xi(A+\alpha B)+\xi\eta(B-\alpha A)}{((A-\eta)^2+(B+\xi)^2)^2}\right.$$

$$\left.+\frac{\alpha^2-1+\alpha(\eta^2-\xi^2)-2\eta\xi-2\xi(A+\alpha B)-2\eta(B-\alpha A)}{((A+\eta)^2+(B+\xi)^2)^2}\right)$$

$$+\frac{i}{4(\alpha^2+1)}\left(\frac{2\alpha+2(\alpha A-B)\xi-2(\alpha B+A)\eta-2\alpha\eta\xi+\eta^2-\xi^2}{((A-\eta)^2+(B+\xi)^2)^2}\right.$$

$$\left.-\frac{2\alpha+2(\alpha A-B)\xi+2(\alpha B+A)\eta+2\alpha\eta\xi+\eta^2-\xi^2}{((A+\eta)^2+(B+\xi)^2)^2}\right)$$

$$p^* = \frac{-1}{4(t\bar{t})^3}\left(\frac{\bar{t}^3(\bar{t}-i\bar{\zeta})}{(t+i\zeta)(\bar{t}-i\bar{\zeta})} + \frac{t^3(t+i\bar{\zeta})}{(\bar{t}-i\zeta)(t+i\bar{\zeta})}\right)$$

$$= \frac{1}{4(\alpha^2+1)^{3/2}}\left(\frac{(\alpha-i)(\alpha-A\eta-B\xi-i(1+A\xi-B\eta))}{(A-\eta)^2+(B+\xi)^2} + \frac{(\alpha+i)(\alpha+A\eta-B\xi+i(1+A\xi+B\eta))}{(A+\eta)^2+(B+\xi)^2}\right)$$

$$= \frac{-1}{4(\alpha^2+1)^{3/2}}\left(\frac{\alpha^2-1-\alpha(A\eta+B\xi)-A\xi+B\eta}{(A-\eta)^2+(B+\xi)^2} + \frac{\alpha^2-1+\alpha(A\eta-B\xi)-A\xi-B\eta}{(A+\eta)^2+(B+\xi)^2}\right)$$

$$\frac{-i}{4(\alpha^2+1)^{3/2}}\left(\frac{A\eta+B\xi-2\alpha-\alpha(A\xi-B\eta)}{(A-\eta)^2+(B+\xi)^2} + \frac{A\eta-B\xi+2\alpha+\alpha(A\xi+B\eta)}{(A+\eta)^2+(B+\xi)^2}\right)$$

$$q^* = \frac{i}{2t\bar{t}}\left(\frac{\bar{t}(\bar{t}-i\bar{\zeta})}{(t+i\zeta)(\bar{t}-i\bar{\zeta})} - \frac{t(t+i\bar{\zeta})}{(\bar{t}-i\zeta)(t+i\bar{\zeta})}\right)$$

$$= -\frac{f_3}{2} = \frac{-1}{2\sqrt{\alpha^2+1}}\left(\frac{1+A\xi-B\eta}{(A-\eta)^2+(B+\xi)^2} + \frac{1+A\xi+B\eta}{(A+\eta)^2+(B+\xi)^2}\right)$$

$$- \frac{1}{2\sqrt{\alpha^2+1}}\left(\frac{\alpha-A\eta-B\xi}{(A-\eta)^2+(B+\xi)^2} - \frac{\alpha+A\eta-B\xi}{(A+\eta)^2+(B+\xi)^2}\right)$$

$$A = B(\sqrt{\alpha^2+1} + \alpha)$$

$$B = A(\sqrt{\alpha^2+1} - \alpha)$$

$$p^* = -\frac{1}{4(\alpha^2+1)^{3/2}}\left(\frac{\alpha^2-1+B\eta(1-\alpha^2-\alpha\sqrt{\alpha^2+1})-A\xi(1-\alpha^2+\alpha\sqrt{\alpha^2+1})}{(A-\eta)^2+(B+\xi)^2}\right.$$

$$\left.+\frac{\alpha^2-1-B\eta(1-\alpha^2-\alpha\sqrt{\alpha^2+1})-A\xi(1-\alpha^2+1+\alpha\sqrt{\alpha^2+1})}{(A+\eta)^2+(B+\xi)^2}\right)$$

$$\frac{-i}{4(\alpha^2+1)^{3/2}}\left(\frac{-2\alpha+A\eta(1-\alpha^2+\alpha\sqrt{\alpha^2+1})+B\xi(1-\alpha^2-\alpha\sqrt{\alpha^2+1})}{(A-\eta)^2+(B+\xi)^2}\right.$$

$$\left.\frac{-2\alpha-A\eta(1-\alpha^2+\alpha\sqrt{\alpha^2+1})+B\xi(1-\alpha^2-\alpha\sqrt{\alpha^2+1})}{(A+\eta)^2+(B+\xi)^2}\right)$$

$$= -\frac{\alpha^2-1}{4(\alpha^2+1)}f_3+\frac{\alpha}{4(\alpha^2+1)}\left(\frac{B\eta+A\xi}{(A-\eta)^2+(B+\xi)^2}+\frac{A\xi-B\eta}{(A+\eta)^2+(B+\xi)^2}\right)$$

$$+\frac{i\alpha}{4(\alpha^2+1)}\left(\frac{B\xi-A\eta+\sqrt{\alpha^2+1}}{(A-\eta)^2+(B+\xi)^2}-\frac{B\xi+A\eta+\sqrt{\alpha^2+1}}{(A+\eta)^2+(B+\xi)^2}\right)$$

Let $r^* = q^* - \frac{\alpha^2-1}{4(\alpha^2+1)}f_3 = -f_3\left(\frac{3\alpha^2+1}{4(\alpha^2+1)}\right)$

$$s^* = p^* + \frac{\alpha^2-1}{4(\alpha^2+1)}f_3$$

Therefore $f_4 = f^* + r^* + s^*$

Special Cases

1) $$\mathrm{Re}(f^*) = -\frac{1}{4(\alpha^2+1)}\left(\frac{\alpha^2-1+\alpha^2-1+2\alpha^2-2}{16(\alpha^2+1)} + \frac{\alpha^2-1+\alpha^2+1-2\alpha\sqrt{\alpha^2+1}}{4(\sqrt{\alpha^2+1}-\alpha)^2}\right)$$

$$= -\frac{1}{4(\alpha^2+1)}\left(\frac{\alpha^2-1}{4(\alpha^2+1)} - \frac{\alpha}{2(\sqrt{\alpha^2+1}-\alpha)}\right)$$

$$= \frac{1-\alpha^2}{16(\alpha^2+1)^2} + \frac{\alpha(\sqrt{\alpha^2+1}+\alpha)}{8(\alpha^2+1)}$$

$$\mathrm{Re}(s^*) = \frac{\alpha}{4(\alpha^2+1)}\left(\frac{1}{2(\sqrt{\alpha^2+1}-\alpha)}\right)$$

$$= \frac{\alpha(\sqrt{\alpha^2+1}+\alpha)}{8(\alpha^2+1)}$$

$$\mathrm{Re}(r^*) = -\left(\frac{3\alpha^2+1}{4(\alpha^2+1)}\right)\left(\frac{1}{2(\alpha^2+1)} + \frac{1}{2\sqrt{\alpha^2+1}\,(\sqrt{\alpha^2+1}-\alpha)}\right)$$

$$= -\frac{3\alpha^2+1}{8(\alpha^2+1)}\left(\frac{1}{\alpha^2+1} + 1 + \frac{\alpha}{\sqrt{\alpha^2+1}}\right)$$

$$\tilde{\psi}_4 = \frac{\alpha(\sqrt{\alpha^2+1}+\alpha)}{4(\alpha^2+1)} - \frac{7\alpha^2+1}{16(\alpha^2+1)^2} - \frac{3\alpha^2+1}{8(\alpha^2+1)} - \frac{\alpha(3\alpha^2+1)}{8(\alpha^2+1)^{3/2}}$$

$$= \frac{\alpha(1-\alpha^2)}{8(\alpha^2+1)^{3/2}} - \frac{1}{8} - \frac{1}{4(\alpha^2+1)} - \frac{3(\alpha^2-1)}{16(\alpha^2+1)^2}$$

$$= \frac{1-\alpha^2}{16}\left(\frac{2\alpha}{(\alpha^2+1)^{3/2}} + \frac{3}{(\alpha^2+1)^2} - \frac{2}{\alpha^2+1}\right) - \frac{1}{4}$$

$$\mathrm{Im}(f^*) = \frac{1}{4(\alpha^2+1)}\left(\frac{2\alpha+\alpha-\sqrt{\alpha^2+1}+\alpha+\sqrt{\alpha^2+1}+\alpha+\alpha+\alpha+\alpha}{16(\alpha^2+1)}\right.$$

$$\left.-\frac{2\alpha+\alpha-\sqrt{\alpha^2+1}+\alpha-\sqrt{\alpha^2+1}-\alpha-\alpha+\alpha-\alpha}{4(\sqrt{\alpha^2+1}-\alpha)^2}\right)$$

$$= \frac{1}{8(\alpha^2+1)}\left(\frac{\alpha}{\alpha^2+1}+\frac{1}{\sqrt{\alpha^2+1}-\alpha}\right)$$

$$= \frac{1}{8(\alpha^2+1)}\left(\frac{\alpha}{\alpha^2+1}+\sqrt{\alpha^2+1}+\alpha\right)$$

$$\mathrm{Im}(s^*) = \frac{\alpha}{4(\alpha^2+1)}\left(\frac{2\sqrt{\alpha^2+1}}{4\sqrt{\alpha^2+1}}-\frac{\sqrt{\alpha^2+1}-\alpha}{2(\sqrt{\alpha^2+1}-\alpha)}\right) = 0$$

$$\mathrm{Im}(r^*) = -\frac{(3\alpha^2+1)}{8(\alpha^2+1)^2}(\alpha+\sqrt{\alpha^2+1})$$

$$\phi_4 = -\frac{3\alpha^3}{8(\alpha^2+1)^2}+\frac{\alpha}{8(\alpha^2+1)}-\frac{\alpha^2}{4(\alpha^2+1)^{3/2}}$$

2) $$\mathrm{Re}(f^*) = -\frac{1}{4(\alpha^2+1)}\left(\frac{\alpha^2-1+\alpha\eta^2+2\eta(B-\alpha A)}{((A-\eta)^2+B^2)^2}+\frac{\alpha^2-1+\alpha\eta^2-2\eta(B-A\alpha)}{((A+\eta)^2+B^2)^2}\right)$$

$$((A-\eta)^2+B^2)(A+\eta)^2+B^2) = B^4+(A^2-\eta^2)^2+2B^2(A^2+\eta^2)$$

$$= \alpha^2+1+\eta^4-2\alpha\eta^2$$

$$\mathrm{Re}(f^*) = -\frac{(\alpha^2-1+2\eta^2)((\sqrt{\alpha^2+1}+\eta^2)^2+2(\sqrt{\alpha^2+1}+\alpha)\eta^2+2\eta(B-2A)4A\eta(\sqrt{\alpha^2+1}+\eta^2)}{2(\alpha^2+1)(\alpha^2+1+\eta^4-2\alpha\eta^2)^2}$$

$$= \frac{-(\alpha^2-1+\alpha\eta^2)(\alpha^2+1+2(2\sqrt{\alpha^2+1}+\alpha)\eta^2+\eta^4)+4\eta^2(1-\alpha^2-\alpha\sqrt{\alpha^2+1})(\sqrt{\alpha^2+1}+\eta^2)}{2(\alpha^2+1)(\alpha^2+1+\eta^4-2\alpha\eta^2)^2}$$

$$= -\frac{\alpha^2-1+\alpha\eta^2}{2(\alpha^2+1)(\alpha^2+1+\eta^4-2\alpha\eta^2)}$$

$$-2\,\frac{\eta^2\left[(\alpha^2-1+\alpha\eta^2)(\sqrt{\alpha^2+1}+\alpha)+(1-\alpha^2-\alpha\sqrt{\alpha^2+1})(\sqrt{\alpha^2+1}+\eta^2)\right]}{(\alpha^2+1)(\alpha^2+1+\eta^4-2\alpha\eta^2)^2}$$

$$= -\frac{\alpha^2-1+\alpha\eta^2}{2(\alpha^2+1)(\alpha^2+1+\eta^4-2\alpha\eta^2)} - \frac{2\eta^2(\eta^2+2\alpha)}{(\alpha^2+1)(\alpha^2+1+\eta^4-2\alpha\eta^2)^2}$$

$$= \frac{2}{(\alpha^2+1+\alpha^4-2\alpha\eta^2)^2} - \frac{\alpha^2+3+\alpha\eta^2}{2(\alpha^2+1)(\alpha^2+1+\eta^4-2\alpha\eta^2)}$$

$$\mathrm{Re}(s^*) = \frac{\alpha B\eta}{4(\alpha^2+1)}\left(\frac{1}{(A-\eta)^2+B^2} - \frac{1}{(A+\eta)^2+B^2}\right)$$

$$= \frac{\alpha B\eta(4A\eta)}{4(\alpha^2+1)(\alpha^2+1+\eta^4-2\alpha\eta^2)}$$

$$= \frac{\alpha\eta^2}{2(\alpha^2+1)(\alpha^2+1+\eta^4-2\alpha\eta^2)}$$

$$\mathrm{Re}(r^*) = -\frac{3\alpha^2+1}{4(\alpha^2+1)}\;\frac{2}{\eta^4-2\alpha\eta^2+\alpha^2+1}$$

$$\tilde{\psi}_4 = \frac{2}{(\alpha^2+1+\eta^4-2\alpha\eta^2)^2} - \frac{2}{\alpha^2+1+\eta^4-2\alpha\eta^2}$$

$$= \frac{1}{2}\left(\frac{x+\ell}{\ell}\right)^2 - \frac{x+\ell}{\ell}$$

$$\mathrm{Im}(f^*) = \frac{1}{4(\alpha^2+1)}\left(\frac{2\alpha-2(A+B\alpha)\eta+\eta^2}{((A-\eta)^2+B^2)^2} - \frac{2\alpha+2(A+B\alpha)\eta+\eta^2}{((A+\eta)^2+B^2)^2}\right)$$

$$= \frac{(2\alpha+\eta^2)4A\eta(\sqrt{\alpha^2+1}+\eta^2)-2(A+B\alpha)\eta(\alpha^2+1+2(2\sqrt{\alpha^2+1}+\alpha)\eta^2+\eta^4)}{2(\alpha^2+1)(\eta^4-2\alpha\eta^2+\alpha^2+1)^2}$$

$$= \frac{-(A+B\alpha)\eta}{(\alpha^2+1)(\eta^4-2\alpha\eta^2+\alpha^2+1)}$$

$$+ \frac{2A\eta(\eta^4+(\sqrt{\alpha^2+1}+2\alpha)\eta^2+2\alpha\sqrt{\alpha^2+1}) - 4(A+B\alpha)\eta(\sqrt{\alpha^2+1}+\alpha)\eta^2}{(\alpha^2+1)(\eta^4-2\alpha\eta^2+\alpha^2+1)^2}$$

$$(A+B\alpha)(\sqrt{\alpha^2+1}+\alpha) = A(1+\alpha\sqrt{\alpha^2+1}-\alpha^2)(\sqrt{\alpha^2+1}+\alpha)$$

$$= A(\sqrt{\alpha^2+1}+2\alpha)$$

$$\mathrm{Im}(f^*) = \frac{-(A+B\alpha)\eta}{(\alpha^2+1)(\eta^4-2\alpha\eta^2+\alpha^2+1)} + \frac{2A\eta(\eta^4-(\sqrt{\alpha^2+1}+2\alpha)\eta^2+2\alpha\sqrt{\alpha^2+1})}{(\alpha^2+1)(\eta^4-2\alpha\eta^2+\alpha^2+1)^2}$$

$$= \frac{(A-B\alpha)\eta}{(\alpha^2+1)(\eta^4-2\alpha\eta^2+\alpha^2+1)} + \frac{2A\eta(-\eta^2+2\alpha-\sqrt{\alpha^2+1})}{\sqrt{\alpha^2+1}(\eta^2-2\alpha\eta^2+\alpha^2+1)^2}$$

$$\mathrm{Im}(s^*) = \frac{\alpha}{4(\alpha^2+1)} \left(\frac{-An + \sqrt{\alpha^2+1}}{(A-n)^2+B^2} - \frac{A\eta + \sqrt{\alpha^2+1}}{(A+n)^2 + B^2} \right)$$

$$= \frac{-\alpha(A\eta(\sqrt{\alpha^2+1} + \eta^2) - 2A\eta\sqrt{\alpha^2+1})}{2(\alpha^2+1)(\eta^4-2\alpha\eta^2+\alpha^2+1)}$$

$$= \frac{-\alpha A\eta(\eta^2 - \sqrt{\alpha^2+1})}{2(\alpha^2+1)(\eta^4-2\alpha\eta^2+\alpha^2+1)}$$

$$\mathrm{Im}(r^*) = -\frac{(3\alpha^2+1)}{2(\alpha^2+1)} \frac{A\eta(2\alpha - \sqrt{\alpha^2+1} - \eta^2)}{\sqrt{\alpha^2+1}(\eta^4-2\alpha\eta^2+\alpha^2+1)}$$

$$\phi_4 = \frac{-2A\eta(\eta^2-2\alpha + \sqrt{\alpha^2+1})}{\sqrt{\alpha^2+1}\,(\eta^4-2\alpha\eta^2+\alpha^2+1)^2}$$

$$+ \frac{A\eta(2\sqrt{\alpha^2+1}(1+\alpha^2-\alpha\sqrt{\alpha^2+1}) - \alpha(\eta^2 - \sqrt{\alpha^2+1}) - (3\alpha^2+1)(2\alpha-\sqrt{\alpha^2+1}-\eta^2))}{2(\alpha^2+1)^{3/2}(\eta^4-2\alpha\eta^2+\alpha^2+1)}$$

$$= \frac{-2A\eta(\eta^2-2\alpha+\sqrt{\alpha^2+1})}{\sqrt{\alpha^2+1}(\eta^4-2\alpha\eta^2+\alpha^2+1)^2} + \frac{A\eta((3\alpha^2+1-\alpha\sqrt{\alpha^2+1})\eta^2 + \sqrt{\alpha^2+1}(5\alpha^2+3) - \alpha(7\alpha^2+3))}{2(\alpha^2+1)^{3/2}(\eta^4-2\alpha\eta^2+\alpha^2+1)}$$

3) $$\mathrm{Re}(f^*) = -\frac{1}{2(\alpha^2+1)}\left(\frac{\alpha^2-1-\alpha\xi^2-2(A+\alpha B)\xi}{(A^2+(B+\xi)^2)^2}\right)$$

$$(\sqrt{\alpha^2+1}+\xi^2-2B\xi)^2=\alpha^2+1+\xi^4+4B^2\xi^2+2\sqrt{\alpha^2+1}\,\xi^2-4B\xi(\sqrt{\alpha^2+1}+\xi^2)$$

$$=\alpha^2+1+\xi^4+2\xi^2(2\sqrt{\alpha^2+1}-\alpha)-4B\xi(\sqrt{\alpha^2+1}+\xi^2)$$

$$\mathrm{Re}(f^*) = -\frac{(\alpha^2-1-\alpha\xi^2)(\xi^4+2\xi^2(2\sqrt{\alpha^2+1}-\alpha)+\alpha^2+1)+4\xi^2(\sqrt{\alpha^2+1}+\xi^2)(1+\alpha\sqrt{\alpha^2+1}-\alpha^2)}{2(\alpha^2+1)(\xi^4+2\alpha\xi^2+\alpha^2+1)^2}$$

$$+\frac{2(A+\alpha B)\xi(\alpha^2+1+\xi^4+2\xi^2(2\sqrt{\alpha^2+1}-\alpha))+(\alpha^2-1-\alpha\xi^2)4B\xi(\sqrt{\alpha^2+1}+\xi^2)}{2(\alpha^2+1)(\xi^4+2\alpha\xi^2+\alpha^2+1)^2}$$

$$=f_E^*+f_o^*$$

$$f_E^* = \frac{\alpha\xi^2+1-\alpha^2}{2(\alpha^2+1)(\xi^4+2\alpha\xi^2+\alpha^2+1)}-\frac{2\xi^2((\alpha^2-1-\alpha\xi^2)(\sqrt{\alpha^2+1}-\alpha)+(\xi^2+\sqrt{\alpha^2+1})(1-\alpha^2+\alpha\sqrt{\alpha^2+1}))}{(\alpha^2+1)(\xi^4+2\alpha\xi^2+\alpha^2+1)^2}$$

$$=\frac{\alpha\xi^2+1-\alpha^2}{2(\alpha^2+1)(\xi^4+2\alpha\xi^2+\alpha^2+1)}-\frac{2\xi^2(\xi^2+2\alpha)}{(\alpha^2+1)(\xi^4+2\alpha\xi^2+\alpha^2+1)^2}$$

$$=\frac{\alpha\xi^2-3-\alpha^2}{2(\alpha^2+1)(\xi^4+2\alpha\xi^2+\alpha^2+1)}+\frac{2}{(\xi^4+2\alpha\xi^2+\alpha^2+1)^2}$$

$$\mathrm{Re}(s^*) = \frac{\alpha A\xi}{2(\alpha^2+1)}\left(\frac{1}{A^2+(B+\xi)^2}\right)$$

$$=\frac{\alpha A\xi(\sqrt{\alpha^2+1}+\xi^2-2B\xi)}{2(\alpha^2+1)(\xi^4+2\alpha\xi^2+\alpha^2+1)}$$

$$s_E^* = \frac{-\alpha\xi^2}{2(\alpha^2+1)(\xi^4+2\alpha\xi^2+\alpha^2+1)}$$

$$r_E^* = -\left(\frac{3\alpha^2+1}{4(\alpha^2+1)}\right)\frac{2}{\xi^4+2\alpha\xi^2+\alpha^2+1}$$

$$\tilde{\psi}_{4E} = \frac{2}{(\xi^4+2\alpha\xi^2+\alpha^2+1)^2} - \frac{-4\alpha^2-4}{2(\alpha^2+1)(\xi^4+2\alpha\xi^2+\alpha^2+1)}$$

$$= \frac{1}{2}\left(\frac{x+\ell}{\ell}\right)^2 - \frac{x+\ell}{\ell}$$

$$f_o^* = \frac{A\xi}{(\alpha^2+1)(\xi^4+2\alpha\xi^2+\alpha^2+1)^2}\Big((1+\alpha\sqrt{\alpha^2+1}-\alpha^2)(\xi^4+2\xi^2(2\sqrt{\alpha^2+1}-\alpha)+\alpha^2+1)$$

$$-2(\sqrt{\alpha^2+1}-\alpha)(\alpha\xi^4+(\alpha\sqrt{\alpha^2+1}+1-\alpha^2)\xi^2+(1-\alpha^2)\sqrt{\alpha^2+1})\Big)$$

$$= \frac{A\xi((1+\alpha^2-\alpha\sqrt{\alpha^2+1})\xi^4+2(1-\alpha^2+\alpha\sqrt{\alpha^2+1})\sqrt{\alpha^2+1}\xi^2+(\alpha^2+1)(1-\alpha^2+\alpha\sqrt{\alpha^2+1})+2(\alpha^2-1)(\alpha^2+1-\alpha\sqrt{\alpha^2+1}))}{(\alpha^2+1)(\xi^4+2\alpha\xi^2+\alpha^2+1)^2}$$

$$= \frac{A\xi(\sqrt{\alpha^2+1}-\alpha)}{\sqrt{\alpha^2+1}(\xi^4+2\alpha\xi^2+\alpha^2+1)} + \frac{A\xi(2\sqrt{\alpha^2+1}\xi^2+4\alpha\sqrt{\alpha^2+1}-2(\alpha^2+1))}{(\alpha^2+1)(\xi^4+2\alpha\xi^2+\alpha^2+1)^2}$$

$$s_o^* = \frac{A\alpha\xi(\xi^2+\sqrt{\alpha^2+1})}{2(\alpha^2+1)(\xi^4+2\alpha\xi^2+\alpha^2+1)}$$

$$r_o^* = \frac{-(3\alpha^2+1)}{4(\alpha^2+1)}\psi_3^*$$

$$\psi_4^* = \frac{2A\xi(\xi^2+2\alpha-\sqrt{\alpha^2+1})}{\sqrt{\alpha^2+1}(\xi^4+2\alpha\xi^2+\alpha^2+1)^2} + \frac{A\xi(\alpha\xi^2-\alpha\sqrt{\alpha^2+1}+2(\alpha^2+1))}{2(\alpha^2+1)(\xi^4+2\alpha\xi^2+\alpha^2+1)} - \frac{3\alpha^2+1}{4(\alpha^2+1)}\psi_3^*$$

$$4)\quad -4(\alpha^2+1)f^* = \frac{1}{D_1^2(\eta)D_1^2(-\eta)}\left((\alpha^2-1+\alpha(\eta^2-\xi^2)-2\xi(A+\alpha B))(D_1^2(\eta)+D_1^2(-\eta))\right.$$

$$\left. + 2\eta(\xi+B-\alpha A)(D_1^2(\eta)-D_1^2(-\eta))\right)$$

where

$$D_1^2(\eta) = (A^2+\eta^2+(B+\xi)^2+2A\eta)^2$$

$$= (A^2+\eta^2+(B+\xi)^2)^2 + 4A^2\eta^2+4A\eta(A^2+\eta^2+(B+\xi)^2)$$

$$-2(\alpha^2+1)f^* = \frac{1}{D_1^2(\eta)D_1^2(-\eta)}\left((2\alpha^2-1-2\xi(A+\alpha B))((\sqrt{\alpha^2+1}+\alpha+2\xi^2+2B\xi)^2\right.$$

$$\left. + 2(\sqrt{\alpha^2+1}+\alpha)(\xi^2+\alpha)) + 8A(\xi^2+\alpha)(\xi+B-\alpha A)(\sqrt{\alpha^2+1}+\alpha+2\xi^2+2B\xi)\right)$$

$$D_1(\eta)D_1(-\eta) = (\sqrt{\alpha^2+1}+\alpha+2\xi^2+2B\xi+2A\eta)(\sqrt{\alpha^2+1}+\alpha+2\xi^2+2B\xi-2A\eta)$$

$$= (\sqrt{\alpha^2+1}+\alpha+2\xi^2+2B\xi)^2 - 2(\sqrt{\alpha^2+1}+\alpha)(\xi^2+\alpha)$$

$$= 4\xi^4+8B\xi^3+4\sqrt{\alpha^2+1}\;\xi^2+4(\sqrt{\alpha^2+1}+\alpha)B\xi+1$$

$$-2(\alpha^2+1)f^* = \frac{2\alpha^2-1-2\xi(A+\alpha B)}{D_1(\eta)D_1(-\eta)} + \frac{4}{D_1^2(\eta)D_1^2(-\eta)}\left((2\alpha^2-1-2\xi(A+\alpha B))(\sqrt{\alpha^2+1}+\alpha)(\xi^2+\alpha)\right.$$

$$\left. + (\xi^2+\alpha)(\sqrt{\alpha^2+1}+\alpha+2\xi^2+2B\xi)(2A\xi+1-\alpha(\sqrt{\alpha^2+1}+\alpha))\right)$$

$$= \frac{2\alpha^2-1-2\xi(A+\alpha B)}{D_1(\eta)D_1(-\eta)} + \frac{4}{D_1^2(\eta)D_1^2(-\eta)}\left((\xi^2+\alpha)(\sqrt{\alpha^2+1}+\alpha)(\alpha(\alpha-\sqrt{\alpha^2+1})-2\alpha B\xi)\right.$$

$$\left. + 2(\xi^2+\alpha)(\xi^2+B\xi)(2A\xi+1-\alpha(\sqrt{\alpha^2+1}+\alpha))\right)$$

$$-2(\alpha^2+1)f^* = \frac{2\alpha^2-1-2\xi(A+\alpha B)}{D_1(\eta)D_1(-\eta)} + \frac{4(\xi^2+\alpha)}{D_1^2(\eta)D_1^2(-\eta)}\left(-\alpha(1+2(\sqrt{\alpha^2+1}+\alpha)B\xi)\right.$$

$$\left. + 2(2A\xi^3+(2-\alpha(\sqrt{\alpha^2+1}+\alpha))\xi^2+(1-\alpha(\sqrt{\alpha^2+1}+\alpha))B\xi)\right)$$

$$= \frac{2\alpha^2-1-2\xi(A+\alpha B)}{D_1(\eta)D_1(-\eta)} + \frac{4(\xi^2+\alpha)}{D_1^2(\eta)D_1^2(-\eta)}\left(4\xi^2+2B\xi-\alpha+(\sqrt{\alpha^2+1}+\alpha)(4B\xi^2-2\alpha\xi^2-4\alpha B\xi)\right)$$

$$= \frac{2\alpha^2-1-2\xi(A+\alpha B)}{D_1(\eta)D_1(-\eta)} + \frac{4}{D_1^2(\eta)D_1^2(-\eta)}(4\xi^4+2B\xi^3+3\alpha\xi^2+2B\alpha\xi-\alpha^2)$$

$$+ \frac{4\xi(\sqrt{\alpha^2+1}+\alpha)}{D_1^2(\eta)D_1^2(-\eta)}(4B\xi^4-2\alpha\xi^3-2\alpha^2\xi-4\alpha^2B)$$

$$= \frac{2\alpha^2-1-2\xi(A+\alpha B)+4+4(\sqrt{\alpha^2+1}+\alpha)B\xi}{D_1(\eta)D_1(-\eta)}$$

$$+ \frac{4}{D_1^2(\eta)D_1^2(-\eta)}(-6B\xi^3+(3\alpha-4\sqrt{\alpha^2+1})\xi^2-(4\sqrt{\alpha^2+1}+2\alpha)B\xi-(\alpha^2+1))$$

$$+ \frac{4\xi(\sqrt{\alpha^2+1}+\alpha)}{D_1^2(\eta)D_1^2(-\eta)}((2\alpha-4\sqrt{\alpha^2+1})\xi^3-4B\sqrt{\alpha^2+1}\,\xi^2-2(\alpha^2+1)\xi-(1+4\alpha^2)B)$$

$$= \frac{2\alpha^2-1-2\xi(A+\alpha B)+4+4(\sqrt{\alpha^2+1}+\alpha)B\xi}{D_1(\eta)D_1(-\eta)}$$

$$- \frac{4}{D_1^2(\eta)D_1^2(-\eta)}(4\xi^4+6B\xi^3+(4\sqrt{\alpha^2+1}-3\alpha)\xi^2+(4\sqrt{\alpha^2+1}+2\alpha)B\xi+(\alpha^2+1))$$

$$- \frac{4(\sqrt{\alpha^2+1}+\alpha)}{D_1^2(\eta)D_1^2(-\eta)}(2\alpha\xi^4+4B\sqrt{\alpha^2+1}\,\xi^3+2(\alpha^2+1)\xi^2+(4\alpha^2+1)B\xi)$$

$$-2(\alpha^2+1)f^* = \frac{2\alpha^2-1-2\xi(A+\alpha B)+4(\sqrt{\alpha^2+1}+\alpha)B\xi-2\alpha(\sqrt{\alpha^2+1}+\alpha)}{D_1(\eta)D_1(-\eta)}$$

$$- \frac{4}{D_1^2(\eta)D_1^2(-\eta)}\,(-2B\xi^3-3\alpha\xi^2-2\alpha B\xi+\alpha^2)$$

$$- \frac{4(\sqrt{\alpha^2+1}+\alpha)}{D_1^2(\eta)\;D_1^2(-\eta)}\,(4B(\sqrt{\alpha^2+1}-\alpha)\xi^3+2\sqrt{\alpha^2+1}(\sqrt{\alpha^2+1}-\alpha)\xi^2$$

$$+ \;(1+2\alpha(\alpha-\sqrt{\alpha^2+1}))B\xi-\frac{\alpha}{2}\;)$$

$$= \frac{-1-2\alpha\sqrt{\alpha^2+1}+2\sqrt{\alpha^2+1}\;B\xi}{D_1(\eta)D_1(-\eta)}$$

$$+ \frac{-4}{D_1^2(\eta)D_1^2(-\eta)}\,(2B\xi^3+(2\sqrt{\alpha^2+1}-3\alpha)\xi^2+(\sqrt{\alpha^2+1}-3\alpha)B\xi-\frac{\alpha}{2}(\sqrt{\alpha^2+1}-\alpha))$$

Let $\qquad D_1(\eta)D_1(-\eta) = D_2(\xi)$

$$-2(\alpha^2+1)f^* = \frac{2\sqrt{\alpha^2+1}(B\xi-\alpha)-1}{D_2(\xi)} - \frac{2}{D_2^2(\xi)D_2(-\xi)}\,(4B\xi^3+2(2(\sqrt{\alpha^2+1}-\alpha)\xi^2$$

$$+ 2((\sqrt{\alpha^2+1}-\alpha)-2\alpha)B\xi-(\sqrt{\alpha^2+1}-\alpha)\alpha)D_2(-\xi)$$

$$= \frac{2\sqrt{\alpha^2+1}(B\xi-\alpha)-1}{D_2(\xi)} - \frac{4\xi D_2(-\xi)}{D_2^2(\xi)D_2(-\xi)}\,(2B\xi^2-\alpha\xi-2\alpha B)$$

$$- \frac{2D_2(-\xi)}{D_2^2(\xi)D_2(-\xi)}\,(\sqrt{\alpha^2+1}-\alpha)(4\xi^2+2B\xi-\alpha)$$

$$D_2(-\xi) = D_2(\xi) - 16B\xi^3 - 8(\sqrt{\alpha^2+1}+\alpha)B\xi$$

$$-2(\alpha^2+1)f^* = \frac{2\sqrt{\alpha^2+1}(B\xi+\alpha)-1}{D_2(\xi)} - \frac{4\xi(2B\xi^2-\alpha\xi-2\alpha B)}{D_2(\xi)D_2(-\xi)}$$

$$+ \frac{32B\xi^2(2\xi^2+(\sqrt{\alpha^2+1}+\alpha))}{D_2^2(\xi)D_2(-\xi)}(2B\xi^2-\alpha\xi-2\alpha B)$$

$$- \frac{2(\sqrt{\alpha^2+1}-\alpha)(4\xi^2+2B\xi-\alpha)}{D_2(\xi)D_2(-\xi)} + \frac{16B\xi(\sqrt{\alpha^2+1}-\alpha)(2\xi^2+\sqrt{\alpha^2+1}+\alpha)(4\xi^2+2B\xi-\alpha)}{D_2^2(\xi)D_2(\xi)}$$

$$N_1 \equiv 32\xi^2(2\xi^2+(\sqrt{\alpha^2+1}+\alpha))((\sqrt{\alpha^2+1}-\alpha)\xi^2-\alpha B\xi-\alpha(\sqrt{\alpha^2+1}-\alpha))$$

$$= 32\xi^2(2(\sqrt{\alpha^2+1}-\alpha)\xi^4-2\alpha B\xi^3+(1-2\alpha(\sqrt{\alpha^2+1}-\alpha))\xi^2-\alpha(\sqrt{\alpha^2+1}+\alpha)B\xi-\alpha)$$

$$= 16\xi^2(\sqrt{\alpha^2+1}-\alpha)(4\xi^4-4(\sqrt{\alpha^2+1}+\alpha)\alpha B\xi^3+2(\sqrt{\alpha^2+1}-\alpha)\xi^2$$

$$-2\alpha(\sqrt{\alpha^2+1}+\alpha)^2B\xi-2\alpha(\sqrt{\alpha^2+1}+\alpha))$$

$$N_2 \equiv 16B\xi(\sqrt{\alpha^2+1}-\alpha)(2\xi^2+\sqrt{\alpha^2+1}+\alpha)(4\xi^2+2B\xi-\alpha)$$

$$= 16B\xi(\sqrt{\alpha^2+1}-\alpha)(8\xi^4+4B\xi^3+(-2\alpha+4(\sqrt{\alpha^2+1}+\alpha))\xi^2+2(\sqrt{\alpha^2+1}+\alpha)B\xi-\alpha(\sqrt{\alpha^2+1}+\alpha))$$

$$N_1+N_2 = 16(\sqrt{\alpha^2-1}-\alpha)(\xi^2(4\xi^4+8B\xi^3-4\alpha(\sqrt{\alpha^2+1}+\alpha)B\xi^3+4\sqrt{\alpha^2+1}\xi^2-4\alpha\xi^2$$

$$+ 4(\sqrt{\alpha^2+1}+\alpha)B\xi-2\alpha(1+(\sqrt{\alpha^2+1}+\alpha)^2)B\xi+1-2\alpha(\sqrt{\alpha^2+1}+\alpha))-B\xi\alpha(\sqrt{\alpha^2+1}+\alpha))$$

$$= 16(\sqrt{\alpha^2+1}-\alpha)(\xi^2(D_2(\xi)-4\alpha\xi^2-2\alpha B\xi)$$

$$+ (\sqrt{\alpha^2+1}+\alpha)(\xi^2(-4\alpha B\xi^3-2\alpha(\sqrt{\alpha^2+1}+\alpha)B\xi-2\alpha)-\alpha B\xi))$$

$$= 16((\sqrt{\alpha^2+1}-\alpha)(\xi^2D_2(\xi)-4\alpha\xi^4-2\alpha B\xi^3)-\alpha\xi(4B\xi^4+2B(\sqrt{\alpha^2+1}+\alpha)\xi^2+2\xi+B))$$

$$N_1 + N_2 = 16((\sqrt{a^2+1}-a)(\xi^2 D_2(\xi)-4a\xi^4-2aB\xi^3)$$

$$-a\xi(BD_2(\xi)-4(\sqrt{a^2+1}-a)\xi^3+2(a-\sqrt{a^2+1})B\xi^2))$$

$$= 16(((\sqrt{a^2+1}-a)\xi^3-aB\xi)D_2(\xi))$$

$$-2(a^2+1)f^* = \frac{2\sqrt{a^2+1}(B\xi-a)-1}{D_2(\xi)}$$

$$+ \frac{-4\xi(2B\xi^2-a\xi-2aB)-2(\sqrt{a^2+1}-a)(4\xi^2+2B\xi-a)+16((\sqrt{a^2+1}-a)\xi^2-aB\xi)}{D_2(\xi)D_2(-\xi)}$$

$$-2(a^2+1)f^*D_2(\xi)D_2(-\xi) = (4\xi^4-8B\xi^3+4\sqrt{a^2+1}\xi^2-4(\sqrt{a^2+1}+a)B\xi+1)(2\sqrt{a^2+1}B\xi-2a\sqrt{a^2+1}-1$$

$$- 8B\xi^3+4a\xi^3+8(\sqrt{a^2+1}-a)\xi^2-4(\sqrt{a^2+1}+a)B\xi+2a(\sqrt{a^2+1}-a)$$

Let $Q = 4\xi^4 + 4a\xi^2-1$

$$-2(a^2+1)f^*Q^2 = Q(2\sqrt{a^2+1}B\xi-2a\sqrt{a^2+1}-1)+2\sqrt{a^2+1}(-4(\sqrt{a^2+1}-a)\xi^4$$

$$+ 4(\sqrt{a^2+1}-a)B\xi^3-2\xi^2+2B\xi)+16a\sqrt{a^2+1}B\xi^3$$

$$+ 4\sqrt{a^2+1}(1-2a(\sqrt{a^2+1}-a))\xi^2+8a\sqrt{a^2+1}(\sqrt{a^2+1}+a)B\xi$$

$$- 2\sqrt{a^2+1}(a+\sqrt{a^2+1})$$

$$= Q(2\sqrt{a^2+1}B\xi-1-2(a^2+1))+8\sqrt{a^2+1}(\sqrt{a^2+1}+a)B\xi^3$$

$$+ 4\sqrt{a^2+1}(1+2a(\sqrt{a^2+1}+a))B\xi-4(a^2+1)$$

$$-2(\alpha^2+1)f^*Q^2 = Q(2\sqrt{\alpha^2+1}B\xi-1-2(\alpha^2+1))+8\sqrt{\alpha^2+1}A\xi^3$$

$$+ 4\sqrt{\alpha^2+1}(\sqrt{\alpha^2+1}+\alpha)A\xi-4(\alpha^2+1)$$

$$\frac{4(\alpha^2+1)}{\alpha} s^* = A\xi\left(\frac{1}{D_1(\eta)} + \frac{1}{D_1(-\eta)}\right) + B\eta\left(\frac{1}{D_1(-\eta)} - \frac{1}{D_1(\eta)}\right)$$

$$= \frac{A\xi(D_1(\eta) + D_1(-\eta))+ B\eta(D_1(\eta) - D_1(-\eta))}{D_2(\xi)}$$

$$\frac{2(\alpha^2+1)}{\alpha} s^* = \frac{A\xi(\sqrt{\alpha^2+1}+\alpha+2\xi^2+2B\xi)+\xi^2+\alpha}{D_2(\xi)}$$

$$= \frac{2A\xi^3+2\xi^2+A(\sqrt{\alpha^2+1}+\alpha)\xi+\alpha}{D_2(\xi)}$$

$$\frac{2(\alpha^2+1)}{\alpha} s^*Q^2 = (2A\xi^3+2\xi^2+A(\sqrt{\alpha^2+1}+\alpha)\xi+\alpha)(Q-8B\xi^3+4(\sqrt{\alpha^2+1}-\alpha)\xi^2-4A\xi+2)$$

$$= (2A\xi^3+2\xi^2+A(\sqrt{\alpha^2+1}+\alpha)\xi+\alpha)Q-8\xi^6+8((\sqrt{\alpha^2+1}-\alpha)A-2B)\xi^5$$

$$+ (8(\sqrt{\alpha^2+1}-\alpha)-8((\sqrt{\alpha^2+1}+\alpha))\xi^4+(4A-8A+4A-8\alpha B)\xi^3$$

$$+ (4+4\alpha(\sqrt{\alpha^2+1}-\alpha)-2(\sqrt{\alpha^2+1}+\alpha)^2)\xi^2$$

$$+ (2(\sqrt{\alpha^2+1}+\alpha)-4\alpha)A\xi+2\alpha$$

$$= (2A\xi^3+2\xi^2+A(\sqrt{\alpha^2+1}+\alpha)\xi+\alpha)Q-8\xi^6-8B\xi^5-16\alpha\xi^4-8\alpha B\xi^3$$

$$+ (2-8\alpha^2)\xi^2+2(\sqrt{\alpha^2+1}-\alpha)A\xi+2\alpha$$

$$\frac{2(\alpha^2+1)}{\alpha}\, s^* Q^2 = (2A\xi^3+(A(\sqrt{\alpha^2+1}+\alpha)-2B)\xi+\alpha)Q-8\alpha\xi^4-8\alpha^2\xi^2+(2(\sqrt{\alpha^2+1}-\alpha)A-2B)\xi+2\alpha$$

$$= (2A\xi^3+(3\alpha-\sqrt{\alpha^2+1})A\xi-\alpha)Q$$

$$2(\alpha^2+1)(f^*+s^*)Q^2 = (2A\alpha\xi^3+((3\alpha-\sqrt{\alpha^2+1})\alpha A-2\sqrt{\alpha^2+1}B)\xi-\alpha^2+1+2(\alpha^2+1))Q$$

$$+ 8\sqrt{\alpha^2+1}A\xi^3-4\sqrt{\alpha^2+1}(\sqrt{\alpha^2+1}+\alpha)A\xi+4(\alpha^2+1)$$

$$= (2A\alpha\xi^3+(\alpha(\sqrt{\alpha^2+1}+\alpha)-2)A\xi+\alpha^2+3)Q-8\sqrt{\alpha^2+1}A\xi^3$$

$$+ 4\sqrt{\alpha^2+1}(\sqrt{\alpha^2+1}+\alpha)A\xi+4(\alpha^2+1)$$

$$2(\alpha^2+1)r^* = (3\alpha^2+1)\,\frac{1}{Q}\,\left(1-\frac{2A\xi^3}{\sqrt{\alpha^2+1}}-\left(1+\frac{\alpha}{\sqrt{\alpha^2+1}}\right)A\xi\right)$$

$$2(\alpha^2+1)\tilde{\psi}_4 Q^2 = \left(2\left(\alpha-\frac{(3\alpha^2+1)}{\sqrt{\alpha^2+1}}\right)A\xi^3+\left(\alpha(\sqrt{\alpha^2+1}+\alpha)-3(\alpha^2+1)-\frac{\alpha(3\alpha^2+1)}{\sqrt{\alpha^2+1}}\right)A\xi\right.$$

$$\left.+\ 4\alpha^2+4\right)Q-4\sqrt{\alpha^2+1}(2A\xi^3+(\sqrt{\alpha^2+1}+\alpha)A\xi)+4(\alpha^2+1)$$

$$= (2(\alpha\sqrt{\alpha^2+1}-3\alpha^2-1)\,\frac{A\xi^3}{\sqrt{\alpha^2+1}}-(2\alpha^3+\sqrt{\alpha^2+1}(2\alpha^2+3))\,\frac{A\xi}{\sqrt{\alpha^2+1}}$$

$$+\ 4(\alpha^2+1))Q-4\sqrt{\alpha^2+1}(2A\xi^3+(\sqrt{\alpha^2+1}+\alpha)A\xi)+4(\alpha^2+1)$$

APPENDIX II - $I_n(\alpha)$, $J_n(\alpha)$

Let $$I_n(\alpha) = \int_{-\infty}^{\infty} \frac{du}{P^n}$$

Let $$J_n(\alpha) = \int_{-\infty}^{\infty} \frac{u^2 du}{P^n}$$

where $P(u,\alpha) = 1 + (u^2+\alpha)^2$

$$\frac{\partial P}{\partial u} = P' = 4u(u^2+\alpha)$$

Integrate by parts

$$I_n = n \int_{-\infty}^{\infty} \frac{u}{P^{n+1}} P' \, du$$

$$= 4n \int_{-\infty}^{\infty} \frac{u^2(u^2+\alpha)}{P^{n+1}} \, du$$

$$u^2(u^2+\alpha) = P - \alpha u^2 - 1 - \alpha^2$$

Therefore

$$\frac{I_n}{4n} = I_n - \alpha J_{n+1} - (1+\alpha^2) I_{n+1}$$

or

$$\left(1 - \frac{1}{4n}\right) I_n = \alpha J_{n+1} + (1+\alpha^2) I_{n+1}$$

Finally

$$J_n = \frac{n}{3} \int_{-\infty}^{\infty} \frac{u^3}{P^{n+1}} P' \, du$$

$$= \frac{4n}{3} \int_{-\infty}^{\infty} \frac{u^4(u^2 + \alpha)}{P^{n+1}} \, du$$

$$u^4 = P - 2\alpha u^2 - \alpha^2 - 1$$

Therefore

$$\frac{3J_n}{4n} = J_n + \alpha I_n - (\alpha^2+1)J_{n+1} - \alpha(\alpha^2+1)I_{n+1} - 2\alpha \int_{-\infty}^{\infty} \frac{u^2(u^2+\alpha)}{P^{n+1}} \, du$$

Since

$$\int_{-\infty}^{\infty} \frac{u^2(u^2+\alpha)}{P^{n+1}} \, du = \frac{I_n}{4n}$$

and

$$(1+\alpha^2)I_{n+1} = (1 - \frac{1}{4n})I_n - \alpha J_{n+1}$$

therefore

$$(1 - \frac{3}{4n})J_n = J_{n+1} + \frac{\alpha I_n}{4n}$$

For recursion purposes

$$J_{n+1} = - \frac{\alpha I_n}{4n} + (1 - \frac{3}{4n})J_n$$

$$I_{n+1} = \frac{1}{1+\alpha^2} ((1 - \frac{1}{4n})I_n - \alpha J_{n+1})$$

<u>n = 1 Determination</u>

$$P = (u-z)(u-\bar{z})(u+z)(u+\bar{z})$$

Where z and $-\bar{z}$ are in the upper half plane, and $z^2 = -\alpha+i$

$$\int_{-\infty}^{\infty} \frac{(u-z)(u+\bar{z})}{P}\, dz = 0$$

Therefore $J_1 - z\,\bar{z}\, I_1 = 0$

i.e. $J_1 = \sqrt{\alpha^2+1}\; I_1$

$$I_1 = \int_{-\infty}^{\infty} \frac{du}{(u^2-z^2)(u^2-\bar{z}^2)}$$

$$I_1 = 2\pi i \left(\frac{1}{2z(z^2-\bar{z}^2)} - \frac{1}{2\bar{z}(\bar{z}^2-z^2)}\right)$$

Since $z^2 - \bar{z}^2 = 2i$

$$I_1 = \frac{\pi}{2}\left(\frac{1}{z} + \frac{1}{\bar{z}}\right)$$

$$= \frac{\pi(z+\bar{z})}{2\, z\bar{z}}$$

$$I_1(\alpha) = \frac{\pi B}{\sqrt{\alpha^2+1}}$$

$$I_1(-\alpha) = \frac{\pi A}{\sqrt{\alpha^2+1}}$$

APPENDIX III - Steady State Integrals

$$\text{Let } L_k(\alpha) = \int_{-\infty}^{\infty} \frac{x^{2(k-1)}(x^2+\alpha)}{P^2}\, dx$$

where $P = P(x,\alpha)$ is defined in Appendix II.

We wish to evaluate L_1 and L_2.

Using Appendix II and suppressing the argument α in L_k, I_k, J_k.

$$L_1 = J_2 + \alpha I_2$$

$$= J_2 + \frac{\alpha}{1+\alpha^2}\left(\frac{3I_1}{4} - \alpha J_2\right)$$

$$= \frac{J_2}{1+\alpha^2} + \frac{3\alpha I_1}{4(1+\alpha^2)}$$

$$= \frac{-\alpha I_1}{4(1+\alpha^2)} + \frac{J_1}{4(1+\alpha^2)} - \frac{3\alpha I_1}{4(1+\alpha^2)}$$

$$= \frac{2\alpha + \sqrt{\alpha^2+1}}{4(1+\alpha^2)}\, I_1$$

$$= \frac{\pi B}{4(1+\alpha^2)}\left(1 + \frac{2\alpha}{\sqrt{\alpha^2+1}}\right)$$

$$L_2 = \frac{I_1}{4}$$

$$= \frac{\pi B}{4\sqrt{1+\alpha^2}}$$

APPENDIX IV - Condition (4) Integrals

Let $P(\xi,\alpha)$ be defined as in appendix II.

Let $M_{c,m} = \int_{-\infty}^{\infty} \frac{1}{P^m} \cos\left(\frac{2\Omega}{P}\right) d\xi$

$$= \sum_{0} \frac{(-1)^n}{(2n)!} (2\Omega)^{2n} \int_{-\infty}^{\infty} \frac{d\xi}{P^{2n+m}}$$

$$= \sum_{0}^{\infty} \frac{(-1)^n (2\Omega)^{2n}}{(2n)!} I_{2n+m}$$

Let

$$N_{c,m} = \int_{-\infty}^{\infty} \frac{\xi^2}{P^m} \cos\left(\frac{2\Omega}{P}\right) d\xi$$

$$= \sum_{0}^{\infty} \frac{(-1)^n (2\Omega)^{2n}}{(2n)!} J_{2n+m}$$

Let

$$M_{s,m} = \int_{-\infty}^{\infty} \frac{1}{P^m} \sin\left(\frac{2\Omega}{P}\right) d\xi$$

$$= \sum_{0}^{\infty} \frac{(-1)^n (2\Omega)^{2n+1}}{(2n+1)!} I_{2n+1+m}$$

Let

$$N_{s,m} = \int_{-\infty}^{\infty} \frac{\xi^2}{P^m} \sin\left(\frac{2\Omega}{P}\right) d\xi$$

$$= \sum_{0}^{\infty} \frac{(-1)^n (2\Omega)^{2n+1}}{(2n+1)!} J_{2n+1+m}$$

$$I_{1,c} = \int_{-\infty}^{\infty} \frac{\xi^2+\alpha}{P^2} \cos \frac{2\Omega}{P} \, d\xi$$

$$= \sum_{0}^{\infty} \frac{(-1)^n}{(2n)!} (2\Omega)^{2n} (J_{2n+2}+\alpha I_{2n+2})$$

$$I_{1,s} = \sum_{0}^{\infty} \frac{(-1)^n (2\Omega)^{2n+1}}{(2n+1)!} (J_{2n+3}+\alpha I_{2n+3})$$

$$I_{2,c} = \int_{-\infty}^{\infty} \frac{\xi^2(\xi^2+\alpha)}{P^2} \cos \frac{2\Omega}{P} \, d\xi$$

$$= \sum_{0}^{\infty} \frac{(-1)^n (2\Omega)^{2n}}{(2n)!} \int_{-\infty}^{\infty} \frac{\xi^2(\xi^2+\alpha)}{P^{2n+2}} \, d\xi$$

$$= \frac{1}{4} \sum_{0}^{\infty} \frac{(-1)^n (2\Omega)^{2n}}{(2n)!} \frac{I_{2n+1}}{2n+1}$$

Similarly

$$I_{2,s} = \frac{1}{4} \sum_{0}^{\infty} \frac{(-1)^n (2\Omega)^{2n+1}}{(2n+1)!} \frac{I_{2n+2}}{2n+2}$$

$$I^*_{1,c} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\xi^2+\alpha}{\xi^2 P^2} \cos \frac{2\Omega}{P} \, d\xi$$

$$= \frac{1}{2} M_{c,2} + \frac{\alpha}{2} \int_{-\infty}^{\infty} \frac{1}{\xi^2 P^2} \cos \frac{2\Omega}{P} \, d\xi$$

$$= \frac{1}{2} M_{c,2} + \alpha \int_{-\infty}^{\infty} \frac{d}{dP} \left(\frac{1}{P^2} \cos \frac{2\Omega}{P}\right) \frac{dP}{d(\xi^2)} \, d\xi$$

Since $P = (\xi^2)^2 + 2\alpha(\xi^2) + \alpha^2 + 1$

$$\frac{dP}{d(\xi^2)} = 2\xi^2 + 2\alpha$$

$$\frac{d}{dP}\left(\frac{1}{P^2}\cos\frac{2\Omega}{P}\right) = -\frac{2}{P^3}\cos\frac{2\Omega}{P} + \frac{2\Omega}{P^4}\sin\frac{2\Omega}{P}$$

Therefore

$$I^*_{1,c} = \frac{1}{2}M_{c,2} + 4\alpha\int_{-\infty}^{\infty}\frac{(\xi^2+\alpha)}{P^3}\left(\frac{\Omega}{P}\sin\frac{2\Omega}{P} - \cos\frac{2\Omega}{P}\right)d\xi$$

$$= \frac{1}{2}M_{c,2} + 4\alpha\left[\Omega(N_{s,4}+\alpha M_{s,4}) - (N_{c,3}+\alpha M_{c,3})\right]$$

$$= \frac{1}{2}\sum_{0}^{\infty}\frac{(-1)^n(2\Omega)^{2n}}{(2n)!}I_{2n+2}$$

$$+ 4\alpha\left(\frac{1}{2}\sum_{0}^{\infty}\frac{(-1)^n(2\Omega)^{2n+2}}{(2n+1)!}(J_{2n+5}+\alpha I_{2n+5})\cdot\frac{2n+2}{2n+2}\right.$$

$$\left. - \sum_{0}^{\infty}\frac{(-1)^n(2\Omega)^{2n}}{(2n)!}(J_{2n+3}+\alpha I_{2n+3})\right)$$

$$= \sum_{0}^{\infty}\frac{(-1)^n(2\Omega)^{2n}}{(2n)!}\left(\frac{I_{2n+2}}{2} - 4\alpha(n+1)(J_{2n+3}+\alpha I_{2n+3})\right)$$

$$\alpha J_{n+1} + \alpha^2 I_{n+1} = \left(1 - \frac{1}{4n}\right)I_n - I_{n+1}$$

$$I^*_{1,c} = \sum_{0}^{\infty}\frac{(-1)^n(2\Omega)^{2n}}{(2n)!}\left((4n+4)I_{2n+3} - (4n+3)I_{2n+2}\right)$$

$$I^*_{1,s} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\xi^2+\alpha}{\xi^2 P^2} \sin\frac{2\Omega}{P}\, d\xi$$

$$= \frac{1}{2} M_{s,2} + \alpha \int_{-\infty}^{\infty} \frac{d}{dP}\left(\frac{1}{P^2}\sin\frac{2\Omega}{P}\right)(2\xi^2 + 2\alpha)\, d\xi$$

$$= \frac{1}{2} M_{s,2} - 4\alpha \int_{-\infty}^{\infty} \frac{(\xi^2+\alpha)}{P^3}\left(\frac{\Omega}{P}\cos\frac{2\Omega}{P} + \sin\frac{2\Omega}{P}\right) d\xi$$

$$= \frac{1}{2} M_{s,2} - 4\alpha(\Omega(N_{c,4} + \alpha M_{c,4}) + M_{s,3} + \alpha M_{s,3})$$

$$= \frac{1}{2}\sum_{0}^{\infty} \frac{(-1)^n (2\Omega)^{2n+1}}{(2n+1)!} I_{2n+3}$$

$$- 4\alpha\left[\frac{1}{2}\sum_{0}^{\infty} \frac{(-1)^n (2\Omega)^{2n+1}}{(2n)!}(J_{2n+4} + \alpha I_{2n+4}) \cdot \frac{2n+1}{2n+1}\right.$$

$$\left. + \sum_{0}^{\infty} \frac{(-1)^n (2\Omega)^{2n+1}}{(2n+1)!}(J_{2n+4} + \alpha I_{2n+4})\right]$$

$$= \sum_{0}^{\infty} \frac{(-1)^n (2\Omega)^{2n+1}}{(2n+1)!}\left(\frac{I_{2n+3}}{2} - 4\alpha\left(n + \frac{3}{2}\right)(J_{2n+4} + \alpha I_{2n+4})\right)$$

$$= \sum_{0}^{\infty} \frac{(-1)^n (2\Omega)^{2n+1}}{(2n+1)!}\left(\frac{I_{2n+3}}{2} - 4\left(n + \frac{3}{2}\right)\left(\left(1 - \frac{1}{4(2n+3)}\right) I_{2n+3} - I_{2n+4}\right)\right)$$

$$= \sum_{0}^{\infty} \frac{(-1)^n (2\Omega)^{2n+1}}{(2n+1)!}\left((4n+6) I_{2n+4} - (4n+5) I_{2n+3}\right)$$

$$I^*_{2,c} = -\frac{1}{2} I_{1,c}$$

$$I^*_{2,s} = -\frac{1}{2} I_{1,s}$$

Let $E = 2\alpha - \sqrt{\alpha^2+1}$

$$I_{3,c} = \frac{2A}{\sqrt{\alpha^2+1}} \int_{-\infty}^{\infty} \frac{\xi^2(\xi^2+\alpha)(\xi^2+E)}{P^3} \cos\left(\frac{2\Omega}{P}\right) d\xi$$

$$= \frac{2A}{\sqrt{\alpha^2+1}} \sum_{0}^{\infty} \frac{(-1)^n(2\Omega)^{2n}}{(2n)!} \int_{-\infty}^{\infty} \frac{\xi^2(\xi^2+\alpha)(\xi^2+E)}{P^{2n+3}} d\xi$$

$$= \frac{2A}{\sqrt{\alpha^2+1}} \sum_{0}^{\infty} \frac{(-1)^n(2\Omega)^{2n}}{2n!} \left(\frac{3J_{2n+2}}{4(2n+2} + \frac{EI_{2n+2}}{4(2n+2)}\right)$$

$$= \frac{A}{2\sqrt{\alpha^2+1}} \sum_{0}^{\infty} \frac{(-1)^n(2\Omega)^{2n}}{(2n)!} \left(\frac{3J_{2n+2} + (2\alpha - \sqrt{\alpha^2+1})I_{2n+2}}{2n+2}\right)$$

Similarly

$$I_{3,s} = \frac{A}{2\sqrt{\alpha^2+1}} \sum_{0}^{\infty} \frac{(-1)^n(2\Omega)^{2n+1}}{(2n+1)} \left(\frac{3J_{2n+3} + (2\alpha - \sqrt{\alpha^2+1})I_{2n+3}}{2n+3}\right)$$

Let $F = \alpha\sqrt{\alpha^2+1} - 2(\alpha^2+1)$

$$I_{4,c} = \frac{2A}{\sqrt{\alpha^2+1}} \int_{-\infty}^{\infty} \frac{\xi^2(\xi^2+\alpha)(\xi^2+E)}{P^4} \cos\left(\frac{2\Omega}{P}\right) d\xi$$

$$+ \frac{A}{2(\alpha^2+1)} \int_{-\infty}^{\infty} \frac{\xi^2(\xi^2+\alpha)(\alpha\xi^2-F)}{P^3} \cos\left(\frac{2\Omega}{P}\right) d\xi - \frac{3\alpha^2+1}{4(\alpha^2+1)} I_{3,c}$$

$$= U_1 + U_2 + U_3$$

$$U_1 = \frac{2A}{\sqrt{\alpha^2+1}} \sum_{o}^{\infty} \frac{(-1)^n (2\Omega)^{2n}}{(2n)!} \int_{-\infty}^{\infty} \frac{\xi^2(\xi^2+\alpha)(\xi^2+E)}{P^{2n+4}} d\xi$$

$$= \frac{2A}{\sqrt{\alpha^2+1}} \sum_{o}^{\infty} \frac{(-1)^n (2\Omega)^{2n}}{(2n)!} \left(\frac{3J_{2n+3}}{4(2n+3)} + \frac{EI_{2n+3}}{4(2n+3)}\right)$$

$$U_2 = \frac{A}{2(\alpha^2+1)} \sum_{o}^{\infty} \frac{(-1)^n (2\Omega)^{2n}}{2n!} \int_{-\infty}^{\infty} \frac{\xi^2(\xi^2+\alpha)(\alpha\xi^2-F)}{P^{2n+3}} d\xi$$

$$= \frac{A}{2(\alpha^2+1)} \sum_{o}^{\infty} \frac{(-1)^n (2\Omega)^{2n}}{2n!} \left(\frac{3\alpha J_{2n+2}}{4(2n+2)} - \frac{FI_{2n+2}}{4(2n+2)}\right)$$

$$U_3 = \frac{-(3\alpha^2+1)A}{8(\alpha^2+1)^{3/2}} \sum_{o}^{\infty} \frac{(-1)^n (2\Omega)^{2n}}{(2n)!} \left(\frac{3J_{2n+2}}{2n+2} + \frac{EI_{2n+2}}{2n+2}\right)$$

$$U_2 + U_3 = \frac{A}{8(\alpha^2+1)^{3/2}} \sum_{o}^{\infty} \frac{(-1)^n (2\Omega)^{2n}}{(2n)!(2n+2)} \left(3(\alpha\sqrt{\alpha^2+1} - (3\alpha^2+1))J_{2n+2}\right.$$

$$\left. - (F\sqrt{\alpha^2+1} + (3\alpha^2+1)E)I_{2n+2}\right)$$

$$F\sqrt{\alpha^2+1} + (3\alpha^2+1)E = \alpha(\alpha^2+1) - 2(\alpha^2+1)\sqrt{\alpha^2+1} + (3\alpha^2+1)(2\alpha - \sqrt{\alpha^2+1})$$

$$= \alpha(7\alpha^2+3) - \sqrt{\alpha^2+1}\,(5\alpha^2+3)$$

$$I_{4,c} = \frac{A}{8(\alpha^2+1)^{3/2}} \sum_{o}^{\infty} \frac{(-1)^n (2\Omega)^{2n}}{(2n)!} \left(\frac{3(\alpha\sqrt{\alpha^2+1} - (3\alpha^2+1))}{2n+2} J_{2n+2}\right.$$

$$+ \frac{(5\alpha^2+3)\sqrt{\alpha^2+1} - \alpha(7\alpha^2+3)}{2n+2} I_{2n+2}$$

$$\left. + \frac{4(\alpha^2+1)}{2n+3} (3J_{2n+3} + (2\alpha - \sqrt{\alpha^2+1})I_{2n+3})\right)$$

Similarly

$$I_{4,s} = \frac{A}{8(\alpha^2+1)^{3/2}} \sum_{o}^{\infty} \frac{(-1)^n (2\Omega)^{2n+1}}{(2n+1)!} \left(\frac{3(\alpha\sqrt{\alpha^2+1} - (3\alpha^2+1))}{2n+3} J_{2n+3} \right.$$

$$+ \frac{(5\alpha^2+3)\sqrt{\alpha^2+1} - \alpha(7\alpha^2+3)}{2n+3} I_{2n+3}$$

$$\left. + \frac{4(\alpha^2+1)}{2n+4} (3J_{2n+4} + (2\alpha - \sqrt{\alpha^2+1}) I_{2n+4}) \right)$$

APPENDIX V - Force and Moment Terms

Part 1.

$$\text{Let } R_k = \int_{-\infty}^{\infty} \phi_k \frac{dx}{d\eta}\, d\eta$$

$$R_k^* = \int_{-\infty}^{\infty} \frac{\partial \phi_k}{\partial \alpha} \frac{dx}{d\eta}\, d\eta$$

$$S_k = \int_{-\infty}^{\infty} (x+\ell)\phi_k \frac{dx}{d\eta}\, d\eta$$

$$S_k^* = \int_{-\infty}^{\infty} (x+\ell) \frac{\partial \phi_k}{\partial \alpha} \frac{dx}{d\eta}\, d\eta$$

Let $P = P(\eta,-\alpha) = 1 + (\eta^2 - \alpha)^2$ (See Appendix II)

$$R_1 = 8\ell \int_{-\infty}^{\infty} \frac{\eta^2 - \alpha}{P^2}\, d\eta$$

$$= 8\ell\, (J_2(-\alpha) - \alpha I_2(-\alpha))$$

Henceforth we will suppress $-\alpha$ in J_n and I_n

$$-\alpha I_2 = \frac{-\alpha}{1+\alpha^2} \left(\frac{3I_1}{4} + \alpha J_2\right)$$

$$\left(1 - \frac{\alpha^2}{1+\alpha^2}\right) J_2 = \frac{1}{1+\alpha^2} \left(\frac{J_1}{4} + \frac{\alpha I_1}{4}\right)$$

$$R_1 = 2\ell \left(\frac{1}{\sqrt{1+\alpha^2}} - \frac{2\alpha}{1+\alpha^2}\right) I_1$$

$$R_1^* = -4\ell \int_{-\infty}^{\infty} \frac{\eta^2-\alpha}{\eta^2 P^2}\, d\eta$$

$$= -4\ell\left(I_2 - \alpha \int_{-\infty}^{\infty} \frac{d\eta}{\eta^2 P^2}\right)$$

$$\int_{-\infty}^{\infty} \frac{d\eta}{\eta^2 P^2} = -\frac{1}{\eta P^2}\Big]_{-\infty}^{\infty} - 2\int_{-\infty}^{\infty} \frac{P'}{\eta P^2}\, d\eta$$

$$= -8\int_{-\infty}^{\infty} \frac{\eta^2-\alpha}{P^3}\, d\eta$$

$$= -8(J_3 - \alpha I_3)$$

$$R_1^* = -4\ell(I_2 + 8\alpha(J_3 - \alpha I_3))$$

$$\alpha I_3 = \frac{\alpha}{1+\alpha^2}\left(\alpha J_3 + \frac{7}{8} I_2\right)$$

$$\frac{J_3}{1+\alpha^2} = \frac{1}{1+\alpha^2}\left(\frac{5J_2}{8} + \frac{\alpha I_2}{8}\right)$$

$$R_1^* = -4\ell\left(I_2 + \frac{\alpha}{1+\alpha^2}(5J_2 - 6\alpha I_2)\right)$$

$$\left(1 - \frac{6\alpha^2}{1+\alpha^2}\right) I_2 = \frac{1-5\alpha^2}{(1+\alpha^2)^2}\left(\alpha J_2 + \frac{3I_1}{4}\right)$$

$$\frac{\alpha}{1+\alpha^2}\left(5 + \frac{1-5\alpha^2}{1+\alpha^2}\right) J_2 = \frac{6\alpha}{(1+\alpha^2)^2}\left(\frac{J_1}{4} + \frac{\alpha I_1}{4}\right)$$

$$R_1^* = -3\ell\left(\frac{2\alpha}{(1+\alpha^2)^{3/2}} + \frac{1-3\alpha^2}{(1+\alpha^2)^2}\right) I_1$$

$$R_2 = -8\ell \int_{-\infty}^{\infty} \frac{\eta^2(\eta^2-\alpha)}{P^2}\, d\eta$$

$$= -2\ell I_1$$

$$R_2^* = -\frac{1}{2} R_1$$

$$= \frac{\ell}{1+\alpha^2}\,(2\alpha - \sqrt{1+\alpha^2})\, I_1$$

$$R_3 = \frac{16A\ell}{\sqrt{\alpha^2+1}} \int_{-\infty}^{\infty} \frac{\eta^2(\eta^2-\alpha)(\eta^2-E)}{P^3}\, d\eta$$

where $E = 2\alpha - \sqrt{\alpha^2+1}$

$$= \frac{16A\ell}{\sqrt{\alpha^2+1}}\left(\frac{3}{8} J_2 - \frac{E}{8} I_2\right)$$

$$\frac{E}{8} I_2 = \frac{E}{8(1+\alpha^2)}\left(\alpha J_2 + \frac{3}{4} I_1\right)$$

$$\left(\frac{3}{8} - \frac{\alpha E}{8(1+\alpha^2)}\right) J_2 = \frac{3(1+\alpha^2)-\alpha E}{8(1+\alpha^2)}\left(\frac{\alpha I_1}{4} + \frac{J_1}{4}\right)$$

$$R_3 = \frac{A\ell}{2(1+\alpha^2)^3}\left((3(1+\alpha^2) - E)J_1 + (3\alpha(1+\alpha^2) - (\alpha^2+3)E)I_1\right)$$

$$(\alpha J_1 + (\alpha^2+3)I_1)E = (\alpha\sqrt{\alpha^2+1} + \alpha^2 + 3)(2\alpha - \sqrt{\alpha^2+1})I_1$$

$$= (2\alpha(\alpha^2+3) - \alpha(\alpha^2+1) + \sqrt{\alpha^2+1}\,(\alpha^2-3))I_1$$

$$= (\alpha(\alpha^2+5) + \sqrt{\alpha^2+1}\,(\alpha^2-3))I_1$$

$$3(1+\alpha^2)(J_1+\alpha I_1) = 3(1+\alpha^2)(\sqrt{\alpha^2+1}+\alpha)I_1$$

$$R_3 = \frac{A\ell I_1}{2(1+a^2)^{3/2}} \left(a(2a^2-2) + \sqrt{a^2+1}\,(2a^2+6)\right)$$

$$= \frac{\pi\ell(\sqrt{a^2+1}+a)}{2(1+a^2)^2} \left(a(a^2-1) + \sqrt{a^2+1}\,(a^2+3)\right)$$

$$= \frac{\pi\ell}{2(1+a^2)^2} \left(a^2(a^2-1) + (a^2+1)(a^2+3) + a\sqrt{a^2+1}(2a^2+2)\right)$$

$$= \frac{\pi\ell}{2(1+a^2)^2} \left(2a^4 + 3a^2 + 3 + 2a(a^2+1)^{3/2}\right)$$

$$R_4 = \frac{16A\ell}{\sqrt{a^2+1}} \int_{-\infty}^{\infty} \frac{\eta^2(\eta^2-a)(\eta^2-E)}{P^4}\, d\eta$$

$$+ \frac{4A\ell}{(a^2+1)} \int_{-\infty}^{\infty} \frac{\eta^2(\eta^2-a)(a\eta^2+F)}{P^3}\, d\eta - \frac{3a^2+1}{4(a+1)} R_3$$

where $F = a\sqrt{a^2+1} - 2(1+a^2)$

$$R_4 = \frac{16A\ell}{\sqrt{a^2+1}} \left(\frac{J_3}{4} - \frac{EI_3}{12}\right) + \frac{4A\ell}{a^2+1} \left(\frac{3aJ_2}{8} + \frac{FI_2}{8}\right) - \frac{3a^2+1}{4(a^2+1)} R_3$$

$$= T_1 + T_2 + T_3$$

$$\frac{EI_3}{12} = \frac{E}{12(1+a^2)} \left(aJ_3 + \frac{7}{8} I_2\right)$$

$$\left(\frac{1}{4} - \frac{aE}{12(1+a^2)}\right) J_3 = \frac{3(1+a^2)-aE}{12(1+a^2)} \left(\frac{aI_2}{8} + \frac{5J_2}{8}\right)$$

$$T_1 = \frac{A\ell}{6(1+\alpha^2)^{3/2}} \left(5(3(1+\alpha^2) - \alpha E)J_2 + (3\alpha(1+\alpha^2) - E(\alpha^2+7))I_2\right)$$

$$T_1 + T_2 = \frac{A\ell}{6(1+\alpha^2)^{3/2}} \left((9\alpha\sqrt{\alpha^2+1} + 5(3(1+\alpha^2) - \alpha E))J_2\right.$$

$$\left. +(3(\sqrt{1+\alpha^2}\,F + \alpha(1+\alpha^2)) - E(\alpha^2+7))I_2\right)$$

$$9\alpha\sqrt{\alpha^2+1} + 5(3(1+\alpha^2) - \alpha E) = 14\alpha\sqrt{\alpha^2+1} + 5(\alpha^2+3)$$

$$3(\sqrt{1+\alpha^2}\,F + \alpha(1+\alpha^2)) - E(\alpha^2+7) = 3(2\alpha(1+\alpha^2) - 2(1+\alpha^2)^{3/2}) + (\sqrt{\alpha^2+1} - 2\alpha)(\alpha^2+7)$$

$$= 2\alpha(2\alpha^2-4) + \sqrt{\alpha^2+1}\,(1-5\alpha^2)$$

$$(2\alpha(2\alpha^2-4) + \sqrt{\alpha^2+1}(1-5\alpha^2))I_2 = \frac{4\alpha(\alpha^2-2) + \sqrt{\alpha^2+1}(1-5\alpha^2)}{\alpha^2+1}\left(\alpha J_2 + \frac{3I_1}{4}\right)$$

$$\left(14\alpha\sqrt{\alpha^2+1} + 5(\alpha^2+3) + \frac{4\alpha^2(\alpha^2-2) + \alpha\sqrt{\alpha^2+1}(1-5\alpha^2)}{\alpha^2+1}\right)J_2$$

$$= \frac{9\alpha^4 + 12\alpha^2 + 15 + \alpha\sqrt{\alpha^2+1}(9\alpha^2+15)}{\alpha^2+1}\left(\frac{\alpha I_1}{4} + \frac{J_1}{4}\right)$$

$$T_1 + T_2 = \frac{A\ell}{8(1+\alpha^2)^{5/2}} \left((3\alpha^4 + 4\alpha^2 + 5 + \alpha\sqrt{\alpha^2+1}\,(3\alpha^2+5))J_1\right.$$

$$\left. + (\alpha(3\alpha^4 + 8\alpha^2 - 3) + \sqrt{1+\alpha^2}\,(3\alpha^4 + 1))I_1\right)$$

$$= \frac{A\ell I_1}{8(1+\alpha^2)^{5/2}} \left(\alpha(6\alpha^4 + 16\alpha^2 + 2) + \sqrt{\alpha^2+1}\,(6\alpha^4 + 4\alpha^2 + 6)\right)$$

$$T_3 = \frac{-(3\alpha^2+1)A\ell I_1}{8(1+\alpha^2)^{5/2}} \left(\alpha(2\alpha^2 - 2) + \sqrt{\alpha^2+1}\ (2\alpha^2+6)\right)$$

$$= \frac{-A\ell I_1}{8(1+\alpha^2)^{5/2}} \left(\alpha(6\alpha^4 - 4\alpha^2 - 2) + \sqrt{\alpha^2+1}\ (6\alpha^4 + 20\alpha^2 + 6)\right)$$

$$R_4 = \frac{A\ell I_1}{8(1+\alpha^2)^{5/2}} \left(\alpha(20\alpha^2 + 4) - 16\alpha^2 \sqrt{\alpha^2+1}\right)$$

$$= \frac{\pi\ell(\sqrt{\alpha^2+1}+\alpha)}{16(1+\alpha^2)^3} \left(16\alpha^2(\alpha - \sqrt{\alpha^2+1}) + 4\alpha(\alpha^2+1)\right)$$

$$= \frac{\pi\ell}{4(1+\alpha^2)^3} \left(\alpha(\alpha^2+1)(\sqrt{\alpha^2+1} + \alpha) - 4\alpha^2\right)$$

$$R_\alpha = \sum_1^2 c_{k0}R_k^*$$

$$c_{10}R_1^* = \frac{3\ell\sigma BI_1}{\alpha^2+1} \left(2\alpha\sqrt{\alpha^2+1} + 1 - 3\alpha^2\right)(\sqrt{\alpha^2+1} - \alpha)$$

$$= \frac{3\ell\sigma BI_1}{\alpha^2+1} \left(\alpha(5\alpha^2+1) + \sqrt{\alpha^2+1}\ (1 - 5\alpha^2)\right)$$

$$c_{20}R_2^* = \frac{\sigma B\ell I_1}{\alpha^2+1} \left(2\alpha - \sqrt{\alpha^2+1}\ (\alpha\sqrt{\alpha^2+1} + 1 - \alpha^2)\right)$$

$$= \frac{\sigma B\ell I_1}{\alpha^2+1} \left(\alpha(1 - 3\alpha^2) + \sqrt{\alpha^2+1}\ (3\alpha^2 - 1)\right)$$

$$R_\alpha = \frac{\sigma B \ell I_1}{\alpha^2+1}\left(\alpha(12\alpha^2+4) + \sqrt{\alpha^2+1}\,(2-12\alpha^2)\right)$$

$$= \frac{\sigma \ell \pi}{(\alpha^2+1)^{3/2}}\left(\alpha(6\alpha^2+2) + \sqrt{\alpha^2+1}\,(1-6\alpha^2)\right)$$

$$= \frac{-U^2 M \ell \pi}{\alpha(\alpha^2+1)^{3/2}}\left(\sqrt{\alpha^2+1}+\alpha\right)^2\left((1-6\alpha^2)(\sqrt{\alpha^2+1}-\alpha) + 3\alpha\right)$$

$$= \frac{-U^2 M \ell \pi}{\alpha(\alpha^2+1)^{3/2}}\left(\sqrt{\alpha^2+1}+\alpha\right)\left(1-3\alpha^2 + 3\alpha\sqrt{\alpha^2+1}\right)$$

$$= \frac{-U^2 M \ell \pi}{\alpha(\alpha^2+1)^{3/2}}\left(4\alpha + \sqrt{\alpha^2+1}\right)$$

$$S_1 = 16\ell^2 \int_{-\infty}^{\infty} \frac{\eta^2-\alpha}{P^3}\, d\eta$$

$$= 16\ell^2(J_3 - \alpha I_3)$$

$$-\alpha I_3 = \frac{-\alpha}{1+\alpha^2}\left(\frac{7}{8} I_2 + \alpha J_3\right)$$

$$\left(1-\frac{\alpha^2}{1+\alpha^2}\right)J_3 = \frac{1}{1+\alpha^2}\left(\frac{5}{8} J_2 + \frac{\alpha I_2}{8}\right)$$

$$S_1 = \frac{2\ell^2}{1+\alpha^2}(5J_2 - 6\alpha I_2)$$

$$= \frac{3\ell}{2(1+\alpha^2)} R_1 - \frac{2\ell^2 J_2}{1+\alpha^2}$$

$$= \frac{3\ell^2}{1+\alpha^2}\left(\frac{1}{\sqrt{1+\alpha^2}} - \frac{2\alpha}{1+\alpha^2}\right)I_1 - \frac{\ell^2}{2(1+\alpha^2)}\left(\sqrt{\alpha^2+1}+\alpha\right)I_1$$

$$= -\ell^2 I_1\left(\frac{1}{2\sqrt{1+\alpha^2}} + \frac{\alpha}{2(1+\alpha^2)} - \frac{3}{(1+\alpha^2)^{3/2}} + \frac{6\alpha}{(1+\alpha^2)^2}\right)$$

$$S_1^* = -8\ell^2 \int_{-\infty}^{\infty} \frac{\eta^2-\alpha}{\eta^2 P^3}\, d\eta$$

$$= -8\ell^2 (I_3 - \alpha \int_{-\infty}^{\infty} \frac{d\eta}{\eta^2 P^3})$$

$$\int_{-\infty}^{\infty} \frac{d\eta}{\eta^2 P^3} = - \frac{1}{\eta P^3} \Big]_{-\infty}^{\infty} - 3 \int_{-\infty}^{\infty} \frac{P' d\eta}{\eta P^4}$$

$$= -12 \int_{-\infty}^{\infty} \frac{(\eta^2-\alpha)\, d\eta}{P^4}$$

$$S_1^* = -8\ell^2 (I_3 + 12\alpha (J_4 - \alpha I_4))$$

$$-\alpha I_4 = \frac{-\alpha}{1+\alpha^2} (\frac{11}{12} I_3 + \alpha J_4)$$

$$(1 - \frac{\alpha^2}{1+\alpha^2}) J_4 = \frac{1}{1+\alpha^2} (\frac{\alpha I_3}{12} + \frac{3}{4} J_3)$$

$$S_1^* = -8\ell^2 (I_3 + \frac{\alpha}{1+\alpha^2} (9J_3 - 10\alpha I_3))$$

$$= \frac{-9\alpha}{2(1+\alpha^2)} S_1 - \frac{8\alpha^2}{1+\alpha^2} I_3$$

$$I_3 = \frac{1}{1+\alpha^2} (\frac{7}{8} I_2 + \alpha J_3)$$

$$= \frac{1}{1+\alpha^2} (\frac{7+\alpha^2}{8} I_2 + \frac{5\alpha}{8} J_2)$$

$$(7+\alpha^2) I_2 = \frac{7+\alpha^2}{1+\alpha^2} (\frac{3}{4} I_1 + \alpha J_2)$$

$$(5 + \frac{7+\alpha^2}{1+\alpha^2})\alpha J_2 = 6(1 + \frac{1}{1+\alpha^2})(\frac{\alpha^2 I_1}{4} + \frac{\alpha J_1}{4})$$

$$I_3 = \frac{1}{32(1+\alpha^2)^2}\,(6(2+\alpha^2)\alpha J_1 + (3(7+\alpha^2) + 6\alpha^2(2+\alpha^2))I_1)$$

$$= \frac{3}{32(1+\alpha^2)^2}\,(2(2+\alpha^2)\alpha\sqrt{\alpha^2+1} + 7 + 5\alpha^2 + 2\alpha^4)I_1$$

$$S_1^* = \ell^2 I_1 \left[\frac{9\alpha}{2(1+\alpha^2)}\,(\frac{1}{2\sqrt{1+\alpha^2}} + \frac{\alpha}{2(1+\alpha^2)} - \frac{3}{(1+\alpha^2)^{3/2}} + \frac{6\alpha}{(1+\alpha^2)^2})\right.$$

$$\left. - \frac{3}{4(1+\alpha^2)^3}\,(2(2+\alpha^2)\alpha\sqrt{\alpha^2+1} + 7 + 5\alpha^2 + 2\alpha^4)\right]$$

$$= \frac{3\ell^2 I_1}{4(1+\alpha^2)^3}\,(3\alpha(1+\alpha^2)^{3/2} + 3\alpha^2(1+\alpha^2) - 18\alpha\sqrt{1+\alpha^2} + 36\alpha^2 - 2\alpha(1+\alpha^2)^{3/2} - 2\alpha\sqrt{1+\alpha^2}$$

$$- 7 - 5\alpha^2 - 2\alpha^4)$$

$$= \frac{3\ell^2 I_1}{4(1+\alpha^2)^3}\,(\alpha(1+\alpha^2)^{3/2} - 20\alpha\sqrt{1+\alpha^2} - 7 + 34\alpha^2 + \alpha^4)$$

$$S_2 = -16\ell^2 \int_{-\infty}^{\infty} \frac{\eta^2(\eta^2-\alpha)}{P^3}\,d\eta$$

$$= -2\ell^2 I_2$$

$$= \frac{-2\ell^2}{1+\alpha^2}\,(\frac{3I_1}{4} + \alpha J_2)$$

$$= \frac{-\ell^2}{2(1+\alpha^2)}\,((3+\alpha^2)I_1 + \alpha J_1)$$

$$= \frac{-\ell^2}{2(1+\alpha^2)}\,(3 + \alpha^2 + \alpha\sqrt{\alpha^2+1})I_1$$

$$S_2^* = -\frac{1}{2} S_1$$

$$= \frac{\ell^2 I_1}{2} \left(\frac{1}{2\sqrt{1+\alpha^2}} + \frac{\alpha}{2(1+\alpha^2)} - \frac{3}{(1+\alpha^2)^{3/2}} + \frac{6\alpha}{(1+\alpha^2)^2}\right)$$

$$S_3 = \frac{32A\ell^2}{\sqrt{\alpha^2+1}} \int_{-\infty}^{\infty} \frac{\eta^2(\eta^2-\alpha)(\eta^2-E)}{P^4}\, d\eta$$

$$= \frac{32A\ell^2}{\sqrt{\alpha^2+1}} \left(\frac{J_3}{4} - \frac{EL_3}{12}\right)$$

$$= 2\ell T_1 \qquad \text{(See } R_4\text{)}$$

$$= \frac{A\ell^2}{3(1+\alpha^2)^{3/2}} \left(5(3(1+\alpha^2)-\alpha E)J_2 + (3\alpha(1+\alpha^2)-E(\alpha^2+7))I_2\right)$$

$$(3\alpha(1+\alpha^2) - E(\alpha^2+7))I_2 = \left(3\alpha - \frac{E(\alpha^2+7)}{\alpha^2+1}\right)\left(\frac{3I_1}{4} + \alpha J_2\right)$$

$$\left(15(1+\alpha^2)-5\alpha E+3\alpha^2 - \frac{\alpha E(\alpha^2+7)}{\alpha^2+1}\right)J_2 = \left(15 + 18\alpha^2 - \frac{6\alpha E(\alpha^2+2)}{\alpha^2+1}\right)\left(\frac{\alpha+\sqrt{\alpha^2+1}}{4}\right)I_1$$

$$S_3 = \frac{A\ell^2 I_1}{4(1+\alpha^2)^{3/2}} \left(3\alpha+(5+6\alpha^2)(\alpha+\sqrt{\alpha^2+1}) - \frac{E}{\alpha^2+1}\left(\alpha^2+7+2\alpha(\alpha^2+2)(\alpha+\sqrt{\alpha^2+1})\right)\right)$$

$$E(\alpha^2+7+2\alpha(\alpha^2+2)(\alpha+\sqrt{\alpha^2+1}))) = \alpha(\alpha^2+7)+2\alpha^2(\alpha^2+2)(\alpha+\sqrt{\alpha^2+1})+(\alpha^2+7)(\alpha-\sqrt{\alpha^2+1})$$

$$-2\alpha(\alpha^2+2)$$

$$= 2\alpha(\alpha^2+7)+2\alpha(\alpha^2+2)(\alpha^2-1)+\sqrt{\alpha^2+1}(2\alpha^2(\alpha^2+2)-\alpha^2-7)$$

$$= 2\alpha(\alpha^4+2\alpha^2+5)+\sqrt{\alpha^2+1}\ (2\alpha^4+3\alpha^2-7)$$

$$S_3 = \frac{A\ell^2 I_1}{4(1+\alpha^2)^{5/2}}\ (2\alpha((3\alpha^2+4)(\alpha^2+1) - \alpha^4 - 2\alpha^2 - 5)$$

$$+ \sqrt{\alpha^2+1}((6\alpha^2+5)(1+\alpha^2) - 2\alpha^4 - 3\alpha^2 + 7))$$

$$= \frac{\pi\ell^2(\sqrt{\alpha^2+1}+\alpha)}{8(1+\alpha^2)^3}\ (2\alpha(2\alpha^4 + 5\alpha^2 - 1) + \sqrt{\alpha^2+1}\ (4\alpha^4 + 8\alpha^2 + 12))$$

$$= \frac{\pi\ell^2}{4(1+\alpha^2)^3}\ (\alpha^2(2\alpha^4 + 5\alpha^2 - 1) + (\alpha^2+1)(2\alpha^4 + 4\alpha^2 + 6)$$

$$+ \alpha\sqrt{\alpha^2+1}\ (2\alpha^4 + 5\alpha^2 - 1 + 2\alpha^4 + 4\alpha^2 + 6))$$

$$= \frac{\pi\ell^2}{4(1+\alpha^2)^3}\ (4\alpha^6 + 11\alpha^4 + 9\alpha^2 + 6 + \alpha\sqrt{\alpha^2+1}\ (4\alpha^4 + 9\alpha^2 + 5))$$

$$S_4 = \frac{32A\ell^2}{\sqrt{\alpha^2+1}} \int_{-\infty}^{\infty} \frac{\eta^2(\eta^2-\alpha)(\eta^2-E)\,d\eta}{P^5} + \frac{8A\ell^2}{(\alpha^2+1)} \int_{-\infty}^{\infty} \frac{\eta^2(\eta^2-\alpha)(\alpha\eta^2+F)}{P^4}\,d\eta - \frac{3\alpha^2+1}{4(\alpha^2+1)}\ S_3$$

$$= \frac{32A\ell^2}{\sqrt{\alpha^2+1}}\ (\frac{3J_4}{16} - \frac{EI_4}{16}) + \frac{8A\ell^2}{(\alpha^2+1)}\ (\frac{\alpha J_3}{4} + \frac{FI_3}{12}) - \frac{3\alpha^2+1}{4(\alpha^2+1)}\ S_3$$

$$= X_1 + X_2 + X_3$$

$$-EI_4 = \frac{-E}{1+\alpha^2}\left(\frac{11}{12} I_3 + \alpha J_4\right)$$

$$\left(3 - \frac{\alpha E}{1+\alpha^2}\right)J_4 = \left(3 - \frac{\alpha E}{1+\alpha^2}\right)\left(\frac{\alpha I_3}{12} + \frac{3J_3}{4}\right)$$

$$X_1 = \frac{A\ell^2}{6\sqrt{\alpha^2+1}}\left(9\left(3 - \frac{\alpha E}{1+\alpha^2}\right)J_3 + \left(3\alpha - \frac{(11+\alpha^2)E}{1+\alpha^2}\right)I_3\right)$$

$$X_1 + X_2 = \frac{A\ell^2}{6(\alpha^2+1)}\left(\left(12\alpha + 9\sqrt{\alpha^2+1}\left(3 - \frac{\alpha E}{1+\alpha^2}\right)\right)J_3\right.$$

$$\left. + \left(4F + \sqrt{\alpha^2+1}\left(3\alpha - \frac{(11+\alpha^2)E}{1+\alpha^2}\right)\right)I_3\right)$$

$$12\alpha + 9\sqrt{\alpha^2+1}\left(3 - \frac{\alpha E}{1+\alpha^2}\right) = 12\alpha + 27\sqrt{\alpha^2+1} - \frac{18\alpha^2}{\sqrt{\alpha^2+1}} + 9\alpha$$

$$= 21\alpha + \frac{9\alpha^2 + 27}{\sqrt{\alpha^2+1}}$$

$$4F + \sqrt{\alpha^2+1}\left(3\alpha - \frac{(11+\alpha^2)E}{1+\alpha^2}\right) = 4\alpha\sqrt{\alpha^2+1} - 8(1+\alpha^2) + 3\alpha\sqrt{\alpha^2+1} - \frac{(11+\alpha^2)2\alpha}{\sqrt{\alpha^2+1}} + 11 + \alpha^2$$

$$= \frac{\alpha}{\sqrt{\alpha^2+1}}(5\alpha^2 - 15) - 7\alpha^2 + 3$$

$$\left(\frac{5\alpha}{\sqrt{\alpha^2+1}}(\alpha^2-3) - 7\alpha^2 + 3\right)I_3 = \left(\frac{5\alpha(\alpha^2-3)}{(\alpha^2+1)^{3/2}} + \frac{3-7\alpha^2}{\alpha^2+1}\right)\left(\frac{7I_2}{8} + \alpha J_3\right)$$

$$(21\alpha + \frac{9(\alpha^2+3)}{\sqrt{\alpha^2+1}} + \frac{5\alpha^2(\alpha^2-3)}{(\alpha^2+1)^{3/2}} + \frac{\alpha(3-7\alpha^2)}{\alpha^2+1})J_3 =$$

$$(\frac{\alpha}{\alpha^2+1}(14\alpha^2+24) + \frac{14\alpha^4+21\alpha^2+27}{(\alpha^2+1)^{3/2}})(\frac{\alpha I_2}{8} + \frac{5J_2}{8})$$

$$X_1 + X_2 = \frac{A\ell^2}{48(\alpha^2+1)^2}((5\alpha(14\alpha^2+24) + \frac{5(14\alpha^4+21\alpha^2+27)}{\sqrt{\alpha^2+1}})J_2$$

$$+ (14\alpha^4 - 25\alpha^2 + 21 + \frac{\alpha}{\sqrt{\alpha^2+1}}(14\alpha^4 + 56\alpha^2 - 78))I_2)$$

$$I_2 = \frac{1}{1+\alpha^2}(\frac{3}{4}I_1 + \alpha J_2)$$

$$J_2 = \frac{\alpha + \sqrt{\alpha^2+1}}{4} I_1$$

$$I_2 = \frac{1}{4(1+\alpha^2)}(3 + \alpha^2 + \alpha\sqrt{\alpha^2+1})I_1$$

$$AI_1 = \frac{\pi(\sqrt{\alpha^2+1} + \alpha)}{2\sqrt{\alpha^2+1}}$$

$$X_1 + X_2 = \frac{\pi\ell^2(\sqrt{\alpha^2+1}+\alpha)}{384(\alpha^2+1)^{5/2}}(5\alpha^2(14\alpha^2+24) + 5(14\alpha^4 + 21\alpha^2 + 27)$$

$$+ \frac{\alpha}{\sqrt{\alpha^2+1}}(5(\alpha^2+1)(14\alpha^2+24) + 5(14\alpha^4+21\alpha^2+27))$$

$$+ \frac{\alpha^2+3}{\alpha^2+1}(14\alpha^4-25\alpha^2+21) + \frac{\alpha^2}{\alpha^2+1}(14\alpha^4+56\alpha^2-78)$$

$$+ \frac{\alpha}{(\alpha^2+1)^{3/2}}((\alpha^2+1)(14\alpha^4-25\alpha^2+21) + (\alpha^2+3)(14\alpha^4+56\alpha^2-78)))$$

$$= \frac{\pi \ell^2 (\sqrt{\alpha^2+1}+\alpha)}{384(\alpha^2+1)^{5/2}} \left(168\alpha^2+256\alpha^2+78+ \frac{14\alpha^4-106\alpha^2+120}{\alpha^2+1}\right.$$

$$\left. + \frac{\alpha}{\sqrt{\alpha^2+1}} \left(168\alpha^4+326\alpha^2+198 + \frac{28\alpha^4+112\alpha^2-156}{\alpha^2+1}\right)\right)$$

$$= \frac{\pi \ell^2 (\sqrt{\alpha^2+1}+\alpha)}{384(\alpha^2+1)^{5/2}} \left(168\alpha^4+270\alpha^2-42 + \frac{240}{\alpha^2+1}\right.$$

$$\left. + \frac{\alpha}{\sqrt{\alpha^2+1}} \left(168\alpha^4+354\alpha^2+282 - \frac{240}{\alpha^2+1}\right)\right)$$

$$= \frac{\pi \ell^2}{384(\alpha^2+1)^{5/2}} \left(\alpha(336\alpha^4 + 624\alpha^2 + 240)\right.$$

$$+ \frac{1}{\sqrt{\alpha^2+1}} \left((\alpha^2+1)(168\alpha^4 + 270\alpha^2 - 42)\right.$$

$$\left.\left. + 240 + \alpha^2 \left(168\alpha^4 + 354\alpha^2 + 282 - \frac{240}{\alpha^2+1}\right)\right)\right)$$

$$= \frac{\pi \ell^2}{384(\alpha^2+1)^{5/2}} \left(48\alpha(7\alpha^4 + 13\alpha^2 + 5)\right.$$

$$\left. + \frac{1}{\sqrt{\alpha^2+1}} \left(336\alpha^6 + 792\alpha^4 + 510\alpha^2 - 42 + \frac{240}{\alpha^2+1}\right)\right)$$

$$X_3 = \frac{-\pi \ell^2 (3\alpha^2+1)}{16(\alpha^2+1)^4} \left(4\alpha^6 + 11\alpha^4 + 9\alpha^2 + 6 + \alpha\sqrt{\alpha^2+1}\,(4\alpha^4 + 9\alpha^2 + 5)\right)$$

$$S_4 = \frac{\pi\ell^2}{384(\alpha^2+1)^4}\,(-24(3\alpha^2+1)(4\alpha^6+11\alpha^4+9\alpha^2+6) + 240$$

$$+ (\alpha^2+1)(336\alpha^6+792\alpha^4+510\alpha^2-42)$$

$$+ \alpha\sqrt{\alpha^2+1}\,(-24(3\alpha^2+1)(4\alpha^4+9\alpha^2+5) + 48(\alpha^2+1)(7\alpha^4+13\alpha^2+5)))$$

$$= \frac{\pi\ell^2}{64(\alpha^2+1)^4}\,(8\alpha^8+40\alpha^6+65\alpha^4-30\alpha^2+9$$

$$+ 4\alpha\sqrt{\alpha^2+1}\,(2\alpha^6+9\alpha^4+12\alpha^2+5))$$

$$S_\alpha = c_{10}S_1^* + c_{20}S_2^*$$

$$c_{10}S_1^* = \frac{-3\sigma B\ell^2 I_1(\sqrt{\alpha^2+1}-\alpha)}{4(\alpha^2+1)^2}\,(\alpha^4+34\alpha^2 - 7 + \alpha\sqrt{\alpha^2+1}\,(\alpha^2-19))$$

$$= \frac{-3\sigma\ell^2\pi}{8(\alpha^2+1)^{5/2}}\,(\alpha((\alpha^2+1)(\alpha^2-19)-\alpha^4-34\alpha^2+7)+\sqrt{\alpha^2+1}(\alpha^4+34\alpha^2-7-\alpha^2(\alpha^2-19)))$$

$$= \frac{-3\sigma\ell^2\pi}{8(\alpha^2+1)^{5/2}}\,(\alpha(-52\alpha^2-12) + \sqrt{\alpha^2+1}\,(53\alpha^2-7))$$

$$c_{20}S_2^* = \frac{\sigma B\ell^2 I_1(\alpha\sqrt{\alpha^2+1}+1-\alpha^2)}{4(\alpha^2+1)^2}\,(\alpha(\alpha^2+13) + \sqrt{\alpha^2+1}\,(\alpha^2-5))$$

$$= \frac{\sigma\ell^2\pi}{8(\alpha^2+1)^{5/2}}(\alpha(\alpha^2+13)(1-\alpha^2)+(\alpha^2+1)(\alpha^2-5))+\sqrt{\alpha^2+1}((\alpha^2-5)(1-\alpha^2)+\alpha^2(\alpha^2+13)))$$

$$= \frac{\sigma\ell^2\pi}{8(\alpha^2+1)^{5/2}}\,(\alpha(-16\alpha^2+8) + \sqrt{\alpha^2+1}\,(19\alpha^2-5))$$

$$S_\alpha = \frac{\sigma \ell^2 \pi}{8(\alpha^2+1)^{5/2}} \left(\alpha(140\alpha^2+44) + \sqrt{\alpha^2+1}\ (-140\alpha^2+16)\right)$$

$$= \frac{-U^2 M \ell^2 \pi}{2\alpha(\alpha^2+1)^{5/2}} \left(\sqrt{\alpha^2+1}+\alpha\right)^2 \left(35\alpha^2(\alpha - \sqrt{\alpha^2+1}) + 11\alpha + 4\sqrt{\alpha^2+1}\right)$$

$$= \frac{-U^2 M \ell^2 \pi}{2\alpha(\alpha^2+1)^{5/2}} \left(\sqrt{\alpha^2+1}+\alpha\right)\left(-20\alpha^2 + 4 + 15\alpha\sqrt{\alpha^2+1}\right)$$

$$= \frac{-U^2 M \ell^2 \pi}{2\alpha(\alpha^2+1)^{5/2}} \left(\alpha(-5\alpha^2+19) + \sqrt{\alpha^2+1}\ (-5\alpha^2+4)\right)$$

Part 2: (Steady Force and Moment)

$$f_o = \frac{-d}{2} \sum_{1}^{2} c_{k0}R_k$$

$$c_{10}R_1 = -2\ell\sigma BI_1(\sqrt{\alpha^2+1} - \alpha)(\sqrt{\alpha^2+1} - 2\alpha)$$

$$= -2\ell\sigma BI_1(3\alpha^2 + 1 - 3\alpha\sqrt{\alpha^2+1})$$

$$c_{20}R_2 = -2\ell\sigma BI_1(\alpha\sqrt{\alpha^2+1} + 1 - \alpha^2)$$

$$f_o = d\ell\sigma BI_1(2\alpha^2+2 - 2\alpha\sqrt{\alpha^2+1})$$

$$= d\ell\sigma\pi(\sqrt{\alpha^2+1} - \alpha)$$

$$= -U^2Md\ell\pi(1 + \frac{\sqrt{\alpha^2+1}}{\alpha})$$

$$m_o = \frac{-d}{2}(c_{10}S_1 + c_{20}S_2)$$

$$c_{10}S_1 = \sigma B\ell^2 I_1 \frac{(\sqrt{\alpha^2+1} - \alpha)}{2(\alpha^2+1)}((\alpha^2+1)(\sqrt{\alpha^2+1} + \alpha) + 12\alpha - 6\sqrt{\alpha^2+1})$$

$$= \frac{\sigma\ell^2\pi}{4(\alpha^2+1)^{3/2}}(\alpha^2+1 - 6(\alpha^2+1) - 12\alpha^2 + 18\alpha\sqrt{\alpha^2+1})$$

$$= \frac{\sigma\ell^2\pi}{4(\alpha^2+1)^{3/2}}(-17\alpha^2 - 5 + 18\alpha\sqrt{\alpha^2+1})$$

$$c_{20}S_2 = -\sigma B\ell^2 I_1 \frac{(\alpha\sqrt{\alpha^2+1} + 1 - \alpha^2)}{2(\alpha^2+1)} (3 + \alpha^2 + \alpha\sqrt{\alpha^2+1})$$

$$= \frac{-\sigma\ell^2\pi}{4(\alpha^2+1)^{3/2}} (\alpha^2(\alpha^2+1) + (1-\alpha^2)(3+\alpha^2) + 4\alpha\sqrt{\alpha^2+1})$$

$$= \frac{-\sigma\ell^2\pi}{4(\alpha^2+1)^{3/2}} (-\alpha^2 + 3 + 4\alpha\sqrt{\alpha^2+1})$$

$$m_o = \frac{\sigma d\ell^2\pi}{8(\alpha^2+1)^{3/2}} (16\alpha^2 + 8 - 14\alpha\sqrt{\alpha^2+1})$$

$$= \frac{-U^2 Md\ell^2\pi}{4\alpha(\alpha^2+1)^{3/2}} (\sqrt{\alpha^2+1} + \alpha)^2 (7\sqrt{\alpha^2+1}\ (\sqrt{\alpha^2+1} - \alpha) + \alpha^2 - 3)$$

$$= \frac{-U^2 Md\ell^2\pi}{4\alpha(\alpha^2+1)^{3/2}} (\sqrt{\alpha^2+1} + \alpha)((\alpha^2+4)\sqrt{\alpha^2+1} + \alpha(\alpha^2 - 3))$$

$$= \frac{-U^2 Md\ell^2\pi}{4\alpha(\alpha^2+1)^{3/2}} ((\alpha^2+4)(\alpha^2+1) + \alpha^2(\alpha^2-3) + (2\alpha^2+1)\ \alpha\sqrt{\alpha^2+1})$$

$$= \frac{-U^2 Md\ell^2\pi}{4\alpha(\alpha^2+1)^{3/2}} (2\alpha^4 + 2\alpha^2 + 4 + (2\alpha^2+1)\ \alpha\sqrt{\alpha^2+1})$$

$$= \frac{-U^2 Md\ell^2\pi}{4\alpha(\alpha^2+1)^{3/2}} (4 + \alpha\sqrt{\alpha^2+1}\ (\sqrt{\alpha^2+1} + \alpha)^2)$$

Table I

Index	R	S
1)	$\dfrac{2\ell(\sqrt{\alpha^2+1}-2\alpha)I_1(-\alpha)}{\alpha^2+1}$	$\dfrac{-\ell^2(\alpha(\alpha^2+13)+\sqrt{\alpha^2+1}(\alpha^2-5))I_1(-\alpha)}{2(\alpha^2+1)^2}$
2)	$-2\ell I_1(-\alpha)$	$\dfrac{-\ell^2(3+\alpha^2+\alpha\sqrt{\alpha^2+1})I_1(-\alpha)}{2(\alpha^2+1)}$
3)	$\dfrac{\pi\ell}{2(\alpha^2+1)^2}(2\alpha^4+3\alpha^2+3+2\alpha(\alpha^2+1)^{3/2})$	$\dfrac{\pi\ell^2}{4(\alpha^2+1)^3}(4\alpha^6+11\alpha^4+9\alpha^2+6 + \alpha\sqrt{\alpha^2+1}\,(4\alpha^4+9\alpha^2+5))$
4)	$\dfrac{\pi\ell}{4(\alpha^2+1)^3}(\alpha(\alpha^2+1)(\sqrt{\alpha^2+1}+\alpha)-4\alpha^2)$	$\dfrac{\pi\ell^2}{64(\alpha^2+1)^4}(8\alpha^8+40\alpha^6+65\alpha^4-30\alpha^2+9 + 4\alpha\sqrt{\alpha^2+1}(2\alpha^6+9\alpha^4+12\alpha^2+5))$
α)	$\dfrac{-U^2M\ell\pi}{\alpha(\alpha^2+1)^{3/2}}(4\alpha+\sqrt{\alpha^2+1})$	$\dfrac{-U^2M\ell^2\pi}{2\alpha(\alpha^2+1)^{5/2}}(\alpha(-5\alpha^2+19)+\sqrt{\alpha^2+1}(-5\alpha^2+4))$

APPENDIX VI - ψ_k, ϕ_k, $\frac{\partial\psi_k}{\partial\alpha}$, $\frac{\partial\phi_k}{\partial\alpha}$ Tabulated

Notes for Tables

$$A = \sqrt{\frac{\sqrt{\alpha^2 + 1} + \alpha}{2}}$$

$$B = \sqrt{\frac{\sqrt{\alpha^2 + 1} - \alpha}{2}}$$

$$\psi_k = \tilde{\psi}_k - \tilde{\psi}_k(\infty)$$

$$\frac{\partial\psi_k}{\partial\alpha} = \frac{\partial\tilde{\psi}_k}{\partial\alpha} - \frac{\partial\tilde{\psi}_k}{\partial\alpha}(\infty)$$

Table I

Harmonic Functions

Index	ϕ_k	$\frac{\partial \phi_k}{\partial \alpha}$	$\tilde{\psi}_k$	$\frac{\partial \tilde{\psi}_k}{\partial \alpha}$
1)	$\frac{-\eta}{\eta^2+\xi^2}$	$\frac{\eta(\eta^2-3\eta^2)}{2(\eta^2+\xi^2)^3}$	$\frac{\xi}{\eta^2+\xi^2}$	$\frac{\xi(\xi^2-3\eta^2)}{2(\eta^2+\xi^2)^3}$
2)	η	$\frac{\eta}{2(\eta^2+\xi^2)}$	ξ	$\frac{-\xi}{2(\eta^2+\xi^2)}$
3)	$\frac{1}{\sqrt{\alpha^2+1}}\left(\frac{\alpha-B\xi-A\eta}{\sqrt{\alpha^2+1}+\eta^2+\xi^2+2B\xi-2A\eta}-\frac{\alpha-B\xi+A\eta}{\sqrt{\alpha^2+1}+\eta^2+\xi^2+2B\xi+2A\eta}\right)$		$\frac{1}{\sqrt{\alpha^2+1}}\left(\frac{1+A\xi-B\eta}{\sqrt{\alpha^2+1}+\eta^2+\xi^2+2B\xi-2A\eta}+\frac{1+A\xi+B\eta}{\sqrt{\alpha^2+1}+\eta^2+\xi^2+2B\xi+2A\eta}\right)$	
4)	$\frac{1}{4(\alpha^2+1)}\left(\frac{\eta^2-\xi^2+2\alpha+2(\alpha A-B)\xi-2(A+\alpha B)\eta-2\alpha\xi\eta}{(\sqrt{\alpha^2+1}-\eta^2+\xi^2+2B\xi-2A\eta)^2}-\frac{\eta^2-\xi^2+2\alpha+2(\alpha A-B)\xi+2(A+\alpha B)\eta+2\alpha\xi\eta}{(\sqrt{\alpha^2+1}+\eta^2+\xi^2+2B\xi+2A\eta)^2}\right)$ $+\frac{1}{4(\alpha^2+1)^{3/2}}\left(\frac{\alpha\sqrt{\alpha^2+1}(\sqrt{\alpha^2+1}+B\xi-A\eta)-(3\alpha^2+1)(\alpha-B\xi-A\eta)}{\sqrt{\alpha^2+1}+\eta^2+\xi^2+2B\xi-2A\eta}-\frac{\alpha\sqrt{\alpha^2+1}(\sqrt{\alpha^2+1}+B\xi+A\eta)-(3\alpha^2+1)(\alpha-B\xi+A\eta)}{\sqrt{\alpha^2+1}+\eta^2+\xi^2+2B\xi+2A\eta}\right)$		$\frac{-1}{4(\alpha^2+1)}\left(\frac{\alpha^2-1+\alpha(\eta^2-\xi^2)-2(\alpha B+A)\xi+2(B-\alpha A)\eta+2\xi\eta}{(\sqrt{\alpha^2+1}+\eta^2+\xi^2+2B\xi-2A\eta)^2}+\frac{\alpha^2-1+\alpha(\eta^2-\xi^2)-2(\alpha B+A)\xi-2(B-\alpha A)\eta-2\xi\eta}{(\sqrt{\alpha^2+1}+\eta^2+\xi^2+2B\xi+2A\eta)^2}\right)$ $+\frac{1}{4(\alpha^2+1)^{3/2}}\left(\frac{\alpha\sqrt{\alpha^2+1}(A\xi+B\eta)-(3\alpha^2+1)(1+A\xi-B\eta)}{\sqrt{\alpha^2+1}+\eta^2+\xi^2+2B\xi-2A\eta}+\frac{\alpha\sqrt{\alpha^2+1}(A\xi-B\eta)-(3\alpha^2+1)(1+A\xi+B\eta)}{\sqrt{\alpha^2+1}+\eta^2+\xi^2+2B\xi+2A\eta}\right)$	

Table II

Harmonic Functions at ∞

Index	ϕ_k	$\frac{\partial \phi_k}{\partial \alpha}$	$\tilde{\psi}_k$	$\frac{\partial \tilde{\psi}_k}{\partial \alpha}$
1)	$\frac{A}{\sqrt{\alpha^2+1}}$	$\frac{-A(2\alpha-\sqrt{\alpha^2+1})}{2(\alpha^2+1)^{3/2}}$	$\frac{B}{\sqrt{\alpha^2+1}}$	$\frac{-B(\sqrt{\alpha^2+1}+2\alpha)}{2(\alpha^2+1)^{3/2}}$
2)	$-A$	$\frac{-A}{2\sqrt{\alpha^2+1}}$	B	$\frac{-B}{2\sqrt{\alpha^2+1}}$
3)	$\frac{\alpha+\sqrt{\alpha^2+1}}{2(\alpha^2+1)}$		$\frac{\alpha^2+2+\alpha\sqrt{\alpha^2+1}}{2(\alpha^2+1)}$	
4)	$\frac{\alpha(1-2\alpha^2-2\alpha\sqrt{\alpha^2+1})}{8(\alpha^2+1)^2}$		$\frac{2(1-\alpha^2)\alpha\sqrt{\alpha^2+1}-(2\alpha^4+11\alpha^2+3)}{16(\alpha^2+1)^2}$	

Table III

Harmonic Functions along Wetted Surface ($\xi=0$)

Index	ϕ_k	$\frac{\partial\phi_k}{\partial\alpha}$	$\tilde{\psi}_k$	$\frac{\partial\tilde{\psi}_k}{\partial\alpha}$
1)	$-\frac{1}{\eta}$	$\frac{1}{2n^3}$	0	0
2)	η	$\frac{1}{2\eta}$	0	0
3)	$-\frac{2A\eta(\eta^2-2\alpha+\sqrt{\alpha^2+1})}{\sqrt{\alpha^2+1}\,((\eta^2-\alpha)^2+1)}$		$\frac{x+\ell}{\ell}$	
4)	$-\frac{2A\eta(\eta^2-2\alpha+\sqrt{\alpha^2+1})}{\sqrt{\alpha^2+1}\,((\eta^2-\alpha)^2+1)^2}$		$\frac{(x+\ell)^2}{2\ell^2}-\frac{x+\ell}{\ell}$	
	$-\frac{A\eta(\alpha\eta^2+\alpha\sqrt{\alpha^2+1}-2(1+\alpha^2))}{2(\alpha^2+1)((\eta^2-\alpha)^2+1))}$			
	$-\frac{3\alpha^2+1}{4(\alpha^2+1)}\phi_3$			

Table IV

Harmonic Functions Along Cavity Surface ($\eta = 0$)

(All $\phi_k = 0$)

Index	$\tilde{\psi}_k$	$\frac{\partial \tilde{\psi}_k}{\partial \alpha}$
1)	$\frac{1}{\xi}$	$\frac{1}{2\xi^3}$
2)	ξ	$-\frac{1}{2\xi}$
3)	$\frac{x+\ell}{\ell} + \psi_3^*$	

where $\psi_3^* = \dfrac{2A\ (\xi^2+2\alpha - \sqrt{\alpha^2+1})}{\sqrt{\alpha^2+1}((\xi^2+\alpha)^2+1)}$

4) $\dfrac{(x+\ell)^2}{2\ell^2} - \dfrac{x+\ell}{\ell} + \psi_4^*$

where $\psi_4^* = \dfrac{2A\xi(\xi^2+2\alpha-\sqrt{\alpha^2+1})}{\sqrt{\alpha^2+1}((\xi^2+\alpha)^2+1)^2} + \dfrac{A\xi(\alpha\xi^2-\alpha\sqrt{\alpha^2+1} + 2(\alpha^2+1))}{2(\alpha^2+1)(\xi^2+\alpha)^2+1)} - \dfrac{3\alpha^2+1}{4(\alpha^2+1)}\psi_3^*$

Table V

Harmonic Functions Along x Axis Outside Hydrofoil ($\eta^2=\xi^2+\alpha$)

Index	$\tilde{\psi}_k$	(ϕ_k not needed) $\frac{\partial\tilde{\psi}}{\partial\alpha}$
1)	$\frac{\xi}{2\xi^2+\alpha}$	$\frac{-\xi(2\xi^2+3\alpha)}{2(2\xi^2+\alpha)^3}$
2)	ξ	$\frac{-\xi}{2(2\xi^2+\alpha)}$
3)	$\frac{x+\ell}{\ell}\left(1-\frac{A\xi}{\sqrt{\alpha^2+1}}(2\xi^2+\sqrt{\alpha^2+1}+\alpha)\right)$	
4)	$\frac{1}{2}\left(\frac{x+\ell}{\ell}\right)^2\left(1-\frac{A\xi}{\sqrt{\alpha^2+1}}(2\xi^2+\sqrt{\alpha^2+1}+\alpha)\right)-\frac{x+\ell}{\ell}\left(1+\frac{A\xi}{4(\alpha^2+1)^{3/2}}\left(2(\alpha\sqrt{\alpha^2+1}-3\alpha^2-1)\xi^2-(2\alpha^3+\sqrt{\alpha^2+1}(2\alpha^2+3))\right)\right)$	

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